σ^* is of weak type $(1,1)$ implies for $\delta > 0$ that

$$\begin{aligned}\mu(\overline{\lim_{n\in\mathbb{P}}}|\sigma_n^\kappa f - f| > \delta) &\le \mu(\overline{\lim_{n\in\mathbb{P}}}|\sigma_n^\kappa f - \sigma_n^\kappa P| > \delta/3)\\ &\quad + \mu(\overline{\lim_{n\in\mathbb{P}}}|\sigma_n^\kappa P - P| > \delta/3) + \mu(\overline{\lim_{n\in\mathbb{P}}}|P - f| > \delta/3)\\ &\le \frac{c}{\delta}\|f-P\|_1 + \mu(\sigma^*(f-P) > \delta/3)\\ &\le \frac{c}{\delta}\|f-P\|_1 \le \frac{c}{\delta}\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we have

$$\mu(\limsup|\sigma_n^\kappa f - f| > \delta) = 0$$

for any $\delta > 0$. This means that $\sigma_n^\kappa f \to f$ a.e. $(n \to \infty)$. ■

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Almost 1-1 extensions of Furstenberg–Weiss type and applications to Toeplitz flows

by

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Abstract. Let (Z,T_Z) be a minimal non-periodic flow which is either symbolic or strictly ergodic. Any topological extension of (Z,T_Z) is Borel isomorphic to an almost 1-1 extension of (Z,T_Z). Moreover, this isomorphism preserves the affine-topological structure of the invariant measures. The above extends a theorem of Furstenberg–Weiss (1989). As an application we prove that any measure-preserving transformation which admits infinitely many rational eigenvalues is measure-theoretically isomorphic to a strictly ergodic Toeplitz flow.

Introduction. In 1989, Furstenberg and Weiss proved a theorem [F-W, Theorem 1] which can be informally expressed as follows: every topological point-transitive flow (X,T_X) which is an extension of a minimal non-periodic flow (Z,T_Z) is in some sense equivalent to a minimal flow (Y,T_Y) which is an almost 1-1 extension of (Z,T_Z). The equivalence is given by a Borel measurable injective map ϕ defined on a subset $X' \subset X$ whose mass is 1 for any T_X-invariant probability measure carried by X. Such a Borel embedding provides a 1-1 affine map ϕ^* (defined as the adjoint map on measures) from the set $P(X)$ of all T_X-invariant probability measures carried by X into the set $P(Y)$ defined analogously for the flow (Y,T_Y). Moreover, for every $\mu \in P(X)$, ϕ is a measure-theoretic isomorphism between the measure-preserving transformations $(X,\mathcal{B}_X,\mu,T_X)$ and $(Y,\mathcal{B}_Y,\phi^*(\mu),T_Y)$ (here $\mathcal{B}_X$ and $\mathcal{B}_Y$ denote the σ-fields of Borel measurable sets in X and Y, respectively).

In this paper we improve the Furstenberg–Weiss theorem. By the methods of symbolic dynamics we obtain a stronger isomorphism under even

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weaker assumptions: dropping transitivity of (X, T_X) we construct (Y, T_Y) and ϕ such that in addition to all previous properties, ϕ^* is a homeomorphism between $P(X)$ and $P(Y)$ for the weak* topology of measures. This is partially achieved by obtaining the image $Y' = \phi(X')$ of mass 1 for every $\nu \in P(Y)$, and partially by controlling the frequencies with which blocks occur in sequences. This kind of isomorphism (which we call "Borel*") is probably the best one can expect to exist between a nearly arbitrary topological flow (the only restriction is that it admits a minimal non-periodic factor) and a minimal one. Clearly, obtaining a topological isomorphism is impossible, nevertheless, our isomorphism behaves like one at the level of invariant measures. In particular, in virtue of the variational principle, topological entropy is preserved (this was not guaranteed by the original version of the theorem).

Our proof is based on combinatorial constructions for symbolic flows (subshifts). Most operations have their direct translations to the general topological case, for instance observing repeating blocks along a sequence corresponds to finding return times of an orbit to a fixed open set. Some tricks, however, like replacing each occurrence of a block by another block of the same length, or permuting certain letters within a block, might lead to a rather complicated description when translated to the general topological language. This is why we decided to state the main result for subshifts. Later we discuss the possibility of extending it to the general case. An additional advantage of such a formulation is that obtaining (Y, T_Y) symbolic in case of (X, T_X) symbolic (and transitive) does not follow directly from the original Furstenberg–Weiss theorem (it can be derived from it via a theorem of Denker–Keane [D-K, Theorem 20], but then it works for a fixed measure on (X, T_X) only).

At the end of the section devoted to the symbolic case we make a digression on the special type of codes we exploit.

In the last section, as an application of the results obtained, we present a measure-theoretic (and Borel*) characterization of Toeplitz flows. In particular, some previous results on possible point spectra of Toeplitz flows obtained in [I-L], [I], and [D-L] are recovered.

The symbolic theorem. By a *topological dynamical system* (*flow*) we mean a pair (X, T_X), where X is a compact metrizable space and T_X is a homeomorphism of X onto itself. We will denote by $P(X)$ the collection of all T_X-invariant Borel probability measures on X. It is known that this set is convex (even a simplex) and compact for the weak* topology of measures. A Borel subset $X' \subset X$ is called a *full set* if every measure $\mu \in P(X)$ assigns mass 1 to it.

By a *Borel* isomorphism* between two flows (X, T_X) and (Y, T_Y) we understand a Borel measurable invertible map $\phi : X' \to Y'$ between full sets $X' \subset X$ and $Y' \subset Y$ such that $\phi \circ T_X = T_Y \circ \phi$ and the adjoint map $\phi^* : P(X) \to P(Y)$, defined by

$$\phi^*(\mu)(A) = \mu(\phi^{-1}(A))$$

(for any Borel set $A \subset Y$), is an affine homeomorphism for the weak* topology. Obviously, a composition of Borel* isomorphisms is a Borel* isomorphism.

In symbolic dynamics one considers *subshifts*, i.e., flows (Z, S), where S denotes the shift transformation on $\Lambda^{\mathbb{Z}}$ and Z is a shift-invariant closed subset of $\Lambda^{\mathbb{Z}}$. The set Λ is called the *alphabet*. Unless otherwise specified, we assume the alphabets appearing in this paper to be finite.

A *block* over the alphabet Λ is a k-tuple $B = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1}) \in \Lambda^k$. We denote by $|B|$ the *length* $k \in \mathbb{N}$ of the block B. We say that B *occurs* in a sequence $z \in \Lambda^{\mathbb{Z}}$ if $(z(n), z(n+1), \ldots, z(n+k-1)) = B$ for some $n \in \mathbb{Z}$. The integer interval $[n, n+k) = \{n, n+1, \ldots, n+k-1\}$ is then called the *domain* of the occurrence. Given finitely many blocks $B_1, \ldots, B_r$ we can build their *concatenation*, i.e., the block $B = B_1 \ldots B_r$. We say that a block C *starts with* B if $C = B$ or $C = BD$ for some block D.

It is well known that for a minimal subshift (Z, S) every block which occurs in some $z \in Z$ occurs in each element of Z *syndetically*, i.e., it occurs arbitrarily far in both directions and the distances between consecutive occurrences are bounded. We say that a block B has *non-overlapping occurrences* if for any $z \in Z$ the domains of any two different occurrences of B in z are disjoint:

$$z = \ldots B..B.....B.B\ldots B....B..BB.....B\ldots B..BB\ldots B..B.....B\ldots$$

Clearly, all blocks of length 1 have this property. If B has non-overlapping occurrences then by a B-*block* we mean any block $B\ldots$ which starts with B, and such that:

- $B\ldots$ cannot be written as a concatenation involving two occurrences of B, and
- $B\ldots B$ occurs in some $z \in Z$.

By minimality, the lengths of all B-blocks are bounded, hence the collection of all B-blocks is finite. Every $z \in Z$ can be represented in a unique way as a concatenation of B-blocks.

The following fact is the starting point of our construction:

LEMMA 1. *Assume (Z, S) is a minimal non-periodic subshift. Let B be a block having non-overlapping occurrences in Z. Then there exist arbitrarily long blocks starting with B and having non-overlapping occurrences.*

Proof. Let $B\ldots$ denote a fixed B-block. Suppose $B\ldots B$ has overlapping occurrences. This implies that $B\ldots B\ldots B$ occurs in Z. If the last block has overlapping occurrences then $B...B...B...B$ occurs in Z, and so on. By minimality and non-periodicity, some block $C = B...B...B......B...B$ (essentially longer than B) has non-overlapping occurrences. Repeating the same argument for C, and so on, we can obtain arbitrarily long blocks of the required form. ∎

It is important to note that

(1) if C starts with B and is sufficiently long then it starts with a concatenation of B-blocks, while

(2) each C-block is a concatenation of B-blocks.

Recall that a *factor map* between two flows (X, T_X) and (Z, T_Z) is a continuous surjective map $\pi : X \to Z$ such that $\pi \circ T_X = T_Z \circ \pi$. For a given factor map π, its *fibers* are the preimages of points. We say that π *provides an almost* 1-1 *extension* if the subset of points of Z having one-point fibers is residual. If (Z, T_Z) is minimal then to establish that the extension is almost 1-1 it suffices to show that a one-point fiber exists. Almost 1-1 extensions play an important role in topological dynamics. Many topological properties pass to almost 1-1 extensions (for instance see [A] for topological disjointness).

The main result of this paper is the following symbolic version of the Furstenberg–Weiss theorem:

THEOREM 1. *Let (Z, S) be a minimal non-periodic subshift over an alphabet Λ, and let (X, S) be a subshift over an alphabet Σ. Suppose there exists a factor map $\pi_X : X \to Z$. Then there exists a minimal subshift (Y, S) (over a new alphabet Ξ) and a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, leftrightarrow, "\phi"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & Z & \end{tikzcd}$$

where ϕ is a Borel isomorphism, and π_Y provides an almost* 1-1 *extension.*

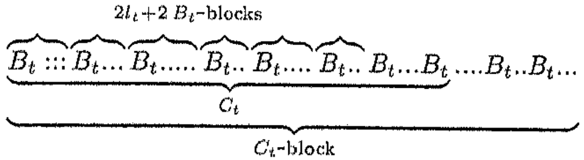
Proof. By Lemma 1 and by (1), we can choose inductively two sequences of blocks B_t and C_t (over Λ) appearing in Z such that for each $t \ge 1$,

(3) B_t and C_t have non-overlapping occurrences in Z,

(4) C_t starts with a concatenation of $2l_t + 2$ B_t-blocks, where l_t is the length of the initial B_t-block in C_t,

(5) B_{t+1} starts with a concatenation of C_t-blocks so long that every existing C_t-block (in Z) is used in it at least $r_t = (\#\Sigma(\#\Sigma + 1))^{m_t}$ times, where m_t is the maximal length of a C_t-block.

We denote by B_t::: the B_t-block with which C_t starts (hence each C_t-block also starts with B_t:::).



For fixed $t \ge 1$, every $z \in Z$ can be represented in a unique way as an infinite concatenation of C_t-blocks, each decomposing into at least $2l_t + 2$ B_t-blocks.

Let $\Xi = \Sigma \times (\Sigma \cup \{\square\}) \times \Lambda$, where $\square$ is an additional symbol. By *letters* we mean the elements of Σ. We view the elements of Ξ as columns of height 3, hence the sequences over Ξ are represented as three sequences (rows): the top row containing letters, the central row containing letters and squares, and the bottom row containing elements of Λ. The positions in the central row will be called *cells*. A cell can be *occupied* or *empty* depending on whether it contains a letter or a square. To start the construction, we treat each element $x \in X$ as the top row and we add two rows below it: the central row consisting entirely of empty cells, and the bottom row identical with $\pi_X(x)$. Since π_X is continuous, this procedure yields a topologically isomorphic representation of (X, S) as a subshift over the alphabet Ξ. From now on (X, S) stands for this representation.

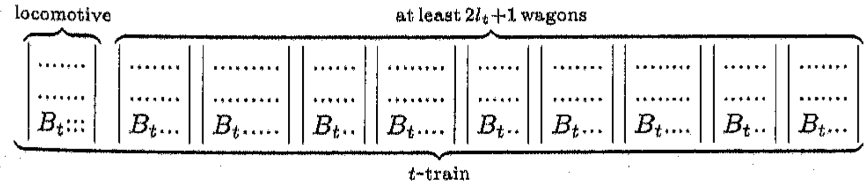
$$x \equiv \begin{array}{cccccccl} \cdots & \sigma_{-1} & \sigma_0 & \sigma_1 & \sigma_2 & \cdots & \leftarrow & \text{top row, letters, } x \\ \cdots & \square & \square & \square & \square & \cdots & \leftarrow & \text{middle row, cells} \\ \cdots & \lambda_{-1} & \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \leftarrow & \text{bottom row, } \pi_X(x) \end{array}$$

Consider a block over the alphabet Ξ such that its bottom row is a C_t-block. We call every such block a *t-train*. Observe that

(6) there exist not more than r_t different t-trains with a common bottom row.

We have the following decomposition of the t-trains:

(7) each t-train decomposes into a concatenation of a *locomotive* having B_t::: in the bottom row (hence of length l_t), followed by at least $2l_t + 1$ *wagons* (having further B_t-blocks in the bottom row).



We must remember that, except for $t = 1$,

(8) the locomotive and the wagons are concatenations of $(t-1)$-trains.

It follows immediately from the representation of (X, S) as a subshift over Ξ that for each $t \geq 1$ every $x \in X$ can be decomposed in a unique way as an infinite concatenation of t-trains. Obviously, by the construction, the positioning of the component t-trains, their locomotives and wagons is determined by the third row $\pi_X(x)$. The t-trains occurring in X will be called *original t-trains*.

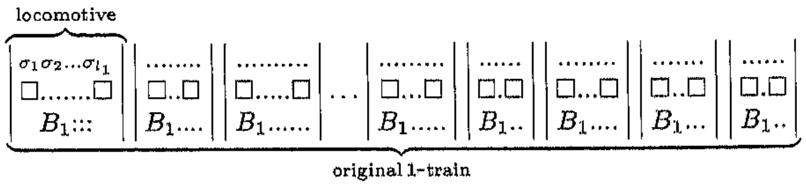
We will soon define a sequence of maps ϕ_t on X into some subshifts over Ξ. Each of the maps ϕ_t will be obtained by a code replacing consecutively the original t-trains by other t-trains. At most coordinates, $\phi_{t+1}(x)$ coincides with $\phi_t(x)$. The only differences are due to the *modifications* described below. The idea is to introduce certain syndetically repeating new blocks (modification (B)) without forgetting the letters which these blocks would overwrite. To achieve this goal we first have to "memorize" these letters by copying them into the empty cells in the middle row (modification (A)).

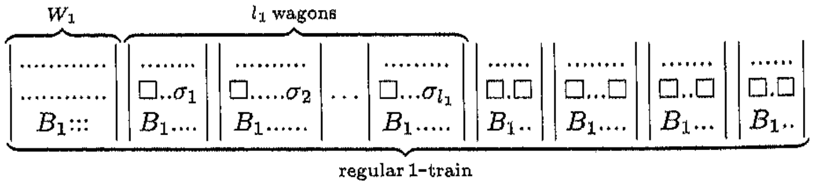
STEP 1. Let W_1 be an arbitrary block over the alphabet Ξ (W_1 need not occur in X) having B_1::: in the bottom row (hence of length l_1). The 1-*code* is defined as a transformation of the original 1-trains by applying the following two modifications:

(A) using consecutively all the letters occurring in the top row of the locomotive we fill in the terminal empty cells in each of the next l_1 wagons,

(B) we replace the locomotive by the new locomotive W_1.

By *regular* 1-*trains* we mean the images of the original 1-trains under the 1-code.





Note that the 1-code does not affect the terminal wagon of the 1-train and thus the terminal cell of each regular 1-train remains empty. Next, since W_1 has B_1::: in the bottom row, the entire bottom row of each 1-train remains

unchanged. Finally, observe that the original 1-train can be reconstructed from its image by removing the upper two rows of the locomotive and filling the top row by the letters occupying the terminal cells of the next l_1 wagons (at the same time we empty these cells). Thus, it is clear that the 1-code is a 1-1 correspondence between the original and regular 1-trains.

INDUCTIVE ASSUMPTION. Let $t \in \mathbb{N}$ and suppose that

(9) a t-code has been defined as a 1-1 correspondence between the original t-trains and their images called *regular t-trains*,

(10) the bottom row is unchanged by the t-code,

(11) the terminal cell of each regular t-train is empty.

STEP $t+1$. We create a block W_{t+1} over the alphabet Ξ (W_{t+1} not necessarily occurring in $\phi_t(X)$) so that:

(12) the bottom row of W_{t+1} is B_{t+1}:::,

(13) W_{t+1} is a concatenation of regular t-trains,

(14) every regular t-train is used at least once in the above concatenation (this is possible by (5) and (6)).

We define the $(t+1)$-code on the original $(t+1)$-trains in the following way: we first replace all the original t-trains into which the given original $(t+1)$-train decomposes (see (8)) by their images under the t-code. The $(t+1)$-train so obtained will be called the *t-coded $(t+1)$-train*. Next we apply the following two modifications:

(A) using consecutively all the letters and squares occurring in the top and middle rows of the locomotive of the t-coded $(t+1)$-train we fill in the terminal empty cells in each of the next $2l_{t+1}$ wagons (by (7) there are enough wagons; observe that each wagon of a t-coded $(t+1)$-train is a concatenation of regular t-trains, hence, by (11) its terminal cell is empty),

(B) we replace the locomotive by W_{t+1}.

Note that the above modifications do not affect the terminal wagon of the t-coded $(t+1)$-train. Thus the terminal cell remains empty, as required in (11). Clearly, by (12), the bottom row is unchanged, as required in (10). We can reverse the modifications (A) and (B) by emptying the two upper rows of the locomotive and filling them back with the letters and squares appearing in the terminal cells of the next $2l_{t+1}$ wagons (at the same time we empty these cells). Next, the original $(t+1)$-train can be recovered from the t-coded $(t+1)$-train by reversing the t-code (use the inductive assumption (9)). Thus the $(t+1)$-code is a 1-1 correspondence, as required in (9).

END OF INDUCTION. The following obvious observations are important:

(15) the modification (B) replaces regular t-trains by other regular t-trains preserving the bottom row (see (13) and (12)),

(16) the modification (B) preserves all the locomotives W_s with $s \le t$ introduced by the t-code

(because the distribution of these locomotives within a regular t-train depends only on the third row, and after step t there are no locomotives other than W_s).

During the modifications (A) and (B) in step $t+1$ each (regular) t-train of the t-coded $(t+1)$-train can be either left unaffected, or replaced by another regular t-train (modification (B)), or it can happen that a letter will be inserted into its terminal cell (modification (A)). A t-train differing from a regular one by having the terminal cell occupied will be called an *irregular t-train*. Thus,

(17) every regular $(t+1)$-train is a concatenation of regular and irregular t-trains.

Later we also use a reverse procedure. A given regular t-train can be *t-decoded*, i.e., replaced by its (unique) preimage under the t-code. Moreover, we can also t-decode an irregular t-train simply disregarding the letter occupying the terminal cell. For instance, we can t-decode a regular (or irregular) $(t+1)$-train by t-decoding all component t-trains. Comparing the t-decoded $(t+1)$-train with the original ($(t+1)$-decoded) $(t+1)$-train we can see that the differences result from applying or not applying the reversed modifications (A) and (B), hence

(18) the t-decoded $(t+1)$-train differs from the original $(t+1)$-train only in having a different locomotive

(in the first case we have t-decoded the component t-trains of W_{t+1}, while in the second case W_{t+1} has been removed and the original locomotive has been recovered from the terminal cells of the wagons—these cells have been ignored in the first case).

The maps ϕ_t. Fix $x \in X$ and $t \ge 1$. As noticed before, x decomposes in a unique way into an infinite concatenation of original t-trains. We define $\phi_t(x)$ as the sequence obtained from x by replacing each original t-train in x by its image under the t-code. Note that the locomotives of all the t-trains of $\phi_t(x)$ are W_t, hence

(19) W_t occurs in $\phi_t(x)$ syndetically.

It is easily seen that ϕ_t is continuous, injective and commutes with the shift transformation. Thus ϕ_t provides a topological isomorphism between (X,S) and $(\phi_t(X),S)$. We will be using the following facts:

(20) $\phi_t(x)$ is a concatenation of regular t-trains,

(21) for each $s > t$, $\phi_s(x)$ is a concatenation of regular and irregular t-trains.

The last statement is obvious for $s = t+1$ (see (17)). For larger s use an inductive argument and the observation that the modification (B) replaces regular $(s-1)$-trains by other regular $(s-1)$-trains (see (15)), while modification (A) inserts letters into terminal cells of some wagons of the $(s-1)$-trains, and these cells are terminal with respect to t-trains, so some more irregular t-trains are produced.

The map ϕ and its domain X'. Let

$$Z_{t,n} = \{z \in Z : n \in \text{the domain of the starting } B_t\text{-block } B_t\!:::\ \text{in a } C_t\text{-block of } z\}.$$

Estimating the mass which an invariant probability measure assigns to a set by the maximal frequency at which this set is visited by some trajectory, we can see that each such measure assigns to $Z_{t,n}$ a mass of at most $1/(2l_t)$. It follows easily from the construction that l_t and r_t grow exponentially, hence

$$\sum_t 1/l_t < \infty \quad \text{and} \quad \sum_t 1/r_t < \infty. \tag{22}$$

Thus

$$Z' = Z \setminus \bigcup_{n\in\mathbb{Z}} \bigcap_{s\ge1} \bigcup_{t\ge s} Z_{t,n}$$

is a full set. This implies that $X' = \pi_X^{-1}(Z')$ is also a full set in X. On the other hand, it is easily seen that

$$X' = \{x \in X : \text{each coordinate falls into the domain of a locomotive of a } t\text{-train for at most finitely many indices } t\}.$$

Observe that

(23) if $x \in X'$ then every block of x is subject to at most finitely many modifications during the construction of the sequence $(\phi_t(x))$ (see argument below).

Modification (B) affects only the locomotives. Modification (A) in step $t+1$ alters the terminal letters of some wagons. By (8), every such letter is followed by a locomotive of a t-train. Thus if a letter were modified infinitely many times, then this or the following coordinate would fall into the domain of a locomotive for infinitely many indices.

It is now clear that the maps ϕ_t converge (coordinatewise) on X'. Thus the map

$$\phi = \lim_t \phi_t$$

is well defined on the full set X'. Obviously, ϕ is Borel measurable and commutes with the shift. Note that, by (21),

(24) for each $x \in X'$ and $t \geq 1$, $\phi(x)$ is a concatenation of regular and irregular t-trains.

Minimal almost 1-1 *extension*. We first need to prove that ϕ provides a Borel isomorphism between (X, S) and a subshift (Y, S) which is a minimal almost 1-1 extension of (Z, S).

Consider an $x \in X'$. Set $y = \phi(x)$. Let $B = y[n, m)$ be a block of y. By the definition of X', we can find t large enough so that

(25) $[n, m)$ does not intersect any domain of a locomotive for any $s > t$.

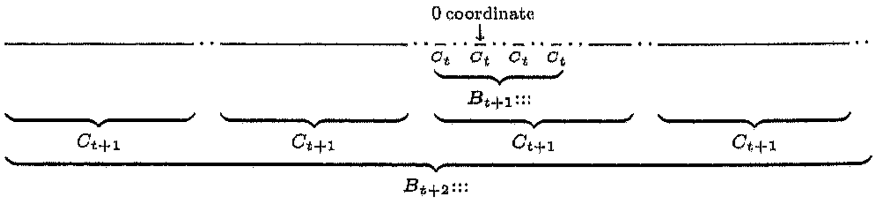
Moreover, by (23), we can assume that

$$\phi_{t+1}(x)[n, m) = \phi(x)[n, m) = B.$$

It is now seen that B is a part of a regular $(t+1)$-train in $\phi_{t+1}(x)$, thus, due to (14), it occurs as a part of W_{t+2}. This implies that B is introduced in a syndetic way in $\phi_{t+2}(x)$ (see (19)). By (16), all these occurrences remain unaltered in further steps, so they occur in y. We have proved that y satisfies the well known criterion for having a minimal orbit-closure. Moreover, by the above argument, any block occurring in y also occurs in $y' = \phi(x')$ for any $x' \in X'$ (because it occurs in W_t for some t). Hence $\phi(X')$ is contained in one minimal subshift (Y, S) over the alphabet Ξ. It is obvious (by (10)) that each element y of (Y, S) has an element of (Z, S) in the bottom row, thus (Y, S) is an extension of (Z, S). Denote by $\pi_Y : Y \to Z$ the projection on the bottom row in Y. By minimality of (Z, S), this projection is surjective.

At this point, we can note that the diagram in the assertion of the theorem commutes, because ϕ preserves the bottom row.

We now prove that π_Y provides an almost 1-1 extension. Let $y = \phi(x)$ ($x \in X'$) and let $z = \pi_Y(y)$. Note that every block of y having C_t in the bottom row starts with W_t. By minimality of (Y, S), this property passes to all elements of Y. Recall that C_t occurs many (more than 3) times in B_{t+1}:::. Thus we can find a $z_0 \in Z$ such that for each $t \geq 1$ the zero coordinate is contained in the domain of a non-extreme (neither initial nor terminal) occurrence of C_t in the starting B_{t+1}::: of some occurrence of C_{t+1} in z_0.



It is now seen that any preimage by π_Y of z_0 has the block W_{t+1} around zero coordinate, and the domains of these blocks expand in both directions

as $t \to \infty$. This determines that the preimage is unique, and the almost 1-1 extension is established.

Define $Y' = \pi_Y^{-1}(Z')$. Being the preimage of a full set, Y' is a full set in (Y, S). We show that ϕ is an invertible map from X' onto Y'. To prove this we construct a map ψ inverse to ϕ on Y'. First note that

(26) $\phi(X')$ is dense in Y.

Obviously, since ϕ preserves the bottom row, we have $\phi(X') \subset Y'$. Let $y \in Y$ be such that $\pi_Y(y) \in Z'$. For each t, y can be decomposed as a concatenation of regular and irregular t-trains (use (24) and (26)). For each $t \geq 1$ define x_t as the element obtained by t-decoding all t-trains of y. Compare x_t with x_{t+1}. The differences may occur only in the locomotives of the $(t+1)$-trains (see (18)). On the other hand, since the bottom row is an element of Z', and since the distribution of the locomotives depends only on the bottom row, y satisfies the condition that every coordinate n falls into the domain of a locomotive for at most finitely many indices t. Combining the last two statements we find that

(27) x_t converges coordinatewise to some x.

We define $\psi(y) = x$. Consider an interval $[n, m)$. For t large enough, $[n, m)$ is contained in the domain of a single t-train (satisfying (25) is possible whenever the bottom row belongs to Z'). Thus the corresponding block of x_t occurs in X (as a part of an original t-train). This implies that $x \in X$, because X is closed. The bottom row of x is the same as $\pi_Y(y) \in Z'$, hence $x \in X'$.

Now, check $\phi(x)$ at a coordinate n. As before, by the definition of Z', we can choose t so large that:

(28) for every $s \geq t$ neither n nor $n+1$ are in the domain of the locomotive of an s-train.

Thus the regular t-train of $\phi_t(x)$ whose domain C contains n (see (20)) coincides with the corresponding t-train of $\phi(x)$, except perhaps for its last cell (by (28) this regular t-train is not a part of a larger locomotive, hence when applying the s-codes for $s > t$ only modification (A) can affect it). Hence

(29) the original t-train of x with domain C can be obtained by t-decoding the corresponding (regular or irregular) t-train of $\phi(x)$.

On the other hand, let $t' \geq t$ be so large that

(30) $x_{t'}$ coincides with x on C.

By the definition of $x_{t'}$, the t'-train of $x_{t'}$ whose domain contains C is obtained by t'-decoding the corresponding t'-train of y. Since C is not in the domain of any locomotive for any indices between t and t', the t'-decoded

t'-train of y coincides on C with the t-decoded t'-train of y (see (18)). But the part with domain C of the above t-decoded t'-train of y is simply the t-decoded t-train of y whose domain is C. In view of (30), we have shown that

(31) the t-train of y whose domain is C, except perhaps for its last cell, can be obtained by t-coding the corresponding t-train of x.

Combining (29) and (31), we conclude that the t-train of y coincides with the t-train of $\phi(x)$ except perhaps for the last cell. But the coordinate of the last cell cannot equal n (in which case $n+1$ would fall into the domain of the following locomotive, contradicting (28)), hence y and $\phi(x)$ agree at n. We have proved that $\phi \circ \psi =$ identity on Y'.

Summarizing, ϕ provides a Borel isomorphism between (X, S) and (Y, S). We will soon prove that in fact it is also Borel*.

Homeomorphism of measures. By the proved properties of ϕ, the map $\phi^* : P(X) \to P(Y)$ is affine, and invertible. By the elementary properties of the weak* topology, it is seen that $\phi^* = \lim \phi_t^*$. We prove that this convergence is uniform. Because all the maps ϕ_t^* are continuous, this will prove the continuity of ϕ^*. The weak* topology of invariant measures in case of a subshift is metrizable by the following metric:

(32) $$d^*(\mu, \nu) = \sum_{B \in \mathcal{B}} c_B |\mu(U_B) - \nu(U_B)|,$$

where $\mathcal{B}$ is the set of all finite blocks over the alphabet Ξ, c_B is a fixed summable normalized (i.e., with sum 1) sequence of positive coefficients indexed by the countable set $\mathcal{B}$, and U_B is the closed and open cylinder defined by the block B, i.e., $U_B = \{x \in X : x[0, |B|) = B\}$.

It is well known (by the Birkhoff Ergodic Theorem) that for each ergodic measure μ on X, μ-almost every point $x \in X$ is *generic* for this measure, which, in case of a subshift, can be expressed as follows: for every $B \in \mathcal{B}$,

(33) $$\mu(U_B) = D_{\mathbb{Z}}\{n : x[n, n + |B|) = B\}$$

($D_{\mathbb{Z}}$ denotes the density in $\mathbb{Z}$ of a subset). In other words, a point x is generic for μ if and only if the mass assigned by μ to each cylinder U_B coincides with the frequency with which B occurs in x. Since for each $t \geq 1$, ϕ_t is a topological isomorphism, the point $\phi_t(x)$ is then generic for $\phi_t^*(\mu)$.

Fix an $\varepsilon > 0$. We can divide $\mathcal{B}$ in two parts: a finite set $\mathcal{B}^\varepsilon$ and $\mathcal{B} \setminus \mathcal{B}^\varepsilon$ such that the sum of the coefficients c_B over the latter set is less than ε. Let $k(\varepsilon) = \max\{|B| : B \in \mathcal{B}^\varepsilon\}$. By (22), we can find t so large that

$$\sum_{s \geq t} \frac{3}{r_t} < \frac{\varepsilon}{k(\varepsilon)}.$$

For every $s \geq t$ the upper density of the set of coordinates where $\phi_t(x)$ differs from $\phi_s(x)$ is hence less than $\varepsilon/k(\varepsilon)$ (because $\phi_{t+1}(x)$ differs from $\phi_t(x)$ along at most $3l_{t+1}$ coordinates within the domain of each $(t+1)$-train and the length of each $(t+1)$-train is at least $r_{t+1}l_{t+1}$). This implies that

(34) the frequencies with which a block $B \in \mathcal{B}^\varepsilon$ occurs in $\phi_t(x)$ and in $\phi_s(x)$ may differ by at most $k(\varepsilon)(\varepsilon/k(\varepsilon)) = \varepsilon$.

Combining (32), (33), the definition of $\mathcal{B}^\varepsilon$, and (34), we obtain

$$d^*(\phi_t(\mu), \phi_s(\mu)) < 2\varepsilon.$$

This yields the desired uniform convergence for ergodic measures, which, by convexity of the metric d^*, extends to all invariant measures. The map ϕ^* has been proved continuous, hence, as an invertible map between compact sets, it is a homeomorphism. This completes the proof of Theorem 1. ■

Remarks on reducing the alphabet. The size of the alphabet used to define the flow (Y, S) is $\#\Sigma(\#\Sigma + 1)\#\Lambda$. It might be interesting to note that the flow (Y, S) can be represented as a subshift over the same alphabet Σ as originally used by (X, S). This is possible thanks to the power of a Borel* isomorphism, more precisely, by the fact that it preserves topological entropy. Namely, we have the following

LEMMA 2. *The subshift (Y, S) of Theorem 1 is topologically isomorphic to a subshift over the alphabet Σ.*

Proof. Set $p = \#\Sigma$. Consider the following two cases:

(a) the topological entropy $h(X)$ of (X, S) is equal to $\ln p$,
(b) $h(X) < \ln p$.

The case (a) is trivial: the flow (X, S) is the full shift over Σ (apply [D-G-S, Theorem 20.11] and some standard arguments). Such a flow contains fixpoints, thus it admits no minimal topological factors except for the one-point flow, hence our theory does not apply.

Assume (b). By a well known formula, we have

$$h(Y) = \lim_n \frac{\ln \#\mathcal{B}_n}{n},$$

where $\mathcal{B}_n$ denotes the collection of all blocks of length n occurring in Y. Because $h(Y) = h(X) < \ln p$, an easy calculation shows that

(35) $$\#\mathcal{B}_{n_0} \leq p^{n_0} - 1 \quad \text{and} \quad \#\mathcal{B}_{n_0+1} \leq p^{n_0+1}$$

for some sufficiently large n_0. Let t_0 be such that $|B_{t_0}| \geq n_0$ and $|C_{t_0}| \geq n_0^2$ (we refer to the objects defined in the proof of Theorem 1). Then every C_{t_0}-block has length at least n_0^2. Every block that long can be decomposed as a concatenation of subblocks whose lengths are either n_0 or n_0+1. Fix one such decomposition starting with a subblock of length n_0 for each C_{t_0}-block, so that all C_{t_0}-blocks of the same length are cut in the same places. This induces a decomposition of all t_0-trains of Y. The starting subblock of each t_0-train consists of the initial n_0 symbols of the locomotive W_{t_0}, hence is common for all t_0-trains. By (35), there exist 1-1 correspondences Ψ from $\mathcal{B}_{n_0}$ into all blocks of length n_0 over the alphabet Σ, and Ψ_1 from $\mathcal{B}_{n_0+1}$ into all blocks of length n_0+1 over Σ. Moreover, there remains at least one block B_0 of length n_0 over Σ unused as a Ψ-image.

The desired topological isomorphism between (Y,S) and a subshift over Σ is obtained by a code replacing each subblock of each t_0-train in Y by its image under Ψ or Ψ_1 (depending on whether its length is n_0 or n_0+1), except for the starting subblock of each t_0-train which we replace by B_0. It is clear that the above code yields a continuous map commuting with the shift. Its injectivity is immediate: we can determine the positioning of the t_0-trains in the preimage from the positioning of the occurrences of B_0 in the image. Then, knowing the lengths of consecutive t_0-trains, we can determine where they are cut into subblocks. Finally, reversing Ψ and Ψ_1, we can determine the preimage. ■

We take this opportunity to make a general comment concerning block codes. In 1969 G. Hedlund proved that every factor map $\pi : X \to Z$ between two subshifts (the first over Σ, the second over Λ) is induced by a *block code*, i.e., there exists a map $\Pi : \Sigma^{2r+1} \to \Lambda$ such that $\pi(x)$ at position n is equal to $\Pi(x[n-r,n+r])$. The parameter r is often called the *radius* of the code. Because all the codes appearing it this paper have a slightly different form, it might be interesting to see how general this form is.

Definition 1. By a *length-preserving code* we mean any function Ψ defined on some finite collection $\mathcal{B}$ of blocks over Σ into the blocks over Λ such that $|\Psi(B)| = |B|$ for each $B \in \mathcal{B}$.

We say that a map π between two subshifts (X,S) and (Z,S) is *induced* by a length-preserving code if there exists a length-preserving code Ψ such that

- each $x \in X$ can be decomposed in a unique way as an infinite concatenation of blocks belonging to the domain of Ψ, and
- $z = \pi(x)$ coincides with the sequence obtained from x by replacing all blocks in the above concatenation by their images under Ψ.

It is not hard to see that any map induced by a length-preserving code is continuous and commutes with the shift transformation, hence is a factor map. It is not true that every factor map between two subshifts is induced by a length-preserving code. However, we now prove that it is always so whenever the factor is minimal. A similar result has been obtained for factor maps between Toeplitz flows in [D-K-L, Theorem 1].

Proposition 1. *Let $\pi : X \to Z$ be a factor map between two subshifts, (X,S) and (Z,S), where (Z,S) is minimal and non-periodic. Then π is induced by a length-preserving code Ψ.*

Proof. Let Π be the classical block code inducing π and let r denote its radius. Let B be a block of length at least $2r$, having non-overlapping occurrences in Z (see Lemma 1). For each $x \in X$ let $(n_i(x))_{i\in\mathbb{Z}}$ denote the starting positions of the consecutive occurrences of B in $\pi(x)$. Consider the family $\mathcal{B}$ of blocks occurring in X as $x[n_i(x)+r, n_{i+1}(x)+r)$ for some $x \in X$ and $i \in \mathbb{Z}$. Since B occurs syndetically in Z, this family is finite. To define the length-preserving code Ψ on $\mathcal{B}$ we first apply the block code Π, from which we can determine all letters of the image blocks except for the extreme r positions at both ends. But we know that each of these image blocks ends with the initial subblock of B of length r and starts with the remaining part of B of length $|B|-r \ge r$. This covers the missing r positions on both sides, thus the image blocks are fully determined. It is obvious that the length-preserving code Ψ so obtained induces π, as desired. ■

The general case. The construction used in the proof of Theorem 1 can be easily generalized to the case where (X,T_X) is an arbitrary (non-symbolic) flow. Temporarily, we maintain the assumption that (Z,S) is a subshift. Clearly, the resulting almost 1-1 extension, (Y,T_Y), will no longer be symbolic.

Theorem 2. *Let (X,T_X) be an arbitrary extension of a minimal non-periodic subshift (Z,S). Then (X,T_X) is Borel* isomorphic to some minimal almost 1-1 extension (Y,T_Y) of (Z,S). The corresponding diagram commutes* (see formulation of Theorem 1).

Proof. We represent (X,T_X) as a subshift over the infinite alphabet X, i.e., we identify each $x \in X$ with the sequence $(\ldots,T_X^{-1}(x),x,T_X(x),T_X^2(x),\ldots) \in X^{\mathbb{Z}}$. Obviously, such a representation is a topological isomorphism. From this point on, we repeat the whole proof of Theorem 1, which leads to obtaining (Y,S) as a subshift over the infinite alphabet $\Xi = X \times (X \cup \{\square\}) \times \Lambda$. The minor differences are the following:

- Before we start, we fix a sequence (ε_t) decreasing to zero. The number of all possible regular t-trains is infinite, nevertheless, by compactness of $\phi_t(X)$, there exists a finite collection $\mathcal{T}_t$ of regular t-trains such that every regular t-train is close to some of the t-trains from $\mathcal{T}_t$, where by "close" we understand that the distance at each coordinate is less than ε_t. In (5) we define $r_t = \#\mathcal{T}_t$.

• In (14) we demand that every t-train from $\mathcal{T}_t$ occurs in W_{t+1}.

• In the proof of minimality of Y, we observe that B is ε_t-close to a block occurring syndetically in $\phi_{t+2}(x)$.

• In (27) we add that for each $n \in \mathbb{N}$ the sequence $(x_t(n))$ is eventually constant.

A bit more complicated is the adaptation of the part of the proof concerning the homeomorphism of measures. We only outline the most essential changes:

• In (32) the cylinders U_B are replaced by an appropriate countable family of continuous functions, each depending on finitely many coordinates.

• In (33) frequencies are replaced by appropriate averages.

From that point on we conduct analogous estimations of the averages calculated for the functions chosen. ∎

The next case which we discuss is where (Z, T_Z) is a non-symbolic flow, but it is strictly ergodic. In fact, most of the known examples of non-uniquely ergodic minimal flows are obtained as extensions of certain strictly ergodic flows. In this situation we can apply our theory by finding a "symbolic replacement" for (Z, T_Z). We begin with the finite entropy case.

THEOREM 3. *Let (Z, T_Z) be a strictly ergodic non-periodic flow having finite topological entropy and let (X, T_X) be an extension of (Z, T_Z). Then the assertion of Theorem 2 holds. If (X, T_X) is symbolic then so is (Y, T_Y).*

Proof. Applying to (Z, T_Z) a theorem of Denker–Keane [D-K, Theorem 20], we can construct a subshift $(\widetilde{Z}, S)$ and a finitary isomorphism between these flows, i.e., a continuous injective map $\varphi : Z' \to \widetilde{Z}$ with continuous inverse, where Z' is a residual full subset of Z. Moreover, by [D-K, Corollary 8], $(\widetilde{Z}, S)$ is strictly ergodic, thus φ is also a Borel* isomorphism.

We introduce a system of commutative diagrams involving two additional intermediate flows. We use the method of obtaining topological extensions by joinings. In all product spaces the actions are defined coordinatewise and we omit their notation. Therefore the flows are denoted by the corresponding spaces only. Define $X' = \pi_X^{-1}(Z')$ and note that X' is a full set in X.

(a) Let $\widetilde{X} = \overline{\{(x, \varphi\circ\pi_X(x)) : x \in X'\}} \subset X \times \widetilde{Z}$. The projection π_1 on the first coordinate provides a Borel* isomorphism between $\widetilde{X}$ and X. To see this, note that $\pi_1^{-1}(X')$ is a full set in $\widetilde{X}$, and consider an element $(x, \widetilde{z})$ of this set, i.e., such that $x \in X'$. By the definition of $\widetilde{X}$, there exists a sequence (x_n) of elements of X' such that $(x_n, \varphi\circ\pi_X(x_n)) \to (x, \widetilde{z})$. By continuity of φ on Z', we have $\widetilde{z} = \varphi\circ\pi_X(x)$. We have proved that π_1 is 1-1 on the full set $\pi_1^{-1}(X')$. Continuity of the adjoint map π_1^* follows immediately from the continuity of π_1.

We can now apply our Theorem 2 to the extension $\pi_2 : \widetilde{X} \to \widetilde{Z}$ (or Theorem 1 if X is a subshift, because then $\widetilde{X}$ is obviously also a subshift). Let $\widetilde{Y}$, $\pi_{\widetilde{Y}} : \widetilde{Y} \to \widetilde{Z}$, and $\widetilde{\phi} : \widetilde{X} \to \widetilde{Y}$ denote the flow, the almost 1-1 extension, and the Borel* isomorphism obtained, respectively.

(b) Let $Y = \overline{\{(\widetilde{y}, z) : z \in Z',\ \pi_{\widetilde{Y}}(\widetilde{y}) = \varphi(z)\}} \subset \widetilde{Y} \times Z$. Now, π_1 provides a Borel* isomorphism between Y and $\widetilde{Y}$ (use the same argument as for $\widetilde{X}$ and X, but this time with continuity of φ^{-1}). On the other hand, π_2 provides an almost 1-1 extension of Z. Indeed, if $z \in Z'$ is such that $\varphi(z)$ has a one-point fiber for $\pi_{\widetilde{Y}}$, then z has a one-point fiber for π_2 (similar argument again). It could be proved that the Y so defined is minimal, but we can avoid proving this by letting Y be a minimal subset of the set previously defined. By minimality of $\widetilde{Y}$ and Z, both projections remain onto, hence their required properties remain satisfied.

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, leftrightarrow, "\widetilde{\phi}"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & & \widetilde{Y} \arrow[dl, "\pi_{\widetilde{Y}}"'] \\
X \arrow[dr, "\pi_X"'] & \widetilde{Z} & Y \arrow[u, "\pi_1"'] \arrow[dl, "\pi_2"] \\
& Z \arrow[u, "\varphi"'] &
\end{tikzcd}$$

Once this is done, our assertion holds for X, Z and Y with ϕ defined on (an appropriate subset of) X as $\pi_1^{-1}\circ\widetilde{\phi}\circ\pi_1^{-1}$ (see diagram). ∎

A similar method involving infinite products leads to the following

THEOREM 4. *Theorem 2 also holds if (Z, S) is a subshift over the countable alphabet $\mathbb{N}\cup\{\infty\}$ (this time we do not assume strict ergodicity).*

Proof. Let $z \in Z$. For each $n \in \mathbb{N}$ denote by z_n the sequence over the finite alphabet $\{1, \ldots, n\}$ obtained from z by replacing all letters of the alphabet $\mathbb{N}\cup\{\infty\}$ which are larger than n (including ∞) by n. Let Z_n be the corresponding factor of Z.

If for each n, Z_n is periodic, then Z represents the rotation of a compact monothetic group (so-called p-adic adding machine, see next section on Toeplitz flows). Such a flow has topological entropy zero and is strictly ergodic. This case has been dealt with in Theorem 3.

So suppose the flows Z_n are non-periodic (for n sufficiently large). We can apply Theorem 2 to X and each Z_n, which produces a sequence of flows Y_n. The flow Y will be defined as an appropriate joining within the infinite product $\prod Y_n$. We omit the details of the definition of Y, and proving its required properties. The arguments are similar to those used for joinings in the proof of Theorem 3. ∎

THEOREM 5. *Theorem* 3 *also holds if* (Z, T_Z) *is strictly ergodic and has infinite entropy.*

Proof. By [D-K, Theorem 18], we can find a subshift $(\widetilde{Z}, S)$ over the countable alphabet $\mathbb{N} \cup \{\infty\}$, finitarily (and hence Borel*) isomorphic to (Z, T_Z). The assertion follows by the same proof as in Theorem 3, with Theorem 4 applied instead of Theorem 2. ■

The problem with generalizing our Theorem 3 (and 5) to the non-strictly ergodic case lies in finding an appropriate symbolic representation $(\widetilde{Z}, S)$ for (Z, T_Z).

QUESTION. Let (Z, T_Z) be a minimal non-periodic topological flow with finite topological entropy. Does there exist a subshift representation $\varphi : (Z, T) \to (\widetilde{Z}, S)$ which is both a Borel* isomorphism and a universally (for each invariant measure) finitary isomorphism?

COMMENT. The starting point in the quoted construction of [D-K] is finding a Rokhlin tower with an open base U whose boundary ∂U is a null set. This is done for a single invariant measure. It can also be easily done for at most countably many ergodic measures. But even then we do not know whether the universally finitary isomorphism obtained induces a continuous map on invariant measures (this problem does not appear in the case of finitely many ergodic measures, because any affine map defined on a finite-dimensional simplex is continuous). Also without strict ergodicity there is a danger that some unwanted invariant measures might be supported by $\widetilde{Z} \setminus \widetilde{Z}'$.

REMARK 1. Theorem 1 can also be proved for $\mathbb{Z}^2$-actions. A proof based on the same principles works in the case where both horizontal and vertical shifts on $\mathbb{Z}$ are minimal non-periodic. As a consequence, theorems analogous to Theorems 2 through 5 are valid. Our Theorems 2 through 5 may be useful in producing concrete examples of topological dynamical systems with prescribed properties, for instance, as was done in [B-G-K].

Characterization of Toeplitz flows. Toeplitz sequences have been introduced in 1969 by Jacobs and Keane [J-K], although particular examples were known much earlier (see e.g. [Ga-H], [O], [G-H]). Some general topological dynamical properties such as minimality and strict ergodicity (for the regular case) were established in these earlier works. The maximal equicontinuous factor was identified in [E] (1970) for regular Toeplitz flows. Topological characterization of all Toeplitz flows as minimal almost 1-1 symbolic extensions over the so-called p-adic adding machines is stated (without proof) in [M-P] (1979). Because of the importance of this characterization for our further investigations, a simple proof of this fact is presented below.

Since 1984 there have appeared various constructions of Toeplitz flows exhibiting a variety of topological and, to some extent, spectral invariants, such as the set of invariant measures, topological centralizer, topological entropy, topological coalescence, point spectrum (see e.g. [Wi], [D1,2,3], [B-K1,2], [D-I], [D-K-L], [I-L], [I], [D-L]).

Much less was known about possible realizations within this class of measure-theoretic invariants such as rank, covering number, spectral multiplicity or order of the quotient group (of the measure-theoretic centralizer). However, multiple realizations of these invariants were obtained in a larger class of flows including Morse sequences and other extensions over the rational point spectrum (see e.g. [L], [F-K-M], [K-L]).

Our original desire, motivated by several discussions with some other mathematicians interested in this subject, was to fully characterize Toeplitz flows from the measure-theoretic point of view. The missing link was a "symbolic version" of the Furstenberg–Weiss theorem. In view of the results of the preceding sections, such a characterization is now possible even at the level of a Borel* isomorphism.

DEFINITION 2 [J-K]. A *Toeplitz sequence* is a non-periodic element $x \in \Sigma^{\mathbb{Z}}$ such that

$$(\forall n \in \mathbb{Z})(\exists p \in \mathbb{N})(\forall k \in \mathbb{Z}) \quad x(kp + n) = x(n),$$

i.e., each position in x is a periodic position.

A subshift (X, S) is called a *Toeplitz flow* if it is the orbit-closure of some Toeplitz sequence. Toeplitz flows are well known to be minimal.

The topological maximal equicontinuous factor (see e.g. [A] for the definition) of a Toeplitz flow is known to have the form of a so-called *p-adic adding machine* $(G_p, 1)$ (see e.g. [Wi]). One of the possible ways of viewing the group G_p is the following: its elements are sequences $(j_t)_{t\ge1} \in \prod_{t\ge1}\{0, \ldots, p_t - 1\}$ such that for each t, $j_{t+1} \equiv j_t \bmod p_t$, where $(p_t)_{t\ge1}$ is a fixed increasing sequence of positive integers satisfying $p_t \mid p_{t+1}$. Addition is defined coordinatewise modulo p_t. Then $1 := (1, 1, 1, \ldots)$ is a topological generator of the compact monothetic group G_p (by the same letter 1 we also denote the rotation by the generator 1 in G_p; see [H-R] for more details on G_p). We view G_p as a compactification of $(\mathbb{Z}, +)$ by writing k instead of $k1$ (multiplication by integers is well defined in G_p). Recall that the sets $p_t G_p$ form a base for the topology at 0 in G_p. The flow $(G_p, 1)$ is strictly ergodic and the invariant measure is the Haar measure λ.

THEOREM 6 [M-P]. *A subshift* (X, S) *is a Toeplitz flow if and only if it is a minimal almost* 1-1 *extension of some p-adic adding machine.*

Proof. If (X, S) is a Toeplitz flow then the required properties are fulfilled for the maximal equicontinuous factor $(G_p, 1)$ of (X, S) (see [Wi]).

Conversely, suppose (X,S) satisfies the above conditions with some $(G_p,1)$ and let x be a one-point fiber of the factor $\pi_X: X\to G_p$. We show that x itself is a Toeplitz sequence (by minimality, this will complete the proof). Suppose the converse, i.e., that there exists a non-periodic position n in x. In particular, for each t we can find a k_t such that

$$x(k_tp_t+n)\neq x(n).$$

Choosing if necessary a convergent subsequence we define

$$x'=\lim_{t\to\infty}S^{k_tp_t}x.$$

Of course, $x\neq x'$, because they differ at position n. On the other hand,

$$\pi_X(x')=\lim k_tp_t+\pi_X(x),$$

by the properties of π_X. But k_tp_t converges to 0 in G_p, from which it follows that $\pi_X(x')=\pi_X(x)$, a contradiction. ∎

The above characterization allows us to apply the theorems previously obtained on almost 1-1 extensions:

THEOREM 7. *Any symbolic topological extension (X,S) of a p-adic adding machine is Borel* isomorphic to a Toeplitz flow.*

Proof. Use Theorem 3. ∎

REMARK 2. A similar passage (in a very particular case) can be found in [D2], where a Borel* representation in the form of a Toeplitz flow is constructed for a (non-transitive) flow obtained as the closure of a union of many Toeplitz flows factoring to the dyadic integers.

REMARK 3. The statement reversing Theorem 7 is false. For example there exists a strictly ergodic flow having an adding machine as a measurable but not topological factor. Such a flow is measure-theoretically isomorphic to a Toeplitz flow (see Theorem 8). By strict ergodicity, this isomorphism is Borel*.

Finally, we state the measure-theoretic characterization of Toeplitz flows, as a consequence of which all the measure-theoretic information that was known for systems factoring to some p-adic adding machines is now known to be realizable within the class of Toeplitz flows. To pass from measure-preserving transformations to topological flows we apply a strengthening of the famous Jewett–Krieger Theorem, due to Weiss (1985), in which the entire diagram of a measure-theoretic factor is replaced by a strictly ergodic topological model.

THEOREM 8. *An ergodic dynamical system (X,μ,T_X) is measure-theoretically isomorphic to a strictly ergodic Toeplitz flow if and only if it has finite entropy and its set of eigenvalues contains infinitely many rationals.*

Proof. Clearly, each Toeplitz flow has finite entropy and it admits infinitely many rational eigenvalues, and so does every dynamical system (X,μ,T_X) measure-theoretically isomorphic to a Toeplitz flow.

For the converse, first note that having infinitely many rational eigenvalues is equivalent to having a p-adic adding machine $(G_p,\lambda,1)$ as a measure-theoretic factor. By a theorem of Weiss [W], (X,μ,T_X) is measure-theoretically isomorphic to some topological extension $(\widetilde{X},\widetilde{\mu},S)$ of $(G_p,1)$, where $(\widetilde{X},S)$ is a strictly ergodic subshift. By Theorem 7, $(\widetilde{X},S)$ is Borel* (hence measure-theoretically) isomorphic to a strictly ergodic Toeplitz flow. ∎

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Two-sided estimates for the approximation numbers of Hardy-type operators in L^∞ and L^1

by

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Abstract. In [2] and [3] upper and lower estimates and asymptotic results were obtained for the approximation numbers of the operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by $(Tf)(x) := v(x)\int_0^\infty u(t)f(t)\,dt$ when $1 < p < \infty$. Analogous results are given in this paper for the cases $p = 1, \infty$ not included in [2] and [3].

1. Introduction. In [2] and [3] the operator $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$Tf(x) = v(x)\int_0^\infty u(t)f(t)\,dt \tag{1.1}$$

was studied in the case $1 < p < \infty$, with u, v real-valued functions and $u \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^+)$, $v \in L^p(\mathbb{R}^+)$, $p' = p/(p-1)$. Estimates for the approximation numbers $a_n(T)$ of T were obtained in [2], but the procedure for extracting the upper and lower bounds from the results is rather cumbersome to apply. This deficiency was overcome in [3] where asymptotic bounds for the approximation numbers which are easy to check in practice were determined. Specifically, it was proved that

$$\lim_{n\to\infty} na_n(T) = \frac{1}{\pi}\int_0^\infty |u(t)v(t)|\,dt \tag{1.2}$$

when $p = 2$; and when $p \neq 2$,

$$\begin{aligned}\frac{1}{4}\alpha_p \int_0^\infty |u(t)v(t)|\,dt &\le \liminf_{n\to\infty} na_n(T)\\ &\le \limsup_{n\to\infty} na_n(T) \le \alpha_p \int_0^\infty |u(t)v(t)|\,dt\end{aligned} \tag{1.3}$$

for some constant α_p depending on p. Further in [3], two-sided estimates

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