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Sequential topological groups of any sequential order under CH

by

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Abstract. For any $\alpha < \omega_1$ a countable sequential topological group of sequential order α is constructed using CH.

1. Introduction. In [N] P. Nyikos investigated convergence in topological groups and discovered that some convergence properties which are different for general spaces may coincide in the class of sequential topological groups. He asked whether the sequential order of a sequential topological group can be nontrivial (i.e. between 2 and ω_1). The question was answered consistently in [S1] where a countable sequential topological group was constructed for which the sequential order is known to be between 2 and ω but is otherwise undetermined (see also [DP], [P] and [F] for examples of sequential spaces of intermediate sequential order which are "close" to topological groups). Although answering the initial question raised by Nyikos, the result left open many other interesting questions about sequential order in topological groups.

In an e-mail correspondence, R. Pierone asked whether it is possible to construct a sequential topological group of sequential order 2. In this paper we answer this question affirmatively constructing sequential topological groups of any given sequential order.

We use some techniques from [S1] to construct a sequential topology on $\mathbb{Q}$ that makes it a topological group of a given sequential order. The main idea is to construct the topology by induction "killing" witnesses of high sequential order on the way by adding new convergent sequences to the topology. As we need our group to have a nontrivial sequential order some witnesses should be left untouched however. The major technical difficulty lies in adding new sequences so that to leave the witnesses of the second

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kind intact. Therefore, we introduce subspaces of special kind as our witnesses. These subspaces turn out to have nice convergence properties. For example, their sequential order does not rise after adding a compact subspace (Lemma 3.8). This property allows one to carry out the main inductive construction in Lemmas 3.9 and 3.10.

2. Definitions and preliminary results. Let us now give some definitions. A space X is called a *k_ω-space* if there is an increasing family $\mathcal{K} = \{K_i : i \in \omega\}$ of compact subsets of X such that $U \subseteq X$ is open if and only if $U \cap K$ is relatively open for every $K \in \mathcal{K}$. If $S \subseteq X$ and $x \in X$ then we say that S *converges to* x if for any neighborhood U of x the set $S \setminus U$ is finite. Define $[A]_{\mathrm{seq}} = \{x : x \text{ is the limit point of a sequence in } A\}$. Then $[A]^0 = A$, $[A]^{\alpha+1} = [[A]^\alpha]_{\mathrm{seq}}$, $[A]^\alpha = \bigcup_{\beta<\alpha} [A]^\beta$ for limit α. If $[A]^\alpha = \overline{A}$ for any $A \subseteq X$, then we write $\mathrm{so}(X) \le \alpha$. Now $\mathrm{so}(X) = \min\{\alpha : \mathrm{so}(X) \le \alpha\}$. A space X is called *sequential* if $[A]^{\omega_1} = \overline{A}$ for any $A \subseteq X$. $\mathrm{is}(X)$ denotes the set of all isolated points of X. The space S_ω is obtained by identifying the limit points of countably many convergent sequences.

Let $\mathbb{Q}$ be the set of rationals. Let $\mathcal{K} = \{K_\alpha\}_{\alpha\in A}$ be an arbitrary family of subsets of $\mathbb{Q}$. Suppose $\vec{a} \in \mathbb{Q}^n$, $\vec{K} \in \mathcal{K}^n$, $n \in \omega \setminus \{0\}$. Let us write $\langle \vec{a}, \vec{K}\rangle = \langle (a_1, \ldots, a_n), (K_{\alpha_1}, \ldots, K_{\alpha_n})\rangle = a_1 \cdot K_{\alpha_1} + \ldots + a_n \cdot K_{\alpha_n} \subseteq \mathbb{Q}$, where $a_i \in \mathbb{Q}$. Define $\mathbb{Q}^\infty = \bigcup_{n\in\omega} \mathbb{Q}^n$, $\mathbb{Q}^0 = \{0\}$. If $K \subseteq \mathbb{Q}$, $\vec{a} \in \mathbb{Q}^n$ we set $\vec{a}\langle K\rangle = a_1 \cdot K + \ldots + a_n \cdot K$. If $\vec{a} \in \mathbb{Q}^0$ then $\vec{a}\langle K\rangle = 0$. Let $\mathbb{Q} = \{b_i : i \in \omega\}$, $b_i \ne b_j$ if $i \ne j$, $\mathbb{Q}(i) = \{b_j : j \le i\}$ and $\mathbb{Q}_k = \bigcup_{i,j\le k} (\mathbb{Q}(i))^j$. If $a \in \mathbb{Q} \setminus \{0\}$ let $n_{\mathbb{Q}}(a) = n$ provided $a = b_n$, $n_{\mathbb{Q}}(0) = \infty > k$ for any $k \in \omega$.

To analyze the intermediate topologies in the construction we will need the following lemmas which have been proved in [S1].

LEMMA 2.1. *Let $\mathcal{K}$ be a countable family of compact subsets of $\mathbb{Q}$. Then there exists a countable family $C(\mathcal{K}) \supseteq \mathcal{K}$ of compact subsets of $\mathbb{Q}$ such that*:

(1) $\{a\} \in C(\mathcal{K})$ *for any* $a \in \mathbb{Q}$,
(2) *if* $\vec{a} \in \mathbb{Q}^n$ *and* $\vec{K} \in C(\mathcal{K})^n$ *then* $\langle \vec{a}, \vec{K}\rangle \in C(\mathcal{K})$,
(3) *if* $K^1 \in C(\mathcal{K}), \ldots, K^n \in C(\mathcal{K})$ *then* $\bigcup_{i\le n} K^i \in C(\mathcal{K})$,
(4) *if* $\mathcal{K} \subseteq \mathcal{K}'$ *and* $\mathcal{K}'$ *has properties* (1)–(3) *then* $C(\mathcal{K}) \subseteq \mathcal{K}'$.

LEMMA 2.2. *If* $\mathcal{K} = \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ *and* $\mathcal{K}_\beta \subseteq \mathcal{K}_{\beta'}$ *for* $\beta \le \beta'$ *then* $C(\mathcal{K}) = \bigcup_{\beta<\alpha} C(\mathcal{K}_\beta)$.

LEMMA 2.3. *Let $\mathcal{K}$ be a countable family of compact subsets of $\mathbb{Q}$. Let us introduce a new topology on $\mathbb{Q}$ with $U \subseteq \mathbb{Q}$ open if and only if $U \cap K$ is relatively open for every $K \in C(\mathcal{K})$. Denote $\mathbb{Q}$ with this topology by $G(\mathcal{K})$. Then*:

(5) *if $\vec{a} \in \mathbb{Q}^n$ then the mapping $p : G(\mathcal{K})^n \to G(\mathcal{K})$, $p(\vec{b}) = \langle \vec{a}, \vec{b} \rangle$, is continuous*,

(6) *$G(\mathcal{K})$ is a k_ω-space.*

Lemma 2.4. *For any countable family $\mathcal{K}$ of compact subsets of $\mathbb{Q}$ and any countable family $\mathcal{U}$ of open subsets of $G(\mathcal{K})$ one can fix a topology $\tau(\mathcal{U}, \mathcal{K})$ such that*:

(7) *the mapping $p : \mathbb{Q}^n \to \mathbb{Q}$ where $p(\vec{a}) = \langle \vec{b}, \vec{a} \rangle$, $\vec{b} \in \mathbb{Q}^n$, is continuous in $\tau(\mathcal{U}, \mathcal{K})$*,

(8) *$\tau(\mathcal{U}, \mathcal{K})$ is a Hausdorff group topology with a countable base*,

(9) *$U \in \tau(\mathcal{U}, \mathcal{K})$ for any $U \in \mathcal{U}$*,

(10) *$\tau(\mathcal{U}, \mathcal{K})$ is finer than the usual topology of $\mathbb{Q}$ and coarser than the topology of $G(\mathcal{K})$, and*

(11) *if $\mathcal{U} \supseteq \tau_0(\mathcal{U}', \mathcal{K}')$ then $\tau(\mathcal{U}, \mathcal{K})$ is finer than $\tau(\mathcal{U}', \mathcal{K}')$ where $\mathcal{K}$ and $\mathcal{K}'$ are countable families of compact subsets of $\mathbb{Q}$ and $\tau_0(\mathcal{U}, \mathcal{K})$ is a fixed countable base at $0 \in \mathbb{Q}$ in $\tau(\mathcal{U}, \mathcal{K})$.*

The first part of the following lemma is a corollary of (10). The second part follows from the definition of $C(\mathcal{K})$.

Lemma 2.5. *Let $\mathcal{K}$ be a countable family of compact subsets of $\mathbb{Q}$. Then*:

(12) *if $K \in C(\mathcal{K})$ then the topology of K as a subspace of $G(\mathcal{K})$ coincides with the topology induced by $\mathbb{Q}$ or $\tau(\mathcal{U}, \mathcal{K})$ for any countable family $\mathcal{U}$ of open subsets of $G(\mathcal{K})$*,

(13) *if $K' \in C(\mathcal{K} \cup \{K\})$ then there are $K'' \in \mathcal{K}$ and $\vec{a} \in \mathbb{Q}^\infty$ such that $K' \subseteq \vec{a}\langle K \rangle + K''$.*

We will also need the following lemma which is an immediate corollary of [S2, Theorem 2.4].

Lemma 2.6. *Let G be a sequential topological group in which every compact subset is metrizable. If G is a k_ω-space then either G is metrizable or* $\mathrm{so}(G) = \omega_1$.

3. Example. The object defined below will be used as a measure of the sequential order of the subsets of the topological groups that appear in the construction.

Definition 3.1. Let C be a closed subset of X and $x \in C$. The pair (x, C) is called an *α-pair in X* for some $\alpha < \omega_1$ if either $\alpha = 0$ and $C = \{x\}$ or there exists a sequence $\langle (x_i, C_i) : i \in \omega \rangle$ such that

(14) each (x_i, C_i) is an α_i-pair in X for some $\alpha_i < \alpha$,

(15) $\sup\{\alpha_i : i \in \lambda\} + 1 = \alpha$ for any infinite $\lambda \subseteq \omega$,

(16) $C_i \cap C_j = \emptyset$ if $i \neq j$,

(17) $\langle x_i : i \in \omega \rangle$ converges to x,

(18) $\bigcup_{i\in\omega} C_i = C\setminus\{x\}$,
(19) for any infinite $\lambda\subseteq\omega$ any subset $\{d_i : i\in\lambda\}$ such that $d_i\in C_i\setminus\{x_i\}$ is a closed discrete subset of X.

For brevity such a sequence will be called a *decomposition sequence for* (x, C).

Using Definition 3.1 one can prove the following easy fact.

(20) If $\langle (x_i, C_i) : i\in\omega\rangle$ is a decomposition sequence for an α-pair (x, C) in X then for any $i\in\omega$ the set $F=\bigcup_{j\neq i} C_j\cup\{x\}$ is closed and $C_i\subseteq X\setminus F$.

The next lemma lists some elementary properties of α-pairs.

LEMMA 3.2. *Let (x, C) be an α-pair in a sequential space X for some $1\le\alpha<\omega_1$, $\langle (x_i, C_i) : i\in\omega\rangle$ be its decomposition sequence and K be a closed subset of X. Then*:

(21) $\mathrm{is}(C)=\bigcup_{i\in\omega}\mathrm{is}(C_i)$,
(22) $x\in[\mathrm{is}(C)]^\alpha$ *and* $x\notin[\mathrm{is}(C)]^\beta$ *for any* $\beta<\alpha$,
(23) *if $x\notin K$ then there is $C'\subseteq C\setminus K$ such that (x, C') is an α-pair in X and* $\mathrm{is}(C')\subseteq\mathrm{is}(C)$,
(24) *if the set $\lambda=\{i : x_i\notin K\}$ is infinite then there is $C'\subseteq (C\setminus K)\cup\{x\}$ such that (x, C') is an α-pair in X and* $\mathrm{is}(C')\subseteq\mathrm{is}(C)$.

Proof. (21) follows from (20); (22) and (23) are easily proved by induction on α. To prove (24) one can apply (23) to every (x_i, C_i) with $i\in\lambda$ to find $C_i'\subseteq C_i\setminus K$ such that (x_i, C_i') is an α_i-pair and $\mathrm{is}(C_i')\subseteq\mathrm{is}(C_i)$. Put $C'=\bigcup_{i\in\lambda} C_i'$. It is easy to check that (x, C') is an α-pair in X and $\mathrm{is}(C')\subseteq\mathrm{is}(C)$. ∎

One of the consequences of Lemma 3.2 is that if (x, C) is an α-pair then x and α are uniquely determined by C. The following lemma shows that α-pairs can be used in analyzing the sequential order in a natural way.

LEMMA 3.3. *Let X be a countable k_ω-space, $x\in X$, $E\subseteq X$, $x\in[E]^\alpha$ and $x\notin[E]^\beta$ if $\beta<\alpha$. Then there is an α-pair (x, C) such that* $\mathrm{is}(C)\subseteq E$.

Proof. Suppose that the lemma is true for all ordinals less than α. Let $\langle x_i : i\in\omega\rangle$ be a sequence converging to x such that $x_i\in[E]^{\alpha_i}$ and $x_i\notin[E]^\beta$ if $\beta<\alpha_i$ where $\alpha_i<\alpha$. Choose C_i, $i\in\omega$, so that (x_i, C_i) are α_i-pairs in X and $\mathrm{is}(C_i)\subseteq E$. Using (23) and Hausdorffness of X we may assume that

$(*)$ for any $i\in\omega$ there is an open $U_i\subseteq X$ such that $C_i\subseteq U_i$ and $U_i\cap U_j=\emptyset$ if $i\neq j$.

For each (x_i, C_i) let $\langle (x_i^j, C_i^j) : j\in\omega\rangle$ be its decomposition sequence such that (x_i^j, C_i^j) is an α_i^j-pair in X. Let $\mathcal{K}=\{K_i : i\in\omega\}$ be an increasing family of compact subsets of X determining the topology of X. Let us consider two cases.

CASE 1. *There are $K \in \mathcal{K}$ and $N \in \omega$ such that $\{x_i^j : j \in \omega\} \setminus K$ is finite for any $i \geq N$.*

It follows from (19) that for any K_i and any $k \in \omega$ there is $n_k^i \in \omega$ such that $K_i \cap (C_k^j \setminus \{x_k^j\}) = \emptyset$ for $j \geq n_k^i$. Since K is a metrizable compact space there is a sequence $\langle x_{i(n)}^{j(n)} : n \in \omega \rangle$ converging to x such that for every $n \in \omega$ we have $i(n+1) > i(n)$ and $j(n) \geq \max\{n_{i(n)}^i : i \leq i(n)\}$. Put $C'_n = C_{i(n)}^{j(n)}$, $x'_n = x_{i(n)}^{j(n)}$, $\alpha'_n = \alpha_{i(n)}^{j(n)}$, $C = \bigcup_{n \in \omega} C'_n \cup \{x\}$.

Let us show that (x, C) is an α-pair with a decomposition sequence $\langle (x'_n, C'_n) : n \in \omega \rangle$ satisfying the conclusion of the lemma. By $(*)$, the fact that $C'_n \subseteq C_{i(n)}$ and (21) one has $\mathrm{is}(C) = \bigcup_{n \in \omega} \mathrm{is}(C'_n) \subseteq E$ and by (22), $x'_n \in [\mathrm{is}(C'_n)]^{\alpha'_n}$ and $x \in [\mathrm{is}(C)]^{\sup\{\alpha'_n : n \in \lambda\}+1}$ for any infinite $\lambda \subseteq \omega$. So $\sup\{\alpha'_n : n \in \lambda\} + 1 = \alpha$ for any infinite $\lambda \subseteq \omega$ and (15) is satisfied. It is now enough to check property (19). Suppose $d_n \in C'_n \setminus \{x'_n\}$ for every $n \in \lambda$ for some infinite $\lambda \subseteq \omega$ and the set $\{d_n : n \in \lambda\}$ is not a closed discrete subset of X. Then without loss of generality we may assume that $\{d_n : n \in \lambda\} \subseteq K_i$ for some $i \in \omega$. Pick $n \in \lambda$ such that $i(n) > i$. Then $j(n) \geq n_{i(n)}^i$ and $K_i \cap (C_{i(n)}^{j(n)} \setminus \{x_{i(n)}^{j(n)}\}) = \emptyset$, which contradicts the fact that $d_n \in K_i \cap (C'_n \setminus \{x'_n\}) = K_i \cap (C_{i(n)}^{j(n)} \setminus \{x_{i(n)}^{j(n)}\})$.

CASE 2. *For any $K \in \mathcal{K}$ and any $N \in \omega$ there is $i \geq N$ such that $\{x_i^j : j \in \omega\} \setminus K$ is infinite.*

Let $\langle i(n) : n \in \omega \rangle$ be such that $i(n+1) > i(n)$ and $\{x_{i(n)}^j : j \in \omega\} \setminus K_n$ is infinite. Using (24) choose $C'_n \subseteq (C_{i(n)} \setminus K_n) \cup \{x_{i(n)}\}$ so that $(x_{i(n)}, C'_n)$ is an $\alpha_{i(n)}$-pair and $\mathrm{is}(C'_n) \subseteq \mathrm{is}(C_{i(n)})$. Put $C = \bigcup_{n \in \omega} C'_n$. Using the fact that $\{K_n : n \in \omega\}$ is an increasing family and an argument similar to that of Case 1 one can prove that (x, C) is an α-pair in X with a decomposition sequence $\langle (x_{i(n)}, C'_n) : n \in \omega \rangle$ satisfying the conclusion of the lemma. ∎

In the lemma above the restriction of being a k_ω-space cannot be dropped for there exists a regular countable sequential space X of sequential order $\omega + 1$ such that X is a quotient image of a separable metric space and there is no $C \subseteq X$, $x \in X$ such that (x, C) is an $(\omega + 1)$-pair in X.

LEMMA 3.4. *Let (x, C) be an α-pair in a sequential space X for some $\alpha < \omega_1$ and $y \in C$ be a nonisolated point of C. Then there exists a sequence $S_C(y) \subseteq C$ converging to y such that for any β-pair (y, D) in X such that $\beta \geq 1$ and $D \subseteq C$ the set $S_C(y) \cap D$ is infinite.*

P r o o f. It is enough to prove that for any $y \in C$ there is $S_C(y) \subseteq C$ converging to y such that for any infinite $D \subseteq C$ converging to y the set $S_C(y) \cap D$ is infinite. Suppose we have proved this for every β-pair (x', C') where $\beta < \alpha$ and let (x, C) be an α-pair. Let $\langle (x^i, C^i) : i \in \omega \rangle$ be a decomposition sequence for (x, C) such that each (x^i, C^i) is an α_i-pair for some $\alpha_i < \alpha$. Put $S_C(x) = \{x_i : i \in \omega\}$ and for any $y \in C^i$ where $i \in \omega$ put $S_C(y) = S_{C^i}(y)$. Let $D \subseteq C$ be an infinite set converging to some $y \in C$. It follows from (19) that we may assume that either $D \subseteq C^i$ for some $i \in \omega$ or $D \subseteq \{x_i : i \in \omega\}$. In either case it is easy to check that the set $D \cap S_C(y)$ is infinite. ■

The next lemma is used in the proof of Lemma 3.7 below.

Lemma 3.5. *Let (x, C) be an α-pair in a sequential space X for some $\alpha < \omega_1$. Let $\langle (x_i, C_i) : i \in \omega \rangle$ be such that each (x_i, C_i) is an α_i-pair for some $\alpha_i \le \alpha$, $C_i \subseteq C$, the sequence $\langle x_i : i \in \omega \rangle$ converges to some $y \in C$ and $x_i \ne x_j$ if $i \ne j$. If $\beta < \omega_1$ is such that $\{\alpha_i : i \in \lambda\} + 1 = \beta$ for any infinite $\lambda \subseteq \omega$ then for every $i \in \omega$ there is $C'_i \subseteq C_i$ such that (x_i, C'_i) is an α_i-pair in X for all but finitely many $i \in \omega$ and $\langle (x_i, C'_i) : i \in \omega \rangle$ is a decomposition sequence for the β-pair $(y, \bigcup_{i\in\omega} C'_i \cup \{y\})$.*

P r o o f. By induction on $\alpha < \omega_1$. Suppose that the lemma is proved for all $\alpha < \gamma$ and let (x, C) be a γ-pair. Let $\langle (x^i, C^i) : i \in \omega \rangle$ be a decomposition sequence for (x, C) such that each (x^i, C^i) is a β_i-pair for some $\beta_i < \gamma$. Using (20) and (23) we may assume that for any $n \in \omega$ if $x_i \in C^n$ then $C_i \subseteq C^n$. Using property (19) we may assume that either there is $n \in \omega$ such that $x_i \in C^n$ for every $i \in \omega$ or $x_i = x^{n(i)}$ for every $i \in \omega$. In the first case each $C_i \subseteq C^n$ and the induction hypothesis gives the desired result. In the second case the conclusion follows from the fact that $\langle (x^i, C^i) : i \in \omega \rangle$ is a decomposition sequence and $C_i \subseteq C^{n(i)}$. Part (15) of the definition of an α-pair is satisfied by the property of $\{\alpha_i : i \in \omega\}$. ■

The following lemma easily follows from the fact that if $\langle x_i : i \in \omega \rangle$ converges to x and $\langle y_i : i \in \omega \rangle$ converges to y in a topological group G then $\langle x_i + y_i : i \in \omega \rangle$ converges to $x + y$ in G.

Lemma 3.6. *Let G be a sequential topological group, K be a metrizable compact subset of G, and A be a convergent sequence with a limit point $x \in A$ such that $A \subseteq B + K$ for some closed $B \subseteq G$. Then there is an infinite $A' \subseteq A$ such that either $A' \cup \{x\} \subseteq b + K$ for some $b \in B$ or there is a homeomorphism $f : B' \to A' \cup \{x\}$ for some $B' \subseteq B$ such that $(f(b) - b) \in K$ for any $b \in B'$.*

Lemma 3.7. *Let G be a sequential topological group, K be a metrizable compact subset of G, (x, A) be an α-pair in G for some $2 \le \alpha < \omega_1$ and*

(y, B) be a β-pair in G for some $\beta < \omega_1$ such that $A \subseteq B + K$. Then there exist α-pairs (x, C) and (z, D) and a homeomorphism $f : D \to C$ such that:

(25) $C \subseteq A$, $D \subseteq B$,

(26) $f(z) = x$,

(27) $(f(d) - d) \in K$ *for any* $d \in D$.

Proof. Suppose that the lemma is true for all ordinals less than α. Let $\langle (x_i, A_i) : i \in \omega \rangle$ be a decomposition sequence for (x, A) where each (x_i, A_i) is an α_i-pair where $1 \le \alpha_i < \alpha$. Let us first consider the case $\alpha = 2$. Then each A_i is a converging sequence with the limit point x_i. Using Lemma 3.6 for every $i \in \omega$ choose an infinite $C_i \subseteq A_i$ and $D_i \subseteq B$ so that $x_i \in C_i$ and either D_i consists of a single point and $C_i \subseteq D_i + K$ or there is a homeomorphism $f_i : D_i \to C_i$ such that $(f_i(d) - d) \in K$ for any $d \in D_i$. Suppose the set $\lambda = \{i \in \omega : |D_i| = 1\}$ is infinite. Let $D_i = \{d_i'\}$ for every $i \in \lambda$. Then $x_i = d_i' + p_i$ for every $i \in \lambda$ where $p_i \in K$. Using the fact that K is a metrizable compact space one can choose an infinite $\lambda' \subseteq \lambda$ so that $\langle p_i : i \in \lambda' \rangle$ converges to some $p \in K$. Then $\langle d_i' : i \in \lambda' \rangle$ converges to some $d \in B$. Since $C_i \subseteq d_i' + K$ one can choose for every $i \in \lambda'$ a point $a_i \in C_i \setminus \{x_i\}$ so that $\{a_i : i \in \lambda'\} \subseteq (\{d_i' : i \in \lambda'\} \cup \{d\}) + K$. But the set $(\{d_i' : i \in \lambda'\} \cup \{d\}) + K$ is compact while by (19) the set $\{a_i : i \in \lambda'\}$ is a closed discrete subset of X. Hence λ is finite and we may assume without loss of generality that each f_i is a homeomorphism.

For $i \in \omega$ let $C_i \subseteq A_i$, $D_i \subseteq B$, $z_i \in D_i$ and $f_i : D_i \to C_i$ be such that:

(28) (x_i, C_i) and (z_i, D_i) are α_i-pairs,

(29) $f_i(z_i) = x_i$,

(30) f_i is a homeomorphism,

(31) $(f_i(d) - d) \in K$ for any $d \in D_i$.

Such C_i, D_i, z_i and f_i have been constructed above for $\alpha = 2$. In the general case the induction hypothesis is used. Suppose that there is $z' \in B$ such that the set $\lambda = \{i \in \omega : z_i = z'\}$ is infinite. Using Lemma 3.4 choose $S(z')$ such that $S(z')$ is a sequence converging to z' and for any $i \in \lambda$ the set $D_i \cap S(z')$ is infinite. Since each f_i is a 1-1 map one can choose a point $d_i \in D_i \cap S(z')$ for every $i \in \lambda$ so that $f_i(d_i) \in C_i \setminus \{x_i\} \subseteq A_i \setminus \{x_i\}$. By (19) the set $\{f_i(d_i) : i \in \lambda\}$ is a closed discrete subset of X. But $\{f_i(d_i) : i \in \lambda\} \subseteq K + S(z')$ and the set $K + S(z')$ is compact. Hence for every $i \in \omega$ the set $\{z_j : z_j = z_i\}$ is finite.

Using the fact that K is compact and metrizable one can choose for every $i \in \omega$ a point $p_i \in K$ and $n_i \in \omega$ so that $\langle p_i : i \in \omega \rangle$ converges to some $p \in K$, $f_{n_i}(z_{n_i}) - z_{n_i} = p_i$ and $z_{n_i} \neq z_{n_j}$ if $i \neq j$. Then since $f_{n_i}(z_{n_i}) = x_{n_i}$ and $\langle x_i : i \in \omega \rangle$ converges to x, the sequence $\langle z_{n_i} : i \in \omega \rangle$ converges to some $w \in B$. Using Lemma 3.5 for every $i \in \omega$ choose $D_i' \subseteq D_{n_i}$ so that each

(z_{n_i}, D'_i) is an α_{n_i}-pair and $\langle (z_{n_i}, D'_i) : i \in \omega \rangle$ is a decomposition sequence for an α-pair $(w, \bigcup_{i\in\omega} D'_i \cup \{w\})$. Set $D = \bigcup_{i\in\omega} D'_i \cup \{w\}$ and define $f : D \to C$ as follows: $f(w) = x$; if $d \in D'_i$ then $f(d) = f_{n_i}(d)$. Now f is a 1-1 map from D onto $f(D)$. That f maps D onto $f(D)$ homeomorphically follows from the easily proved fact that if $\langle (x^i, C^i) : i \in \omega \rangle$ is a decomposition sequence of an α-pair (x', C'), $\langle (y^i, D^i) : i \in \omega \rangle$ is a decomposition sequence of a β-pair (y', D') and $g : D' \to C'$ is a 1-1 map such that $g(y^i) = x^i$ and $g|D^i$ is a homeomorphism for every $i \in \omega$, then g is a homeomorphism. Now (25)–(27) are easy to prove. ∎

As a consequence of the previous lemma and Lemma 3.3 one has the following statement.

LEMMA 3.8. *Let G be a topological group which is a countable k_ω-space, (x, C) be an α-pair in G for some $\alpha < \omega_1$, K be a compact subset of G. Then* $\mathrm{so}(C + K) = \alpha$.

The lemma above says that the sequential order of an α-pair is stable under addition of a compact subset. This is not true for arbitrary subsets of a topological group for it is easy to prove that if A is a copy of S_ω in G and K is a nontrivial compact subset of G where G is a sequential topological group then $\mathrm{so}(A + K) \geq 2$.

The stability of the sequential order of an α-pair is used in the next lemma to add a new converging sequence to a k_ω-topology without touching the existing α-pairs.

LEMMA 3.9. *Let $\mathcal{K}$ be a countable family of compact subsets of $\mathbb{Q}$, $\mathcal{U}$ be a countable family of open subsets of $G(\mathcal{K})$, $\langle (0, C_i) : i \in \omega \rangle$ be such that each $(0, C_i)$ is a θ_i-pair in $G(\mathcal{K})$ for some $\theta_i < \omega_1$ and* $\sup\{\theta_i : i \in \omega\} = \theta$, *and $E \subseteq G(\mathcal{K})$ be such that $0 \in [E]^\delta$ for some $\delta > \theta$ and $0 \notin [E]^\gamma$ for any $\gamma < \delta$. Then there is $S \subseteq E$ such that S converges to 0 in $\tau(\mathcal{K}, \mathcal{U})$ and for any $K \in C(\mathcal{K} \cup \{S \cup \{0\}\})$ and any $i \in \omega$ there is $P_i \in C(\mathcal{K})$ such that $K \cap C_i \subseteq P_i \cap C_i$.*

Proof. Let $C(\mathcal{K}) = \{K_i : i \in \omega\}$ and $\tau_0(\mathcal{U}, \mathcal{K}) = \{U_i : i \in \omega\}$. Put $p_{-1} = 0$ and choose $p_k \in \mathbb{Q}$ for $k \in \omega$ by induction so that

$$p_k \notin \bigcup_{\substack{n_{\mathbb{Q}}(a) \leq k \\ \vec{a} \in \mathbb{Q}_k,\, j \leq k}} \Big(C_j - \Big(\bigcup_{i \leq k} K_i + \vec{a} \langle \{p_i : i < k\} \rangle \Big) \Big) \cdot a^{-1} \tag{32}$$

$$p_k \in U_k \cap E. \tag{33}$$

The set in (32) is a union of sets of the form $(C_j + K) \cdot a^{-1}$ where K is compact in $G(\mathcal{K})$. Therefore its sequential order is less than or equal to θ by Lemma 3.8. Since each U_k is open in $G(\mathcal{K})$, it follows that $0 \in [U_k \cap E]^\delta$

and $0 \notin [U_k \cap E]^\gamma$ for any $\gamma < \delta$ where $\delta > \theta$. Hence $U_k \cap E$ is not a subset of the set in (32) and it is possible to choose p_k satisfying (32) and (33).

Put $S = \{p_k : k \in \omega\}$. Then it follows from (33) that S converges to 0 in $\tau(\mathcal{U}, \mathcal{K})$. Let $K \in C(\mathcal{K} \cup \{S \cup \{0\}\})$ and $i \in \omega$. Then by (13) we may assume that $K = \vec{a}\langle S \cup \{0\}\rangle + K_n$ for some $n \in \omega$. The following argument is very similar to that of [S1, Lemma 2.10].

One has $\vec{a} = (a_1, \ldots, a_k)$ for some $k \in \omega$ and $\vec{a} \in \mathbb{Q}_N$ for some $N \in \omega$. Put $A = \{\langle \vec{a}, \vec{b}\rangle : \vec{b} \in \{0,1\}^k\} \setminus \{0\}$ and $r = \max\{n_{\mathbb{Q}}(a) : a \in A\}$. Let $M_i = \max\{N, r, n, i\}$. Let us call a point $(i_1, \ldots, i_k) \in (\omega \cup \{-1\})^k$ *essential* if for every $m \le k$ we have $\sum_{i_\nu = i_m} a_\nu \ne 0$ or $p_{i_m} = 0$. Let $\Omega \subseteq (\omega \cup \{-1\})^k$ be the set of all essential points. Since $p_{-1} = 0$ it follows that

$$K = \bigcup_{(i_1, \ldots, i_k) \in \Omega} a_1 \cdot p_{i_1} + \ldots + a_k \cdot p_{i_k} + K_n.$$

Put

$$L_i = \bigcup_{(i_1, \ldots, i_k) \in \Omega \setminus \{j : j \le M_i\}^k} a_1 \cdot p_{i_1} + \ldots + a_k \cdot p_{i_k} + K.$$

Suppose $C_i \cap L_i \ne \emptyset$. Then there is a point $c \in C_i$ such that

$$c = a_1 \cdot p_{i_1} + \ldots + a_k \cdot p_{i_k} + f$$

where $(i_1, \ldots, i_k) \in \Omega \setminus \{j : j \le M_i\}^k$ and $f \in K_n$. Let $i_m = \max\{i_j : j \le k\}$. Then $i_m > M_i$ and one has

$$p_{i_m} = (c - (a_1 \cdot p_{j_1} + \ldots + a_k \cdot p_{j_k} + f) \cdot a^{-1}$$

where $a = \sum_{i_\nu = i_m} a_\nu \ne 0$ (because $(i_1, \ldots, i_k) \in \Omega$) and $j_\nu = i_\nu$ if $i_\nu \ne i_m$ and $j_\nu = -1$ otherwise for $\nu \le k$. So $j_\nu < i_m$ for $\nu \le k$, $n_{\mathbb{Q}}(a) \le r \le M_i < i_m$, $f \in K_n$, $n \le M_i < i_m$, $c \in C_i$, $i \le M_i < i_m$ and $\vec{a} \in \mathbb{Q}_N$, $N \le M_i < i_m$, which contradicts (32). So $C_i \cap L_i = \emptyset$.

Now $K = (\vec{a}\langle\{p_j : j \le M_i\}\rangle + K_n) \cup L_i$ so $P_i = \vec{a}\langle\{p_j : j \le M_i\}\rangle + K_n$ is as desired. ∎

Let $\{O_\alpha : \alpha \in \omega_1\}$ list all the subsets of $\mathbb{Q}$ such that $O_0 = \emptyset$ and each O_α repeats ω_1 times. Let $S_1 \subseteq \mathbb{Q}$ be a nontrivial convergent sequence and put $\mathcal{S} = \{S_1\}$. The proof of the next lemma is very similar to the proof of [S1, Lemma 2.11] with some simplifications.

Lemma 3.10 (CH). *Let $\langle (0, C_i) : i \in \omega\rangle$ be such that each $(0, C_i)$ is a θ_i-pair in $G(\mathcal{S})$ for some $\theta_i < \omega_1$ and* $\sup\{\theta_i : i \in \omega\} = \theta$. *Then for every $\alpha < \omega_1$ there exist a countable family $\mathcal{K}_\alpha$ of compact subsets of $\mathbb{Q}$ and a countable family $\mathcal{U}_\alpha$ of subsets of $\mathbb{Q}$ such that*

(34) *$\mathcal{K}_\beta \subseteq \mathcal{K}_\alpha$ if $\beta \le \alpha$, and $S_1 \in \mathcal{K}_\alpha$,*

(35) *if $0 \in [O_\alpha]^\delta$ for some $\delta > \theta$ and $0 \notin [O_\alpha]^\gamma$ for any $\gamma < \delta$ in the topology of $G(\bigcup_{\beta<\alpha} \mathcal{K}_\beta)$ then there is $S \subseteq O_\alpha$ such that S converges to 0 in $G(\mathcal{K}_\alpha)$,*

(36) *the topology of* $G(\mathcal{K}_\alpha)$ *is finer than* $\tau(\mathcal{U}_\beta, \mathcal{K}_\beta)$ *for* $\beta \le \alpha$,

(37) *if* O_α *is open in* $G(\mathcal{K}_\alpha)$ *then* $O_\alpha \in \mathcal{U}_\alpha$,

(38) *for any* $K \in C(\mathcal{K}_\alpha)$ *and any* $i \in \omega$ *there is* $P_i \in C(\mathcal{S})$ *such that* $K \cap C_i \subseteq P_i \cap C_i$.

Proof. Set $\mathcal{K}_0 = \mathcal{S}$ and $\mathcal{U}_0 = \{\emptyset\}$. Then (34)–(38) are easy to check. Suppose that $\mathcal{K}_\beta$ and $\mathcal{U}_\beta$ have already been constructed for all $\beta < \alpha$ so that they satisfy (34)–(38). Define

$$\mathcal{U} = \bigcup_{\beta<\alpha} \tau_0(\mathcal{U}_\beta, \mathcal{K}_\beta), \qquad \mathcal{K} = \bigcup_{\beta<\alpha} \mathcal{K}_\beta. \tag{39}$$

If O_α is open in $C(\mathcal{K})$ put

$$\mathcal{U}_\alpha = \mathcal{U} \cup \{O_\alpha\}. \tag{40}$$

Otherwise put $\mathcal{U}_\alpha = \mathcal{U}$. Let us prove that any $U \in \mathcal{U}_\alpha$ is open in $G(\mathcal{K})$. It is enough to show that $U \cap K$ is relatively open in $\mathbb{Q}$ for any $K \in C(\mathcal{K})$. By induction, (36), Lemma 2.2 and (34) we may assume that $U \in \tau_0(\mathcal{U}_\beta, \mathcal{K}_\beta)$ and $K \in C(\mathcal{K}_\beta)$ for some $\beta < \alpha$. Then it follows from (12) that $U \cap K$ is relatively open.

Suppose that $0 \in [O_\alpha]^\delta$ for some $\delta > \theta$ and $0 \notin [O_\alpha]^\gamma$ for any $\gamma < \delta$ in $G(\mathcal{K})$. Using Lemma 3.9 find $S \subseteq O_\alpha$ converging to 0 in $\tau(\mathcal{U}_\alpha, \mathcal{K})$ and such that for any $K \in C(\mathcal{K} \cup \{S \cup \{0\}\})$ and any $i \in \omega$ there is $P_i \in C(\mathcal{K})$ such that $K \cap C_i \subseteq P_i \cap C_i$. Put $\mathcal{K}_\alpha = \mathcal{K} \cup \{S \cup \{0\}\}$. Then (34) and (35) are satisfied. Let us show that any $U \in \mathcal{U}_\alpha$ is open in $G(\mathcal{K}_\alpha)$. Let $K \in C(\mathcal{K}_\alpha)$. By (13), $K \subseteq \vec{a}\langle S \cup \{0\}\rangle + K'$ for some $\vec{a} \in \mathbb{Q}^\infty$ and $K' \in C(\mathcal{K})$. So K is compact in $\tau(\mathcal{U}_\alpha, \mathcal{K})$ by (7) and the choice of S. Since U is open in $\tau(\mathcal{U}_\alpha, \mathcal{K})$ as has been proved above the set $K \cap U$ is relatively open in $\mathbb{Q}$. By (39) and (11) the topology $\tau(\mathcal{U}_\alpha, \mathcal{K}_\alpha)$ is finer than $\tau(\mathcal{U}_\beta, \mathcal{K}_\beta)$ for any $\beta \le \alpha$ and by (10) the topology of $G(\mathcal{K}_\alpha)$ is finer than $\tau(\mathcal{U}_\alpha, \mathcal{K}_\alpha)$. So (36) is satisfied.

Now (37) follows from (40). If $K \in C(\mathcal{K}_\alpha)$ and $i \in \omega$ then by the choice of $\mathcal{K}_\alpha$ there is $P' \in C(\mathcal{K})$ such that $P' \cap C_i \subseteq K \cap C_i$. By Lemma 2.2, $P' \in C(\mathcal{K}_\beta)$ for some $\beta < \alpha$. So by the induction hypothesis and (38) there is $P_i \in C(\mathcal{S})$ such that $P' \cap C_i \subseteq P_i \cap C_i$. Hence (38) holds. ∎

LEMMA 3.11. *For any countable family* $\mathcal{K}$ *of nontrivial compact subsets of* $\mathbb{Q}$ *and any* $\theta + 1 < \omega_1$ *there is* $C \subseteq \mathbb{Q}$ *such that* $(0, C)$ *is a* $(\theta+1)$*-pair in* $G(\mathcal{K})$.

Proof. It is enough to prove that $G(\mathcal{K})$ is not metrizable since then the conclusion will follow from Lemmas 2.6 and 3.3. Now if $G(\mathcal{K})$ were metrizable it would be homeomorphic to the space of rationals, being a countable space without isolated points. Since $\mathbb{Q}$ is not a k_ω-space [M] we get a contradiction. ∎

Let $\theta < \omega_1$. Using the lemma above fix a sequence $\langle (0, C_i) : i \in \omega \rangle$ of θ_i-pairs in $G(\mathcal{S})$ so that $\sup\{\theta_i : i \in \omega\} = \theta$.

EXAMPLE 3.12. Let $\mathcal{K} = \bigcup_{\alpha<\omega_1} C(\mathcal{K}_\alpha)$ where $\mathcal{K}_\alpha$ have been constructed in Lemma 3.10. Define a topology on $\mathbb{Q}$ as follows: $U \subseteq \mathbb{Q}$ is open if and only if $U \cap K$ is relatively open for every $K \in \mathcal{K}$. Denote this space by $\mathfrak{G}$. It follows from (36), (37), (9) and the definition of the topology of $\mathfrak{G}$ that $O \subseteq \mathfrak{G}$ is open in $\mathfrak{G}$ if and only if $O \in \tau(\mathcal{U}_\alpha, \mathcal{K}_\alpha)$ for some $\alpha < \omega_1$. So the topology of $\mathfrak{G}$ is the common refinement of the topologies $\tau(\mathcal{U}_\alpha, \mathcal{K}_\alpha)$. Since every $\tau(\mathcal{U}_\alpha, \mathcal{K}_\alpha)$ is a group topology, $\mathfrak{G}$ is a topological group. That $\mathfrak{G}$ is sequential follows from the fact that $\mathfrak{G}$ is a quotient space of the topological sum of $\mathcal{K}$ under the obvious quotient map. It easily follows from (38) that each $(0, C_i)$ is a θ_i-pair in $\mathfrak{G}$ so by (22), $\mathrm{so}(\mathfrak{G}) \geq \theta$. If $\mathrm{so}(\mathfrak{G}) > \theta$ let $E \subseteq \mathfrak{G}$ be such that $0 \in [E]^{\theta+1}$ and $0 \notin [E]^\theta$. Then it follows from countability of $\mathfrak{G}$ and the definition of the topology of $\mathfrak{G}$ that there is $\alpha < \omega_1$ such that $0 \in \overline{E}$ in the topology of $G(\bigcup_{\beta<\alpha} \mathcal{K}_\beta)$. We may also assume that $E = O_\alpha$. Since the topology of $G(\mathcal{K}_\alpha)$ is finer than the topology of $\mathfrak{G}$, we have $0 \notin [E]^\theta_{G(\mathcal{K}_\alpha)}$. Then by (35) there is $S \subseteq E$ such that S convereges to 0 in $G(\mathcal{K}_\alpha)$ and thus in a coarser topology of $\mathfrak{G}$. A contradiction.

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