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Spectrum of commutative Banach algebras and isomorphism of C^*-algebras related to locally compact groups

by

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Abstract. Let A be a semisimple commutative regular tauberian Banach algebra with spectrum Σ_A. In this paper, we study the norm spectra of elements of $\overline{\operatorname{span}}\,\Sigma_A$ and present some applications. In particular, we characterize the discreteness of Σ_A in terms of norm spectra. The algebra A is said to have property (S) if, for all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$, φ has a nonempty norm spectrum. For a locally compact group G, let $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ denote the C^*-algebra generated by left translation operators on $L^2(G)$ and G_{d} denote the discrete group G. We prove that the Fourier algebra $A(G)$ has property (S) iff the canonical trace on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ is faithful iff $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$. This provides an answer to the isomorphism problem of the two C^*-algebras and generalizes the so-called "uniqueness theorem" on the group algebra $L^1(G)$ of a locally compact abelian group G. We also prove that G_{d} is amenable iff G is amenable and the Figà-Talamanca–Herz algebra $A_p(G)$ has property (S) for all p.

1. Introduction. Let A be a semisimple commutative regular tauberian Banach algebra with spectrum Σ_A. In this paper, elements of Σ_A are considered as multiplicative functionals on A and Σ_A has the Gelfand topology induced by $\sigma(A^*, A)$. Let I be a proper closed ideal of A with the zero set $Z(I) = F$. The ideal I is said to be synthesizable if I is the largest closed ideal of A whose zero set is F. De Vito proved in [9] that synthesizable ideals of $L^1(\mathbb{R})$ are exactly the ideals of the form $I_\varphi = \{a \in L^1(\mathbb{R}) : \varphi * a = 0\}$ for some nonzero almost periodic function φ on $\mathbb{R}$. It is well known that the algebra of almost periodic functions on $\mathbb{R}$ is identified with $\overline{\operatorname{span}}\,\Sigma_{L^1(\mathbb{R})}$. To study synthesizable ideals for general algebras, Ülger defined in [32] the norm spectrum $\sigma(\varphi)$ for elements φ of $\overline{\operatorname{span}}\,\Sigma_A$ by $\sigma(\varphi) = \overline{\{\varphi \cdot a : a \in A\}} \cap \Sigma_A$, which coincides with the definition given, for instance, by Katznelson [25, p. 159] in the case where $A = L^1(\mathbb{R})$. It is also known that $\sigma(\varphi) \neq \emptyset$ for

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all $\varphi \in \overline{\operatorname{span}}\, \Sigma_A \setminus \{0\}$ when $A = L^1(G)$ for some locally compact abelian group G (the so-called "uniqueness theorem"; see the books by Benedetto [5, p. 110] and Katznelson [25, p. 163]). An algebra A with this property is said to have property (S). Among other results on the space Σ_A, under the assumption that A has property (S) plus the "separating ball property" (SBP for short; see §2), Ülger gave the following generalization of De Vito's result: the ideal I is synthesizable with a separable zero set iff $I = I_\varphi$ for some $\varphi \in \overline{\operatorname{span}}\, \Sigma_A \setminus \{0\}$ (see [32, Theorem 5.5]).

As stated in Question (h) of [32], it would therefore be important to decide when the algebra A has property (S). In particular, it is interesting to consider this problem for the Fourier algebra $A(G)$ and the Figà-Talamanca–Herz algebra $A_p(G)$ of a locally compact group G. We will provide in this paper a complete answer to the above question.

Let $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ denote the C^*-algebra generated by left translation operators on $L^2(G)$ and G_{d} denote the group G considered as a discrete group. Then $\mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ is the reduced group C^*-algebra of G_{d}. A natural question is when we have $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ (which is also written as $C_\delta^*(G) \cong C_r^*(G_{\mathrm{d}})$ in the literature). Obviously, $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) = \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ if G is discrete. It is known that $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ if G is abelian. Zeller-Meier proved in [33] that the same conclusion is true whenever G_{d} is amenable (see also [12], [3], and [4]). Recently, Bédos complemented Zeller-Meier's result by showing that G_{d} is amenable iff G is amenable and $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ (see [1, Theorem 3]). Another problem tackled in our paper is the existence of any characterization for $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ to hold. We show that this is intrinsically related to the property (S) of $A(G)$.

After the results presented in this paper were obtained, we received a preprint of Bekka, Kaniuth, Lau, and Schlichting [2], where it is showed that $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ iff G contains an open subgroup H such that H_{d} is amenable ([2, Theorem 1]). This nice result is further reformulated in terms of weak containment of unitary representations of G and in terms of inclusion of Fourier and Fourier–Stieltjes algebras of G (see [2, Theorem 2 and Theorem 2′]).

Here are some details on the organization of the paper. §2 consists of some notations and preliminaries used throughout.

We investigate in §3 some basic properties of the norm spectrum and present some applications. Assume the algebra A has the SBP. Ülger [32, Theorem 3.6] showed that the space Σ_A is discrete iff, for each a in A, the map $A \to A$ given by $x \mapsto ax$ is (weakly) compact, which is also known to be equivalent to A being an ideal in its second dual A^{**} equipped with either of the Arens multiplications (see Duncan and Husseiniun [11, Lemma 3]). We present in this section another equivalent property: the space Σ_A is discrete iff $\sigma(\varphi) = Z(I_\varphi)$ for all $\varphi \in \overline{\operatorname{span}}\, \Sigma_A$ (Theorem 3.4). The following are also obtained:

(1) The ideal I is synthesizable with a separable zero set iff $I = I_\varphi$ for some $\varphi \in \overline{\operatorname{span}}\, \Sigma_A \setminus \{0\}$ satisfying the condition "$\varphi \cdot a \neq 0$ implies $\sigma(\varphi \cdot a) \neq \emptyset$" (Theorem 3.11);

(2) If the algebra $A_p(G)$ has property (S), then either $\mathcal{M}_p^{\mathrm{d}}(\widehat{G}) = PF_p(G)$ or $\mathcal{M}_p^{\mathrm{d}}(\widehat{G}) \cap PF_p(G) = \{0\}$ (Corollary 3.15(f)), where $\mathcal{M}_p^{\mathrm{d}}(\widehat{G})$ and $PF_p(G)$ denote the norm closures of $l^1(G)$ and $L^1(G)$ in $A_p(G)^*$, respectively.

The proofs are primarily motivated by some results in Ülger [32] and our understanding of the relation between the norm spectrum and invariant mean.

§4 concerns the property (S) for $A = A(G)$ and $A = A_p(G)$. Let G be a locally compact group with unit e. Let tr be the finite trace on the C^*-algebra $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ defined by $\operatorname{tr}(\varphi) = \varphi(\{e\})$ ($\varphi \in l^1(G)$). We prove that $A(G)$ has property (S) if and only if tr is faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ if and only if $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ (Theorem 4.3). This result provides an answer to the isomorphism problem of the two C^*-algebras from a point of view different from that of [2]. As we know, when G is abelian with dual group $\widehat{G}$, then $A(G) \cong L^1(\widehat{G})$, $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong AP(\widehat{G})$ (the algebra of almost periodic functions on $\widehat{G}$), and tr is always faithful on $AP(\widehat{G})$. Therefore, Theorem 4.3 generalizes the "uniqueness theorem" mentioned above on the group algebra $L^1(G)$ of a locally compact abelian group G. We also prove that G_{d} is amenable if and only if G is amenable and $A_p(G)$ has property (S) for all p (Theorem 4.5). Our approach depends heavily on the well-developed theories of Fourier algebras, C^*-algebras, and amenability.

This paper is mainly inspired by Ülger [32]. It is a pleasure to thank Professor Ali Ülger for his encouragement and valuable suggestions and for providing a copy of [32]. The author is also indebted to Professor Anthony T. Lau and Professor Ali Ülger for their helpful comments on the early draft of the paper and for bringing to her attention the preprint of [2] and references [1, 5], respectively.

2. Preliminaries and notations. In this paper, we assume that all spaces are over the complex field $\mathbb{C}$. For a Banach space E, we denote by E^* and E_1 the Banach space dual of E and the closed unit ball of E, respectively. If $\varphi \in E^*$ and $x \in E$, the value of φ at x will be written as $\langle \varphi, x \rangle$ or $\langle x, \varphi \rangle$. We always regard E as being naturally embedded into its second dual E^{**}.

Let A be a semisimple commutative regular tauberian Banach algebra with the spectrum Σ_A. We consider each element of Σ_A as a multiplicative functional on A. The usual (Gelfand) topology of Σ_A is the relative weak* topology on Σ_A induced by $\sigma(A^*, A)$. Further, $\overline{\operatorname{span}}\,\Sigma_A$ denotes the norm-closed linear subspace of A^* spanned by Σ_A. For $a \in A$ and $f \in A^*$, we define $f \cdot a \in A^*$ by $\langle f \cdot a, b\rangle = \langle f, ab\rangle$, $b \in A$. If $f \in A^*$ and the set $\{f \cdot a : a \in A_1\}$ is relatively compact, f is said to be *almost periodic*. Note that, for all $\varphi \in \Sigma_A$ and $a \in A$, we have $\varphi \cdot a = \langle \varphi, a\rangle\varphi$. Thus every $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$ is almost periodic.

For a closed ideal I of A, $Z(I)$ denotes the zero set of I, that is, $Z(I) = \{f \in \Sigma_A : I \subseteq \ker f\}$. A proper closed ideal I of A is said to be *synthesizable* if $I = \bigcap_{f\in Z(I)} \ker f$ (see, for instance, De Vito [9] for the case $A = L^1(\mathbb{R})$ and Ülger [32] for general A). In other words, if $F = Z(I)$, then I is synthesizable iff I is the largest closed ideal of A whose zero set is F. Note that I is synthesizable if $Z(I)$ is a set of spectral synthesis in the usual sense (that is, there exists a unique closed ideal of A with zero set equal to $Z(I)$; cf. Hewitt and Ross [24, §39]). But the converse is not true (see Remark 3.9(i) below). It is well known that, even in $L^1(\mathbb{R})$, not every closed ideal is synthesizable (Malliavin's theorem). De Vito [9] proved that synthesizable ideals of $L^1(\mathbb{R})$ are exactly the ideals of the form $I_\varphi = \{a \in L^1(\mathbb{R}) : \varphi * a = 0\}$ for some nonzero almost periodic function φ on $\mathbb{R}$ (i.e. $\varphi \in \overline{\operatorname{span}}\,\Sigma_{L^1(\mathbb{R})} \setminus \{0\}$). To study synthesizable ideals for general algebras, Ülger defined in [32] the norm spectrum $\sigma(\varphi)$ for $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$, which coincides with the definition given, for instance, by Katznelson [25] for $A = L^1(\mathbb{R})$.

DEFINITION 2.1 ([32]). Let $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$. The *norm spectrum* of φ is defined by

$$\sigma(\varphi) = \overline{\{\varphi \cdot a : a \in A\}} \cap \Sigma_A.$$

Note that $\sigma(\varphi)$ is different from the usual "w^*-spectrum" of φ, which is always nonempty if φ is nonzero (see, for example, Hewitt and Ross [24, §40]). As mentioned in the introduction, $\sigma(\varphi) \neq \emptyset$ for all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$ when A is the group algebra $L^1(G)$ of a locally compact abelian group G. But this is not the case for general algebras A. Therefore, we would like to give the following.

DEFINITION 2.2. The algebra A is said to have *property* (S) if $\sigma(\varphi) \neq \emptyset$ for all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$.

In [32], Ülger introduced the concept of "*separating ball property*" (SBP for short) which played an important role in his discussion on the discreteness of Σ_A under the weak topology of A^*. The algebra A is said to have the SBP if, given any two distinct elements f and g in Σ_A, there exists an $a \in A_1$ such that $\langle f, a\rangle = 1$ and $\langle g, a\rangle = 0$. Many algebras of harmonic analysis have this property. For easy reference, we would like to quote the following results from [32] on algebras with the SBP.

LEMMA 2.3 ([32, Lemma 5.1]). *Assume A has the SBP. Then, for each $f \in \Sigma_A$, there exists $m_f \in A^{**}$ such that $\langle m_f, f\rangle = 1$ and $\langle m_f, g\rangle = 0$ for all $g \in \Sigma_A \setminus \{f\}$.*

LEMMA 2.4 ([32, Lemma 5.2 and 5.3]). *Assume A has the SBP. Let $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$, $f \in \Sigma_A$, and $a \in A$. Then*

(i) $\langle \varphi \cdot a, m_f\rangle = \langle f, a\rangle\langle \varphi, m_f\rangle$.
(ii) $f \in \sigma(\varphi)$ *iff* $\langle \varphi, m_f\rangle \neq 0$.
(iii) $\sigma(\varphi \cdot a) = \sigma(\varphi) \cap \{g \in \Sigma_A : \langle g, a\rangle \neq 0\}$.
(iv) $\sigma(\varphi)$ *is a countable subset of* Σ_A.

Throughout this paper, G denotes a locally compact group with unit e and a fixed left Haar measure λ. For any subset U of G, we denote by 1_U the characteristic function of U. The symbol $L^p(G)$ $(1 \le p \le \infty)$ has the usual meaning. The group G is said to be *amenable* if there exists $m \in L^\infty(G)^*$ such that $\|m\| = \langle m, 1_G\rangle = 1$ and $\langle m, {}_x f\rangle = \langle m, f\rangle$ for all $f \in L^\infty(G)$ and $x \in G$, where ${}_x f$ is the left translate of f by x. We denote by G_{d} the algebraic group G endowed with the discrete topology. Then G is amenable if G_{d} is amenable. All solvable groups and all compact groups are known to be amenable. However, the free group on two generators is not amenable. For more information on this subject, see Greenleaf's book [21] and the books by Pier [30] and Paterson [29].

For $1 < p < \infty$, we denote by $A_p(G)$ the Figà-Talamanca–Herz algebra of G. Elements of $A_p(G)$ can be represented, nonuniquely, as

$$(*) \qquad a = \sum_{n=1}^{\infty} v_n * \check{u}_n$$

with $u_n \in L^p(G)$, $v_n \in L^q(G)$ (where $1/p + 1/q = 1$), $\check{u}_n(x) = u_n(x^{-1})$, and $\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$. The norm of a is defined by

$$\|a\| = \inf \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q,$$

where the infimum is taken over all the possible representations of a of the form $(*)$. It is known that $A_p(G)$ is a subspace of $C_0(G)$ (the space of all continuous functions on G vanishing at infinity) and, equipped with the above norm and the pointwise multiplication, is a semisimple commutative regular tauberian Banach algebra whose spectrum is G (via Dirac measures). For $p = 2$, we have $A_p(G) = A(G)$, the Fourier algebra of G, which is isometrically isomorphic to $L^1(\widehat{G})$ for commutative G with dual group $\widehat{G}$. See Eymard [14] and Herz [23] for details on the algebras $A(G)$ and $A_p(G)$, respectively. Furthermore, $A_p(G)$ has the SBP for all $1 < p < \infty$ (see Ülger [32, Proposition 2.5]).

Let $M(G)$ denote the measure algebra of G and $M_{\mathrm{d}}(G)$ the space of discrete measures in $M(G)$. Then $M(G)$ can be considered as a subspace of $A_p(G)^*$ by

$$\langle \mu, u\rangle = \int_G u(x)\,d\mu(x), \quad u \in A_p(G),$$

with $\|\mu\|_{A_p(G)^*} \le \|\mu\|_{M(G)}$. In particular, $\langle \delta_x, u\rangle = u(x)$, $x \in G$, $u \in A_p(G)$, where δ_x denotes the point measure at x. By definition, $\mathcal{M}_p^{\mathrm{d}}(\widehat{G})$ and $PF_p(G)$ are the norm closures of $M_{\mathrm{d}}(G)$ and $L^1(G)$ in $A_p(G)^*$, respectively (see Granirer [17]). For $p=2$, we have $PF_2(G) = C_r^*(G)$, the reduced group C^*-algebra of G; and $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ is also denoted as $C_\delta^*(G)$ (see Lau [26]). Under the identification $G = \Sigma_{A_p(G)}$, we have $\overline{\operatorname{span}}\,\Sigma_{A_p(G)} = \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$. Therefore, $A = A_p(G)$ has property (S) iff $\sigma(\varphi) \ne \emptyset$ for all $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G}) \setminus \{0\}$.

3. Norm spectrum, discreteness of Σ_A, and synthesizable ideals.

Throughout this section, A will be a semisimple commutative regular tauberian Banach algebra and Σ_A the spectrum of A with the Gelfand topology. Then, for any proper closed ideal I of A, the zero set $Z(I)$ of I is nonempty. We also assume in this section that A has the SBP.

For $f \in \Sigma_A$, let $m_f \in A^{**}$ be the same as in Lemma 2.3. The following lemma is a direct consequence of Lemma 2.4(ii).

LEMMA 3.1. (i) *For* $\varphi = \sum_{f\in\Sigma_A} c^f f \in \operatorname{span}\Sigma_A$, *we have* $\sigma(\varphi) = \{f \in \Sigma_A : c^f \ne 0\}$.

(ii) *If* $\varphi_n = \sum_{f\in\Sigma_A} c_n^f f \in \operatorname{span}\Sigma_A$ *and* $\varphi_n \to \varphi \in \overline{\operatorname{span}}\,\Sigma_A$, *then* $\langle\varphi, m_f\rangle = \lim_{n\to\infty} c_n^f$ *for all* $f \in \Sigma_A$. *In particular*, $\sigma(\varphi) = \{f \in \Sigma_A : \lim_{n\to\infty} c_n^f \ne 0\}$.

For $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$, let

$$I_\varphi = \{a \in A : \varphi\cdot a = 0\}.$$

Then I_φ is a closed ideal of A. If $\varphi \ne 0$, then I_φ is a proper closed ideal of A.

LEMMA 3.2. *For any* $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$, *we have* $\sigma(\varphi) \subseteq Z(I_\varphi)$.

Proof. Suppose $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$. Let $f \in \sigma(\varphi)$. We need to show that $f \in Z(I_\varphi)$. Let $a \in I_\varphi$. Then $\varphi\cdot a = 0$ and hence $0 = \langle \varphi\cdot a, m_f\rangle = \langle f, a\rangle\langle\varphi, m_f\rangle$ (by Lemma 2.4(i)). But $\langle\varphi, m_f\rangle \ne 0$ (Lemma 2.4(ii)). It follows that $\langle f, a\rangle = 0$ for all $a \in I_\varphi$, i.e., $f \in Z(I_\varphi)$. Therefore, $\sigma(\varphi) \subseteq Z(I_\varphi)$. ∎

The following simple lemma is well known.

LEMMA 3.3. *If X is a nondiscrete locally compact Hausdorff space, then X contains a countable nonclosed subset.*

Now, we first observe the following description of the discreteness of Σ_A in terms of norm spectra.

THEOREM 3.4. *The space Σ_A is discrete if and only if* $\sigma(\varphi) = Z(I_\varphi)$ *for all* $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$.

Proof. Suppose the space Σ_A is discrete. Let $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$. By Lemma 3.2, we only need to show that $Z(I_\varphi) \subseteq \sigma(\varphi)$. Let $f \in Z(I_\varphi)$. Then $I_\varphi \subseteq \ker f$, that is, $\varphi\cdot a = 0$ implies $\langle f, a\rangle = 0$, for all $a \in A$. Since A is regular and Σ_A is discrete, there exists $a \in A$ such that $\langle f, a\rangle = 1$ and $\langle g, a\rangle = 0$ for all $g \in \Sigma_A \setminus \{f\}$. So we can now take $m_f = a$ (m_f is the same as in Lemma 2.3). Since $\varphi\cdot a \ne 0$ (otherwise, $\langle f, a\rangle = 0$), there exists $b \in A$ such that $\langle \varphi\cdot a, b\rangle \ne 0$, that is,

$$0 \ne \langle \varphi\cdot a, b\rangle = \langle \varphi\cdot b, m_f\rangle = \langle f, b\rangle\langle\varphi, m_f\rangle \quad \text{(by Lemma 2.4(i))}.$$

Thus, $\langle\varphi, m_f\rangle \ne 0$. That $f \in \sigma(\varphi)$ follows readily from Lemma 2.4(ii). Therefore, $Z(I_\varphi) \subseteq \sigma(\varphi)$ and hence $\sigma(\varphi) = Z(I_\varphi)$.

Conversely, suppose $\sigma(\varphi) = Z(I_\varphi)$ for all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$. Assume that the space Σ_A is not discrete. By Lemma 3.3, Σ_A contains a countable nonclosed subset $(f_n)_{n\ge1}$. Let $\varphi = \sum_{n=1}^\infty 2^{-n} f_n \in \overline{\operatorname{span}}\,\Sigma_A$. By Lemma 3.1, we have $\sigma(\varphi) = (f_n)_{n\ge1}$, which is not closed. But the zero set $Z(I_\varphi)$ is always closed in the space Σ_A. So $\sigma(\varphi) \ne Z(I_\varphi)$, a contradiction. Therefore, the space Σ_A is discrete. ∎

An element a in A is said to be (*weakly*) *compact* if the map $\tau_a : A \to A$, defined by $\tau_a(x) = ax$, is (weakly) compact. It is well known that each a in A is weakly compact iff A is an ideal in A^{**} equipped with either of the Arens multiplications (see Duncan and Husseiniun [11, Lemma 3]). One of important results of Ülger [32] is that the space Σ_A is discrete iff each a in A is (weakly) compact (see [32, Theorem 3.6]). Our Theorem 3.4 provides another characterization for the discreteness of Σ_A. In summary, we now have the following.

COROLLARY 3.5. *The following assertions are equivalent*:

(a) *Each a in A is compact.*
(b) *Each a in A is weakly compact.*
(c) *The space Σ_A is discrete.*
(d) $\sigma(\varphi) = Z(I_\varphi)$ *for all* $\varphi \in \overline{\operatorname{span}}\,\Sigma_A$.
(e) *A is an ideal in A^{**} equipped with either of the Arens multiplications.*

As an immediate consequence of Theorem 3.4, we have

COROLLARY 3.6. *If the space Σ_A is discrete, then the algebra A has property* (S).

Proof. Let $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$. Since $\varphi \neq 0$ and A is regular tauberian, I_φ is a proper closed ideal of A. Thus, $Z(I_\varphi) \neq \emptyset$. By Theorem 3.4, $\sigma(\varphi) = Z(I_\varphi) \neq \emptyset$. Therefore, A has property (S). ■

REMARK 3.7. The converse of Corollary 3.6 is generally not true. For example, $A = A(\mathbb{R}) \cong L^1(\mathbb{R})$ has property (S) (see Katznelson's book [25, p. 163]), but $\Sigma_A = \mathbb{R}$ is not discrete. In §4, we will present some characterizations for $A(G)$ to have property (S) for all locally compact groups G.

Next, we investigate the structure of synthesizable ideals of A by using norm spectra of elements in $\overline{\operatorname{span}}\,\Sigma_A$. For any $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$, we consider the following conditions on φ:

(1) $\varphi = \sum_{n=1}^\infty c_n f_n$ for some $c_n \in \mathbb{C}$ and $f_n \in \operatorname{span}\Sigma_A$ with $(\sigma(f_n))_{n\ge1}$ pairwise disjoint.

(2) $\varphi \cdot a \neq 0$ implies that $\sigma(\varphi \cdot a) \neq \emptyset$, for all $a \in A$.

(3) $\sigma(\varphi)$ is (weak*) dense in $Z(I_\varphi)$.

(4) $\sigma(\varphi) \neq \emptyset$.

(5) The ideal I_φ is synthesizable.

The following result shows some implications among these conditions.

PROPOSITION 3.8. *Let $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$. Then* (1)⇒(2)⇒(3)⇒(4) *and* (2)⇒(5).

Proof. (1)⇒(2) follows from Lemma 3.1 and (3)⇒(4) is trivial.

We now follow an argument of Ülger [32, Theorem 5.5] to prove that (2)⇒(3) and (2)⇒(5).

(2)⇒(3). Suppose (2) holds for φ. Assume that $\sigma(\varphi)$ is not dense in $Z(I_\varphi)$. Then there exists $f \in Z(I_\varphi)$ such that f is not in the (weak*) closure of $\sigma(\varphi)$. By the regularity of A, there is $a \in A$ such that $\langle f, a\rangle \neq 0$ and $\langle g, a\rangle = 0$ for all $g \in \sigma(\varphi)$. Thus, $\varphi \cdot a \neq 0$. By the assumption of condition (2), $\sigma(\varphi \cdot a) \neq \emptyset$. However, by Lemma 2.4(iii), we have $\sigma(\varphi \cdot a) = \sigma(\varphi) \cap \{g \in \Sigma_A : \langle g, a\rangle \neq 0\}$. So, there exists $g \in \sigma(\varphi)$ such that $\langle g, a\rangle \neq 0$, a contradiction. Therefore, $\sigma(\varphi)$ is dense in $Z(I_\varphi)$.

(2)⇒(5). Suppose φ satisfies condition (2). Let $J = I_\varphi$. We need to prove that $J = \bigcap_{f\in Z(J)} \ker f$. Clearly, $J \subseteq \bigcap_{f\in Z(J)} \ker f$. To prove $\bigcap_{f\in Z(J)} \ker f \subseteq J$, let $a \in \bigcap_{f\in Z(J)} \ker f$. Then $\langle f, a\rangle = 0$ for all $f \in Z(J)$.

We claim that $\varphi \cdot a = 0$. Otherwise, by condition (2), $\sigma(\varphi \cdot a) = \{g \in \Sigma_A : \langle g, a\rangle \neq 0\} \neq \emptyset$. Thus, there is $g \in \sigma(\varphi)$ such that $\langle g, a\rangle \neq 0$. But $\sigma(\varphi) \subseteq Z(I_\varphi)$ from Lemma 3.2. We have $\langle g, a\rangle = 0$, a contradiction. Hence, $\varphi \cdot a = 0$, that is, $a \in I_\varphi = J$. Therefore, $J = \bigcap_{f\in Z(J)} \ker f$. It follows that I_φ is synthesizable. ■

REMARK 3.9. (i) Let E be a closed subset of Σ_A. Define $I(E) = \bigcap_{f\in E} \ker f$. Then $I(E)$ is the largest closed ideal of A whose zero set is E. The set E is said to be a *set of spectral synthesis* (s-set for short) if $I(E)$ is the only closed ideal of A with zero set E (cf. Hewitt and Ross [24, §39]). Let I be a proper closed ideal of A. By definition, if $Z(I)$ is an s-set, then $I = I(Z(I)) = \bigcap_{f\in Z(I)} \ker f$ is synthesizable.

The converse is not true even for $A = A(\mathbb{R})$ and ideals of the form I_φ. It is well known that $\mathbb{R}$ has a closed subset E which is not an s-set for $A(\mathbb{R})$ (Malliavin's theorem). Suppose $(x_n)_{n\ge1}$ is a dense subset of E and let $\varphi = \sum_{n=1}^\infty 2^{-n}\delta_{x_n} \in \overline{\operatorname{span}}\,\Sigma_{A(\mathbb{R})}$. By Proposition 3.8, I_φ is synthesizable, but $Z(I_\varphi) = \overline{\sigma(\varphi)} = E$ is not an s-set. On the other hand, it is true that each proper closed ideal of A is synthesizable iff each closed subset of Σ_A is an s-set.

(ii) Let G be an infinite compact group with unit e and the normalized Haar measure λ. Chou, Lau, and Rosenblatt [7] called G having *property* (A) if $\lambda \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G}) = \overline{\operatorname{span}}\,\Sigma_{A(G)}$. Suppose that G has property (A) (for example, $G = SO(n)$, $n \ge 3$; see Chou, Lau, and Rosenblatt [7] (for $n \ge 5$) and Drinfel'd [10] (for $n = 3, 4$), see also Chou [6]). The closed ideal I_λ of $A(G)$ is $\{0\}$ and hence I_λ is synthesizable. However, $\sigma(\lambda) = \emptyset$ (see Lemma 3.14 below). Therefore, the synthesizability of I_φ does not imply that $\sigma(\varphi) \neq \emptyset$.

If, in the above, we take $\varphi = \lambda + \delta_e$, then $\sigma(\varphi) = \{e\} \neq \emptyset$ but $\sigma(\varphi)$ is not dense in $Z(I_\varphi)$ $(= G)$. If we further assume that G is separable with dense subset $(x_n)_{n\ge1}$ and let $\varphi = \lambda + \sum_{n=1}^\infty 2^{-n}\delta_{x_n}$, then φ satisfies (2) but not (1). Therefore, we do not have (4)⇒(3) or (2)⇒(1). The implication (1) ⇒ $\varphi \in l^1(\Sigma_A)$ is not true either; see the example given by Cowling and Fournier in [8, pp. 64–65]. We do not know whether the implication (3)⇒(2) is true. We are only able to show that [(3) and (5)] ⇒ (2).

However, as stated in the following corollary, (2), (3), and (4) are equivalent if they hold for all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$. This follows readily from Proposition 3.8 and the proof is essentially included in that of Ülger [32, Theorem 5.5].

COROLLARY 3.10. *The following assertions are equivalent*:

(a) *The algebra A has property* (S).

(b) *For all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$, $\sigma(\varphi)$ is* (*weak**) *dense in $Z(I_\varphi)$*.

(c) *For all $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$, φ satisfies condition* (2).

Let I be a proper closed ideal of A. Under the assumption that A has property (S), Ülger [32, Theorem 5.5] proved that I is synthesizable with (weak*) separable zero set iff $I = I_\varphi$ for some $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$. This generalizes De Vito's result on synthesizable ideals of $L^1(\mathbb{R})$ (see [9]). We observe that only condition (2) was used in Ülger's proof (not the property (S) on the whole algebra A). Therefore, we have the following slightly stronger assertion.

THEOREM 3.11. *Let I be a proper closed ideal of A. Then the following statements are equivalent*:

(a) *I is synthesizable and $Z(I)$ is (weak*) separable.*

(b) *$I = I_\varphi$ for some $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$ satisfying condition* (2).

Proof. (b)⇒(a). This follows from Proposition 3.8 and Lemma 2.4(iv).

(a)⇒(b). Assume that the ideal I is synthesizable and $Z(I)$ is weak* separable. Let $(f_n)_{n\ge1}$ be a weak* dense sequence in $Z(I)$. Let $\varphi = \sum_{n=1}^{\infty} 2^{-n} f_n$. Then $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$ and φ satisfies condition (1) (hence condition (2)). Now $\sigma(\varphi) = (f_n)_{n\ge1}$ (by Lemma 3.1). By Proposition 3.8, I_φ is synthesizable and $Z(I_\varphi) = \overline{\sigma(\varphi)}^{w^*} = Z(I)$. Therefore, $I = \bigcap_{f\in Z(I)} \ker f = \bigcap_{f\in Z(I_\varphi)} \ker f = I_\varphi$. ∎

COROLLARY 3.12 ([32]). *Assume that the algebra A has property* (S) *and Σ_A is (weak*) separable. Then a proper closed ideal I of A is synthesizable if and only if $I = I_\varphi$ for some $\varphi \in \overline{\operatorname{span}}\,\Sigma_A \setminus \{0\}$.*

In the rest of this section, we present some applications of the results obtained so far to the Figà-Talamanca–Herz algebra $A_p(G)$. Let G be a locally compact group with unit e and a fixed left Haar measure λ. Let $1 < p < \infty$. Recall that $A_p(G)$ is a semisimple commutative regular tauberian Banach algebra with spectrum G (via Dirac measures) and $\overline{\operatorname{span}}\,\Sigma_{A_p(G)} = \mathcal{M}_p^{\mathrm d}(\widehat G)$. Also, $A_p(G)$ has the SBP. For $x \in G$, the set of topologically invariant means on $A_p(G)^*$ at x is defined by $TIM_p(x) = \{m \in A_p(G)^{**} : \|m\| = \langle m, \delta_x\rangle = 1 \text{ and } \langle m, T\cdot u\rangle = \langle m, T\rangle \text{ for all } T \in A_p(G)^*,\ u \in A_p(G) \text{ with } \|u\| = u(x) = 1\}$ (see Granirer [18] and [19]). In particular, $TIM_p(e) = TIM_p(\widehat G)$, the set of topologically invariant means on $A_p(G)^*$. It is well known that $TIM_p(\widehat G) \ne \emptyset$ (see Renaud [31, p. 287] for $p = 2$ and Granirer [17, Theorem 5] for general p). And, it is also easy to see that, for all $x \in G$,

$$TIM_p(x) = \{{}_x m \in A_p(G)^{**} : m \in TIM_p(\widehat G)\},$$

where $\langle {}_x m, T\rangle = \langle m, {}_{x^{-1}}T\rangle$ and $\langle {}_x T, u\rangle = \langle T, {}_x u\rangle$ for all $T \in A_p(G)^*$ and $u \in A_p(G)$ (${}_x u$ denotes the left translate of u by x).

For $A = A_p(G)$, concerning the functional m_f in Lemma 2.3, we have the following observation.

LEMMA 3.13. *Let $A = A_p(G)$ $(1 < p < \infty)$ and $x \in G$. Then, for each $m \in TIM_p(\widehat G)$, we can choose ${}_x m$ as the functional m_x in Lemma* 2.3.

Proof. Let $m \in TIM_p(\widehat G)$ and $x \in G$. Then ${}_x m \in TIM_p(x)$. Thus, $\langle {}_x m, \delta_x\rangle = 1$. We only need to show that $\langle {}_x m, \delta_y\rangle = 0$ for all $y \in G\setminus\{x\}$. To prove this, let $y \in G \setminus \{x\}$. Choose a compact neighbourhood U of e such that $xU \cap yU = \emptyset$. Let

$$u = \frac{1}{\lambda(U)} 1_{xU} * 1_U^{\vee} \in A_p(G).$$

Then $\|u\| = u(x) = 1$ and $u(y) = 0$. Now, $\delta_x \cdot u = u(y)\delta_y = 0$. Therefore,

$$\langle {}_x m, \delta_y\rangle = \langle {}_x m, \delta_y \cdot u\rangle = 0. \ ∎$$

The following lemma will be useful in the sequel. It shows that, if $\mu \in M(G)\cap\mathcal{M}_p^{\mathrm d}(\widehat G)$, then the norm spectrum $\sigma(\mu)$ of μ is completely determined by the discrete part of the measure μ and is independent of the number p.

LEMMA 3.14. *Let $A = A_p(G)$ $(1 < p < \infty)$ and $m \in TIM_p(\widehat G)$.*

(a) *If $\mu \in M(G)$, then, for all $x \in G$, $\langle \mu, {}_x m\rangle = \mu(\{x\})$. In particular, if $\mu \in M(G)\cap \mathcal{M}_p^{\mathrm d}(\widehat G)$, then*

$$\sigma(\mu) = \{x \in G : \mu(\{x\}) \ne 0\}.$$

(b) *If G is nondiscrete, then $\sigma(\varphi) = \emptyset$ for all $\varphi \in PF_p(G) \cap \mathcal{M}_p^{\mathrm d}(\widehat G)$.*

Proof. (a) Let $\mu \in M(G)$ and $x \in G$. Then ${}_{x^{-1}}\mu \in M(G)$ is the measure given by ${}_{x^{-1}}\mu(E) = \mu(xE)$ for all measurable sets E. By Granirer [17, Proposition 10],

$$\langle \mu, {}_x m\rangle = \langle m, {}_{x^{-1}}\mu\rangle = {}_{x^{-1}}\mu(\{e\}) = \mu(\{x\}).$$

If $\mu \in M(G)\cap\mathcal{M}_p^{\mathrm d}(\widehat G)$, then $x \in \sigma(\mu)$ iff $\langle \mu, {}_x m\rangle \ne 0$ (by Lemmas 2.4(ii) and 3.13) iff $\mu(\{x\}) \ne 0$. The second statement follows.

(b) Suppose G is nondiscrete and $\varphi \in PF_p(G)\cap\mathcal{M}_p^{\mathrm d}(\widehat G)$. Then there exists a sequence $(f_n)_{n\ge1}$ in $L^1(G)$ such that $f_n \to \varphi$ in the $\|\cdot\|_{A_p(G)^*}$-norm. For all $x \in G$, we have

$$\langle \varphi, {}_x m\rangle = \lim_{n\to\infty} \langle f_n, {}_x m\rangle = 0 \quad \text{(by part (a))}.$$

It follows, from Lemmas 2.4(ii) and 3.13 again, that $\sigma(\varphi) = \emptyset$. ∎

For any $T \in A_p(G)^*$, the support of T is defined as follows: for $x \in G$, we let $x \notin \operatorname{supp} T$ iff there is a neighbourhood U of x such that $\langle T, u\rangle = 0$ for all $u \in A_p(G)$ with $\operatorname{supp} u \subseteq U$. An equivalent definition for $\operatorname{supp} T$ is that $x \in \operatorname{supp} T$ iff $T \cdot u = 0$ implies $u(x) = 0$ for all $u \in A_p(G)$ (see Herz [23]). Let $\varphi \in \mathcal{M}_p^{\mathrm d}(\widehat G)$. By definition, $I_\varphi = \{u \in A_p(G) : \varphi\cdot u = 0\}$, and hence $Z(I_\varphi) = \operatorname{supp}\varphi$.

To conclude this section, we would like to present the following corollary as a summary of the applications to $A_p(G)$ of 3.2, 3.5, 3.6, 3.10, 3.12, and 3.14.

COROLLARY 3.15. *Let G be a locally compact group and $A = A_p(G)$ $(1 < p < \infty)$. Then the following assertions hold:*

(a) *For all* $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$, *we have* $\sigma(\varphi) \subseteq \operatorname{supp}\varphi$.

(b) *Each* u *in* $A_p(G)$ *is (weakly) compact iff* G *is discrete iff* $\sigma(\varphi) = \operatorname{supp}\varphi$ *for all* $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$ *iff* $A_p(G)$ *is an ideal in* $A_p(G)^{**}$.

(c) *If* G *is discrete, then* $A_p(G)$ *has property* (S).

(d) $A_p(G)$ *has property* (S) *iff* $\sigma(\varphi)$ *is dense in* $\operatorname{supp}\varphi$ *for all* $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$.

(e) *Suppose* G *is second countable and* $A_p(G)$ *has property* (S). *Then a proper closed ideal* I *of* $A_p(G)$ *is synthesizable iff* $I = I_\varphi$ *for some* $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G}) \setminus \{0\}$.

(f) *If* $A_p(G)$ *has property* (S), *then either* $\mathcal{M}_p^{\mathrm{d}}(\widehat{G}) = PF_p(G)$ *or* $\mathcal{M}_p^{\mathrm{d}}(\widehat{G}) \cap PF_p(G) = \{0\}$.

REMARK 3.16. The equivalence of G being discrete and $A_p(G)$ being an ideal in $A_p(G)^{**}$ was given by Lau [27] for $p = 2$ and Forrest [15] for all p. The following characterizations for the discreteness of G are also known.

(1) $A_p(G)$ has the bounded power property, i.e., for $f \in A_p(G)$, $\|f\|_{sp} = \lim_n \|f^n\|^{1/n} \le 1$ implies that $\sup_n \|f^n\| \le 1$ (see Granirer [16]).

(2) $A_p(G)^*$ has a unique topologically invariant mean (see Renaud [31] (for metrizable G) and Lau and Losert [28] (for general G) for $p = 2$ and Granirer [20] for all p).

4. Property (S), faithful trace, and $*$-isomorphism. From the discussion in §3, we see that it would be interesting to consider when an algebra A has property (S). In this section, we will investigate this question for $A = A_p(G)$. Recall that $\Sigma_{A_p(G)} = G$ and $\overline{\operatorname{span}}\,\Sigma_{A_p(G)} = \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$. See the text after Lemma 2.4 for more information on the algebra $A_p(G)$.

Since $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ is a C^*-algebra, we start our discussion with $p = 2$.

Let $m \in TIM_2(\widehat{G})$, a topologically invariant mean on $VN(G)$. It is known that $\langle m, \mu\rangle = \mu(\{e\})$ for all $\mu \in M(G)$ (see Dunkl and Ramirez [13, Theorem 2.12 and Section 8]). Let $\mathrm{tr} = m|_{\mathcal{M}_2^{\mathrm{d}}(\widehat{G})}$, the restriction of m to $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. The functional tr has the following properties:

(1) $\mathrm{tr}(\mu\nu) = \mathrm{tr}(\nu\mu) = \sum_{x\in G} \mu(\{x\})\nu(\{x^{-1}\})$ for $\mu, \nu \in M_{\mathrm{d}}(G)$,

(2) $\mathrm{tr}(\mu^*\mu) = \sum_{x\in G} |\mu(\{x\})|^2 \ge 0$ for $\mu \in M_{\mathrm{d}}(G)$.

Therefore, tr is the unique finite trace on the C^*-algebra $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ with $\mathrm{tr}(\mu) = \mu(\{e\})$, $\mu \in M_{\mathrm{d}}(G)$. The trace tr is said to be *faithful* on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ if $\mathrm{tr}(\varphi^*\varphi) = 0$ implies that $\varphi = 0$ for all $\varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G})$, where φ^* denotes the adjoint of φ as a bounded operator on $L^2(G)$.

Recall that G_{d} denotes the algebraic group G endowed with the discrete topology. Then $\mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ is the reduced group C^*-algebra of G_{d}. Dunkl and Ramirez showed in [12, Theorem 2.1] that $\|\mu\|_{\mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})} \le \|\mu\|_{\mathcal{M}_2^{\mathrm{d}}(\widehat{G})}$ for all $\mu \in M_{\mathrm{d}}(G)$. Thus, the map $\mu \mapsto \mu$, for $\mu \in M_{\mathrm{d}}(G) = M_{\mathrm{d}}(G_{\mathrm{d}})$, extends to a surjective $*$-homomorphism $\Gamma : \mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \to \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$. A natural question is: when is Γ a $*$-isomorphism (or $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$)? We answer this question as follows.

LEMMA 4.1. *Let* G *be a locally compact group. Then* $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ *if and only if the trace* tr *is faithful on* $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$.

Proof. Dunkl and Ramirez used Tr in [12] to denote the finite trace on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ defined by $\mathrm{Tr}(\mu) = \mu(\{e\})$, for $\mu \in M(G_{\mathrm{d}}) = M_{\mathrm{d}}(G_{\mathrm{d}}) = M_{\mathrm{d}}(G)$. Then Tr is continuous because it is also the restriction of a topologically invariant mean. It is known that Tr is always faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ (see [12, Theorem 2.3]).

Now suppose $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$. Then $\mathrm{tr} = \mathrm{Tr}$ and hence tr is faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$.

Conversely, suppose tr is faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$.

First, we observe that $\mathrm{Tr}(\Gamma\varphi) = \mathrm{tr}(\varphi)$ for all $\varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. In fact, if $\mu \in M_{\mathrm{d}}(G)$, then $\Gamma(\mu) = \mu$ and hence $\mathrm{Tr}(\Gamma\mu) = \mu(\{e\}) = \mathrm{tr}(\mu)$. The assertion follows from the continuity of Γ, Tr, and tr.

Next, let $\varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \setminus \{0\}$. Then

$$\mathrm{Tr}((\Gamma\varphi)^*(\Gamma\varphi)) = \mathrm{Tr}(\Gamma(\varphi^*\varphi)) = \mathrm{tr}(\varphi^*\varphi) > 0.$$

Thus, $\Gamma\varphi \ne 0$. It follows that $\Gamma : \mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \to \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ is injective and hence is a $*$-isomorphism. ∎

We show in the following that the faithfulness of tr is also intimately related to property (S).

LEMMA 4.2. *The trace* tr *is faithful on* $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$ *if and only if* $A(G)$ *has property* (S).

Proof. Suppose the trace tr is faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. Then $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ by Lemma 4.1. Let $\varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. According to the proof of Dunkl and Ramirez [12, Theorem 2.3], there exists a square-summable function F_φ on G_{d} such that

$$(\varphi f)(x) = \sum_{y\in G} F_\varphi(xy^{-1})f(y), \quad f \in l^2(G_{\mathrm{d}}),\ x \in G,$$

and

$$\mathrm{tr}(\varphi^*\varphi) = \sum_{x\in G} |F_\varphi(x)|^2.$$

In particular, $(\varphi 1_{\{e\}})(x) = F_\varphi(x)$ for all $x \in G$. By Lemma 3.14(a), it is easy to see that if $\varphi \in M_{\mathrm{d}}(G)$, then

$$\langle \varphi, m_x\rangle = \varphi(\{x\}) = (\varphi 1_{\{e\}})(x), \quad x \in G,$$

where m_x is the same functional as in Lemma 2.3. Hence, for all $x \in G$ and for all $\varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G})$, we have $\langle \varphi, m_x \rangle = F_\varphi(x)$. Therefore,

$$\operatorname{tr}(\varphi^*\varphi) = \sum_{x\in G} |\langle \varphi, m_x\rangle|^2 \quad \text{for all } \varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G}).$$

Property (S) of $A(G)$ follows readily from the above equality and Lemma 2.4(ii).

Conversely, suppose $A(G)$ has property (S).

Assume that the trace tr is not faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. Then there exists $\varphi \in \mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \setminus \{0\}$ such that $\operatorname{tr}(\varphi^*\varphi) = 0$. By the assumption of property (S), we can take an $x_0 \in \sigma(\varphi)$. Let $\varphi_n = \sum_{x\in G} c_n^x \delta_x \in \operatorname{span} \Sigma_{A(G)}$ and $\varphi_n \to \varphi$. By Lemma 3.1, we have $\lim_{n\to\infty} c_n^{x_0} \neq 0$. Thus,

$$\operatorname{tr}(\varphi^*\varphi) = \lim_{n\to\infty} \operatorname{tr}(\varphi_n^*\varphi_n) \quad \Big(\text{where } \varphi_n^* = \sum_{x\in G} \overline{c_n^x}\delta_{x^{-1}}\Big)$$

$$= \lim_{n\to\infty} \operatorname{tr}\Big(\sum_{x,y\in G} c_n^x \overline{c_n^y} \delta_{y^{-1}x}\Big) = \lim_{n\to\infty} \sum_{x\in G} |c_n^x|^2 \geq \lim_{n\to\infty} |c_n^{x_0}|^2 > 0,$$

contradicting that $\operatorname{tr}(\varphi^*\varphi) = 0$. Therefore, tr is faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. ■

Combining Lemmas 4.1 and 4.2, we are ready to present one of the main results of this paper.

THEOREM 4.3. *Let G be a locally compact group. Then the following assertions are equivalent*:

(1) *$A(G)$ has property* (S).

(2) *The trace* tr *is faithful on $\mathcal{M}_2^{\mathrm{d}}(\widehat{G})$.*

(3) $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$.

In particular, $A(G)$ has property (S) *if either G is discrete or G_{d} is amenable.*

Next, we consider property (S) for $A_p(G)$ with $1 < p < \infty$. Let G be amenable and $1 < p < \infty$. Herz showed that the identification of functions gives a contraction $i : A(G) \to A_p(G)$; dually, there is a contraction $i^* : A_p(G)^* \to A(G)^*$ (see Herz [22]). In this case, $i^*(\mathcal{M}_p^{\mathrm{d}}(\widehat{G})) \subseteq \mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. If we use $\sigma_p(\varphi)$ to denote the norm spectrum of φ in $A_p(G)^*$, then $\sigma_p(\varphi) \subseteq \sigma_2(i^*\varphi)$ for all $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$ since $A(G)\cap C_{00}(G)$ is $\|\cdot\|_{A_p(G)}$-dense in $A_p(G)$. We can further prove that $\sigma_p(\varphi) = \sigma_2(i^*\varphi)$ as follows.

LEMMA 4.4. *Let G be an amenable locally compact group and $1 < p < \infty$. Then $\sigma_p(\varphi) = \sigma_2(i^*\varphi)$ for all $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$.*

In particular, if $A(G)$ has property (S), *then so does $A_p(G)$.*

Proof. We only need to prove that $\sigma_2(i^*\varphi) \subseteq \sigma_p(\varphi)$ for all $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$.

Let $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G})$ and $x_0 \in \sigma_2(i^*\varphi)$. Let $m \in TIM_2(\widehat{G})$. By Lemma 3.13, we have $\langle i^*\varphi, {}_{x_0}m\rangle \neq 0$, that is, $\langle \varphi, i^{**}({}_{x_0}m)\rangle \neq 0$. It is easy to see that $i^{**}({}_{x_0}m) = {}_{x_0}(i^{**}m)$ and hence $\langle \varphi, {}_{x_0}(i^{**}m)\rangle \neq 0$. Since $i^{**}m \in TIM_p(\widehat{G})$, by Lemma 3.13 again, $x_0 \in \sigma_p(\varphi)$. It follows that $\sigma_2(i^*\varphi) \subseteq \sigma_p(\varphi)$.

The second statement follows from the above argument and the inclusion $i^*(\mathcal{M}_p^{\mathrm{d}}(\widehat{G})) \subseteq \mathcal{M}_2^{\mathrm{d}}(\widehat{G})$. ■

Finally, we discuss the relation between property (S) and amenability. As mentioned in the introduction, Bédos [1, Theorem 3] showed that G_{d} is amenable iff G is amenable and $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cong \mathcal{M}_2^{\mathrm{d}}(\widehat{G_{\mathrm{d}}})$ (see also [2]). Combining this result with Theorem 4.3 and Lemma 4.4, we can conclude the following.

THEOREM 4.5. *Let G be locally compact group. Then the following assertions are equivalent*:

(1) *G_{d} is amenable.*

(2) *G is amenable and $A(G)$ has property* (S).

(3) *G is amenable and $A_p(G)$ has property* (S) *for all $1 < p < \infty$.*

REMARK 4.6. (i) We know that the Fourier algebra $A(G)$ has property (S) if either G is discrete or G_{d} is amenable (see Theorem 4.3). But the converse does not hold in general. In fact, let $\mathbf{F}_2$ denote the free group on two generators and let G be any nondiscrete locally compact group such that $A(G)$ has property (S). Then $\mathbf{F}_2 \times G$ is neither discrete nor amenable (hence it is not amenable as a discrete group). However, by Theorem 4.3 and [2, Theorem 1], it is easy to see that $A(\mathbf{F}_2 \times G)$ has property (S).

(ii) Recall the property (A) mentioned in Remark 3.9(ii). Chou, Lau, and Rosenblatt [7] proved, among other characterizations, that an infinite compact group G has property (A) iff $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cap PF_2(G) \neq \{0\}$. For any nondiscrete locally compact group G, the fact that $A(G)$ has porperty (S) implies $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cap PF_2(G) = \{0\}$ (by Corollary 3.15(f)). Meanwhile, it is also possible that $\mathcal{M}_2^{\mathrm{d}}(\widehat{G}) \cap PF_2(G) = \{0\}$ for some compact group G (i.e., G does not have property (A)) and $A(G)$ also fails to have property (S) (hence G_{d} is not amenable). See [7, Remark 1.4] for such groups. Therefore, the converse of Corollary 3.15(f) is not true.

(iii) There was a gap in the proof of [32, Proposition 5.4]. It can be seen that the scalar λ there is equal to $\langle \varphi, m_f\rangle$. Generally, we were unable to draw that $\lambda = 1$ or even $\lambda \neq 0$. It is clear now that Proposition 5.4 of [32] may not hold when G_{d} is not amenable.

Since any abelian group is amenable as a discrete group, we have the following result as a simple application of Theorem 4.5.

COROLLARY 4.7. *Let G be a locally compact abelian group. Then $A_p(G)$ has property* (S) *for all* $1 < p < \infty$.

We conclude this paper with the following immediate consequence of Corollary 3.15(e) and Theorem 4.3.

COROLLARY 4.8. *Let G be a second countable locally compact group and $1 < p < \infty$. If G is either discrete or amenable as a discrete group, then a proper closed ideal I of $A_p(G)$ is synthesizable if and only if $I = I_\varphi$ for some $\varphi \in \mathcal{M}_p^{\mathrm{d}}(\widehat{G}) \setminus \{0\}$.*

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