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THE AUSLANDER TRANSLATE OF A SHORT EXACT SEQUENCE

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1. Introduction. Let Λ be an artin algebra over a commutative artin ring R and let $\operatorname{mod}\Lambda$ be the category of finitely generated (right) Λ-modules. A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1}$$

in $\operatorname{mod}\Lambda$ induces a left exact sequence

$$0 \to \tau A \xrightarrow{(p,q)} \tau B \oplus I \xrightarrow{\binom{r}{t}} \tau C, \tag{2}$$

where τ is the Auslander translate DTr (see Section 2 for the definitions of D and Tr) and I is a direct summand of the injective envelope of τA.

The main aim of this paper is to study the circumstances in which this left exact sequence is a short exact sequence of the form

$$0 \to \tau A \xrightarrow{p} \tau B \xrightarrow{r} \tau C \to 0. \tag{3}$$

We show that the condition for the map $\binom{r}{t}$, occurring in (2), to be an epimorphism is that any map from A to a projective module factors through f. Further, the map p is a monomorphism if and only if $I = 0$, whereas r is a monomorphism if and only if $I = I(\tau A)$ where, for any module X (over any ring), $I(X)$ denotes its injective envelope.

Let l be a positive integer. We shall say that the short exact sequence (1) *belongs to the class* $\mathcal{F}_l$ if, for all indecomposable modules X with length $l(X) < l$, every map $\phi : A \to X$ factors through f. If g is irreducible, then (1) is in $\mathcal{F}_{l(A)}$ (see [2]). Let $\mathcal{X}$ be the set of isomorphism classes of indecomposable modules which are either a direct summand of the radical of a projective module or a direct summand of the socle factor of an injective module and let

$$L(\Lambda) = \max_{X\in\mathcal{X}} l(X)+1 \le \max\{l(P) : P \text{ is indecomposable projective}\} \le l(\Lambda).$$

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Our main result is the following theorem.

THEOREM 1. *If the short exact sequence* (1) *belongs to the class* $\mathcal{F}_{L(\Lambda)}$, *and a fortiori*, *if* (1) *belongs to* $\mathcal{F}_{l(\Lambda)}$, *and* A *has no projective direct summand*, *then the sequence* (1) *induces an exact sequence of the form* (3).

This result (with g irreducible) is used in [5] in the course of proving that, if Λ is an algebra over an algebraically closed field, and there is an almost split sequence of the form

$$0 \to A \to B \oplus B \oplus B' \to C \to 0$$

in which neither B nor B' is the zero module and B' is not both projective and injective, then Λ is wild. In the same paper, a class of short exact sequences which belong to $\mathcal{F}_{l(\Lambda)}$, but which do not have irreducible cokernel term, is constructed and used in another proof.

Suppose now that g is irreducible and $r = \tau g$ (see Section 4) is a monomorphism. In Section 4 we establish the remarkable fact that, in this case, A has a simple top, that $\operatorname{soc}(\operatorname{coker} \tau g) \cong \operatorname{top} A$ and that exactly one of A and $\operatorname{coker} \tau g$ is simple.

The reference [4] contains the material cited from the original references [1], [2] and [3].

2. Construction and simple consequences. Let J be the radical of Λ and denote by t the natural transformation from $\mathrm{id}_{\operatorname{mod} \Lambda}$ to $- \otimes_\Lambda (\Lambda/J)$. Suppose that

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \to 0$$

is a right exact sequence. We obtain an exact commutative diagram

$$\begin{tikzcd}
 & & X \arrow[r, "\phi"] \arrow[d, "t_X"] & Y \arrow[r, "\psi"] \arrow[d, "t_Y"] & Z \arrow[r] \arrow[d, "t_Z"] & 0 \\
0 \arrow[r] & E_\phi \arrow[r, "\mu"] & \operatorname{top} X \arrow[r, "t_\phi"] \arrow[d] & \operatorname{top} Y \arrow[r, "t_\psi"] \arrow[d] & \operatorname{top} Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mu = \ker t_\phi$. We may write

$$\operatorname{top} X = E_\phi \oplus F_\phi$$

where $F_\phi \cong \operatorname{im} t_\phi$.

It is easy to verify the following lemma.

LEMMA 2. *Let* σ *be a map from* X *to a semi-simple module* Σ. *There is a unique map* $\varrho : \operatorname{top} X \to \Sigma$ *such that* $\sigma = t_X \varrho$ *and* σ *factors through* ϕ *if and only if* $\mu\varrho = 0$.

If $X \in \operatorname{mod} \Lambda$, we write $\pi_X : P(X) \to X$ for a projective cover of X and $\iota_X : \Omega(X) \to P(X)$ for the kernel of π_X.

We can now use the notation above to obtain from the exact sequence (1) an exact commutative diagram of the form

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega(A) \arrow[r, "(\iota_1,\psi)"] \arrow[d, "(\iota_1,\iota_2)"] & P(E) \oplus \Omega(B) \arrow[r] \arrow[d] & \Omega(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(E) \oplus P(F) \arrow[r] \arrow[d] & P(E) \oplus P(F) \oplus P(C) \arrow[r] \arrow[d] & P(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{4}
$$

in which $E = E_f$, $F = F_f$, $P(A) = P(E) \oplus P(F)$ and $P(B) = P(F) \oplus P(C)$. Using similar notation to write the projective cover of $\Omega(A)$ as a direct sum, we get an exact commutative diagram of the form

$$
\begin{tikzcd}
0 \arrow[r] & P(U) \oplus P(V) \arrow[r, "\chi"] \arrow[d] & P(U) \oplus P(V) \oplus P(E) \oplus P \arrow[r] \arrow[d, "\pi_1"] & P(E) \oplus P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(E) \oplus P(F) \arrow[r] \arrow[d] & P(E) \oplus P(F) \oplus P(C) \arrow[r] \arrow[d, "\pi"] & P(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{5}
$$

in which $U = E_{(\iota_1,\psi)}$, $V = F_{(\iota_1,\psi)}$, P is a projective module, the left and right hand columns are minimal projective presentations of A and C, respectively, and the middle column is isomorphic to

$$
P(U) \oplus P(V) \oplus P(E) \oplus P \xrightarrow{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \pi_{11} & \pi_{12} \\ \mathrm{id} & 0 & 0 \\ 0 & \pi_{21} & \pi_{22} \end{pmatrix}} P(E) \oplus P(F) \oplus P(C) \xrightarrow{\begin{pmatrix} 0 \\ \pi' \\ \pi'' \end{pmatrix}} B \to 0,
\tag{6}
$$

where

$$P(V)\oplus P \xrightarrow{\left(\begin{smallmatrix}\pi_{11} & \pi_{12}\\ \pi_{21} & \pi_{22}\end{smallmatrix}\right)} P(F)\oplus P(C) \xrightarrow{\left(\begin{smallmatrix}\pi'\\ \pi''\end{smallmatrix}\right)} B \to 0$$

is a minimal projective presentation of B.

It is not hard to see that we may arrange (by using appropriate automorphisms of projectives, if necessary) that the map π_1 in diagram (5) can be written in the form given by (6) and the map χ in diagram (5) can be written in the form

$$\chi = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{pmatrix}, \tag{7}$$

where we have written $\cdot$ for a map which we do not need to know.

Note that, if A is projective, we have $P(U)\oplus P(V) = 0$. All our calculations remain valid in this case and we shall only comment when it is essential to do so.

Let $P_1 \xrightarrow{p} P \to X$ be a minimal projective presentation of a module $X \in \operatorname{mod} \Lambda$. The cokernel of the map p^* induced by the functor $^* = \hom_\Lambda(-,\Lambda)$ is called the *transpose* of X and denoted by $\operatorname{Tr} X$ (see [1]). If X is projective, then $\operatorname{Tr} X = 0$.

We apply the functor $^* = \hom_\Lambda(-,\Lambda)$ to diagram (5), and take cokernels of the columns, to obtain an exact commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^* \arrow[r] \arrow[d] & B^* \arrow[r, "f^*"] \arrow[d] & A^* \arrow[d] & \\
0 \arrow[r] & P(C)^* \arrow[r] \arrow[d] & P(E)^*\oplus P(F)^*\oplus P(C)^* \arrow[r] \arrow[d, "\pi_1^*"] & P(E)^*\oplus P(F)^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(E)^*\oplus P^* \arrow[r] \arrow[d] & P(U)^*\oplus P(V)^*\oplus P(E)^*\oplus P^* \arrow[r, "\chi^*"] \arrow[d] & P(U)^*\oplus P(V)^* \arrow[r] \arrow[d, "\left(\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right)"] & 0 \\
 & \operatorname{Tr} C \arrow[r, "(\alpha,\beta)"] \arrow[d] & P(U)^*\oplus \operatorname{Tr} B \arrow[r, "\left(\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right)"] \arrow[d] & \operatorname{Tr} A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{8}$$

The commutativity of the bottom right hand square of diagram (8) and the forms given by (6) and (7) for the maps π_1 and χ imply $\gamma = \nu_1$ in (8).

The exactness of the bottom row of (8) implies that β is an epimorphism if and only if γ is. Since the right hand column of (8) is a minimal projective presentation of $\operatorname{Tr} A$ (see [1]), it follows that β is an epimorphism if and only if $V = 0$.

Similarly, δ is an epimorphism if and only if α is. Since τC has no projective direct summand, this implies that δ is an epimorphism if and only if $U = 0$.

Application of the functor $\mathrm{D} = \hom_R(-, I(R/\operatorname{rad} R))$ to the bottom row of (8) gives the left exact sequence

$$0 \to \tau A \xrightarrow{(p,q)} \tau B \oplus I \xrightarrow{\binom{r}{t}} \tau C$$

where $I = \mathrm{D}P(U)^* \cong I(U)$, $p = \mathrm{D}\delta$, $r = \mathrm{D}\beta$, etc. Hence the above discussion establishes the following proposition.

Proposition 3. *The short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\operatorname{mod} \Lambda$ *induces a left exact sequence*

$$0 \to \tau A \xrightarrow{(p,q)} \tau B \oplus I \xrightarrow{\binom{r}{t}} \tau C,$$

where I is a direct summand of $I(\tau A)$. The map p is a monomorphism if and only if $I = 0$ and the map r is a monomorphism if and only if $I = I(\tau A)$.

Remark. The maps p, q, r and t in the exact sequence (2) depend on the initial choice of projective presentations for A, B and C. However (up to isomorphism) I does not.

3. Proof of Theorem 1. We establish first that the conditions $I = 0$ and $I = I(\tau A)$ are equivalent to the conditions (C1) and (C2), respectively, defined below.

(C1) For every simple Λ-module S, every non-zero map $s : A \to I(S)/S$ which does not factor through the natural epimorphism $I(S) \to I(S)/S$ factors through f.

(C2) For every simple Λ-module S, no non-zero map $s : A \to I(S)/S$ factors through f.

First we need the following lemma.

Lemma 4. *Suppose that there is an exact commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \Omega(A) \arrow[r, "\iota_A"] \arrow[d, "f_\Omega"] & P(A) \arrow[r, "\pi_A"] \arrow[d, "f_P"] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega \arrow[r, "\iota"] & P \arrow[r, "\pi"] & B \arrow[r] & 0
\end{tikzcd}$$

in which π_A is a projective cover and P is projective. An epimorphism $\sigma : \Omega(A) \to \Sigma$, where Σ is semi-simple, factors through f_Ω if and only if there is an exact commutative diagram of the form

$$
(9)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(A) & \xrightarrow{\iota_A} & P(A) & \xrightarrow{\pi_A} & A & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma'} & & \big\downarrow{\scriptstyle\sigma''} & & \\
0 & \longrightarrow & \Sigma & \xrightarrow{\mu} & I(\Sigma) & \xrightarrow{\nu} & I(\Sigma)/\Sigma & \longrightarrow & 0
\end{array}
$$

such that σ'' *factors through* f.

Proof. Suppose first that $\sigma = f_\Omega\lambda$ for some $\lambda : \Omega \to \Sigma$. Since ι is a monomorphism and $I(\Sigma)$ is injective, there exists a map $\lambda' : P \to I(\Sigma)$ such that $\iota\lambda' = \lambda\mu$. Then $\iota\lambda'\nu = 0$ and so there exists $\lambda'' : B \to I(\Sigma)/\Sigma$ such that $\lambda'\nu = \pi\lambda''$. Let $\sigma' = f_p\lambda'$ and $\sigma'' = f\lambda''$. Then $\iota_A\sigma' = \iota_A f_p\lambda' = f_\Omega\iota\lambda' = f_\Omega\lambda\mu = \sigma\mu$ and $\pi_A\sigma'' = \pi_A f\lambda'' = f_p\pi\lambda'' = f_p\lambda'\nu = \sigma'\nu$. Hence we have an exact commutative diagram of form (9) such that $\sigma'' = f\lambda''$.

Now suppose, conversely, that we have an exact commutative diagram of form (9) and that $\sigma'' = f\lambda''$. Then, since P is projective and ν is an epimorphism, there is a map $\lambda' : P \to I(\Sigma)$ such that $\pi\lambda'' = \lambda'\nu$. Since $\iota\lambda'\nu = \iota\pi\lambda'' = 0$, there is a map $\lambda : \Omega \to \Sigma$ such that $\lambda\mu = \iota\lambda'$. Now $f_P\lambda'\nu = f_P\pi\lambda'' = \pi_A f\lambda'' = \pi_A\sigma'' = \sigma'\nu$ and so $f_P\lambda' - \sigma' = \zeta\mu$ for some $\zeta : P(A) \to \Sigma$. Since Σ is semi-simple and $\operatorname{im}\iota_A \subseteq \operatorname{rad} P(A)$, it follows that $\iota_A\zeta = 0$. Now $f_\Omega\lambda\mu = f_\Omega\iota\lambda' = \iota_A f_P\lambda' = \iota_A\sigma' = \sigma\mu$ and so, since μ is a monomorphism, we have $f_\Omega\lambda = \sigma$ as required.

LEMMA 5. *The conditions* $I = 0$ *and* $I = I(\tau A)$ *is equivalent to the conditions* (C1) *and* (C2), *respectively.*

Proof. Let S be a simple module and suppose that there is a non-zero map $s : A \to I(S)/S$. This induces an exact commutative diagram of form (9) with $\Sigma = S$ and $\sigma'' = s$. Furthermore, $\sigma = 0$ only if $s = \sigma''$ factors through $\nu : I(S) \to I(S)/S$. Now it follows from Lemma 2 that $U = 0$ if and only if every map from $\Omega(A)$ to a simple module factors through the map (i_1, ψ) of diagram (4). Similarly, $V = 0$ if and only if no map from $\Omega(A)$ to a simple module factors through (i_1, ψ). Hence it follows from Lemma 4 that the conditions (C1) and (C2) are equivalent to the statements $U = 0$ and $V = 0$, respectively. These, in turn, are equivalent to the conditions $I = 0$ and $I = I(\tau A)$, respectively.

The map $\binom{r}{t}$ is an epimorphism if and only if the map (α, β) in the bottom line of the commutative diagram (8) is a monomorphism. By the Serpent Lemma and the construction of the top line of (8), this is the case if and only if every map from A to a projective factors through f. Now, if (1) is in $\mathcal{F}_{L(\Lambda)}$, then every map from A to the socle factor of an injective module, or to the radical of a projective module, factors through f. It follows that $I = 0$ and, if A has no projective direct summand, the map r in (3) is an epimorphism. This completes the proof of Theorem 1.

4. Irreducible cokernels. If, in the short exact sequence (1), g is irreducible, then [3, Proposition 2.2] the map r in the left exact sequence (2) is also irreducible and we shall denote it by τg, although in the case where B has a projective direct summand we shall have to be a little cautious with this notation. (Of course, τg depends on the choice of projective presentations for B and C. However, it is well defined modulo $\mathrm{rad}^2(\tau B, \tau C)$.)

We shall make frequent use of the following easily proved lemma and its dual.

LEMMA 6. *Suppose $h : K \to L$ is an irreducible monomorphism. Then* $\mathrm{coker}\, h$ *is simple if and only if $I(K) \cong I(L)$. If* $\mathrm{coker}\, h$ *is not simple, then $I(L) \cong I(K) \oplus I(\mathrm{coker}\, h)$.*

THEOREM 7. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in which g is irreducible. Suppose that τg is a monomorphism. Then A has simple top, $\mathrm{top}\, A \cong \mathrm{soc}(\mathrm{coker}\, \tau g)$ *and exactly one of A and* $\mathrm{coker}\, \tau g$ *is simple.*

Proof. We use the notation introduced in Section 2.

Since g is irreducible, A is indecomposable [3].

Since $r = \tau g$ is a monomorphism, it follows from Proposition 3 that $V = 0$. Since g is irreducible, it follows from the dual of Lemma 6 that either A is not simple and $E = 0$ or A is simple and $F = 0$.

Consider first the case in which A is not simple. Then, from diagram (8), we see that $P(\mathrm{Tr}\, C) = P^* = P(\mathrm{Tr}\, B)$ and hence $I(\tau C) = I(\tau B)$. It follows from Lemma 6 that $\mathrm{coker}\, \tau g$ is simple. Write $\mathrm{coker}\, \tau g = S$. Then the kernel of the map $\beta = \mathrm{D}(\tau g)$ is $\mathrm{D}S$. Now either A is projective and then $P(U)^* = 0 = \mathrm{Tr}\, A$, or (α, β) is a monomorphism, which implies $\ker \gamma = \ker \beta = \mathrm{D}S$. In the first case, it follows from the Serpent Lemma applied to diagram (8) that $A^* = P(F)^*$ maps onto $\mathrm{D}S = \ker(\beta)$ and so $A = P(F) = P(S)$. In the second case the right hand column of the diagram (8) induces (remember that $\nu_1 = \gamma$) the exact sequence

$$0 \to A^* \to P(F)^* \to \mathrm{D}S \to 0, \tag{10}$$

and it follows (since $E = 0$) that $P(A) = P(F) = P(S)$ and so $\mathrm{top}\, A = S$.

We now consider the case in which $A = S$ is simple. Then $P(E) = P(S)$ and so, from diagram (8), $P(\mathrm{Tr}\, C) = P(\mathrm{D}S) \oplus P(\mathrm{Tr}\, B)$. This is equivalent to $I(\tau C) = I(\tau B) \oplus I(S)$ and it follows from Lemma 6 that $\mathrm{coker}\, \tau g$ is not simple and has socle S.

This completes the proof of the theorem.

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