COLLOQUIUM MATHEMATICUM

VOL. 77 1998 NO. 2

A COUNTEREXAMPLE TO A CONJECTURE OF BASS, CONNELL AND WRIGHT

BY

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Let $F = X - H : k^n \to k^n$ be a polynomial map with H homogeneous of degree 3 and nilpotent Jacobian matrix $J(H)$. Let $G = (G_1, \ldots, G_n)$ be the formal inverse of F. Bass, Connell and Wright proved in [1] that the homogeneous component of G_i of degree $2d + 1$ can be expressed as $G_i^{(d)} = \sum_T \alpha(T)^{-1}\sigma_i(T)$, where T varies over rooted trees with d vertices, $\alpha(T) = \operatorname{Card}\operatorname{Aut}(T)$ and $\sigma_i(T)$ is a polynomial defined by (1) below. The Jacobian Conjecture states that, in our situation, F is an automorphism or, equivalently, $G_i^{(d)}$ is zero for sufficiently large d. Bass, Connell and Wright conjecture that not only $G_i^{(d)}$ but also the polynomials $\sigma_i(T)$ are zero for large d.

The aim of the paper is to show that for the polynomial automorphism (4) and rooted trees (3), the polynomial $\sigma_2(T_s)$ is non-zero for any index s (Proposition 4), yielding a counterexample to the above conjecture (see Theorem 5).

1. Preliminaries. Throughout the paper k is a field of characteristic zero. A polynomial map from k^n to k^n is called a *polynomial automorphism* if it has an inverse that is also a polynomial map. The sequence $X = (X_1, \ldots, X_n)$ denotes the identity automorphism and $J(F)$ denotes the Jacobian matrix of F.

CONJECTURE 1 (Jacobian Conjecture). If $F = (F_1, \ldots, F_n) : k^n \to k^n$ is a polynomial map and $\det J(F) \in k \setminus \{0\}$, then F is a polynomial automorphism.

For a historical survey and detailed introduction to the subject see [1]. The Jacobian Conjecture is still open for all $n \ge 2$.

Yagzhev [4] and Bass, Connell and Wright in [1] proved that it suffices to prove the Jacobian Conjecture for all $n \ge 2$ and polynomial maps of the

1991 *Mathematics Subject Classification*: Primary 13B25; Secondary 13B10, 14E09, 05C05.

form $F_i = X_i - H_i$, where for $i = 1, \ldots, n$ the polynomial H_i is homogeneous of degree 3.

Note that if $F = X - H$, where $H_1, \ldots, H_n$ are homogeneous of degree ≥ 2, then the condition $\det J(F) \in k \setminus \{0\}$ is equivalent to the nilpotency of $J(H)$ ([1, Lemma 4.1]).

2. The tree expansion of the formal inverse. We recall some definitions and facts from [1] (see also [3]).

Let $F : k^n \to k^n$ be a polynomial map of the form $F_i = X_i - H_i$, where each H_i is homogeneous of degree $\delta \geq 2$ ($i = 1, \ldots, n$). It is well known ([1, Chapter III]) that for F there exist unique formal power series $G_1, \ldots, G_n \in k[[X_1, \ldots, X_n]]$ defined by the conditions $G_i(F_1, \ldots, F_n) = X_i$ for $i = 1, \ldots, n$. We call $G = (G_1, \ldots, G_n)$ the *formal inverse* of F.

One can write $G_i = \sum_{d \geq 0} G_i^{(d)}$, where the component $G_i^{(d)}$ is a homogeneous polynomial of degree $d(\delta - 1) + 1$.

It is obvious that the Jacobian Conjecture is true if and only if G_i is a polynomial for $i = 1, \ldots, n$.

If T is a non-directed tree, then $V(T)$ denotes its set of vertices and (the symmetric subset) $E(V) \subseteq V(T) \times V(T)$ is the set of edges. A *rooted tree* T is defined as a tree with a distinguished vertex $\mathrm{rt}_T \in V(T)$ called a *root*. We define, by induction on j, the sets $V_j(T)$ of vertices of *height* j. Let $V_0(T) = \{\mathrm{rt}_T\}$ and for $j > 0$ let $v \in V_j(T)$ iff there exists $w \in V_{j-1}(T)$ such that $(w, v) \in E(T)$ and $v \notin V_i(T)$ for $i < j$.

For $v \in V_j(T)$ we set

$$v^+ = \{w \in V_{j+1}(T) : (w, v) \in E(T)\}.$$

Rooted trees form a category in which a morphism $T \to T'$ is a map $f : V(T) \to V(T')$ such that $f(\mathrm{rt}_T) = \mathrm{rt}_{T'}$ and $(f \times f)(E(T)) \subseteq E(T')$. For a rooted tree T we denote by $\mathrm{Aut}(T)$ the group of all automorphisms of T, and $\alpha(T) = \mathrm{Card}\,\mathrm{Aut}(T)$. Moreover, $\mathbb{T}_d$ denotes the set of representatives of isomorphism classes of rooted trees with d vertices.

Suppose now that $H = (H_1, \ldots, H_n)$ and $H_1, \ldots, H_n \in k[X_1, \ldots, X_n]$ are homogeneous of degree $\delta \geq 2$. For a particular $i \in \{1, \ldots, n\}$, a rooted tree T and an *i-rooted labeling* f of T (that is, by definition, a function $f : V(T) \to \{1, \ldots, n\}$ such that $f(\mathrm{rt}_T) = i$) we define polynomials

$$P_{T,f} = \prod_{v \in V(T)} \Big(\Big(\prod_{w \in v^+} D_{f(w)} \Big) H_{f(v)} \Big)$$

and

$$\sigma_i(T) = \sum_f P_{T,f} \tag{1}$$

(f varies over all i-rooted labelings of T).

Using the above assumptions and definitions we can quote the following theorem ([1, Ch. III, Theorem 4.1], [3, Theorem 4.3]).

THEOREM 2 (Bass, Connell, Wright). *If the matrix* $J(H)$ *is nilpotent, then* $G_i^{(0)} = X_i$, *and for* $d \geq 1$,

$$G_i^{(d)} = \sum_{T \in \mathbb{T}_d} \frac{1}{\alpha(T)} \sigma_i(T). \tag{2}$$

Let $[J(H)^e]$ denote the differential ideal of $k[X_1, \ldots, X_n]$ generated by all entries of $J(H)^e$, that is, the ideal generated by elements of the form $D_1^{p_1} \ldots D_n^{p_n} f$ for any $(p_1, \ldots, p_n) \in \mathbb{N}^n$ and any entry f of $J(H)^e$.

Let us formulate the following conjecture which is the main object of interest in our paper ([1, Ch. III, Conjecture 5.1], [4, 5.2]).

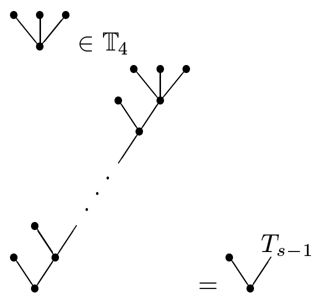
CONJECTURE 3 (Bass, Connell, Wright). If $e \geq 1$, then there is an integer $d(e)$ such that for all $d \geq d(e)$, $T \in \mathbb{T}_d$ and $i = 1, \ldots, n$ we have $\sigma_i(T) \in [J(H)^e]$.

If Conjecture 3 is true for $\delta = 3$, then the Jacobian Conjecture is also true. Indeed, if $F = X - H : k^n \to k^n$, $\det J(H) = 1$ and H_i are homogeneous of degree 3, then the matrix $J(H)$ is nilpotent. Hence $J(H)^n = 0$ and, by Conjecture 3, for all $T \in \mathbb{T}_d$, $d \geq d(n)$ and $i = 1, \ldots, n$, we have $\sigma_i(T) = 0$. Substituting this into (2) we get $G_i^{(d)} = 0$ for $d \geq d(n)$, so G_i are polynomials and F is an automorphism.

3. A counterexample. Let us define the following sequence of rooted trees:

(3)

$T_0 =$ … $\in \mathbb{T}_4$



$T_s =$ … $=$ … $\in \mathbb{T}_{2s+4}$ for $s \geq 1$,

where always the lowest vertex is a root.

PROPOSITION 4. *For the polynomial endomorphism* $F : k^4 \to k^4$ *defined by*

$$F = (X_1 + X_4(X_1X_3 + X_2X_4),\; X_2 - X_3(X_1X_3 + X_2X_4), X_3 + X_4^3, X_4) \tag{4}$$

and rooted trees T_s, $s \geq 0$, *defined by* (3), *we have*

$$\sigma_1(T_s) = 0, \quad \sigma_2(T_s) = (-1)^{s+1} \cdot 6 \cdot X_4^{4s+7}(X_1X_3 + X_2X_4),$$
$$\sigma_3(T_s) = 0, \quad \sigma_4(T_s) = 0.$$

Proof. The endomorphism F has the form $X - H$, where

$$H_1 = -X_1X_3X_4 - X_2X_4^2, \quad H_2 = X_1X_3^2 + X_2X_3X_4, \quad H_3 = -X_4^3, \quad H_4 = 0. \tag{5}$$

We proceed by induction on s.

Let $s = 0$. Let $V(T_0) = \{\mathrm{rt}_{T_0} = 0, 1, 2, 3\}$. Then, for $i = 1, 2, 3, 4$,

$$\begin{aligned} \sigma_i(T_0) &= \sum_{\substack{f: V(T_0) \to \{1,2,3,4\} \\ f(\mathrm{rt}_{T_0}) = i}} \prod_{v \in V(T_0)} \Big(\Big(\prod_{w \in v^+} D_{f(w)} \Big) H_{f(v)} \Big) \\ &= \sum_{f:\{1,2,3\} \to \{1,2,3,4\}} D_{f(1)}D_{f(2)}D_{f(3)}H_i \cdot H_{f(1)} \cdot H_{f(2)} \cdot H_{f(3)}. \end{aligned}$$

It is obvious that $D_{a_1}D_{a_2}D_{a_3}X_{b_1}X_{b_2}X_{b_3}$ can be non-zero only if the sequences (a_1, a_2, a_3) and (b_1, b_2, b_3) have the same elements up to order. Hence, by (5), we have

$$\begin{aligned} \sigma_1(T_0) &= 6 \cdot D_1D_3D_4H_1 \cdot H_1H_3H_4 + 3 \cdot D_2D_4D_4H_1 \cdot H_2H_4^2 = 0, \\ \sigma_2(T_0) &= 3 \cdot D_1D_3D_3H_2 \cdot H_1H_3^2 + 6 \cdot D_2D_3D_4H_2 \cdot H_2H_3H_4 \\ &= -6 \cdot X_4(X_1X_3 + X_2X_4) \cdot (-X_4^3)^2 \\ &= (-1)^1 \cdot 6 \cdot X_4^7(X_1X_3 + X_2X_4), \\ \sigma_3(T_0) &= D_4D_4D_4H_3 \cdot H_4^3 = 0, \\ \sigma_4(T_0) &= 0. \end{aligned}$$

Let $s \geq 0$ and assume that the statement of the proposition holds for s. Then (it is a particular case of "tree surgery"; see [1] or [3])

$$\sigma_i(T_{s+1}) = \sum_{a=1}^{4} \Big(\sum_{j=1}^{4} D_jD_aH_i \cdot H_j \Big) \cdot \sigma_a(T_s).$$

By assumption, $\sigma_a(T_s) = 0$ for $a \neq 2$. Therefore

$$\sigma_i(T_{s+1}) = \Big(\sum_{j=1}^{4} D_jD_2H_i \cdot H_j \Big) \cdot \sigma_2(T_s)$$

and hence, by (5) and the assumption,

$$\begin{aligned}
\sigma_1(T_{s+1}) &= D_4D_2H_1\cdot H_4\cdot\sigma_2(T_s)=0,\\
\sigma_2(T_{s+1}) &= (D_3D_2H_2\cdot H_3+D_4D_2H_2\cdot H_4)\cdot\sigma_2(T_s)\\
&= X_4\cdot(-X_4^3)\cdot(-1)^{s+1}\cdot 6\cdot X_4^{4s+7}(X_1X_3+X_2X_4)\\
&= (-1)^{(s+1)+1}\cdot 6\cdot X_4^{4(s+1)+7}(X_1X_3+X_2X_4),\\
\sigma_3(T_{s+1}) &= 0,\\
\sigma_4(T_{s+1}) &= 0,
\end{aligned}$$

which completes the proof. ∎

REMARK. A. van den Essen [2] proved that the endomorphism $F:\mathbb{C}^4\to\mathbb{C}^4$ defined by (4) is a counterexample to a conjecture of Meisters.

THEOREM 5. *Conjecture* 3 *is false for* $\delta=3$ *and* $e\ge 4$.

Proof. Let F be the endomorphism defined by (4). Then $F=X-H$, where H is homogeneous of degree $\delta=3$. One can verify that F is an automorphism and its inverse is

$$F^{-1}=G=X+H+G^{(2)}+G^{(3)},$$

where

$$G^{(2)}=(X_1X_4^4,-X_4^3(2X_1X_3+X_2X_4),0,0),\qquad G^{(3)}=(0,X_1X_4^6,0,0).$$

Therefore $G^{(d)}=0$ for $d\ge 4$.

Moreover, $J(H)^3\ne 0$ and $J(H)^4=0$. Hence $[J(H)^e]=0$ for $e\ge 4$.

On the other hand, by Proposition 4, we have $\sigma_2(T_s)\ne 0$ for $s\ge 0$. Therefore $\sigma_2(T_s)\notin[J(H)^e]$ for $s\ge 0$ and $e\ge 4$.

Since $T_s\in\mathbb{T}_{2s+4}$ and $\lim_{s\to\infty}(2s+4)=\infty$, for $e\ge 4$ there is no $d(e)$ as in Conjecture 3. ∎

4. Final remarks. In [1, Proposition 5.3] it was shown that Conjecture 3 is true for $e=1$ with $d(1)=1$ and for $e=2$ with $d(2)=2$. We have proved in Theorem 5 that Conjecture 3 is false for $e\ge 4$. The case $e=3$ remains open but the author's computer calculations show that the following conjecture is plausible.

CONJECTURE 6. There is an integer $d(3)$ with the following property. If $H=(H_1,\ldots,H_n)$, the polynomials $H_1,\ldots,H_n\in k[X_1,\ldots,X_n]$ are homogeneous of degree 3, and $J(H)^3=0$, then for $d\ge d(3)$, a rooted tree $T\in\mathbb{T}_d$ and all $i=1,\ldots,n$, the polynomial $\sigma_i(T)$ equals zero.

It is evident that for $e=3$ Conjecture 3 implies Conjecture 6.

Computer calculations show that $d(3)\ge 19$.

REFERENCES

[1] H. Bass, E. H. Connell and D. Wright, *The Jacobian conjecture: Reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 287–330.

[2] A. van den Essen, *A counterexample to a conjecture of Meisters*, in: Automorphisms of Affine Spaces, Proc. Internat. Conf. on Invertible Polynomial Maps (Curaçao, 1994), Kluwer, 1995, 231–233.

[3] D. Wright, *Formal inverse expansion and the Jacobian conjecture*, J. Pure Appl. Algebra 48 (1987), 199–219.

[4] A. V. Yagzhev, *On Keller's problem*, Sibirsk. Mat. Zh. 21 (1980), no. 5, 141–150 (in Russian).

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Received 29 October 1997;
revised 9 January 1998