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Multiplier transformations on H^p spaces

by

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Abstract. The authors obtain some multiplier theorems on H^p spaces analogous to the classical L^p multiplier theorems of de Leeuw. The main result is that a multiplier operator $(Tf)^\wedge(x) = \lambda(x)\widehat{f}(x)$ $(\lambda \in C(\mathbb{R}^n))$ is bounded on $H^p(\mathbb{R}^n)$ if and only if the restriction $\{\lambda(\varepsilon m)\}_{m\in\Lambda}$ is an $H^p(\mathbb{T}^n)$ bounded multiplier uniformly for $\varepsilon > 0$, where Λ is the integer lattice in $\mathbb{R}^n$.

1. Introduction. Consider the n-dimensional Euclidean space $\mathbb{R}^n$; let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz test functions on $\mathbb{R}^n$ and λ be any function on $\mathbb{R}^n$. *The multiplier operator* T associated with λ is defined by $(Tf)^\wedge(\xi) = \lambda(\xi)\widehat{f}(\xi)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Let X, Y be two function spaces on $\mathbb{R}^n$ with norms $\|\ \|_X$ and $\|\ \|_Y$, respectively. If $\mathcal{S}(\mathbb{R}^n)$ is dense in both X and Y, and if there exists a constant C such that

$$\|Tf\|_Y \le C\|f\|_X$$

uniformly for $f \in \mathcal{S}(\mathbb{R}^n)$, then we say that T is a bounded operator from X to Y with finite norm

$$\|T\| = \sup_{\|f\|_X=1} \|Tf\|_Y \le C.$$

We denote this by writing $T \in (X, Y)$.

The n-torus $\mathbb{T}^n$ can be identified with $\mathbb{R}^n/\Lambda$, where Λ is the unit lattice which is the additive group of points in $\mathbb{R}^n$ having integral coordinates. The multiplier operator $\widetilde{T}_\varepsilon$ on $\mathbb{T}^n$ associated with a function λ on $\mathbb{R}^n$ is defined by

$$\widetilde{T}_\varepsilon \widetilde{f}(x) \sim \sum_{m\in\Lambda} \lambda(\varepsilon m) a_m e^{2\pi i m\cdot x}$$

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for all $\widetilde{f} \in \mathcal{S}(\mathbb{T}^n)$, where
$$\sum_{m\in\Lambda} a_m e^{2\pi i m\cdot x}$$
is the Fourier series of $\widetilde{f}$.

Suppose that $(X, \|\ \|_X)$ and $(Y, \|\ \|_Y)$ are two function spaces on $\mathbb{T}^n$ such that the Schwartz space $\mathcal{S}(\mathbb{T}^n)$ is dense in both X and Y. We write $\widetilde{T}_\varepsilon \in (X,Y)$ if $\widetilde{T}_\varepsilon$ is a bounded operator from X to Y:
$$\|\widetilde{T}_\varepsilon \widetilde{f}\|_Y \le C\|\widetilde{f}\|_X$$
for all $\widetilde{f} \in \mathcal{S}(\mathbb{T}^n)$ with a constant C independent of $\widetilde{f}$.

In [8], de Leeuw proved that if $p \ge 1$ and if λ is continuous on $\mathbb{R}^n$, then T is bounded in $L^p(\mathbb{R}^n)$ if and only if $\widetilde{T}_\varepsilon$ is uniformly bounded in $L^p(\mathbb{T}^n)$ for $\varepsilon > 0$. Kenig and Thomas [6] extended this result to the related maximal operators. Auscher and Carro [1] studied a discrete version of de Leeuw's theorem on relations between multipliers on $\mathbb{R}^n$, $\mathbb{T}^n$, and $\mathbb{Z}^n$. But all their considerations are in L^p spaces for $p \ge 1$. However, in the last twenty years, H^p spaces have played an important role in harmonic analysis. Therefore an interesting and natural question is whether de Leeuw's theorem is still true in Hardy space H^p, particularly when $0 < p \le 1$. This paper will give an affirmative answer to this question. Our proofs will be based on the theory of S-functions, and the atomic decomposition of H^p spaces.

2. Basic notation.

In this section we introduce some basic definitions and notation, most of which can be found in Stein and Weiss' book [9]. Let Λ be the unit lattice. The Poisson kernel $P_t(x)$ on $\mathbb{R}^n$ is defined by
$$P_t(x) = C_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}}, \quad C_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}.$$
The Poisson kernel on $\mathbb{T}^n$ is defined by
$$\widetilde{P}_t(x) = \sum_{m\in\Lambda} e^{-2\pi|m|t} e^{2\pi i m\cdot x}.$$
By the Poisson summation formula we easily get
$$\widetilde{P}_t(x) = \sum_{m\in\Lambda} P_t(x+m).$$
Using these two Poisson kernels, we can now give the definitions of the Hardy spaces H^p $(p>0)$ on both $\mathbb{R}^n$ and $\mathbb{T}^n$:
$$H^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^p(\mathbb{R}^n)} = \|\sup_{t>0}|P_t * f|\,\|_{L^p(\mathbb{R}^n)} < \infty\},$$
$$H^p(\mathbb{T}^n) = \{\widetilde{f} \in \mathcal{S}'(\mathbb{T}^n) : \|\widetilde{f}\|_{H^p(\mathbb{T}^n)} = \|\sup_{t>0}|\widetilde{P}_t * \widetilde{f}|\,\|_{L^p(\mathbb{T}^n)} < \infty\}.$$

A few remarks are in order concerning these definitions. First of all, $\|\ \|_{H^p}$ is not a norm in general if $p \le 1$. But $\|\ \|_{H^p}$ can be used to introduce a topology in H^p, and with this topology H^p is a complete metric space. Secondly, since the Hardy–Littlewood maximal function is a bounded operator from L^p to L^p if $1 < p < \infty$, and since it majorizes the maximal functions
$$\sup_{t>0}|P_t * f| \quad \text{and} \quad \sup_{t>0}|\widetilde{P}_t * \widetilde{f}|,$$
we can easily see that $H^p = L^p$ if $1 < p < \infty$. For this reason in the sequel we restrict our attention to $0 < p \le 1$. Thirdly, many different equivalent definitions of these H^p spaces have been introduced and studied. Particularly, in their celebrated 1972 paper [4], Fefferman and Stein gave an S-function characterization of $H^p(\mathbb{R}^n)$:
$$\|f\|^p_{H^p(\mathbb{R}^n)} \approx \int_{\mathbb{R}^n}\Big\{\int_{|x-y|<t} |f * \nabla P_t(y)|^2 t^{1-n}\,dy\,dt\Big\}^{p/2} dx \tag{1}$$
where $\nabla = (\partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ is the gradient.

In [3], we proved an analogue of Fefferman and Stein's result on semisimple compact Lie groups. By analogous arguments, for the Abelian compact Lie group $\mathbb{T}^n$, we have
$$\|\widetilde{f}\|_{H^p(\mathbb{T}^n)} \approx \int_Q \Big\{\int_{\substack{y\in Q\\ |x-y|<t\le 1}} |\widetilde{f} * \nabla\widetilde{P}_t(x)|^2 t^{1-n}\,dy\,dt\Big\}^{p/2} dx, \tag{2}$$
$$\approx \int_Q \Big\{\int_{\substack{y\in Q\\ |x-y|<t<\infty}} |\widetilde{f} * \nabla\widetilde{P}_t(x)|^2 t^{1-n}\,dy\,dt\Big\}^{p/2} dx. \tag{3}$$
Here, $Q = \{x \in \mathbb{R}^n : -1/2 \le x_j < 1/2,\ j = 1,\ldots,n\}$ is the fundamental cube on which
$$\int_{\mathbb{T}^n} \widetilde{f}(x)\,dx = \int_Q \widetilde{f}(x)\,dx \quad \text{for all functions } \widetilde{f} \text{ on } \mathbb{T}^n.$$

Finally, throughout this paper, the letter "C" will denote (possibly different) constants that are independent of the essential variables in the argument; this independence will be clear from the context.

3. The "if part" of the main theorem.

In this section, using the S-function characterization (1)–(3) of H^p spaces, we prove $T \in (H^p(\mathbb{R}^n), H^p(\mathbb{R}^n))$ provided $\widetilde{T}_\varepsilon \in (H^p(\mathbb{T}^n), H^p(\mathbb{T}^n))$ uniformly for $\varepsilon > 0$. Moreover, some weak type theorems related to this theorem will be mentioned.

The main result in this section is the following theorem:

THEOREM 3.1. *Suppose that $0 < p \leq 1$, and that λ is a continuous function on $\mathbb{R}^n$ such that for every $\varepsilon > 0$, there exists an operator*
$$\widetilde{T}_\varepsilon \in (H^p(\mathbb{T}^n), H^p(\mathbb{T}^n))$$
given by
$$\widetilde{T}_\varepsilon \widetilde{f}_\varepsilon(x) \sim \sum_{m\in\Lambda} \lambda(\varepsilon m) a_m e^{2\pi i m\cdot x}.$$
Furthermore, suppose that the norms $\|\widetilde{T}_\varepsilon\|$ are uniformly bounded. Then λ is a multiplier of type $(H^p(\mathbb{R}^n), H^p(\mathbb{R}^n))$ and the associated multiplier operator T satisfies
$$\|T\| \leq C \sup_{\varepsilon>0} \|\widetilde{T}_\varepsilon\|.$$

Proof. We first show that

$$|\lambda(\varepsilon m)| \leq \sup_{\varepsilon>0} \|\widetilde{T}_\varepsilon\|. \tag{4}$$

In fact, consider the functions $\widetilde{f}_m(x) = e^{2\pi i m\cdot x}$, $m \in \Lambda$. Then $\widetilde{T}_\varepsilon \widetilde{f}_m(x) = \lambda(\varepsilon m) e^{2\pi i m\cdot x}$ and
$$\|\widetilde{T}_\varepsilon \widetilde{f}_m\|^p_{H^p(\mathbb{T}^n)} = |\lambda(\varepsilon m)|\, \|\widetilde{f}_m\|^p_{H^p(\mathbb{T}^n)} \leq \|\widetilde{T}_\varepsilon\|\, \|\widetilde{f}_m\|^p_{H^p(\mathbb{T}^n)}.$$
But it is easy to check that $\|\widetilde{f}_m\|_{H^p(\mathbb{T}^n)} = 1$ for all $m \in \Lambda$. Thus
$$|\lambda(\varepsilon m)| \leq \|\widetilde{T}_\varepsilon\| \leq \sup_{\varepsilon>0} \|\widetilde{T}_\varepsilon\|$$
for all $\varepsilon > 0$ and $m \in \Lambda$.

Since the set $\{\varepsilon m : \varepsilon > 0,\ m \in \Lambda\}$ is dense in $\mathbb{R}^n$ it follows that λ is bounded. Therefore, when $f \in L^2(\mathbb{R}^n)$, $\lambda\widehat{f}$ also belongs to $L^2(\mathbb{R}^n)$; hence, $\lambda\widehat{f}$ is the Fourier transform of a square integrable function. In particular, this allows us to define Tf, for $f \in \mathcal{D}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : f$ has compact support$\}$, to be the function whose Fourier transform is $\lambda\widehat{f}$. Now it is enough to show that

$$\|Tf\|_{H^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)} \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^n). \tag{5}$$

In order to do so, define $\widetilde{f}_\varepsilon$ for $\varepsilon > 0$ to be the dilation and periodized version of f, that is,
$$\widetilde{f}_\varepsilon(x) = \varepsilon^{-n} \sum_{m\in\Lambda} f\left(\frac{x+m}{\varepsilon}\right).$$
Then by the Poisson summation formula we obtain
$$\widetilde{f}_\varepsilon(x) = \sum_{m\in\Lambda} \widehat{f}(\varepsilon m) e^{2\pi i m\cdot x}.$$

From [9], we know that these $\widetilde{T}_\varepsilon$ and $\widetilde{f}_\varepsilon$ satisfy

$$\lim_{\varepsilon\to0} \varepsilon^n \widetilde{T}_\varepsilon \widetilde{f}_\varepsilon(\varepsilon x) = Tf(x). \tag{6}$$

Also by [9] there exists a nonnegative continuous function η having compact support in $\mathbb{R}^n$ and satisfying

$$\eta(0) = 1, \tag{7}$$
$$\sum_{m\in\Lambda} \eta(x+m) \equiv 1. \tag{8}$$

Thus, using Fatou's lemma, we have
$$\|Tf\|^p_{H^p(\mathbb{R}^n)}$$
$$\leq \lim_{\varepsilon\to0} \int_{\mathbb{R}^n} \eta(\varepsilon x) \Big\{ \int_0^{1/\varepsilon} \int_{|x-y|<t} \Big| \int_{\mathbb{R}^n} \varepsilon^n \widetilde{T}_\varepsilon \widetilde{f}_\varepsilon(\varepsilon y - \varepsilon z) \nabla P_t(z)\, dz \Big|^2 t^{1-n} dy\, dt \Big\}^{p/2} dx$$
$$= \lim_{\varepsilon\to0} I^p_\varepsilon.$$
Now we only have to show that

$$I^p_\varepsilon \leq C\, \|f\|^p_{H^p(\mathbb{R}^n)} \tag{9}$$

uniformly for $\varepsilon \to 0$.

In fact, using the equality
$$\nabla P_t(x/\varepsilon) = \varepsilon^{n+1} \nabla P_{\varepsilon t}(x),$$
we easily see that I^p_ε is equal to
$$\int_{\mathbb{R}^n} \eta(\varepsilon x) \Big\{ \int_0^{1/\varepsilon} \int_{|\varepsilon x - \varepsilon y|<\varepsilon t} \Big| \varepsilon^n \widetilde{T}_\varepsilon \widetilde{f}_\varepsilon(\varepsilon y - \varepsilon z) \nabla P_t(z)\, dz \Big|^2 t^{1-n}\, dy\, dt \Big\}^{p/2} dx$$
$$= \varepsilon^{-n} \int_{\mathbb{R}^n} \eta(x) \Big\{ \varepsilon^{-n} \int_0^{1/\varepsilon} \int_{|x-y|<t\varepsilon} \Big| \varepsilon^{n+1} \int_{\mathbb{R}^n} \widetilde{T}_\varepsilon \widetilde{f}_\varepsilon(y-z) \nabla P_{\varepsilon t}(z)\, dz \Big|^2$$
$$\times\, t^{1-n} dy\, dt \Big\}^{p/2} dx.$$
The above identity is obtained by changing the variables $\varepsilon x \to x$, $\varepsilon y \to y$, and $\varepsilon z \to z$.

Next, changing variables $\varepsilon t \to t$ and $y - z \to y$, we can further see that I^p_ε is equal to

$$\varepsilon^{-n}\int_{\mathbb{R}^n}\eta(x)\Big\{\varepsilon^{-2}\int_0^1\int_{|x-y|<t}\Big|\varepsilon^{n+1}\int_{\mathbb{R}^n}\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(y-z)\nabla P_t(z)\,dz\Big|^2 t^{1-n}\,dy\,dt\Big\}^{p/2}dx$$

$$=\varepsilon^{-n}\sum_{k\in\Lambda}\int_Q\eta(x+k)\Big\{\varepsilon^{-2}\int_0^1\int_{|x-y|<t}\Big|\varepsilon^{n+1}\int_{\mathbb{R}^n}\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(y-z)\nabla P_t(z)\,dz\Big|^2 \times t^{1-n}\,dy\,dt\Big\}^{p/2}dx$$

$$=\varepsilon^{-n}\int_Q\Big\{\varepsilon^{-2}\int_0^1\int_{|x-y|<t}\Big|\varepsilon^{n+1}\int_Q\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(y-z)\sum_{k\in\Lambda}\nabla P_t(z+k)\,dz\Big|^2 \times t^{1-n}dy\,dt\Big\}^{p/2}dx$$

$$=\varepsilon^{-n-p+p(n+1)}\int_Q\Big\{\int_0^1\int_{|x-y|<t}\Big|\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon*\nabla\widetilde{P}_t(z)\Big|^2 t^{1-n}\,dy\,dt\Big\}^{p/2}dx$$

$$=\varepsilon^{-n-p+p(n+1)}\|\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon\|^p_{H^p(\mathbb{T}^n)}\le\varepsilon^{-n-p+p(n+1)}\|\widetilde{f}_\varepsilon\|^p_{H^p(\mathbb{T}^n)}$$

$$\approx\varepsilon^{-n-p+p(n+1)}\int_Q\Big\{\int_0^1\int_{|x-y|<t}\Big|\int_Q\widetilde{f}_\varepsilon(z)\sum_{k\in\Lambda}\nabla P_t(y-z-k)\,dz\Big|^2 \times t^{1-n}\,dy\,dt\Big\}^{p/2}dx.$$

For ε sufficiently small, the support of $\varepsilon^{-n}f(x/\varepsilon)$ lies entirely in Q and, in this case,

$$\varepsilon^{-n}f(x/\varepsilon)=\widetilde{f}_\varepsilon(x).$$

Thus, for small ε, the right side of the above inequality is bounded by

$$\varepsilon^{-n-p+p(n+1)}\int_{\mathbb{R}^n}\Big\{\int_0^1\int_{|x/\varepsilon-y/\varepsilon|<t/\varepsilon}\Big|\int_{\mathbb{R}^n}\varepsilon^{-n}f\Big(\frac{y-z}{\varepsilon}\Big)\nabla P_t(z)\,dz\Big|^2 \times t^{1-n}\,dy\,dt\Big\}^{p/2}dx$$

$$\le\int_{\mathbb{R}^n}\Big\{\int_0^{1/\varepsilon}\int_{|x-y|<t/\varepsilon}\varepsilon^{n-2}\Big|\int_{\mathbb{R}^n}f(z-y)\nabla P_{t/\varepsilon}(z)\,dz\Big|^2 t^{1-n}\,dy\,dt\Big\}^{p/2}dx$$

$$=\int_{\mathbb{R}^n}\Big\{\int_0^{1/\varepsilon}\int_{|x-y|<t}|f*\nabla P_t(y)|^2 t^{1-n}\,dy\,dt\Big\}^{p/2}dx\le C\|f\|^p_{H^p(\mathbb{R}^n)}.$$

This proves the desired inequality (5), thus finishing the proof of Theorem 3.1. From the argument in the proof of (5), we easily get the following:

COROLLARY 3.2. *Suppose that $\widetilde{T}_\varepsilon$ is a multiplier operator in $(H^p(\mathbb{T}^n), H^p(\mathbb{T}^n))$ for $0<p\le 1$. Then there exists a bounded complex-valued function λ on the lattice Λ such that*

$$\widetilde{T}_\varepsilon\widetilde{f}\sim\sum_{m\in\Lambda}\lambda(\varepsilon m)e^{2\pi i m\cdot x}.$$

Thus, a necessary condition for a multiplier operator $\widetilde{T}_\varepsilon$ to be bounded from $H^p(\mathbb{T}^n)$ to $H^p(\mathbb{T}^n)$ is that $\widetilde{T}_\varepsilon$ must be bounded in $L^2(\mathbb{T}^n)$.

Also following the same argument as for Theorem 3.1, we easily get the following weak type theorem:

THEOREM 3.3. *Suppose that $0<p\le 1$ and that λ is a bounded continuous function on $\mathbb{R}^n$. If for each $\varepsilon>0$, there exists an operator $\widetilde{T}_\varepsilon$ such that*

$$|\{x\in Q:|\widetilde{T}_\varepsilon\widetilde{f}(x)|>\alpha\}|\le C\{\|\widetilde{f}\|^p_{H^p(\mathbb{T}^n)}\}/\alpha^p$$

for all $\alpha>0$, where C is independent of $\varepsilon>0$, then

$$|\{x\in\mathbb{R}^n:|Tf(x)|>\alpha\}|\le C\{\|f\|^p_{H^p(\mathbb{R}^n)}\}/\alpha^p$$

for all $\alpha>0$.

THEOREM 3.4. *Suppose λ is a bounded continuous function on $\mathbb{R}^n$. If $\widetilde{T}_\varepsilon$ is of weak type (p,p) uniformly for $\varepsilon>0$, then T is of weak type (p,p) on $\mathbb{R}^n$ $(p\ge 1)$.*

Proof. Here we only give the proof for Theorem 3.4, because the proof of Theorem 3.3 easily follows by using the same ideas as in the proof of Theorems 3.1 and 3.4. Let $\eta(x)=\chi_Q(x)$ be the characteristic function of Q. Then for any $\alpha>0$,

$$\begin{aligned}|\{x\in\mathbb{R}^n:|Tf(x)|>\alpha\}|&\le\lim_{\varepsilon\to0}|\{x\in\mathbb{R}^n:\eta(\varepsilon x)|\varepsilon^n\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(\varepsilon x)|>\alpha\}|\\&=\lim_{\varepsilon\to0}\varepsilon^{-n}|\{x\in\mathbb{R}^n:\eta(x)|\varepsilon^n\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(x)|>\alpha\}|\\&=\lim_{\varepsilon\to0}\varepsilon^{-n}|\{x\in Q:|\varepsilon^n\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(x)|>\alpha\}|\\&=\lim_{\varepsilon\to0}\varepsilon^{-n}|\{x\in Q:|\widetilde{T}_\varepsilon\widetilde{f}_\varepsilon(x)|>\alpha/\varepsilon^n\}|\\&\le\lim_{\varepsilon\to0}\varepsilon^{-n+np}\|\widetilde{f}_\varepsilon\|^p_p/\alpha^p.\end{aligned}$$

But

$$\|\widetilde{f}_\varepsilon\|^p_p=\int_Q|\varepsilon^{-n}f(x/\varepsilon)|^p\,dx=\varepsilon^{n-np}\int_{Q/\varepsilon}|f(x)|^p\,dx.$$

Therefore,

$$|\{x\in\mathbb{R}^n : |Tf(x)|>\alpha\}| \le \lim_{\varepsilon\to 0}\alpha^{-p}\int_{Q/\varepsilon}|f(x)|^p\,dx = \int_{\mathbb{R}^n}|f(x)|^p\,dx/\alpha^p.$$

4. Some lemmas. To prove the converse theorems of Theorems 3.1, 3.3 and 3.4, we need several lemmas.

LEMMA 4.1. *If the operator T defined by $(Tf)^\wedge(\xi)=\lambda(\xi)\widehat{f}(\xi)$ is bounded in $H^p(\mathbb{R}^n)$, then the family of operators*

$$T_\varepsilon : (T_\varepsilon f)^\wedge(\xi)=\lambda(\varepsilon\xi)\widehat{f}(\xi),\qquad \varepsilon>0,$$

is uniformly bounded in $H^p(\mathbb{R}^n)$. Moreover, $\|T\|=\|T_\varepsilon\|$.

Proof. Let $f^\varepsilon(x)=f(\varepsilon x)$. By taking the Fourier transform we have $T_\varepsilon f(x)=Tf^\varepsilon(x/\varepsilon)$. So the lemma follows easily by the definition and a simple computation.

LEMMA 4.2. *Suppose that Ψ is a continuous function on $\mathbb{R}^n$ satisfying*

$$\int_{|x|\le B}\Psi(x)\,dx = A\ne 0,$$

where A and B are constants. Then for any bounded periodic function f, we have

$$\lim_{\varepsilon\to 0}A^{-1}\int_{|x|\le B/\varepsilon}\varepsilon^n\Psi(\varepsilon x)f(x)\,dx=\int_Q f(x)\,dx. \tag{10}$$

Proof. We can assume that $f(x)$ is a trigonometric polynomial because an arbitrary bounded periodic function can be approximated uniformly on Q by such polynomials. Thus, we need to consider the functions

$$f_k(x)=e^{2\pi i k\cdot x}\qquad\text{for } k\in\Lambda.$$

If $k=0$, then $e^{2\pi ik\cdot x}=1$ and

$$\lim_{\varepsilon\to 0}A^{-1}\int_{|x|<B/\varepsilon}\varepsilon^n\Psi(\varepsilon x)\,dx = A^{-1}\int_{|x|\le B}\Psi(x)\,dx = 1=\int_Q 1\,dx.$$

If $k\ne 0$, then

$$\lim_{\varepsilon\to 0}A^{-1}\int_{|x|<B/\varepsilon}\varepsilon^n\Psi(\varepsilon x)e^{2\pi ik\cdot x}\,dx$$
$$=\lim_{\varepsilon\to 0}A^{-1}\int_{|x|<B}\Psi(x)e^{2\pi i(k/\varepsilon)\cdot x}\,dx = 0=\int_Q f_k(x)\,dx$$

by the Riemann–Lebesgue lemma.

LEMMA 4.3. *Suppose that $\Psi(x)=\prod_{j=1}^n(1-x_j^2)_+^\alpha$, $\alpha>0$, where*

$$f_+(x)=\begin{cases} f(x) & \text{if } f(x)\ge 0,\\ 0 & \text{if } f(x)<0.\end{cases}$$

Write $\widehat{\Psi}(\xi)=\Phi(\xi)$ and $\Psi^{1/N}(\xi)=\Psi(\xi/N)$. Let λ be a bounded continuous function on $\mathbb{R}^n$ and let T_ε and $\widetilde{T}_\varepsilon$ be the families of operators on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively, associated with the function λ. Then for any $g\in\mathcal{S}(\mathbb{T}^n)$, there is a constant $M>0$ such that for sufficiently large integer N,

$$|\widetilde{T}_\varepsilon g(y)|\le M|T_\varepsilon(g\Psi^{1/N})(y)|+J_N(y)$$

for any $y\in(N/2)Q$, where Q is the fundamental cube and $J_N(y)$ tends to zero uniformly for $y\in\mathbb{R}^n$ as $N\to\infty$.

Proof. Since $g(x)$ equals its Fourier series $\sum a_ke^{2\pi ik\cdot x}$ with $\{a_k\}$ rapidly decaying, it suffices to prove the lemma for $g(x)$ being a trigonometric polynomial $\sum a_ke^{2\pi ik\cdot x}$. We write

$$|\widetilde{T}_\varepsilon g(y)|\le|T_\varepsilon(g\Psi^{1/N})(y)|+|\widetilde{T}_\varepsilon g(y)-T_\varepsilon(g\Psi^{1/N})(y)|. \tag{11}$$

Then by the Plancherel theorem, we have

$$\begin{aligned}
&|\widetilde{T}_\varepsilon g(y)-T_\varepsilon(g\Psi^{1/N})(y)|\\
&\quad=\Big|\sum a_k\int_{\mathbb{R}^n}\lambda(\varepsilon x)N^n\Phi(N(x-k))e^{2\pi ix\cdot y}\,dx-\sum a_k\lambda(\varepsilon k)e^{2\pi ik\cdot y}\Big|\\
&\quad=\Big|\sum a_k\int_{\mathbb{R}^n}N^n\Phi(N(x-k))\{\lambda(\varepsilon x)e^{2\pi ix\cdot y}-\lambda(\varepsilon k)e^{2\pi ik\cdot y}\}\,dx\Big|\\
&\quad\le\Big|\sum a_k\int_{\mathbb{R}^n}N^n\Phi(N(x-k))\{\lambda(\varepsilon x)-\lambda(\varepsilon k)\}e^{2\pi ix\cdot y}\,dx\Big|\\
&\qquad+\Big|\sum a_k\lambda(\varepsilon k)\int_{\mathbb{R}^n}N^n\Phi(N(x-k))\{e^{2\pi ix\cdot y}-e^{2\pi ik\cdot y}\}\,dx\Big|\\
&\quad=J_N(y)+I_N(y).
\end{aligned}$$

We easily see that as $N\to\infty$, the above $J_N(y)$ goes to zero uniformly for all $y\in\mathbb{R}^n$.

But $I_N(y)$ is equal to

$$\Big|\sum a_k\lambda(\varepsilon k)e^{2\pi ik\cdot y}\int_{\mathbb{R}^n}N^n\Phi(Nx)\{1-e^{2\pi ix\cdot y}\}\,dx\Big|$$
$$=\Big(1-\prod_{j=1}^n(1-y_j^2N^{-2})_+^\alpha\Big)|\widetilde{T}_\varepsilon g(y)|.$$

So from the inequality (11), we obtain

$$\prod_{j=1}^{n}(1-y_j^2N^{-2})_+^{\alpha}|\widetilde{T}_\varepsilon g(y)| \le |T_\varepsilon(\Psi^{1/N}g)(y)| + J_N(y).$$

Therefore for $y \in (N/2)Q$,

$$|\widetilde{T}_\varepsilon g(y)| < 4^{-\alpha}|T_\varepsilon(\Psi^{1/N}g)(y)| + J_N(y).$$

The lemma is proved.

5. The "only if" part of the main theorems

THEOREM 5.1. *Suppose that λ is continuous on $\mathbb{R}^n$ and $0<p\le 1$. If T is bounded in $H^p(\mathbb{R}^n)$, then the operators $\widetilde{T}_\varepsilon$ are bounded in $H^p(\mathbb{T}^n)$ uniformly for $\varepsilon>0$.*

Proof. By Lemma 4.1, without loss of generality, we can assume $\varepsilon = 1$ and write $\widetilde{T}=\widetilde{T}_1$. To prove this theorem, we need the atomic decomposition of the Hardy spaces $H^p(\mathbb{T}^n)$. An *exceptional atom* is an $L^\infty(\mathbb{T}^n)$ function bounded by 1. A *regular* (p,∞,s) *atom* is a function a supported in some ball $B(x,\varrho)$ satisfying:

(i) $\|a\|_\infty \le \varrho^{-n/p}$,

(ii) $\int_Q a(x)P(x)\,dx = 0$ for all polynomials P of degree less than or equal to s.

The space $H_{\mathrm{a}}^{p,s}(\mathbb{T}^n)$, $0<p\le 1$, is the space of all distributions $\widetilde{f}\in \mathcal{S}'(\mathbb{T}^n)$ having the form

$$\widetilde{f}=\sum c_k a_k \tag{12}$$

and satisfying

$$\sum |c_k|^p < \infty, \tag{13}$$

where each a_k is either a regular (p,∞,s) atom or an exceptional atom. The "norm" $\|\widetilde{f}\|_{H_{\mathrm{a}}^{p,s}(\mathbb{T}^n)}$ is the infimum of the expressions $(\sum|c_k|^p)^{1/p}$ for all representations (12) of $\widetilde{f}$. In [3], we proved that

$$H_{\mathrm{a}}^{p,s}(\mathbb{T}^n)=H^p(\mathbb{T}^n) \quad\text{and}\quad \|\cdot\|_{H^p(\mathbb{T}^n)} \cong \|\cdot\|_{H_{\mathrm{a}}^{p,s}(\mathbb{T}^n)} \tag{14}$$

if $s\ge [n(1/p-1)]$.

Clearly, to prove the theorem, by (12)–(14) we only need to show

$$\|\widetilde{T}a\|_{H^p(\mathbb{T}^n)} \le C \tag{15}$$

with a constant C independent of the atom a.

Since T is bounded in $H^p(\mathbb{R}^n)$, it is bounded in $L^2(\mathbb{R}^n)$. Hence λ is a bounded function and $\widetilde{T}$ is a bounded operator in $L^2(\mathbb{T}^n)$. If a is an exceptional atom, then using Hölder's inequality, we easily see that

$$\|\widetilde{T}a\|_{H^p(\mathbb{T}^n)} \le C\|\widetilde{T}a\|_{L^2(\mathbb{T}^n)} \le C\|a\|_{L^2(\mathbb{T}^n)} \le C$$

with a constant C independent of a. So it remains to show (15) for any regular atom.

Let a be a regular atom on $\mathbb{T}^n$ with support in $B(0,\varrho)$. We can consider it as an atom a' on $\mathbb{R}^n$ with support in the fundamental cube Q. Then the function $b=Ta'$ is in $H^p(\mathbb{R}^n)$ and

$$\|b\|_{H^p(\mathbb{R}^n)} = \|\sup_{t>0}|P_t*b|\|_{L^p} \le \|T\|\,\|a'\|_{H^p(\mathbb{R}^n)} \cong \|T\|\,\|a\|_{H^p(\mathbb{T}^n)}. \tag{16}$$

It is easy to observe that if we view $\widetilde{T}a$ as a periodic function on $\mathbb{R}^n$, then

$$\widetilde{T}a(x)=\sum_{k\in\Lambda} b(x+k). \tag{17}$$

The above sum is well-defined because $b\in L^p(\mathbb{R}^n)$, $0<p\le 1$, and it is trivial to see that $\int_Q|\sum b(x+k)|^p dx \le \int_Q \sum|b(x+k)|^p\,dx = \int_{\mathbb{R}^n}|b(x)|^p\,dx$. So we let $\widetilde{b}(x)=\sum_{k\in\Lambda} b(x+k)$ and define, for $y\in Q$,

$$\gamma(y)=\sup_{t>0}\Big|\int_Q \widetilde{b}(x)\widetilde{P}_t(y-x)\,dx\Big|,$$

where $\widetilde{P}_t$ is the Poisson kernel on $\mathbb{T}^n$ which is considered to be a periodic function on $\mathbb{R}^n$.

Now all we have to show is that

$$\|\widetilde{T}a\|_{H^p(\mathbb{T}^n)} = \Big(\int_Q |\gamma(y)|^p\,dy\Big)^{1/p} \le \|T\|\,\|a\|_{H^p(\mathbb{T}^n)}. \tag{18}$$

But

$$\begin{aligned}\sup_{t>0}\Big|\int_Q \widetilde{b}(x)\widetilde{P}_t(y-x)\,dx\Big| &= \sup_{t>0}\Big|\int_{\mathbb{R}^n}\widetilde{b}(x)P_t(y-x)\,dx\Big|\\ &= \sup_{t>0}\Big|\int_{\mathbb{R}^n}\sum_{k\in\Lambda} b(x+k)P_t(y-x)\,dx\Big| = \sup_{t>0}\Big|\sum_{k\in\Lambda}\int_{\mathbb{R}^n} b(x)P_t(y+k-x)\,dx\Big|\\ &\le \sup_{t>0}\sum_{k\in\Lambda}|b*P_t(y+k)| \le \sum_{k\in\Lambda}\sup_{t>0}|b*P_t(y+k)|.\end{aligned}$$

This means that

$$\|\gamma\|_{L^p(\mathbb{T}^n)} \le \|\sup_{t>0}|b*P_t(y)|\|_{L^p(\mathbb{R}^n)}.$$

Now (16) gives (18). It is well known that $\|a\|_{H^p(\mathbb{T}^n)}\le C$ with a constant C independent of a. So (15) is proved for any atom centered at 0. But $\widetilde{T}$ commutes with shift operators, hence (15) is valid for all atoms.

Using a duality argument and a well-known fact $(H^1)^* = \mathrm{BMO}$, we easily obtain the following corollary:

COROLLARY 5.2. *T is in $(\mathrm{BMO}(\mathbb{R}^n), \mathrm{BMO}(\mathbb{R}^n))$ if and only if the $\widetilde{T}_\varepsilon$ are in $(\mathrm{BMO}(\mathbb{T}^n), \mathrm{BMO}(\mathbb{T}^n))$ uniformly for $\varepsilon > 0$.*

The following theorem is the converse version of Theorem 3.4.

THEOREM 5.3. *Let λ be a continuous bounded function on $\mathbb{R}^n$ and $1 \le p < \infty$. Suppose that T and $\widetilde{T}_\varepsilon$ are the operators on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively, associated with λ. If T is of weak type (p,p) on $\mathbb{R}^n$, then $\widetilde{T}_\varepsilon$ is of weak type (p,p) on $\mathbb{T}^n$ uniformly for $\varepsilon > 0$.*

Proof. Because the space $C^\infty(\mathbb{T}^n)$ is dense in $L^p(\mathbb{T}^n)$, we only need to prove that there exists a constant M such that for all $\alpha > 0$ and $g \in C^\infty(\mathbb{T}^n)$,

$$|\{x \in Q : |\widetilde{T}_\varepsilon g(x)| > \alpha\}| \le M \|g\|^p_{L^p(\mathbb{T}^n)}/\alpha^p. \tag{19}$$

Notice that $\widetilde{T}_\varepsilon g$ is a periodic function, which implies that for any positive even number N,

$$\begin{aligned}|\{x \in Q : |\widetilde{T}_\varepsilon g(x)| > \alpha\}| &\cong 2^n N^{-n} |\{x \in (N/2)Q : |\widetilde{T}_\varepsilon g(x)| > \alpha\}| \\ &\le 2^n N^{-n} |\{x \in (N/2)Q : |T_\varepsilon(\Psi^{1/N} g)(x)| > \alpha/(2M)\}| \\ &\quad + 2^n N^{-n} |\{x \in (N/2)Q : |J_N(x)| > \alpha/2\}|.\end{aligned}$$

In the above formula, the function Ψ is as in Lemma 4.3.

Lemma 4.3 implies that $\lim_{N\to\infty} 2^n N^{-n} |\{x \in (N/2)Q : |J_N(x)| > \alpha/2\}| = 0$. Thus by the above formula we have

$$\begin{aligned}|\{x \in Q : |\widetilde{T}_\varepsilon g(x)| > \alpha\}| \\ \le 2^n \lim_{N\to\infty} N^{-n} |\{x \in \mathbb{R}^n : |T_\varepsilon(\Psi^{1/N} g)(x)| > \alpha/(2M)\}|.\end{aligned} \tag{20}$$

Using a similar argument to the proof of Lemma 4.1, one can easily see that $T_\varepsilon(\Psi^{1/N} g)(x) = T(\Psi^{1/N} g)^\varepsilon (x/\varepsilon)$. Therefore

$$\begin{aligned}|\{x \in Q : |\widetilde{T}_\varepsilon g(x)| > \alpha\}| \\ &\le 2^n \lim_{N\to\infty} N^{-n} |\{x \in \mathbb{R}^n : |T(\Psi^{1/N} g)^\varepsilon (x/\varepsilon)| > \alpha/(2M)\}| \\ &= 2^n \lim_{N\to\infty} N^{-n} \varepsilon^n |\{x \in \mathbb{R}^n : |T(\Psi^{1/N} g)^\varepsilon (x)| > \alpha/(2M)\}| \\ &\le 2^{n+p} M^p \lim_{N\to\infty} N^{-n} \varepsilon^n \|(\Psi^{1/N} g)^\varepsilon\|^p_{L^p(\mathbb{R}^n)}/\alpha^p \\ &= A\alpha^{-p} \lim_{N\to\infty} N^{-n} \int_{\mathbb{R}^n} \Psi^p(x/N) |g(x)|^p \, dx,\end{aligned} \tag{21}$$

with a constant $A > 0$, so the formula (19) follows by using Lemma 4.1.

Finally, we prove a slightly weaker converse version of Theorem 3.3. We need to introduce the local Hardy space $h^p(\mathbb{R}^n)$ which was studied by Goldberg in [5]. Let f be a distribution on $\mathbb{R}^n$. We say that f is in the *local Hardy space* $h^p(\mathbb{R}^n)$ provided that

$$f^*_{\mathrm{loc}}(x) = \sup_{|x-y|<t\le 1} |P_t * f(y)|$$

satisfies $f^*_{\mathrm{loc}} \in L^p(\mathbb{R}^n)$, $0 < p < \infty$. We set $\|f\|_{h^p} = \|f^*_{\mathrm{loc}}\|_{L^p}$. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int \Phi \, dx \ne 0$. Then Goldberg proved that $f \in h^p$ if and only if

$$f^+_{\mathrm{loc}}(x) = \sup_{0<t\le 1} |f * \Phi_t(x)|$$

satisfies $\|f\|_{h^p} \cong \|f^+_{\mathrm{loc}}\|_{L^p} < \infty$. A well-known fact is that $H^p = L^p = h^p$ for $1 < p < \infty$. More details on h^p and its applications can be found in [5] and [7].

THEOREM 5.4. *Let λ be a continuous and bounded function on $\mathbb{R}^n$ and $0 < p \le 1$. Suppose that T and $\widetilde{T}_\varepsilon$ are operators on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively, associated with the multiplier λ. If for any $\alpha > 0$ and $f \in h^p(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \le B\alpha^{-p} \|f\|^p_{h^p(\mathbb{R}^n)}, \tag{22}$$

then for any $\alpha > 0$ and $g \in H^p(\mathbb{T}^n)$, we have

$$|\{x \in Q : |\widetilde{T}_\varepsilon g(x)| > \alpha\}| \le A\alpha^{-p} \|g\|^p_{H^p(\mathbb{T}^n)}. \tag{23}$$

In (22) and (23), B and A are constants independent of α, f and g.

Proof. Without loss of generality, we assume that $g \in C^\infty(\mathbb{T}^n) \cap H^p(\mathbb{T}^n)$. By [3] and (14), we know that

$$g(x) = \sum c_k e_k(x) + \sum \nu_k a_k(x) = E(x) + A(x),$$

where e_k's are exceptional atoms and a_k's are regular $(p, \infty, [n(1/p-1)] + 2n^2\gamma)$ atoms with $\gamma = [1/p] + 1$, $\sum |c_k|^p + \sum |\nu_k|^p \cong \|g\|^p_{H^p(\mathbb{T}^n)}$. Thus if we write

$$E_\alpha = \{x \in Q : |\widetilde{T}_\varepsilon E(x)| > \alpha\}, \quad A_\alpha = \{x \in Q : |\widetilde{T}_\varepsilon A(x)| > \alpha\},$$

then we need only prove that

$$|E_\alpha| \le M\alpha^{-p} \sum |c_k|^p, \tag{24}$$

$$|A_\alpha| \le M\alpha^{-p} \sum |\nu_k|^p. \tag{25}$$

Notice that our assumption on λ implies that the $\widetilde{T}_\varepsilon$ are uniformly bounded in $L^2(\mathbb{T}^n)$, so one easily sees that

$$|E_\alpha| = \int\limits_{\{x\in Q:|\widetilde{T}_\varepsilon E(x)|>\alpha\}} dx \le \alpha^{-p}\int\limits_Q |\widetilde{T}_\varepsilon E(x)|^p\,dx$$
$$\le \alpha^{-p}\sum |c_k|^p \int\limits_Q |\widetilde{T}_\varepsilon e_k(x)|^p\,dx.$$

Now using Hölder's inequality, we have a constant M independent of e_k such that

$$\int\limits_Q |\widetilde{T}_\varepsilon e_k|^p\,dx \le \Big\{\int\limits_Q |\widetilde{T}_\varepsilon e_k(x)|^2\,dx\Big\}^{p/2} \le M\|e_k\|^p_{L^2(\mathbb{T}^n)} \le M.$$

Thus (24) is proved.

It remains to prove (25). Let γ be the integer $[1/p]+1$ and let

$$\Psi(x) = \prod_{j=1}^n (1-x_j^2)_+^{n\gamma}.$$

By Lemma 4.3, the proof of (21) now implies that

$$|A_\alpha| \le 2^n \lim_{N\to\infty} N^{-n}\varepsilon^n|\{x\in\mathbb{R}^n : |T(\Psi^{1/N}A)^\varepsilon(x)|>\alpha/M\}|,$$

where N runs over even integers. Thus we have

$$|A_\alpha| \le BM^p\alpha^{-p}\lim_{N\to\infty} N^{-n}\varepsilon^n\|(\Psi^{1/N}A)^\varepsilon\|^p_{h^p(\mathbb{R}^n)}$$
$$\le BM^p\alpha^{-p}\sum|c_k|^p\lim_{N\to\infty}N^{-n}\varepsilon^n\|(\Psi^{1/N}a_k)^\varepsilon\|^p_{h^p(\mathbb{R}^n)}.$$

So it suffices to show that

$$\lim_{N\to\infty} N^{-n}\varepsilon^n\|(\Psi^{1/N}a_k)^\varepsilon\|^p_{h^p(\mathbb{R}^n)} \le C \tag{26}$$

with a constant C independent of a_k.

In fact, choose a nonnegative function $\Phi\in\mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\Phi\subseteq B(0,1)$ and $\int\Phi\,dx\ne 0$. Then noticing that $\operatorname{supp}a_k\subset B(x_0,\varrho)\subset Q$, we have

$$N^{-n}\varepsilon^n\|(\Psi^{1/N}a_k)^\varepsilon\|^p_{h^p(\mathbb{R}^n)}$$
$$\cong N^{-n}\varepsilon^n\int\limits_{\mathbb{R}^n}\sup_{0<t\le 1}\Big|\int\limits_{\mathbb{R}^n}\Psi(\varepsilon x/N)a_k(\varepsilon x)\Phi_t(y-x)\,dx\Big|^p dy$$
$$= N^{-n}\int\limits_{\mathbb{R}^n}\sup_{0<t\le\varepsilon}\Big|\int\limits_{\mathbb{R}^n}\Psi(x/N)a_k(x)\Phi_t(y-x)\,dx\Big|^p dy$$

$$= \int\limits_{\mathbb{R}^n}\sup_{0<t\le\varepsilon/N}\Big|\int\limits_{\mathbb{R}^n}\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}a_k(xN)\Phi_t(y-x)\,dx\Big|^p dy$$
$$= N^{-n}\int\limits_{\mathbb{R}^n}\sup_{0<t\le\varepsilon/N}\Big|\int\limits_{|x_j|\le 1}\Big\{\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}a(Nx)N^{n/p}\Big\}\Phi_t(y-x)\,dx\Big|^p dy.$$

We let $\alpha_\nu(Nx)$ be the periodicity of $a(Nx)$ whose support is totally contained in $2Q=\{x:|x_j|\le 1,\ j=1,\dots,n\}$. There are at most 2^nN^n such $\alpha_\nu(Nx)$'s. We let $\beta_i(Nx)$ be the periodicity of $a(Nx)$ whose support meets the boundary of $2Q$. There are at most 4^nN^{n-1} such $\beta_i(Nx)$'s. Therefore

$$N^{-n}\int\limits_{\mathbb{R}^n}\sup_{0<t\le\varepsilon/N}\Big|\int\limits_{|x_j|\le 1}\Big\{\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}a(Nx)N^{n/p}\Big\}\Phi_t(y-x)dx\Big|^p dy$$
$$\le N^{-n}\sum_{\nu=1}^{2^nN^n}\int\limits_{\mathbb{R}^n}\sup_{t>0}\Big|\int\limits_{\operatorname{supp}\alpha_\nu(Nx)}\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}\alpha_\nu(Nx)N^{n/p}\Phi_t(y-x)\,dx\Big|^p dy$$
$$+\sum_{i=1}^{4^nN^{n-1}}\int\limits_{\mathbb{R}^n}\sup_{0<t\le\varepsilon/N}\Big|\int\limits_{|x_j|\le 1}\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}\beta_i(Nx)\Phi_t(y-x)\,dx\Big|^p dy$$
$$= N^{-n}\sum_{\nu=1}^{2^nN^n}I_\nu + \sum_{i=1}^{4^nN^{n-1}}J_i.$$

Now according to the definition of $\alpha_\nu(Nx)$, one easily sees that

$$A_\nu(x) = \prod_{j=1}^n(1-x_j^2)^{n\gamma}\alpha_\nu(Nx)N^{n/p}\quad\text{is a } (p,\infty,[n(1/p-1)]) \text{ atom on } \mathbb{R}^n.$$

Thus

$$N^{-n}\sum_{\nu=1}^{2^nN^n}I_\nu \cong N^{-n}\sum_{\nu=1}^{2^nN^n}\|A_\nu\|^p_{H^p(\mathbb{R}^n)} \le M. \tag{27}$$

To estimate J_i, we notice that when $0<t\le\varepsilon/N$, $\operatorname{supp}\Phi\subseteq B(0,1)$ implies that for any $x\in 2Q$, $\operatorname{supp}\Phi_t(y-x)\subseteq\{|y|\le\varepsilon/N+2=L\}$ for some constant L. Also because the support of β_i is contained in a ball with radius ϱ/N and meets the boundary of the support of $\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}$, it is clear that

$$\Big\|\prod_{j=1}^n(1-x_j^2)_+^{n\gamma}\beta_1\Big\|_\infty \le M\varrho^{-n/p}(\varrho/N)^{n/p}\cong N^{-n/p}\quad\text{as } N\to\infty.$$

Thus

$$\sum_{i=1}^{4^n N^{n-1}} J_i \cong N^{-1} \int_{|y|\le L} \sup_{t>0} \Big\{ \int_{\mathbb{R}^n} |\Phi_t(x)|\,dx \Big\}^p dy \cong N^{-1}.$$

Therefore, (27) is proved.

NOTE. It would be interesting to be able to replace the local Hardy space in Theorem 5.4 by the usual Hardy space. Some extensions of this paper can be found in [2].

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