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A RECIPE FOR FINDING OPEN SUBSETS OF VECTOR SPACES WITH A GOOD QUOTIENT

BY

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The present paper is a continuation of [BBŚw2] ([1]).

The ground field is assumed to be the field $\mathbb{C}$ of complex numbers. Let a reductive group G act on an algebraic variety X and let U be a G-invariant open subset of X. Recall (cf. [S] and [GIT, Chap. I, 1.10 and 1.12), that a morphism $\pi : U \to Y$, where Y is a (complex) algebraic space, is said to be a *good quotient* (*of* U *by* G) if:

1. the inverse image under π of any open affine neighbourhood in the space Y is affine and G-invariant,
2. the restriction of the quotient map to the inverse image of any affine open subset of Y is the classical quotient of an affine variety (by an action of the reductive group G).

In the general case where Y is assumed to be an algebraic space one should understand that in point 1 we consider neighbourhoods in the etale topology.

We consider only separated quotient spaces.

If $\pi : U \to Y$ is a good quotient of U by G, then the space Y is denoted by $U//G$.

Let a reductive group G act linearly on a finite-dimensional complex vector space V. The aim of this paper is to describe all open G-invariant subsets $U \subseteq V$ such that there exists a good quotient $\pi : U \to U//G$. First, notice that, if there exists a good quotient $\pi : U \to U//G$, then, for any G-saturated open subset U' of U, $\pi(U')$ is open in $U//G$ and $\pi|U' : U' \to \pi(U')$ is a good quotient. Therefore, in order to describe all open subsets U with a good quotient, it is enough to describe the family of all subsets of V which are maximal with respect to saturated inclusion in the family of all open subsets U admitting a good quotient $\pi : U \to U//G$. Such subsets will be called G-*maximal* (in V).

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In Section 1 we describe all G-maximal subsets in case where $G = T$ is an algebraic torus. In this case, these subsets can be described by means of some families of polytopes (or of cones) in the real vector space spanned by the characters of T.

In Section 2 we show that T-maximal sets and their quotient spaces are toric varieties, and we describe their fans.

Next in Section 3 we show that, if T is a maximal torus in a reductive group G and U is T-maximal, then $\bigcap_{g\in G} gU$ is open, G-invariant and there exists a good quotient $\bigcap_{g\in G} U \to \bigcap_{g\in G} gU/\!/G$. Moreover, every G-maximal subset of V can be obtained in this way. In this general case, we obtain normal algebraic spaces (not necessarily algebraic varieties) as quotient spaces.

In Section 4 we study the case where the quotient space is quasi-projective. As a corollary of our results, we deduce that, if G is semisimple, then any open G-invariant subset $U \subset V$, with algebraic variety as the quotient space $U/\!/G$, is G-saturated in V. Thus V is the only G-maximal set with algebraic variety as quotient. The paper ends with Section 5 containing some examples.

We frequently use the results obtained in [BBŚw2], where the analogous questions for actions of reductive groups on projective spaces were considered.

The present paper is also related to a paper of D. Cox [C], where it is proved that any toric variety is a good quotient of a canonically defined open subset of a vector space by an action of a diagonalized group.

1. Case of a torus. Let T be a k-dimensional torus and let $X(T)$ be its character group. Let T act linearly on an n-dimensional vector space V. Then the action can be diagonalized, i.e. there exists a basis $\{\alpha_1, \ldots, \alpha_n\}$ of V such that, for every $t \in T$ and $i = 1, \ldots, n$, $t(\alpha_i) = \chi_i(t)\alpha_i$, where $\chi_i \in X(T)$. We fix such a basis. Polytopes in $X(T) \otimes \mathbb{R}$ spanned by 0 and χ_i, where $i \in J \subset \{1, \ldots, n\}$ (possibly $J = \emptyset$), will be called *affinely distinguished*.

The coordinates of a vector $v \in V$ in the basis $\{\alpha_1, \ldots, \alpha_n\}$ are denoted by $v_1, \ldots, v_n$. For any $v \in V$, let $P_{\mathrm{a}}(v)$ be the polytope in $X(T)\otimes\mathbb{R}$ spanned by 0 and all χ_i such that $v_i \neq 0$. Then $P_{\mathrm{a}}(v)$ is an affinely distinguished polytope. If P is an affinely distinguished polytope, then we define

$$V(P) = \{v \in V : P_{\mathrm{a}}(v) = P\}.$$

The closure $\overline{V(P)}$ of $V(P)$ is the T-invariant subspace of V generated by $\{\alpha_j\}_{j\in J}$, where $j \in J$ if and only if $\chi_j \in P$. It follows that $v \in \overline{V(P)}$ if and only if $P_{\mathrm{a}}(v) \subseteq P$.

For any collection Π of affinely distinguished polytopes, let $V(\Pi) = \bigcup_{P\in\Pi} V(P)$. The following lemma follows from the above:

LEMMA 1.1. *For any collection Π of affinely distinguished polytopes, the subset $V(\Pi) \subseteq V$ is T-invariant. Moreover, $V(\Pi)$ is open if and only if Π satisfies the following condition*:

(α) *if an affinely distinguished polytope P contains a polytope belonging to Π, then P also belongs to Π.*

The next lemma will also be useful:

LEMMA 1.2. *Let Π be a collection of affinely distinguished polytopes. Then Π satisfies conditions (α) and (β) if and only if Π satisfies conditions (α) and (γ), where*

(β) *if $P_1, P_2 \in \Pi$ and $P_1 \cap P_2$ is a face of P_1, then $P_1 \cap P_2 \in \Pi$,*

(γ) *if P_1, $P_2 \in \Pi$ and $P_1 \cap P_2$ is contained in a face F of P_1, then $F \in \Pi$.*

Proof. In fact, if Π satisfies (α) and (β) and, for $P_1, P_2 \in \Pi$, $P_1 \cap P_2$ is contained in a face F of P_1, then consider the polytope P_2' spanned by P_2 and F. The intersection $P_1 \cap P_2'$ equals F. But by (α), $P_2' \in \Pi$ and hence by (β), $F \in \Pi$. The converse implication is obvious. ■

DEFINITION 1.3. For any set $U \subset V$, define $A(U) \subset V$ by

$$v \in A(U) \Leftrightarrow P_{\mathrm{a}}(v) \in \{P_{\mathrm{a}}(u) : u \in U\}.$$

$A(U)$ will be called the *affine combinatorial closure* of U.

The main results of the section are the following:

THEOREM 1.4. *Let Π be a set of affinely distinguished polytopes. Then $V(\Pi)$ is open, and there exists a good quotient $V(\Pi) \to V(\Pi)//T$ if and only if Π satisfies (α) and (β).*

THEOREM 1.5. *Let U be an open T-invariant subset of V such that a good quotient $U \to U//T$ exists. Then $A(U)$ is T-invariant, open and there exists a good quotient $A(U) \to A(U)//T$. Moreover, U is T-saturated in $A(U)$.*

THEOREM 1.6. *Let W be a T-maximal subset of V. Then W is affinely combinatorially closed, i.e. there exists a collection Π of affinely distinguished polytopes such that $W = V(\Pi)$.*

EXAMPLE 1.A. Let $p \in \chi(T) \otimes \mathbb{R}$ and let $\Pi(p)$ be the collection of all affinely distinguished polytopes containing p. Then $\Pi(p)$ satisfies (α) and (β) and hence there exists a good quotient $V(\Pi(p)) \to V(\Pi(p))//T$. If $p = 0$, then $V(\Pi(p)) = V$.

We shall reduce the proofs of the above theorems concerning affine spaces to the case of projective spaces.

Consider the inclusion $\imath : V \hookrightarrow P^n = \mathrm{Proj}(\mathbb{C} \oplus V)$ defined by $\imath(v_1, \ldots, v_n) = (1, v_1, \ldots, v_n)$. We identify $v \in V$ and its image $\imath(v)$. Consider the action of T on P^n induced by the trivial action on $\mathbb{C}$ and the given action on V. Then $\imath$ is T-invariant. Notice that the action of T on P^n can be lifted to the above described action on $\mathbb{C} \oplus V$. We fix this lifting and hence we are in the setting considered in [BBŚw2]. The characters corresponding to the homogeneous coordinates are $\chi_0 = 0, \chi_1, \ldots, \chi_n$.

Using the terminology and notation introduced in [BBŚw2], we see that any affinely distinguished polytope is distinguished with respect to the action of T on P^n (i.e. is generated as a convex set by some of the characters χ_i, $i \in \{0, \ldots, n\}$) and any distinguished polytope is affinely distinguished if and only if it contains 0.

Recall that, for any $x = (x_0, \ldots, x_n) \in P^n$, $P(x) = \operatorname{conv}\{\chi_i : x_i \neq 0\}$ and therefore, for any $v \in V$, $P_{\mathrm{a}}(v) = P(\imath(v))$. For any distinguished polytope P, $U(P) = \{x \in P^n : P(x) = P\}$ and for any collection $\mathit{\Pi}$ of distinguished polytopes, $U(\mathit{\Pi}) = \bigcup_{P \in \mathit{\Pi}} U(P)$. Then it is clear that, for any affinely distinguished polytope P, $V(P) = U(P) \cap V$ and, for any collection $\mathit{\Pi}$ of affinely distinguished polytopes, $V(\mathit{\Pi}) = U(\mathit{\Pi}) \cap V$. Moreover, for any $U \subset P^n$ we can define a combinatorial closure $C(U)$ of U in the following way:

$$x \in C(U) \Leftrightarrow P(x) \in \{P(u) : u \in U\}.$$

Notice that, for any $U \subset V$, $A(U) = C(U) \cap V$.

LEMMA 1.7. *$V(\mathit{\Pi})$ is T-saturated in $U(\mathit{\Pi})$.*

P r o o f. Let $v \in V(\mathit{\Pi})$ and $w \in \overline{Tv} \cap U(\mathit{\Pi})$. Then by [BBŚw2, 2.7] there exists $v' \in Tv$ and a one-parameter subgroup $\alpha : \mathbb{C}^* \to T$ such that $w = \lim_{t \to 0} \alpha(t)v'$. Let $(\chi_i \circ \alpha)(t) = t^{n_i}$ and let $m = \min(n_i)$. Then we may assume that, for $i = 0, \ldots, n$, $w_i = v'_i$ if $n_i = m$ and $w_i = 0$ otherwise.

On the other hand, $\operatorname{conv}\{\chi_i : w_i \neq 0\} \in \mathit{\Pi}$. Thus $0 \in \operatorname{conv}\{\chi_i : w_i \neq 0\}$. It follows that $m = 0$ and $v_0 = v'_0 = w_0 = 1$. Hence $w \in U(\mathit{\Pi}) \cap V = V(\mathit{\Pi})$. ∎

Proof of Theorem 1.4. Assume that $\mathit{\Pi}$ satisfies (α) and (β). Then by Lemma 1.1, $V(\mathit{\Pi})$ is open and T-invariant. Moreover, $0 \in P$ for any $P \in \mathit{\Pi}$. Hence, according to Lemma 1.2, $\mathit{\Pi}$ satisfies condition (η) of [BBŚw2, Theorem 7.8] and thus there exists a good quotient $U(\mathit{\Pi}) \to U(\mathit{\Pi})/\!/T$. By Lemma 1.7, $V(\mathit{\Pi})$ is T-saturated in $U(\mathit{\Pi})$. Hence a good quotient $V(\mathit{\Pi}) \to V(\mathit{\Pi})/\!/T$ exists (and is an open subset of $U(\mathit{\Pi})/\!/T$).

Now, assume that there exists a good quotient $V(\mathit{\Pi}) \to V(\mathit{\Pi})/\!/T$. $U(\mathit{\Pi})$ is the combinatorial closure of $V(\mathit{\Pi})$ in P^n. Hence, by [BBŚw2, (AAA), Sec. 6], $U(\mathit{\Pi})$ is open in P^n and there exists a good quotient $U(\mathit{\Pi}) \to U(\mathit{\Pi})/\!/T$. Hence, again by [BBŚw2, Theorem 7.8], $\mathit{\Pi}$ satisfies condition (η) of that theorem and thus $\mathit{\Pi}$ satisfies conditions (α) and (β). ∎

Proof of Theorem 1.5. By [BBŚw2, (AAA), Sec. 6] there exists a good quotient $C(U) \to C(U)//T$. Once again by (AAA), U is T-saturated in $C(U)$. Therefore U is T-saturated in $A(U)$. By Lemma 1.2, $A(U)$ is T-saturated in $C(U)$. Hence there exists a good quotient $A(U) \to A(U)//T$. ∎

Proof of Theorem 1.6. Let $U \subset V$ be T-maximal. By Theorem 1.5, U is T-saturated in $A(U)$ and there exists a good quotient $A(U) \to A(U)//T$. Hence, by maximality of U, $U = A(U)$. Hence U is combinatorially closed. ∎

DEFINITION 1.8. Let Π be a collection of affinely distinguished polytopes and let $\Pi_1 \subseteq \Pi$. We say that Π_1 is *saturated* in Π if any face of a polytope $P \in \Pi_1$ which belongs to Π belongs to Π_1.

The following proposition follows easily from the above:

PROPOSITION 1.9. *Let a collection Π_1 of affinely distinguished polytopes be saturated in Π. Then $U(\Pi_1)$ is T-saturated in $U(\Pi)$.*

COROLLARY 1.10. *Let U be T-maximal. Then $U = V(\Pi)$, where Π is maximal with respect to saturated inclusion in the family of collections of affinely distinguished polytopes satisfying conditions $(\alpha), (\beta)$ (of Lemmas* 1.1 *and* 1.2).

Let P be an affinely distinguished polytope. Let $\mathrm{Cn}(P)$ denote the cone with vertex 0 generated by P. If Π is a set of affinely distinguished polytopes, then $\mathrm{Cn}(\Pi)$ will denote the set of cones $\mathrm{Cn}(P)$, where $P \in \Pi$.

DEFINITION 1.11. Any cone with vertex at 0 generated by an affinely distinguished polytope will be called *distinguished*. Let Λ be a family of distinguished cones. Define $V(\Lambda)$ to be the set of all $v \in V$ such that $P_{\mathrm{a}}(v)$ generates a cone from Λ. Then $V(\Lambda)$ is said to be *determined* (or *defined*) *by* Λ. Let Λ be a collection of affinely distinguished cones and let $\Lambda_1 \subseteq \Lambda$. We say that Λ_1 is *saturated* in Λ if any face of a cone $C \in \Lambda_1$ which belongs to Λ belongs to Λ_1.

If C is a distinguished cone, then $\Pi(C)$ denotes the family of all affinely distinguished polytopes that generate C. For a family Λ of distinguished cones, let $\Pi(\Lambda)$ be the union of all families $\Pi(C)$, where $C \in \Lambda$.

THEOREM 1.12. *Let Λ be a collection of distinguished cones. Then $V(\Lambda)$ is T-invariant. Moreover, $V(\Lambda)$ is open and there exists a good quotient $V(\Lambda) \to V(\Lambda)//T$ if and only if Λ satisfies*:

(A) *if $D \in \Lambda$ and a distinguished cone D' contains D, then $D' \in \Lambda$,*

(B) *if $D_1, D_2 \in \Lambda$ and $D_1 \cap D_2$ is a face of D_1, then $D_1 \cap D_2 \in \Lambda$.*

Proof. First notice (compare Lemma 1.2) that conditions (A) and (B) are equivalent to (A) and the following condition:

(C) if $D_1, D_2 \in \Lambda$ and $D_1 \cap D_2$ is contained in a face D_3 of D_1, then $D_3 \in \Lambda$.

Then consider the set $\Pi = \Pi(\Lambda)$ (of all affinely distinguished polytopes that generate a cone from Λ). Since Λ satisfies (A) and (C), $\Pi(\Lambda)$ satisfies (α) and (β). Moreover, $V(\Pi) = V(\Lambda)$. Thus the theorem follows from Theorem 1.4. ■

THEOREM 1.13. *Let Π be a family of affinely distinguished polytopes satisfying* (α) *and* (β). *Then* $\mathrm{Cn}(\Pi)$ *satisfies* (A) *and* (B). *Moreover*, $V(\Pi)$ *is T-saturated in* $V(\mathrm{Cn}(\Pi))$.

Proof. Obviously $\mathrm{Cn}(\Pi)$ satisfies (A), since Π satisfies (α). Now, if $C_1, C_2 \in \mathrm{Cn}(\Pi)$ and $C_1 \cap C_2$ is a face of C_1, then there exist $P_1, P_2 \in \Pi$ such that $c(P_1) = C_1$, $c(P_2) = C_2$ and $P_1 \cap P_2$ is contained in a face of P_1 generating $C_1 \cap C_2$. It follows from (γ) that the face belongs to Π. Hence $C_1 \cap C_2 \in \mathrm{Cn}(\Pi)$ and thus $\mathrm{Cn}(\Pi)$ satisfies (B).

In order to show that $V(\Pi)$ is T-saturated in $V(\mathrm{Cn}(\Pi))$, it is sufficient to show that Π is saturated (in the sense of Definition 1.8) in $\Pi(\mathrm{Cn}(\Pi))$. If a face F of $P \in \Pi$ belongs to $\Pi(\mathrm{Cn}(\Pi))$, then the face generates a cone from $\mathrm{Cn}(\Pi)$, and hence there exists $P_0 \in \Pi$ such that $\mathrm{Cn}(F) = \mathrm{Cn}(P_0)$. Then $P_0 \cap P \subseteq F$ and hence, by (γ), $F \in \Pi$ and the proof is complete. ■

COROLLARY 1.14. *Let U be a T-maximal subset of V. Then there exists a collection Λ of distinguished cones such that $U = V(\Lambda)$. Moreover*, *Λ is maximal with respect to saturated inclusion.*

EXAMPLE 1.B. Let $p \in X(T) \otimes \mathbb{R}$ and let $\Lambda(p)$ be the collection of all cones C such that $p \in C$. Then $\Lambda(p)$ satisfies conditions (A) and (B). If $p \in P_0 = \mathrm{conv}(\{0\} \cup \{\chi_i : i = 1, \ldots, n\})$, then $\Lambda(p)$ is maximal in the family of collections of affinely distinguished cones ordered by saturated inclusion and hence $V(\Lambda(p))$ is T-maximal.

2. Quotients of combinatorially closed open subsets of vector spaces.

Let, as above, T be a k-dimensional torus acting on an n-dimensional linear space V and let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of V such that, for any $t \in T$ and $i = 1, \ldots, n$, $t(\alpha_i) = \chi_i(t) \cdot \alpha_i$, where $\chi_i \in X(T)$. Moreover, assume that the action of T is effective. Let $S \cong (\mathbb{C}^*)^n$ be a maximal torus of $Gl(n)$ acting diagonally in the basis $\{\alpha_1, \ldots, \alpha_n\}$, i.e. for $(s_1, \ldots, s_n) \in S$, let

$$(s_1, \ldots, s_n)(v_1, \ldots, v_n) = (s_1 v_1, \ldots, s_n v_n).$$

Then V is a toric variety with respect to the action of S and the given action of T is induced by the action of S, where T is embedded in S by $t \mapsto (\chi_1(t), \ldots, \chi_n(t))$ for $t \in T$. Let $x_0 = (1, \ldots, 1)$ and consider the torus

S embedded in V by $s \mapsto s \cdot x_0$. Consider the projective space P^n as a toric variety with respect to the action of S defined by

$$(s_1, \ldots, s_n)(x_0, \ldots, x_n) = (x_0, s_1 x_1, \ldots, s_n x_n).$$

Then V is a toric subvariety of P^n (with respect to the action of S). It was noticed in [BBŚw2] that any open, combinatorially closed subset U in P^n is an open toric subvariety in P^n. Therefore, for any collection Π of affinely distinguished polytopes such that $U(\Pi)$ is open, $V(\Pi) = V \cap U(\Pi)$ is a toric variety. If a good quotient $V(\Pi) \to V(\Pi)//T$ exists, then the torus S acts on the quotient space. Since S has an open orbit in $V(\Pi)$, it also has an open orbit in $V(\Pi)//T$. Since $V(\Pi)//T$ is a normal algebraic variety, it is a toric variety with respect to the action of some quotient of the torus S/T.

To any toric subvariety of V there corresponds a fan of strictly convex cones in the vector space $N(S) \otimes \mathbb{R} \cong \mathbb{R}^n$, where $N(S) \cong \mathbb{Z}^n$ is the group of one-parameter subgroups of S. In this section we describe the fan $\Sigma(\Pi)$ corresponding to the toric variety $V(\Pi)$. Moreover, in the case when a good quotient $V(\Pi) \to V(\Pi)//T$ exists, we describe the fan corresponding to this quotient, considered as a toric variety described as above.

Let ε_i be a one-parameter subgroup $\varepsilon_i : \mathbb{C}^* \to S \cong (\mathbb{C}^*)^n$, the embedding onto the ith coordinate. Then $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a basis of $N(S) \otimes \mathbb{R}$. For any $J \subset \{1, \ldots, n\}$, let $\sigma(J)$ be the cone (with vertex at 0) generated by ε_i with $i \notin J$, i.e.

$$\sigma(J) = \Big\{ \sum_{i \notin J} a_i \varepsilon_i : a_i \geq 0 \Big\}.$$

Moreover, let $P(J)$ denote the affinely distinguished polytope

$$P(J) = \operatorname{conv}(\{0\} \cup \{\chi_j : j \in J\}) \subset X(T) \otimes \mathbb{R}.$$

The definition of $\sigma(J)$ is, in a sense, dual to the definition of $P(J)$: $\sigma(J)$ is spanned (as a cone) by the axes with indices which *do not belong* to J, while $P(J)$ is spanned (as a polytope) by 0 and the characters with indices *belonging* to J.

For any point $v \in V$, let $J(v)$ denote the set $\{i \in I : v_i \neq 0\}$. Notice that then $P_{\mathrm{a}}(v) = P(J(v))$.

It follows from the general theory of toric varieties that to any fan in $N(S) \subset N(S) \otimes \mathbb{R}$ there corresponds an S-toric variety. This toric variety is affine if and only if the fan contains exactly one maximal cone. Moreover, to a subfan of the fan of a toric variety there corresponds a toric subvariety. In particular, to a cone $\sigma(J)$, where $J \subset \{1, \ldots, n\}$, there corresponds an open, affine toric subvariety $V(\sigma(J)) \subset V$. Then $V(\sigma(J))$ can be described as

$$V(\sigma(J)) = \{v \in \mathbb{C}^n : J \subset J(v)\}.$$

Indeed (see [Oda, Prop. 1.6]), $v \in V(\sigma(J))$ if and only if there exists $\alpha \in \sigma(J) \cap N(S)$ such that $v = \lim_{t\to 0} \alpha(t)w$, where w is a point of the open orbit, i.e. $w \in S \cdot x_0$. But, for any $w = (w_1, \ldots, w_n) \in S \cdot x_0$ (i.e. $w_i \neq 0$ for $i = 1, \ldots, n$) and $\alpha = \sum_{j\notin J} a_j \varepsilon_j$, where a_i are non-negative integers, $\lim_{t\to 0} \alpha(t)w = (v_1, \ldots, v_n)$, where $v_i = w_i$ for $i \in J$ and $v_i = 0$ otherwise. Therefore, if $v = (v_1, \ldots, v_n) \in V(\sigma(J))$, then $v_i \neq 0$ for $i \in J$, hence $J \subset J(v)$.

On the other hand, consider any point $v \in V$ such that $J \subset J(v)$. Let $s = (s_1, \ldots, s_n)$, where $s_i = v_i$ for $i \in J(v)$, $s_i = 1$ for $i \notin J(v)$, and $\alpha = \sum_{j\notin J(v)} \varepsilon_j$. Then $s \in S$, $\alpha \in \sigma(J) \cap N(S)$ and for $w = s \cdot x_0$, $v = \lim_{t\to 0} \alpha(t)w$. Therefore $v \in V(\sigma(J))$.

Recall that a collection Σ of strictly convex cones is a *fan* if the following two conditions are satisfied:

1. if $\tau \prec \sigma$ and $\sigma \in \Sigma$ then $\tau \in \Sigma$,
2. if $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2 \prec \sigma_1$,

where, for cones τ, σ, we write $\tau \prec \sigma$ if τ is a face of σ. Notice that $\sigma(J_1) \prec \sigma(J_2)$ if and only if $J_2 \subset J_1$.

In our case, all $\sigma(J)$ are cones of the fan $\Sigma_0 = \{\sigma(J) : J \subset \{1, \ldots, n\}\}$ and hence the second condition is automatically satisfied. The toric variety corresponding to a cone σ spanned by some ε_i, for $i \in \{1, \ldots, n\}$, is a toric subvariety of V and will be denoted by $V(\sigma)$. The toric variety corresponding to a fan $\Sigma \subset \Sigma_0$ will be denoted by $V(\Sigma)$. Then $V(\Sigma) = \bigcup_{\sigma\in\Sigma} V(\sigma)$.

For any collection Π of affinely distinguished polytopes we define a collection of cones by

$$\Sigma(\Pi) = \{\sigma(J) : P(J) \in \Pi\}.$$

PROPOSITION 2.2. *Let Π be a collection of affinely distinguished polytopes satisfying condition* (α) *of Lemma* 1.1. *Then $\Sigma(\Pi)$ is a fan and*

$$V(\Sigma(\Pi)) = V(\Pi).$$

Proof. Consider two cones $\sigma(J_1), \sigma(J_2)$, where $J_1, J_2 \subset \{1, \ldots, n\}$. Assume that $\sigma(J_2) \in \Sigma(\Pi)$, i.e. $P(J_2) \in \Pi$, and let $\sigma_1 \prec \sigma_2$. Then $J_2 \subset J_1$ and hence $P(J_2) \subset P(J_1)$. It follows from condition (α) that $P(J_1) \in \Pi$. Therefore $\Sigma(\Pi)$ is a fan.

Let $v \in V(\Sigma(\Pi))$. Then there exists a set J such that $v \in V(\sigma(J))$ and $P(J) \in \Pi$. It follows that $P(J) \subset P_{\mathrm{a}}(v)$ and $P(J) \in \Pi$. Since Π satisfies condition (α), we see that $P_{\mathrm{a}}(v) \in \Pi$ and therefore $v \in V(\Pi)$.

Let now $v \in V(\Pi)$. Then $P(J(v)) = P_{\mathrm{a}}(v) \in \Pi$ and hence $v \in V(\sigma(J(v)))$ and $\sigma(J(v)) \in \Sigma(\Pi)$. This proves that $v \in V(\Sigma(\Pi))$. ∎

We denote by $\Sigma_{\max}$ the collection of all maximal cones of a fan Σ. Any fan Σ is uniquely determined by its $\Sigma_{\max}$.

REMARK 2.3. Let Π be a collection of affinely distinguished polytopes satisfying condition (α) and let $J_1, \ldots, J_m$ be subsets of $\{1, \ldots, n\}$ minimal in the set of all subsets J_i with $P(J_i) \in \Pi$. Then

$$\Sigma(\Pi)_{\max} = \{\sigma(J_1), \ldots, \sigma(J_m)\}.$$

EXAMPLE 2.A.

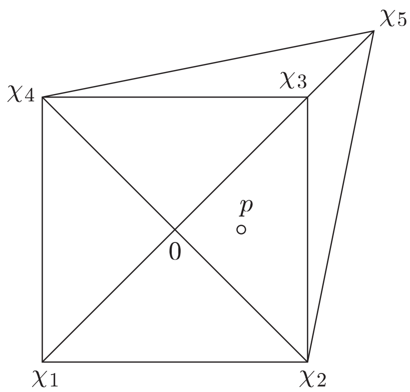

Fig. 1

Let an action of a two-dimensional torus T on $\mathbb{C}^5$ be given by the characters $\chi_1 = (-2, -2)$, $\chi_2 = (2, -2)$, $\chi_3 = (2, 2)$, $\chi_4 = (-2, 2)$, $\chi_5 = (3, 3)$ and let $p = (1, 0)$ (see Fig. 1). Let $J_1 = \{2, 3\}$ and $J_2 = \{2, 5\}$. It is easy to see that J_1, J_2 are subsets of $\{1, \ldots, 5\}$ which are minimal in the collection of subsets J_i such that $p \in P(J_i)$. It follows that $\Sigma(\Pi(p))_{\max} = \{\sigma(J_1), \sigma(J_2)\}$.

We have described the fan $\Sigma(\Pi)$ of any open subvariety $V(\Pi) \subset V$ and now, for a subtorus $T \subset S$, we shall construct a fan of the quotient variety $V(\Pi)/\!/T$ in the case when this good quotient exists.

Let Π be a collection of affinely distinguished polytopes in $\mathbb{R}^k$ such that $V(\Pi)$ is open and a good quotient $V(\Pi) \to V(\Pi)/\!/T$ exists. In order to describe the fan of the quotient variety $V(\Pi)/\!/T$, we first consider the case when S/T acts effectively on $V(\Pi)/\!/T$.

LEMMA 2.4. *Assume that, for a collection of affinely distinguished polytopes Π, $V(\Pi)$ is open and a good quotient $V(\Pi) \to V(\Pi)/\!/T$ exists. Then S/T acts effectively on $V(\Pi)/\!/T$ if and only if no proper face of the polytope*

$$P_0 = \operatorname{conv}(\{0\} \cup \{\chi_i : i = 1, \ldots, n\})$$

belongs to Π.

Proof. We tacitly use the fact that two points have the same image in the (good) quotient space if and only if the closures of their orbits intersect. Let $S \cdot x_0$ be an open orbit of V. Then $S \cdot x_0 \subset V(P_0)$. If no proper face of P_0

belongs to Π, then all T-orbits contained in $V(P_0)$ are closed in $V(\Pi)$ and, in particular, $S \cdot x_0$ is T-saturated in $V(\Pi)$. Therefore $S \cdot x_0//T \simeq S/T$ is an open orbit of S in $V(\Pi)//T$ and hence S/T acts effectively on $V(\Pi)//T$.

If a proper face F of P_0 belongs to Π, then any T-orbit contained in $S \cdot x_0$ has an orbit from $V(F)$ in its closure. Moreover, the fibres of the canonical map $S/T \to V(\Pi)$ are of dimension greater than 0 (since F is a proper face of P_0). Hence $\dim V(\Pi) < \dim S - \dim T$. It follows that the action of S/T on $V(\Pi)$ is not effective. ■

Let $f : N(S)\otimes\mathbb{R} \to N(S/T)\otimes\mathbb{R}$ be the morphism induced by the quotient morphism of the tori. Notice that $N(S/T) \otimes \mathbb{R} \simeq (N(S) \otimes \mathbb{R})/(N(T) \otimes \mathbb{R})$.

Before we state the next theorem first recall that any fan Σ is uniquely determined by the collection $\Sigma_{\max}$ of all cones maximal in Σ.

THEOREM 2.5. *Assume that* $V(\Pi)$ *is open, a good quotient* $\pi : V(\Pi) \to V(\Pi)//T$ *exists and no proper face of* $P_0 \in \Pi$ *belongs to* Π. *Then* $V(\Pi)//T$ *is a toric variety with respect to the action of* S/T *and*

$$\{f(\sigma) \in N(S/T) \otimes \mathbb{R} : \sigma \in \Sigma(\Pi)\}_{\max}$$

is the set of all maximal cones in its fan.

Proof. It follows from Lemma 2.4 that in this case S/T acts effectively on the quotient space $V(\Pi)//T$ and hence the quotient space is a toric variety with respect to the action of S/T. The quotient morphism $V(\Pi) \to V(\Pi)//T$ is then a morphism of an S-toric variety onto an S/T-toric variety consistent with the homomorphism of tori $S \to S/T$. Let Σ_1 be the fan in $N(S/T)\otimes\mathbb{R}$ corresponding to the quotient variety. By [Oda, Theorem 1.13], for every $\sigma \in \Sigma(\Pi)$, $f(\sigma)$ is a strictly convex cone and there exists a cone $\tau \in \Sigma_1$ such that $f(\sigma) \subset \tau$. Since the quotient morphism $V(\Pi) \to V(\Pi)//T$ is an affine morphism, we see that, for any open, S/T-invariant affine set $W \subset V(\Pi)//T$ corresponding to a cone $\eta \in \Sigma_1$, the set $\pi^{-1}(W)$ is an affine, open, S-invariant subset of $V(\Pi)$ and therefore it corresponds to a strictly convex cone from $\Sigma(\Pi)$, and η is the image under f of this cone. It follows that maximal cones of Σ_1 are images of maximal cones of Σ. Moreover, if σ is maximal in Π, then $f(\sigma)$ is maximal in the fan of $V(\Pi)//T$. ■

We now show that the general case can be reduced to the case described in Theorem 2.5.

For any affinely distinguished polytope P, let

$$V_P = \{v \in V : P_{\mathrm{a}}(v) \subset P\} \quad \text{and} \quad J(P) = \{i \in \{1, \ldots, n\} : \chi_i \in P\}.$$

Then $V_P = \{(v_1, \ldots, v_n) \in V : v_i = 0 \text{ for } i \notin J(P)\}$ is a linear subspace of dimension $\dim V_P = \#J(P)$. The subtorus S^P of S generated by the one-parameter subgroups ε_i, $i \notin J(P)$, acts trivially on V_P and the torus S_P defined as S/S^P acts effectively on V_P. Let $T_P = T/T \cap S^P \subset S_P$. The linear subspaces $\operatorname{lin}\{\varepsilon_i : i \in J(P)\} \subset X(S) \otimes \mathbb{R}$ and $\operatorname{lin} P \subset X(T) \otimes \mathbb{R}$ are naturally isomorphic to $X(S_P) \otimes \mathbb{R}$ and $X(T_P) \otimes \mathbb{R}$ respectively. Let $T(P)$ be the subtorus of S generated by T and S^P.

Now, let $\varPi$ be a collection of affinely distinguished polytopes. It follows from Lemma 1.2 that, in the case when a good quotient $\pi : V(\varPi) \to V(\varPi)//T$ exists, for any affinely distinguished polytope $P \in \varPi$, there is exactly one face of P of minimal dimension contained in $\varPi$.

THEOREM 2.6. *Assume that a good quotient* $\pi : V(\varPi) \to V(\varPi)//T$ *exists. Let* P_1 *be a face of* P_0 *of minimal dimension contained in* $\varPi$. *Then a good quotient* $V_{P_1}(\varPi_{P_1}) \to V_{P_1}(\varPi_{P_1})//T_{P_1}$ *exists and* $V_{P_1}(\varPi_{P_1})//T_{P_1}$ *is a toric variety with respect to the induced action of* S_{P_1}/T_{P_1}. *Moreover*, $V(\varPi)//T$ *is a toric variety with respect to the action of the torus* $S/T(P_1)$ *and there is a natural isomorphism* $V(\varPi)//T \simeq V_{P_1}(\varPi_{P_1})//T_{P_1}$ *equivariant with respect to the action of the torus* S.

Proof. Assume first that no proper face of P_0 belongs to $\varPi$. Then $P_1 = P_0$ and therefore $V_{P_1} = V$, $\varPi_{P_1} = \varPi$, $S_1 = S$ and $T(P) = T$. In this case, the theorem follows from Theorem 2.5.

Now, assume that a proper face of P_0 belongs to $\varPi$. Then $\dim P_1 < \dim P_0 = k$. A polytope P_1 is a face of $P_0 = \operatorname{conv}\{\chi_i : i \in I\})$ and hence there exists $\alpha_0 \in N(T) \simeq X(T)^*$ such that $\langle \alpha_0, \chi_i \rangle = 0$ for any $\chi_i \in P_1$ and $\langle \alpha_0, \chi_j \rangle > 0$ for all $\chi_j \notin P_1$. Moreover, we have assumed that a good quotient $\pi : V(\varPi) \to V(\varPi)//T$ exists and therefore condition (β) of Lemma 1.2 is satisfied. It follows that, for any polytope $P \in \varPi$, $P \cap P_1$ is a face of P and $P \cap P_1 \in \varPi$.

Consider any point $v = (v_1, \ldots, v_n) \in V$. It follows from the choice of α_0 that the limit $\lim_{t \to 0} \alpha_0(t)v$ exists in V and equals $(a_1, \ldots, a_n)$, where $a_i = v_i$ for $i \in J(P_1)$ and $a_i = 0$ otherwise. Then, for any $v \in V$ with $P(v) \in \varPi$, $v_0 = \lim_{t \to 0} \alpha(t)v$ exists and $P(v^0) = P(x) \cap P_1 \in \varPi$. Therefore $v^0 \in V(\varPi)$ and $\pi(v) = \pi(v^0)$. It follows that $\pi(V(\varPi)) = \pi(V_{P_1}(\varPi_{P_1}))$.

Notice that V_{P_1} is closed in V and $V_{P_1} \cap V(\varPi)$ is closed in $V(\varPi)$, hence a good quotient $V_{P_1} \cap V(\varPi) \to V_{P_1} \cap V(\varPi)//T$ exists. The torus S acts on V_{P_1} with isotropy group S^{P_1}, and T acts with isotropy group $T \cap S^{P_1}$. Consider now the collection $\varPi_{P_1}$ of distinguished polytopes in $X(T_{P_1} \otimes \mathbb{R})$ defined as

$$\varPi_{P_1} = \{P \in \varPi : P \subset P_1\}.$$

Then $V_{P_1} \cap V(\varPi) = V_{P_1}(\varPi_{P_1})$ and we can now use Theorem 2.5 for the torus S/S^{P_1} and its subtorus $T/(T \cap S^{P_1})$. ∎

EXAMPLE. Let a two-dimensional torus T act on the vector space $\mathbb{C}^5$ with characters $\chi_1 = (-2, -2)$, $\chi_2 = (2, -2)$, $\chi_3 = (2, 2)$, $\chi_4 = (-2, 2)$, $\chi_5 = (3, 3)$ and let $p = (1, 0)$ as in Example 2.3. Obviously no proper face of the polytope $P_0 = \operatorname{conv}\{\chi_1, \ldots, \chi_5\}$ is contained in $\varPi(p)$ and hence we can use Theorem 2.5. Then the fan of the quotient $V(\Pi(p))//T$ has maximal cones $f(\sigma(J_1)), f(\sigma(J_2))$, where $\sigma(J_1)$ is generated by ε_i for $i \neq 2,3$, σ_2 is generated by ε_i for $i \neq 2,5$, and f is the quotient morphism of vector spaces: $f : N(S)\otimes\mathbb{R} \to N(S/T)\otimes\mathbb{R} = (N(S)/N(T))\otimes\mathbb{R}$ (the submodule $N(T)$ is generated in $N(S)$ by $(-2,2,2,-2,3)$ and $(-2,-2,2,2,3)$).

We obtain a somewhat simpler picture by considering distinguished cones instead of affinely distinguished polytopes. This suffices for our purposes, since any T-maximal set is determined by a family of cones as well as by a family of polytopes (see Corollary 1.12). To describe this picture, we define, for any $J \subset \{1,\ldots,n\}$, a distinguished cone

$$\mathrm{Cn}(J) = \Big\{\sum_{j\in J} b_j \cdot \chi_j : b_j \geq 0\Big\} \subset X(T)\otimes\mathbb{R}.$$

PROPOSITION 2.7. *Let Λ be a collection of distinguished cones and assume that $V(\Lambda)$ is open. Let*

$$\Sigma(\Lambda) = \{\sigma(J) : \mathrm{Cn}(J) \in \Lambda\}.$$

Then $V(\Lambda)$ is a toric variety and $V(\Lambda) = V(\Sigma(\Lambda))$.

Proof. The open subvariety $V(\Lambda)$ is defined by a set of affinely distinguished polytopes and hence is a toric variety. Assume that $v \in V(\Sigma(\Lambda))$, i.e. there exists $J \subset \{1,\ldots,n\}$ such that $\mathrm{Cn}(J) \in \Lambda$ and $v \in V(\sigma(J))$. This is equivalent to the existence of $J \subset \{1,\ldots,n\}$ such that $\mathrm{Cn}(J) \in \Lambda$ and $J \subset J(v)$. Therefore $\mathrm{Cn}(J) \in \Lambda$ and $\mathrm{Cn}(J) \subset \mathrm{Cn}(J(v))$. Since $V(\Lambda)$ is open it follows that $\mathrm{Cn}(J(v)) \in \Lambda$ and hence $v \in V(\Lambda)$.

Assume now that $v \in V(\Lambda)$. Then $\mathrm{Cn}(J(v)) \in \Lambda$ and $v \in V(\sigma(J(v)))$ and therefore $v \in V(\Sigma(\Lambda))$. ∎

3. Case of a general reductive group. Let a linear action (representation) of G on a linear space V be given. Let T be a maximal torus of G.

THEOREM 3.1. *Let $U \subseteq V$ be a T-maximal subset of V. Then $\bigcap_{g\in G} gU$ is G-invariant and open. Moreover, there exists a good quotient*

$$\bigcap_{g\in G} gU \to \bigcap_{g\in G} gU//G.$$

Proof. Let U_1 be a T-maximal subset of P^n containing U as a T-saturated subset. Then $U_1 \cap V = U$ and hence

$$\bigcap_{g\in G} gU_1 \cap V = \bigcap_{g\in G} gU.$$

It follows from [BBŚw3, Theorem C] that $\bigcap_{g\in G} gU_1$ is open, G-invariant and there exists a good quotient

$$\bigcap_{g\in G} gU_1 \to \bigcap_{g\in G} gU_1//G.$$

Moreover (since V is affine and G is reductive), there exists a good quotient $V \to V//G$. Hence by [BBŚw4, Proposition 1.1] there exists a good quotient

$$\bigcap_{g\in G} gU \to \bigcap_{g\in G} gU//G. \ \blacksquare$$

THEOREM 3.2. *Let W be a G-maximal set in V. Then there exists a T-maximal subset U of V such that $W = \bigcap_{g\in G} gU$.*

Proof. Since there exists a good quotient $W \to W//G$, there exists (by [BBŚw3, Corollary 2.3]) a good quotient $W \to W//T$. Then W is T-saturated in a T-maximal set U in V and, by Theorem 3.1, there exists a good quotient $\bigcap_{g\in G} gU \to \bigcap_{g\in G} gU//G$. But W is G-saturated in U. In fact, in order to prove this it suffices to show (by [BBŚw1, Proposition 3.2]) that, for any $g \in G$, W is gTg^{-1}-saturated in $\bigcap_{g\in G} gU$. Since both W and $\bigcap_{g\in G} gU$ are G-invariant, it suffices to show that W is T-saturated in $\bigcap_{g\in G} gU$. But W is T-saturated in U and $W \subset \bigcap_{g\in G} gU \subset U$. Thus W is T-saturated in $\bigcap_{g\in G} gU$ and the proof is complete. $\blacksquare$

4. Quasi-projective quotients. In [BBŚw2] we gave a characterization of G-invariant open subsets U of projective space P^n with an action of a reductive group G having a quasi-projective variety as quotient $U//G$. A similar characterization is also valid in the case of an action of G on an affine space V. We first consider the case where G is a torus.

PROPOSITION 4.1. *Let U be an open subset of V such that a good quotient $U \to U//T$ exists and the quotient space $U//T$ is quasi-projective. Then there exists a point $p \in X(T)\otimes\mathbb{R}$ such that U is saturated in $V(\Pi(p))$.*

Proof. As before consider V as an open subset of projective space P^n. Then by [BBŚw2, Proposition 7.13], there exists a point $p \in X(T)\otimes\mathbb{R}$ such that U is T-saturated in $U(p) = \{x \in P^n : p \in P(x)\}$.

But $U(p)\cap V = V(\Pi(p))$. Therefore $U \subset V(\Pi(p))$ is saturated in $V(\Pi(p))$. $\blacksquare$

Recall that, for a given subset $U \subset V$, $A(U)$ and $C(U)$ denote the combinatorial closure of U in V and in P^n, respectively.

PROPOSITION 4.2. *Let U be a T-invariant subset of V such that a good quotient $U//T$ exists and is quasi-projective. Then a good quotient $A(U) \to A(U)//T$ exists and is also quasi-projective.*

P r o o f. It follows from [BBŚw2, Corollary 7.15] that $C(U)//T$ exists and is quasi-projective. But (by Lemma 1.2) $A(U)$ is T-saturated in $C(U)$. Therefore a good quotient $A(U)//T$ is an open subset of $C(U)//T$ and hence is quasi-projective. ■

COROLLARY 4.3. *Let U be a T-invariant open subset of V. Then a good quotient $U//T$ exists and is quasi-projective if and only if U is T-saturated in $V(\Pi(p))$ for some $p \in X(T) \otimes \mathbb{R}$.*

PROPOSITION 4.4. *Let $U \subset V$ be an open T-invariant variety such that a good quotient $U \to U//T$ exists. Then $U//T$ is projective if and only if there exists a point p in* $\operatorname{conv}\{0, \chi_1, \ldots, \chi_n\} \setminus \operatorname{conv}\{\chi_1, \ldots, \chi_n\}$ *such that* $U = V(\Pi(p))$.

P r o o f. As before consider U as an open, T-invariant subset of P^n. Then $C(U) \to C(U)//T$ exists and is projective. But U is T-saturated in $C(U)$, therefore $U = C(U)$. Then, by [BBŚw2, 7.13], $C(U) = \{(x_0, \ldots, x_n) \in P^n : p \in \operatorname{conv}\{\chi_j : x_j \neq 0\}\}$ for some $p \in X(T) \otimes \mathbb{R}$ (as before we assume that $\chi_0 = 0$). It follows that p satisfies, for every $x = (x_0, \ldots, x_n) \in P^n$, the following condition:

$$p \in P(x) \Rightarrow x_0 \neq 0,$$

and this proves the assertion. ■

COROLLARY 4.5. *Assume that a torus T acts on V with characters $\chi_1, \ldots, \chi_n$. There exists an open, T-invariant subset U in V with projective variety as quotient if and only if*

$$\operatorname{conv}\{0, \chi_1, \ldots, \chi_n\} \setminus \operatorname{conv}\{\chi_1, \ldots, \chi_n\} \neq \emptyset.$$

PROPOSITION 4.6. *Let G semisimple. Let U be an open G-invariant subset of V with a good quotient $\pi : U \to U//G$, where $U//G$ is an algebraic variety. Then U is G-saturated in V.*

P r o o f. Assume first that $U//T$ is quasi-projective. It follows from [GIT, 1.12] that there exists a G-linearized invertible sheaf $\mathcal{L}'$ on U such U is equal to the set $X^{\mathrm{ss}}(\mathcal{L}')$ of all semistable points of $\mathcal{L}'$. It follows from the definition of semistable points that there exist a finite family of G-invariant sections $s'_1, \ldots, s'_l \in \mathcal{L}'(U)$ such that the supports $\operatorname{Supp}(s'_i)$, $i = 1, \ldots, l$, are affine, $\bigcup_i \operatorname{Supp}(s'_i) = U$ and $\operatorname{Supp}(s'_i)$ are G-saturated in U.

Let $\mathcal{L}'$ correspond to a divisor $D' = \sum n_i X_i$ and let $\{Y_1, \ldots, Y_k\}$ be all irreducible components of $V \setminus U$ of codimension 1 in X. Now, any divisor $D = \sum n_i \overline{X}_i + \sum m_j Y_j$, where $m_j \in \mathbb{Z}^+$, determines a unique G_0-linearized (where G_0 is the connected component of the identity $e \in G$) invertible sheaf $\mathcal{L}$ on V and we may choose integers m_j so that every section s'_i extends to a G_0-invariant section s_i of $\mathcal{L}$ defined on V with the same support as s'_i

(comp. [GIT, 1.13]). Then U is G_0-saturated (and hence also G-saturated) in $X^{\mathrm{ss}}(\mathcal{L})$.

On the other hand, $\mathcal{L}$ is trivial (any line bundle over V is trivial) and admits a unique (since a connected and semisimple group has no non-trivial characters) G_0-linearization. The G_0-linearization of $\mathcal{L}$ is trivial, i.e. $\mathcal{L} \overset{G_0}{\simeq} V \times \mathbb{C}$, where the action of G_0 on $V \times \mathbb{C}$ is given by

$$g(v, c) = (gv, c)$$

for every $g \in G$, $v \in V$ and $c \in \mathbb{C}$. Hence $X^{\mathrm{ss}}(\mathcal{L}) = V$ and this completes the proof in the case where $U/\!/T$ is quasi-projective.

Now, let $\pi : U \to U/\!/T$ be a good quotient, where the quotient $U/\!/G$ is any algebraic variety. Then $U/\!/T$ can be covered by open quasi-projective subsets, say W_i, for $i = 1, \ldots, s$. It follows from the above that $\pi^{-1}(W_i)$ are G-saturated in V. Since a union of G-saturated subsets is G-saturated and $\bigcup \pi^{-1}(W_i) = U$, U is G-saturated in V. ■

COROLLARY 4.7. *Let G be semisimple. Let U be a G-invariant subset with a good quotient. If the quotient space $U/\!/G$ is an algebraic variety, then $U/\!/G$ is quasi-affine. More exactly, it is an open subset in $V/\!/G$.*

5. Examples

EXAMPLE 5.A. Let T be a one-dimensional torus acting on a linear space V. Let U be a T-maximal subset of V. Then $U = U(\Lambda)$ for a collection Λ of distinguished cones satisfying (A) and (B) (of Theorem 1.12) with vertices at 0 in $X(T) \otimes \mathbb{R} \simeq \mathbb{R}^1$. But there are only four possibilities for distinguished cones: $\{0\}$, $\mathbb{R}^1$, $\mathbb{R}^+ \cup \{0\}$ and $\mathbb{R}^- \cup \{0\}$. If the action of T admits both positive and negative weights (we have fixed an isomorphism $T \simeq \mathbb{C}^*$, hence $X(T) = \mathbb{Z}$), then all these cones are distinguished. Let us consider this case. If all these cones belong to Λ, then $U = V$. If some are not in Λ, then since Λ satisfies conditions (A) and (B), it must be that either

1. $\{0\}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ are not in Λ, or
2. $\{0\}$ and $\mathbb{R}^+$ are not in Λ, or
3. $\{0\}$ and $\mathbb{R}^-$ are not in Λ.

In the first case we obtain $U = V \setminus (V^- \cup V^+ \cup V^T)$ (where V^- (resp. V^+) is the subspace of V spanned by all vectors α_i of the basis $\{\alpha_1, \ldots, \alpha_n\}$ with non-positive (non-negative, respectively) weights χ_i). But then U is T-saturated in V, and hence U is not T-maximal. In the second case $U = V \setminus V^+$, and finally in the third case $U = V \setminus V^-$.

If the weights of the action are all non-positive or all non-negative, then as T-maximal sets we obtain only V and $V \setminus V^T$.

Example 5.B. Let T be a 2-dimensional torus. Consider an action of T on a 6-dimensional linear space determined by the configuration of characters χ_i, $i = 1, \ldots, 6$, as in Fig. 2.

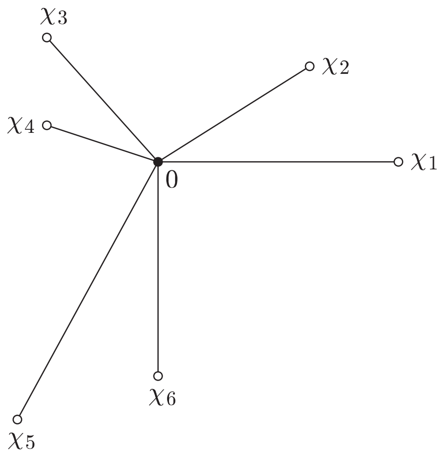

Fig. 2

Then consider the following distinguished cones (with vertices at 0) in $\chi(T) \otimes \mathbb{R} \simeq \mathbb{R}^2$:

1. C_1 spanned by χ_i, $i = 1, 4$,
2. C_2 spanned by χ_i, $i = 2, 5$,
3. C_3 spanned by χ_i, $i = 3, 6$,
4. C_4 spanned by all characters $\chi_1, \ldots, \chi_6$.

Let $\Lambda = \{C_1, C_2, C_3, C_4\}$. Then conditions (A), (B) are satisfied and hence there exists a good quotient $V(\Lambda) \to V(\Lambda)//T$. The open set $V(\Lambda)$ is not saturated in $V = V(\Lambda(0))$ and since $\bigcap_{i=1}^4 C_i = \{0\}$, $p = 0$ is the only point such that Λ is contained in $\Lambda(p)$. It follows that the quotient space $U(\Lambda)//T$ is not quasi projective (but it is an algebraic variety).

Remark 5.1. In constructing examples of open subsets $U \subset V$ with a good quotient $U \to U//T$ the following remark can be useful. Let Λ_0 be a family of distinguished cones. Let Λ be the collection of cones defined by:

$$C \in \Lambda \text{ if and only if there exists a cone } C_0 \in \Lambda_0 \text{ such that } C_0 \subset C.$$

Then Λ satisfies conditions (A) and (B) (and hence also (C)) if and only if Λ_0 satisfies condition (C).

Example 5.C. Let $G = Sl(2)$ act linearly on a vector space V. We show that V is the only $Sl(2)$-maximal set in V. By Theorem 3.2 any $Sl(2)$-maximal set in V is of the form $\bigcap_{g\in G} gU$, where U is T-maximal for a maximal torus T of $Sl(2)$. But $T \simeq \mathbb{C}^*$, $X(T)\otimes\mathbb{R} \simeq \mathbb{R}^1$ and $U = U(\Lambda)$ for a collection Λ of distinguished cones with vertices at 0 in $X(T)\otimes\mathbb{R} \simeq \mathbb{R}^1$. But there are only four such cones: $\mathbb{R}^1, \mathbb{R}^+\cup\{0\}$, $\mathbb{R}^-\cup\{0\}$ and $\{0\}$. If all belong to Λ, then $U = V$ and $\bigcap_{g\in G} gU = V$. If one of them is not in Λ, then either $\{0\}$ and $\mathbb{R}^+$ or $\{0\}$ and $\mathbb{R}^-$ are not in Λ. In both cases $\bigcap_{g\in G} gU(\Lambda)$ is the complement of the null cone of the action. Hence $\bigcap_{g\in G} gU(\Lambda)$ is G-saturated in V. Hence if $\bigcap_{g\in G} gU$ is G-maximal, then $\bigcap_{g\in G} gU = V$. This proves our claim.

Example 5.D. We show that, for $G = Sl(3)$, there exists a linear representation in a linear space V and an open $Sl(3)$-invariant subset $U \subset V$ with a good quotient $U \to U/\!/Sl(3)$ such that the quotient space $U/\!/Sl(3)$ is (an algebraic space but) not an algebraic variety.

Consider the example of Nagata [N], i.e. the action of $Sl(3)$ on the space W_5 of forms of degree 5 in three variables x, y, z induced by the natural action on the 3-dimensional space W_1 of linear forms in these variables. It is known that there exists an open $Sl(3)$-invariant open subset $U_0 \subset \mathrm{Proj}(W_5)$ with a good quotient but such that the quotient space is not an algebraic variety (see [BBŚw2, Example 9.4]). Let U be the inverse image of U_0 in W_5. Then $U_0 = U/\!/\mathbb{C}^*$, where we consider the action of $\mathbb{C}^*$ on W_5 and on U by homotheties. On the other hand, we have an action of $Sl(3)$ on U and both actions commute. Hence we have an action of $Sl(3)\times\mathbb{C}^*$ on U and we may consider the good quotients

$$U \to U/\!/\mathbb{C}^* = U_0 \to U_0/\!/Sl(3) = U/\!/Sl(3)\times\mathbb{C}^*.$$

By [BBŚw3, Corollary 2.3] there exists a good quotient $U \to U/\!/Sl(3)$. Now, $U/\!/Sl(3)$ is an algebraic space but not an algebraic variety since if it were, then its good quotient $U/\!/Sl(3) \to (U/\!/Sl(3))/\!/\mathbb{C}^*$ would have (by [BBŚw1, Corollary 1.3]) an algebraic variety as quotient space. This would contradict the fact that $(U/\!/Sl(3))/\!/\mathbb{C}^* \simeq U_0/\!/Sl(3)$ and that $U_0/\!/Sl(3)$ is not an algebraic variety.

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