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Numbers with a large prime factor

by

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1. Introduction. Let $P(x)$ be the greatest prime factor of the integer $\prod_{x<n\le x+x^{1/2}} n$. It is expected that $P(x)\ge x$ for $x\ge 2$. However, this inequality seems extremely difficult to verify. In 1969, Ramachandra [20, I] obtained a non-trivial lower bound: $P(x)\ge x^{0.576}$ for sufficiently large x. This result has been improved consecutively by many authors. The best estimate known to date is very far from the expected result. The historical records are as follows:

$P(x)\ge x^{0.625}$	by Ramachandra [20, II],
$P(x)\ge x^{0.662}$	by Graham [8],
$P(x)\ge x^{0.692}$	by Jia [16, I],
$P(x)\ge x^{0.7}$	by Baker [1],
$P(x)\ge x^{0.71}$	by Jia [16, II],
$P(x)\ge x^{0.723}$	by Jia [16, III] and Liu [18],
$P(x)\ge x^{0.728}$	by Jia [16, IV],
$P(x)\ge x^{0.732}$	by Baker and Harman [2].

We note that the last two papers are independent. In both, the same estimates for exponential sums were used. But Baker and Harman [2] introduced the alternative sieve procedure, developed by Harman [10] and by Baker, Harman and Rivat [3], to get a better exponent. In this paper we shall prove a sharper lower bound.

THEOREM 1. *We have $P(x)\ge x^{0.738}$ for sufficiently large x.*

As Baker and Harman indicated in [2], it is very difficult to make any progress without new exponential sum estimates. Naturally we first treat

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the corresponding exponential sums

$$S_I := \sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N} b_n e\left(\frac{xh}{mn}\right),$$

$$S_{II} := \sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N} a_m b_n e\left(\frac{xh}{mn}\right),$$

where $e(t) := e^{2\pi it}$, $|a_m| \le 1$, $|b_n| \le 1$ and $m \sim M$ means $cM < m \le c'M$ with some positive unspecified constants c, c'. The improvement in Theorem 1 comes principally from our new bound for S_I (§2, Corollary 2), where we extend the condition $N \le x^{3/8-\varepsilon}$ of Jia [16, III] and Liu [18] to $N \le x^{2/5-\varepsilon}$ (ε is an arbitrarily small positive number). It is noteworthy that we prove this as an immediate consequence of a new estimate on special bilinear exponential sums (§2, Theorem 2). This estimate has other applications, which will be taken up elsewhere. Our results on S_{II} (§3, Theorem 3) improve Theorem 6 of [7] (or [18], Lemma 2) and Lemma 14 of [1]. We need Lemma 9 of [1] only in a very short interval ($3/5 \le \theta \le 11/18$).

If the interval $(x, x + x^{1/2}]$ is replaced by $(x, x + x^{1/2+\varepsilon}]$, one can do much better. In 1973, Jutila [17] proved that the largest prime factor of $\prod_{x<n\le x+x^{1/2+\varepsilon}} n$ is at least $x^{2/3-\varepsilon}$ for $x \ge x_0(\varepsilon)$. The exponent $2/3$ was improved successively to 0.73 by Balog [4, I], to 0.772 by Balog [4, II], to 0.82 by Balog, Harman and Pintz [5], to 11/12 by Heath-Brown [12] and to 17/18 by Heath-Brown and Jia [13]. It should be noted that their methods cannot be applied to treat $P(x)$, and this leads to the comparative weakness of the results on $P(x)$ (cf. [5]).

Throughout this paper, we put $\mathcal{L} := \log x$, $y := x^{1/2}$, $N(d) := |\{x < n \le x + y : d\,|\,n\}|$ and $v := x^{\theta}$. From [16, III], [18] and [2], in order to prove Theorem 1 it is sufficient to show

$$\sum_{x^{0.6-\varepsilon}<p\le x^{0.738}} N(p)\log p < 0.4\, y\mathcal{L}, \tag{1.1}$$

where p denotes a prime number. For this we shall need an upper bound for the quantity $S(\theta) := \sum_{x^{\theta}<p\le ex^{\theta}} N(p)$ $(0.6 \le \theta \le 0.738)$. We write

$$S(\theta) = \sum_{x^{\theta}<p\le ex^{\theta}}\ \sum_{x<mp\le x+y} 1 = \sum_{p\in\mathcal{A}} 1 = S(\mathcal{A}, (ev)^{1/2}), \tag{1.2}$$

where $\mathcal{A} = \mathcal{A}(\theta) := \{n : x^{\theta} < n \le ex^{\theta},\ N(n) = 1\}$, $S(\mathcal{A}, z) := |\{n \in \mathcal{A} : P^-(n) \ge z\}|$ and $P^-(n) := \min_{p|n} p$ ($P^-(1) = \infty$). We would like to give an upper bound for $S(\theta)$ of the form

$$S(\theta) \le \{1 + O(\varepsilon)\}\frac{u(\theta)y}{\theta\mathcal{L}}, \tag{1.3}$$

where $u(\theta)$ is *as small as possible.* Thus in order to prove (1.1), it suffices to show (1.3) and

$$\int_{0.6}^{0.738} u(\theta)\, d\theta < 0.4. \tag{1.4}$$

As in [2], we shall prove (1.3) by the alternative sieve for $0.6 \le \theta \le 0.661$ and by the Rosser–Iwaniec sieve for $0.661 \le \theta \le 0.738$. Thanks to our new estimates for exponential sums, our $u(\theta)$ is strictly smaller than that of Baker and Harman [2].

In the sequel, we use ε_0 to denote a suitably small positive number, ε an arbitrarily small positive number, ε' an unspecified constant multiple of ε and put $\eta := e^{-3/\varepsilon}$.

2. Estimates for bilinear exponential sums and for S_I. First we investigate a special bilinear sum of type II:

$$S(M,N) := \sum_{m\sim M} \sum_{n\sim N} a_m b_n e\biggl(X \frac{m^{1/2} n^{\beta}}{M^{1/2} N^{\beta}}\biggr).$$

Here the exponent 1/2 is important in our method. We have the following result.

Theorem 2. *Let $\beta \in \mathbb{R}$ with $\beta(\beta-1) \ne 0$, $X > 0$, $M \ge 1$, $N \ge 1$, $\mathcal{L}_0 := \log(2 + XMN)$, $|a_m| \le 1$ and $|b_n| \le 1$. Then*

$$\begin{aligned} S(M,N) \ll \{&(X^4 M^{10} N^{11})^{1/16} + (X^2 M^8 N^9)^{1/12} + (X^2 M^4 N^3)^{1/6} \\ &+ (XM^2 N^3)^{1/4} + MN^{1/2} + M^{1/2} N + X^{-1/2} MN\} \mathcal{L}_0. \end{aligned}$$

Proof. In view of Theorem 2 of [7] (or Lemma 3.1 below), we can suppose $X \ge N$. In addition we may also assume $\beta > 0$. Let $Q \in (0, \varepsilon_0 N]$ be a parameter to be chosen later. By the Cauchy–Schwarz inequality and Lemma 2.5 of [9], we have

$$\begin{aligned} |S|^2 \ll{}& \frac{(MN)^2}{Q} \\ &+ \frac{M^{3/2} N}{Q} \sum_{1 \le |q_1| < Q} \biggl(1 - \frac{|q_1|}{Q}\biggr) \sum_{n\sim N} b_{n+q_1} \overline{b}_n \sum_{m\sim M} m^{-1/2} e(Am^{1/2} t), \end{aligned}$$

where $t = t(n, q_1) := (n+q_1)^{\beta} - n^{\beta}$ and $A := X/(M^{1/2} N^{\beta})$. Splitting the range of q_1 into dyadic intervals and removing $1 - q_1/Q$ by partial summation, we get

$$|S|^2 \ll (MN)^2 Q^{-1} + \mathcal{L}_0 M^{3/2} N Q^{-1} \max_{1 \le Q_1 \le Q} |S(Q_1)|, \tag{2.1}$$

where

$$S(Q_1) := \sum_{q_1 \sim Q_1} \sum_{n\sim N} b_{n+q_1} \overline{b}_n \sum_{m\sim M} m^{-1/2}\, e(Am^{1/2} t).$$

If $X(MN)^{-1}Q_1 \geq \varepsilon_0$, by Lemma 1.4 of [18] we transform the innermost sum to a sum over l and then by using Lemma 4 of [16, IV] with $n = n$ we estimate the corresponding error term. As a result, we obtain

$$S(Q_1) \ll \sum_{q_1 \sim Q_1} \sum_{n \sim N} b_{n+q_1} \overline{b}_n \sum_{l \in I(n,q_1)} l^{-1/2}\, e(A_0 t^2) + \{(XM^{-1}N^{-1}Q_1^3)^{1/2} + M^{-1/2}NQ_1 + (X^{-1}MNQ_1)^{1/2} + (X^{-2}MN^4)^{1/2}\}\mathcal{L}_0,$$

where $A_0 := \frac{1}{4}A^2 l^{-1}$, $I(n,q_1) := [c_1 AM^{-1/2}|t|, c_2 AM^{-1/2}|t|]$ and c_j are some constants. Interchanging the order of summation and estimating the sum over l trivially, we find, for some $l \asymp X(MN)^{-1}Q_1$, the inequality

$$S(Q_1) \ll (XM^{-1}N^{-1}Q_1)^{1/2} \Big| \sum\sum_{(n,q_1) \in \mathbf{D}(l)} b_{n+q_1} \overline{b}_n e(A_0 t^2) \Big| + \{(XM^{-1}N^{-1}Q_1^3)^{1/2} + M^{-1/2}NQ_1 + (X^{-1}MNQ_1)^{1/2} + (X^{-2}MN^4)^{1/2}\}\mathcal{L}_0, \tag{2.2}$$

where $\mathbf{D}(l)$ is a subregion of $\{(n, q_1) : n \sim N,\ q_1 \sim Q_1\}$. Let $S_1(Q_1)$ be the double sums on the right-hand side of (2.2). Let $Q_2 \in (0, \varepsilon_0 \min\{Q_1, N^2/X\}]$ be another parameter to be chosen later. Using again the Cauchy–Schwarz inequality and Lemma 2.5 of [9] yields

$$|S_1(Q_1)|^2 \ll (NQ_1)^2 Q_2^{-1} + NQ_1 Q_2^{-1} \sum_{1 \leq q_2 \leq Q_2} |S_2(q_1, q_2)|, \tag{2.3}$$

where

$$S_2(q_1, q_2) := \sum_{n \sim N} \sum_{q_1 \in J_1(n)} b_{n+q_1+q_2} \overline{b}_{n+q_1} e(t_1(n, q_1, q_2)),$$

$J_1(n)$ is a subinterval of $[Q_1, 2Q_1]$ and $t_1(n, q_1, q_2) := A_0\{t(n, q_1 + q_2)^2 - t(n, q_1)^2\}$. Putting $n' := n + q_1$, we have

$$S_2(q_1, q_2) \ll \sum_{n' \sim N} \Big| \sum_{q_1 \in J_2(n')} e(t_1(n' - q_1, q_1, q_2)) \Big|,$$

where $J_2(n')$ is a subinterval of $[Q_1, 2Q_1]$. Noticing

$$t(n' - q_1, q_1 + q_2) - t(n' - q_1, q_1) = t(n', q_2),$$
$$t(n' - q_1, q_1 + q_2) + t(n' - q_1, q_1) = 2t(n' - q_1, q_1) + t(n', q_2),$$

we have

$$t_1(n' - q_1, q_1, q_2) = f(n')q_1 + r(n', q_1) + A_0 t(n', q_2)^2,$$

where $f(n') := 2\beta A_0 t(n', q_2) n'^{\beta-1}$ and $r(n', q_1) := 2A_0 t(n' - q_1, q_1) t(n', q_2) - f(n')q_1$. Since the last term on the right-hand side is independent of q_1, it follows that

$$S_2(q_1,q_2) \ll \sum_{n'\sim N} \Big| \sum_{q_1\in J_2(n')} e(\pm\|f(n')\|q_1 + r(n',q_1))\Big|,$$

where $\|a\| := \min_{n\in\mathbb{Z}} |a-n|$. Since $Q_2 \le \varepsilon_0 N^2/X$, we have

$$\max_{n'\sim N} \max_{q_1\in J_2(n')} |\partial r/\partial q_1| \le c_3 X N^{-2} q_2 \le 1/4.$$

By Lemmas 4.8, 4.2 and 4.4 of [21], the innermost sum on the right-hand side equals

$$\int\limits_{J_2(n')} e(\pm\|f(n')\|s + r(n',s))\,ds + O(1)$$

$$\ll \begin{cases} \|f(n')\|^{-1} & \text{if } \|f(n')\| \ge \varepsilon_0^{-1} X N^{-2} q_2, \\ (XN^{-2}Q_1^{-1}q_2)^{-1/2} & \text{if } \|f(n')\| < \varepsilon_0^{-1} X N^{-2} q_2, \end{cases}$$

which implies

$$\begin{aligned} S_2(q_1,q_2) &\ll \sum_{\|f(n')\|\ge \varepsilon_0^{-1}XN^{-2}q_2} \|f(n')\|^{-1} \\ &\quad + \sum_{\|f(n')\|< \varepsilon_0^{-1}XN^{-2}q_2} (XN^{-2}Q_1^{-1}q_2)^{-1/2} \\ &=: S_2' + S_2''. \end{aligned}$$

As $f'(n') \asymp XN^{-2}Q_1^{-1}q_2$, Lemma 3.1.2 of [14] yields

$$S_2' \ll \mathcal{L}_0 \max_{\varepsilon_0^{-1}XN^{-2}q_2\le\Delta\le 1/2} \sum_{\Delta\le\|f(n')\|<2\Delta} \Delta^{-1} \ll (N + X^{-1}N^2Q_1q_2^{-1})\mathcal{L}_0,$$

$$S_2'' \ll (XQ_1q_2)^{1/2} + (X^{-1}N^2Q_1^3q_2^{-1})^{1/2}.$$

These imply, via (2.3),

$$|S_1(Q_1)|^2 \ll \{(XN^2Q_1^3Q_2)^{1/2} + (NQ_1)^2Q_2^{-1} + (X^{-1}N^4Q_1^5Q_2^{-1})^{1/2}\}\mathcal{L}_0^2,$$

where we have used the fact that

$$N^2Q_1 + X^{-1}N^3Q_1^2Q_2^{-1} \ll (NQ_1)^2Q_2^{-1} \qquad (X\ge N \text{ and } Q_1\ge Q_2).$$

Using Lemma 2.4 of [9] to optimise Q_2 over $(0, \varepsilon_0\min\{Q_1, N^2/X\}]$, we obtain

$$|S_1(Q_1)|^2 \ll \{(XN^4Q_1^5)^{1/3} + (N^3Q_1^4)^{1/2} + N^2Q_1 + XQ_1^2\}\mathcal{L}_0^2,$$

where we have used the fact that $(X^{-1}N^4Q_1^4)^{1/2}$ and $(N^2Q_1^5)^{1/2}$ can be absorbed by $(N^3Q_1^4)^{1/2}$ (since $X\ge N\ge Q_1$). Inserting this inequality into (2.2) yields

$$\begin{aligned} S(Q_1) \ll \{&(X^4M^{-3}NQ_1^8)^{1/6} + (X^2M^{-2}NQ_1^6)^{1/4} + (XM^{-1}NQ_1^2)^{1/2} \\ &+ (X^2M^{-1}N^{-1}Q_1^3)^{1/2} + (X^{-1}MNQ_1)^{1/2} + (X^{-2}MN^4)^{1/2}\}\mathcal{L}_0, \end{aligned}$$

where we have eliminated two superfluous terms $(XM^{-1}N^{-1}Q_1^3)^{1/2}$ and $M^{-1/2}NQ_1$. Replacing Q_1 by Q and inserting the estimate obtained into (2.1), we find

$$(2.4)\quad |S|^2 \ll \{(X^4M^6N^7Q^2)^{1/6} + (X^2M^4N^5Q^2)^{1/4} + (X^2M^2NQ)^{1/2} + (MN)^2Q^{-1} + (XM^2N^3)^{1/2}\}\mathcal{L}_0^2,$$

where we have used the fact that $(X^{-1}M^4N^3Q^{-1})^{1/2}$ and $X^{-1}M^2N^3Q^{-1}$ can be absorbed by $(MN)^2Q^{-1}$ (since $Q \le \varepsilon_0 N \le \varepsilon_0 X$).

If $X(MN)^{-1}Q_1 \le \varepsilon_0$, we first remove $m^{-1/2}$ by partial summation and then estimate the sum over m by the Kuz'min–Landau inequality ([9], Theorem 2.1). Therefore (2.4) always holds for $0 < Q \le \varepsilon_0 N$. Optimising Q over $(0, \varepsilon_0 N]$ yields the desired result. ∎

Next we consider a triple exponential sum

$$S_I^* := \sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_3\sim M_3} a_{m_1}b_{m_2}e\left(X\frac{m_1^{\alpha}m_2m_3^{-1}}{M_1^{\alpha}M_2M_3^{-1}}\right),$$

which is a general form of S_I. We have the following result.

Corollary 1. *Let $\alpha\in\mathbb{R}$ with $\alpha(\alpha-2)\ne 0$, $X>0$, $M_j\ge 1$, $|a_{m_1}|\le 1$, $|b_{m_2}|\le 1$ and let $Y := 2 + XM_1M_2M_3$. Then*

$$S_I^* \ll \{(X^6M_1^{11}M_2^{10}M_3^6)^{1/16} + (X^4M_1^9M_2^8M_3^4)^{1/12} + (X^3M_1^3M_2^4M_3^2)^{1/6} + (XM_1^3M_2^2M_3^2)^{1/4} + (XM_1)^{1/2}M_2 + M_1(M_2M_3)^{1/2} + M_1M_2 + X^{-1}M_1M_2M_3\}Y^{\varepsilon}.$$

Proof. If $M_3' := X/M_3 \le \varepsilon_0$, the Kuz'min–Landau inequality implies $S_I^* \ll X^{-1}M_1M_2M_3$. Next suppose $M_3' \ge \varepsilon_0$. As before using Lemma 1.4 of [18] to the sum over m_3 and estimating the corresponding error term by Lemma 4 of [16, IV] with $n = m_1$, we obtain

$$S_I^* \ll X^{-1/2}M_3S + (X^{1/2}M_2 + M_1M_2 + X^{-1}M_1M_2M_3)\log Y,$$

where

$$S := \sum_{m_1\sim M_1}\sum_{m_2\sim M_2}\sum_{m_3'\sim M_3'} \widetilde{a}_{m_1}\widetilde{b}_{m_2}\xi_{m_3'}e\left(2X\frac{m_1^{\alpha/2}m_2^{1/2}m_3'^{1/2}}{M_1^{\alpha/2}M_2^{1/2}M_3'^{1/2}}\right)$$

and $|\widetilde{a}_{m_1}|\le 1$, $|\widetilde{b}_{m_2}|\le 1$, $|\xi_{m_3'}|\le 1$. Let

$$M_2' := M_2M_3' \quad\text{and}\quad \widetilde{\xi}_{m_2'} := \sum\sum_{m_2m_3'=m_2'} \widetilde{b}_{m_2}\xi_{m_3'}.$$

Then S can be written as a bilinear exponential sum $S(M_2', M_1)$. Estimating it by Theorem 2 with $(M,N)=(M_2',M_1)$, we get the desired result. ∎

Corollary 2. *Let* $x^\theta \le MN \le ex^\theta$ *and* $|b_n|\le 1$. *Then* $S_I \ll_\varepsilon x^{\theta-2\varepsilon}$ *provided* $1/2\le\theta<1$, $H\le x^{\theta-1/2+3\varepsilon}$, $M\le x^{3/4-\varepsilon'}$ *and* $N\le x^{2/5-\varepsilon'}$.

Proof. We apply Corollary 1 with $(X,M_1,M_2,M_3)=(xH/(MN), N,H,M)$. ∎

3. Estimates for exponential sums S_{II}. The main aim of this section is to prove the next Theorem 3. The inequality (3.1) improves Theorem 6 of [7] (or [18], Lemma 2) and the estimate (3.2) sharpens Lemma 14 of [1].

Theorem 3. *Let* $\alpha\in\mathbb{R}$ *with* $\alpha\ne 0,1$, $x>0$, $H\ge 1$, $M\ge 1$, $N\ge 1$, $X:=xH/(MN)$, $|a_m|\le 1$ *and* $|b_n|\le 1$. *Let* (κ,λ) *be an exponent pair. If* $H\le N$ *and* $HN\le X^{1-\varepsilon}$, *then*

$$\begin{aligned}(3.1)\quad S_{II} \ll \{&(X^3H^5M^9N^{15})^{1/14}+(XH^5M^7N^{11})^{1/10}+(XH^2M^3N^6)^{1/5}\\ &+(X^2H^5M^9N^{17})^{1/14}+(H^5M^7N^{13})^{1/10}+(XH^4M^6N^{14})^{1/10}\\ &+(HM^2N)^{1/2}+(X^{-1}HM^2N^3)^{1/2}\}x^\varepsilon,\end{aligned}$$

$$\begin{aligned}(3.2)\quad S_{II} \ll \{&(X^{1+2\kappa}H^{-1-2\kappa+4\lambda}M^{4\lambda}N^{3-2\kappa+4\lambda})^{1/(2+4\lambda)}+(HM^2N)^{1/2}\\ &+(X^{2\kappa-2\lambda}H^{-1-2\kappa+4\lambda}M^{4\lambda}N^{1-2\kappa+8\lambda})^{1/(2+4\lambda)}\\ &+(X^{-1}HM^2N^3)^{1/2}\}x^\varepsilon.\end{aligned}$$

The following corollary will be needed in the proof of Theorem 1.

Corollary 3. *Let* $x^\theta\le MN\le ex^\theta$, $|a_m|\le 1$ *and* $|b_n|\le 1$. *Then* $S_{II}\ll_\varepsilon x^{\theta-2\varepsilon}$ *provided one of the following conditions holds*:

$$(3.3)\quad \tfrac12\le\theta<\tfrac58,\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{2-3\theta-\varepsilon'};$$

$$(3.4)\quad \tfrac12\le\theta<\tfrac23,\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{1/6-\varepsilon'};$$

$$(3.5)\quad \tfrac12\le\theta<\tfrac{11}{16},\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{(9\theta-3)/17-\varepsilon'};$$

$$(3.6)\quad \tfrac12\le\theta<\tfrac{7}{10},\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{(12\theta-5)/17-\varepsilon'};$$

$$(3.7)\quad \tfrac12\le\theta<\tfrac{17}{24},\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{(55\theta-25)/67-\varepsilon'};$$

$$(3.8)\quad \tfrac12\le\theta<\tfrac{5}{7},\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{(59\theta-28)/66-\varepsilon'};$$

$$(3.9)\quad \tfrac12\le\theta<\tfrac{23}{32},\quad H\le x^{\theta-1/2+3\varepsilon},\quad x^{\theta-1/2+3\varepsilon}\le N\le x^{(245\theta-119)/261-\varepsilon'}.$$

Proof. We obtain (3.3) from Lemma 9 of [1]. The result (3.4) is an immediate consequence of (3.1). Let A and B be the classical A-process and B-process. Taking, in (3.2),

$$(\kappa,\lambda) = BA\left(\tfrac{1}{6},\tfrac{4}{6}\right) = \left(\tfrac{2}{7},\tfrac{4}{7}\right),$$
$$(\kappa,\lambda) = BA^2\left(\tfrac{1}{6},\tfrac{4}{6}\right) = \left(\tfrac{11}{30},\tfrac{16}{30}\right),$$
$$(\kappa,\lambda) = BA^3\left(\tfrac{1}{6},\tfrac{4}{6}\right) = \left(\tfrac{13}{31},\tfrac{16}{31}\right),$$
$$(\kappa,\lambda) = BA^4\left(\tfrac{1}{6},\tfrac{4}{6}\right) = \left(\tfrac{57}{126},\tfrac{64}{126}\right),$$
$$(\kappa,\lambda) = BA^5\left(\tfrac{1}{6},\tfrac{4}{6}\right) = \left(\tfrac{60}{127},\tfrac{64}{127}\right),$$

we obtain (3.5)–(3.9). This completes the proof. ■

In order to prove Theorem 3, we need the next lemma. The first inequality is essentially Theorem 2 of [7] with $(M_1, M_2, M_3, M_4) = (H, M, N, 1)$, and the second one is a simple generalisation of Proposition 1 of [22]. It seems interesting that we prove (3.10) by an argument of Heath-Brown [11] instead of the double large sieve inequality ([7], Proposition 1) as in [7].

LEMMA 3.1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \neq 0$, $X > 0$, $H \geq 1$, $M \geq 1$, $N \geq 1$, $\mathcal{L}_0 := \log(2 + XHMN)$, $|a_h| \leq 1$ and $|b_{m,n}| \leq 1$. Let $f(h) \in C^\infty[H, 2H]$ satisfy the condition of exponent pair with $f^{(k)}(h) \asymp F/H^k$ ($h \sim H$, $k \in \mathbb{Z}^+$) and*

$$S = S(H,M,N) := \sum_{h\sim H}\sum_{m\sim M}\sum_{n\sim N} a_h b_{m,n} e\left(X\frac{f(h)m^\alpha n^\beta}{FM^\alpha N^\beta}\right).$$

If (κ,λ) is an exponent pair, then

$$S \ll \{(XHMN)^{1/2} + H^{1/2}MN + H(MN)^{1/2} + X^{-1/2}HMN\}\mathcal{L}_0, \tag{3.10}$$

$$S \ll \{(X^\kappa H^{1+\kappa+\lambda}M^{2+\kappa}N^{2+\kappa})^{1/(2+2\kappa)} + H(MN)^{1/2} + H^{1/2}MN + X^{-1/2}HMN\}\mathcal{L}_0. \tag{3.11}$$

Proof. Let $Q \geq 1$ be a parameter to be chosen later and let $M_0 := CM^\alpha N^\beta$ where C is a suitable constant. Let $T_q := \{(m,n) : m \sim M,\ n \sim N,\ M_0(q-1) < m^\alpha n^\beta Q \leq M_0 q\}$. Then we can write

$$S = \sum_{h\sim H} a_h \sum_{q\leq Q}\sum_{(m,n)\in T_q} b_{m,n} e\left(X\frac{f(h)m^\alpha n^\beta}{FM^\alpha N^\beta}\right).$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|S|^2 &\ll HQ\sum_{q\leq Q}\sum_{(m,n)\in T_q} b_{m,n}\sum_{(\widetilde{m},\widetilde{n})\in T_q}\overline{b}_{\widetilde{m},\widetilde{n}}\sum_{h\sim H} e(g(h))\\ &\ll HQ\sum_{\substack{m,\widetilde{m}\sim M\\ |\sigma|\leq M_0/Q}}\sum_{n,\widetilde{n}\sim N}\Big|\sum_{h\sim H} e(g(h))\Big| =: HQ(E_0+E_1),\end{aligned} \tag{3.12}$$

where $\sigma := m^\alpha n^\beta - \widetilde{m}^\alpha\widetilde{n}^\beta$, $g(h) := X\sigma f(h)/(FM^\alpha N^\beta)$ and E_0, E_1 are the contributions corresponding to the cases $|\sigma| \leq M_0/(MN)$, $M_0/(MN) < |\sigma| \leq M_0/Q$, respectively.

Let $\mathcal{D}(M,N,\Delta) := |\{(m,\widetilde{m},n,\widetilde{n}) : m,\widetilde{m} \sim M;\ n,\widetilde{n} \sim N;\ |\sigma| \le \Delta M_0\}|$. By using Lemma 1 of [7], we find

$$E_0 \ll H\mathcal{D}(M,N,1/(MN)) \ll HMN\mathcal{L}_0. \tag{3.13}$$

We prove (3.10) and (3.11) by using two different methods to estimate E_1. Take $Q := \max\{1, X/(\varepsilon_0 H)\}$. Then $\max_{h\sim H}|g'(h)| = XH^{-1}\Delta \le 1/2$. The Kuz'min–Landau inequality implies

$$E_1 \ll \mathcal{L}_0 \max_{Q\le 1/\Delta\le MN} \mathcal{D}(M,N;\Delta)(XH^{-1}\Delta)^{-1} \ll X^{-1}H(MN)^2\mathcal{L}_0^2. \tag{3.14}$$

Now the inequality (3.10) follows from (3.12)–(3.14).

In view of (3.10), we can suppose $X \ge MN$. Splitting $(M_0/(MN), M_0/Q]$ into dyadic intervals $(\Delta M_0, 2\Delta M_0]$ with $Q \le 1/\Delta \le MN$ and applying the exponent pair (κ,λ) yield

$$\begin{aligned} E_1 &\ll \mathcal{L}_0 \max_{Q\le 1/\Delta\le MN} \mathcal{D}(M,N;\Delta)\{(XH^{-1}\Delta)^{\kappa}H^{\lambda} + (XH^{-1}\Delta)^{-1}\} \\ &\ll (X^{\kappa}H^{-\kappa+\lambda}M^2N^2Q^{-1-\kappa} + X^{-1}HM^2N^2)\mathcal{L}_0^2. \end{aligned} \tag{3.15}$$

Inserting (3.13) and (3.15) into (3.12) and noticing $X^{-1}(HMN)^2Q \le H^2MNQ$, we get

$$|S|^2 \ll \{X^{\kappa}H^{1-\kappa+\lambda}M^2N^2Q^{-\kappa} + H^2MNQ\}\mathcal{L}_0^2.$$

Using Lemma 2.4 of [9] to optimise Q over $[1,\infty)$ yields the required result (3.11). ∎

Next we combine the methods of [1], [7] and [19] to prove Theorem 3.

Let $Q_1 := aH/(bN) \in [100, HN]$ be a parameter to be chosen later with $a,b \in \mathbb{N}$ and let $Q_1^* := NQ_1/(\sqrt{10}H)$. Introducing $T_{q_1} := \{(h,n) : h \sim H,\ n \sim N,\ (q_1-1)/Q_1^* \le hn^{-1} < q_1/Q_1^*\}$, we may write

$$S_{II} = \sum_{q_1\le Q_1} \sum_{m\sim M} \sum\sum_{(h,n)\in T_{q_1}} a_m b_n e\left(\frac{xh}{mn}\right).$$

As before by the Cauchy–Schwarz inequality, we have

$$|S_{II}|^2 \ll MQ_1\Bigg| \mathop{\sum\sum}_{\substack{n_1,n_2\sim N \\ |h_1/n_1-h_2/n_2|<1/Q_1^*}} \mathop{\sum\sum}_{h_1,h_2\sim H} b_{n_1}\overline{b}_{n_2}\delta\left(\frac{h_1}{n_1},\frac{h_2}{n_2}\right) \sum_{m\sim M} e\left(\frac{x(h_1n_2-h_2n_1)}{mn_1n_2}\right)\Bigg|, \tag{3.16}$$

where $\delta(u_1,u_2) := |\{q \in \mathbb{Z}^+ : Q_1^*\max(u_1,u_2) < q \le Q_1^*\min(u_1,u_2)+1\}|$. Without loss of generality, we can suppose $h_1/n_1 \ge h_2/n_2$ in (3.16). Thus we have, with $u_i := h_i/n_i$,

$$\begin{aligned} \delta(u_1,u_2) &= [Q_1^*u_2+1] - [Q_1^*u_1] = 1 + Q_1^*(u_2-u_1) - \psi(Q_1^*u_2) + \psi(Q_1^*u_1) \\ &=: \delta_1 + \delta_2 - \delta_3 + \delta_4, \end{aligned}$$

where $\psi(t) := \{t\} - 1/2$ and $\{t\}$ is the fractional part of t. Inserting into (3.16) yields

$$|S_{II}|^2 \ll MQ_1(|S_1| + |S_2| + |S_3| + |S_4|)$$

with

$$S_j := \sum\sum_{\substack{n_1,n_2\sim N\\ |h_1/n_1-h_2/n_2|<1/Q_1^*}} \sum\sum_{h_1,h_2\sim H} b_{n_1}\overline{b}_{n_2}\delta_j \sum_{m\sim M} e\biggl(\frac{x(h_1n_2-h_2n_1)}{mn_1n_2}\biggr).$$

We estimate $MQ_1|S_3|$ only; the other terms can be treated similarly. We write

$$MQ_1|S_3| \ll MQ_1 \sum\sum_{n_1,n_2\sim N} \biggl| \sum_{0\le k\ll HN/Q_1} \sum\sum_{\substack{h_1,h_2\sim H\\ h_1n_2-h_2n_1=k}} \delta_3 \sum_{m\sim M} e\biggl(\frac{xk}{mn_1n_2}\biggr)\biggr|.$$

Since $|\delta_3| \le 1$, the terms with $k = 0$ contribute trivially $O(HM^2NQ_1\mathcal{L}_0)$. After dyadic split, we see that for some K with $1 \le K \ll HN/Q_1$ and some D with $1 \le D \le \min\{K, N\}$,

$$\begin{aligned} MQ_1|S_3|\mathcal{L}_0^{-2} \ll MQ_1 \sum_{d\sim D} \sum\sum_{\substack{n_1,n_2\sim N'\\ (n_1,n_2)=1}} \biggl| \sum_{r\sim R} \omega_d(n_1,n_2;r) \sum_{m\sim M} e\biggl(\frac{xr}{dmn_1n_2}\biggr)\biggr| \\ + HM^2NQ_1, \end{aligned}$$

where $N' := N/D$, $R := K/D$ and

$$\omega_d(n_1,n_2;r) := \sum\sum_{\substack{h_1,\,h_2\sim H\\ h_1n_2-h_2n_1=r}} \psi(Q_1^*h_2/(dn_2)).$$

In view of $H \le N$, Lemma 4 of [19] gives

$$\begin{aligned} |\omega_d(n_1,n_2;r)| &= \biggl|\int_0^1 \widehat{\omega}_d(n_1,n_2;\vartheta)e(r\vartheta)\,d\vartheta\biggr| \\ &\le \int_0^1 |\widehat{\omega}_d(n_1,n_2;\vartheta)|\,d\vartheta \ll D\mathcal{L}_0^3, \end{aligned} \tag{3.17}$$

where

$$\widehat{\omega}_d(n_1,n_2;\vartheta) := \sum_{|m|\le 8HN} \omega_d(n_1,n_2;m)e(-m\vartheta).$$

If $L := XK/(HMN) \ge \varepsilon_0$, by Lemma 1.4 of [18] we transform the sum over m into a sum over l, then we interchange the order of summations (r, l), finally by Lemma 1.6 of [18] we relax the condition of summation of r. The contribution of the main term of Lemma 1.4 of [18] is

$$(X^{-1}HM^4NK^{-1}Q_1^2)^{1/2} \times \sum_{d\sim D} \sum_{\substack{n_1,n_2\sim N' \\ (n_1,n_2)=1}}\sum \sum_{l\sim L} \Big| \sum_{r\sim R} g(r)e(rt)\omega_d(n_1,n_2;r)e(W\sqrt{r/R})\Big|,$$

where $g(r) = (r/R)^{1/4}$, $W := 2(XK/(HN))(l/L)^{1/2}(dn_1n_2/(DN'^2))^{-1/2}$, t is a real number independent of variables. Let $J := N^2/D$ and $\tau_3(j) := \sum_{dn_1n_2=j} 1$. Let c_i be some constants and

$$T_i(j) := \min\{(X^{-1}HM^2N^{-1}jr^{-1})^{1/2},\, 1/\|c_iXH^{-1}M^{-1}Nr/j\|\}.$$

By Lemma 4 of [16, IV], the contribution of the error term of Lemma 1.4 of [18] is

$$\ll D\mathcal{L}_0^4MQ_1\Big\{D^{-1}N^2R+X^{-1}D^{-2}HMN^3+\sum_{r\sim R}\sum_{j\sim J}\tau_3(j)(T_1(j)+T_2(j))\Big\}$$

$$\ll (HMN^3+X^{-1}HM^2N^3Q_1+X^{1/2}HMNQ_1^{-1/2}+X^{-1/2}HM^2NQ_1^{1/2})x^{\varepsilon}.$$

Combining these and noticing $X^{-1/2}HM^2NQ_1^{1/2} \le HM^2NQ_1$, we obtain

$$(3.18)\qquad MQ_1|S_3|x^{-\varepsilon} \ll (X^{-1}HM^4NK^{-1}Q_1^2)^{1/2}S_{3,1}+HM^2NQ_1 + X^{-1}HM^2N^3Q_1 + X^{1/2}HMNQ_1^{-1/2} + HMN^3,$$

where

$$S_{3,1} := \sum_{d\sim D} \sum_{\substack{n_1,n_2\sim N' \\ (n_1,n_2)=1}}\sum \sum_{l\sim L} \Big| \sum_{r\sim R} g(r)e(rt)\omega_d(n_1,n_2;r)e(W\sqrt{r/R})\Big|.$$

Let $S_{3,2}$ be the innermost sum. Using the Cauchy–Schwarz inequality and (3.17), we deduce

$$|S_{3,2}|^2 \ll D\mathcal{L}_0^3\int_0^1 |\widehat{\omega}_d(n_1,n_2;\vartheta)|\Big|\sum_{r\sim R} g(r)e(rt-r\vartheta)e(W\sqrt{r/R})\Big|^2\,d\vartheta.$$

By Lemma 2 of [7], we have, for any $Q_2 \in (0,\, R^{1-\varepsilon}]$,

$$\Big|\sum_{r\sim R} g(r)e(rt-r\vartheta)e(W\sqrt{r/R})\Big|^2 \le C\Big\{R^2Q_2^{-1}+RQ_2^{-1}\sum_{1\le q_2\le Q_2}\eta\sum_{r\sim R}a_{r,q_2}e\Big(\frac{Wt(r,q_2)}{\sqrt{R}}\Big)\Big\},$$

where C is a positive constant, $\eta = \eta_{q_2,\vartheta,t} = e^{4\pi i q_2(t-\vartheta)}(1-|q_2|/Q_2)$, $a_{q_2,r} = g(r+q_2)g(r-q_2)$, $t(r,q_2) := (r+q_2)^{1/2}-(r-q_2)^{1/2}$. Splitting the range of q_2 into dyadic intervals and inserting the preceding estimates into the definition of $S_{3,1}$, we find, for some $Q_{2,0} \le Q_2$,

$$|S_{3,1}|^2 \ll JL \sum_{d\sim D} \sum_{\substack{n_1,n_2\sim N' \\ (n_1,n_2)=1}}\sum \sum_{l\sim L} |S_{3,2}|^2 \tag{3.19}$$
$$\ll D^2\mathcal{L}_0^7\{(JLR)^2Q_2^{-1} + JLRQ_2^{-1}S_{3,3}\},$$

where $Z := 2XK/(HN)$ and

$$S_{3,3} := \sum_{q_2\sim Q_{2,0}} \sum_{j\sim J} \tau_3(j) \Big| \sum_{l\sim L}\sum_{r\sim R} a_{r,q_2} e\Big(Z\frac{(l/j)^{1/2}t(r,q_2)}{(LR/J)^{1/2}}\Big)\Big|.$$

Applying (3.10) of Lemma 3.1 with $(X,H,M,N) = (ZR^{-1}q_2, R, J, L)$ to the inner triple sums and summing trivially over q_2, we find

$$S_{3,3} \ll \{(ZJLQ_{2,0}^3)^{1/2} + (JL)^{1/2}RQ_{2,0} + JLR^{1/2}Q_{2,0} + (Z^{-1}J^2L^2R^3Q_{2,0})^{1/2}\}x^{\varepsilon}.$$

Replacing $Q_{2,0}$ by Q_2 and inserting the estimate obtained into (3.19) yield

$$S_{3,1} \ll \{(ZJ^3L^3R^2Q_2)^{1/4} + JLRQ_2^{-1/2} + (Z^{-1}J^4L^4R^5Q_2^{-1})^{1/4} + (JL)^{3/4}R + JLR^{3/4}\}Dx^{\varepsilon}.$$

Using Lemma 2.4 of [9] to optimise Q_2 over $(0, R^{1-\varepsilon}]$, we find

$$|S_{3,1}| \ll \{(ZJ^5L^5R^4)^{1/6} + (JL)^{3/4}R + JLR^{3/4}\}Dx^{\varepsilon},$$

where for simplifying we have used the fact that $JLR^{1/2} \le JLR^{3/4}$, $(JLR)^{7/8} = \{(JL)^{3/4}R\}^{1/2}\{JLR^{3/4}\}^{1/2}$, $Z^{-1/4}JLR \le JLR^{3/4}$. Inserting $J = D^{-1}N^2$, $L = XK/(HMN)$, $R = D^{-1}K$, $Z = 2XK/(HN)$, we obtain an estimate for $S_{3,1}$ in terms of (X, D, H, M, N, K). Noticing that all exponents of D are negative, we can replace D by 1 to write

$$|S_{3,1}| \ll \{(X^6H^{-6}M^{-5}N^4K^{10})^{1/6} + (X^3H^{-3}M^{-3}N^3K^7)^{1/4} + (X^4H^{-4}M^{-4}N^4K^7)^{1/4}\}x^{\varepsilon}.$$

Inserting into (3.18) and replacing K by HN/Q_1 yield

$$\begin{aligned} MQ_1|S_3| &\ll \{(X^3H^4M^7N^{14}Q_1^{-1})^{1/6} + (XH^4M^5N^{10}Q_1^{-1})^{1/4} \\ &\quad + (X^2H^3M^4N^{11}Q_1^{-1})^{1/4} \\ &\quad + HM^2NQ_1 + X^{-1}HM^2N^3Q_1\}x^{\varepsilon} \\ &=: E(Q_1)x^{\varepsilon}, \end{aligned} \tag{3.20}$$

where we have used the fact that

$$X^{1/2}HMNQ_1^{-1/2} + HMN^3 \ll (X^2H^3M^4N^{11}Q_1^{-1})^{1/4}.$$

If $L \le \varepsilon_0$, using the Kuz'min–Landau inequality and (3.17) yields

$$MQ_1|S_3|\mathcal{L}_0^{-2} \ll MQ_1D^{-1}N^2RD\mathcal{L}_0^3/L \ll X^{-1}HM^2N^3Q_1\mathcal{L}_0^3 \ll E(Q_1)\mathcal{L}_0^3.$$

Therefore the estimate (3.20) always holds. Similarly we can establish the same bound for $MQ_1|S_j|$ $(j = 1, 2, 4)$. Hence we obtain, for any $Q_1 \in [100, HN]$,

$$|S_{II}|^2 \ll E(Q_1)x^{\varepsilon}.$$

In view of the term HM^2NQ_1, this inequality is trivial when $Q_1 \ge HN$. By using Lemma 2.4 of [9], we see that there exists some $\widetilde{Q}_1 \in [100, \infty)$ such that

$$\begin{aligned} E(\widetilde{Q}_1) \ll{}& (X^3H^5M^9N^{15})^{1/7} + (XH^5M^7N^{11})^{1/5} + (X^2H^4M^6N^{12})^{1/5} \\ &+ (X^2H^5M^9N^{17})^{1/7} + (H^5M^7N^{13})^{1/5} + (XH^4M^6N^{14})^{1/5} \\ &+ HM^2N + X^{-1}HM^2N^3. \end{aligned}$$

Now taking $Q_1 := 100[\widetilde{Q}_1]H(1 + [N])/((1 + [H])N)$ and noticing that $E(Q_1) \ll E(\widetilde{Q}_1)$, we obtain the desired result (3.1).

In order to prove (3.2), we first write

$$S_{3,1} = \sum_{d\sim D} \mathop{\sum\sum}_{\substack{n_1,n_2\sim N' \\ (n_1,n_2)=1}} \sum_{l\sim L} \Big| \int_0^1 \widehat{\omega}_d(n_1, n_2; \vartheta)S_{d,n_1,n_2,l}(\vartheta)\, d\vartheta \Big|,$$

where $S_{d,n_1,n_2,l}(\vartheta) = \sum_{r\sim R} g(r)e(f(r))$, $f(r) = W\sqrt{r/R} + (t + \vartheta)r$ $(t, \vartheta \in [0, 1])$. Since $HN \le X^{1-\varepsilon}$, we have

$$f'(r) \asymp W/R + t + \vartheta \asymp LM/R + t + \vartheta \ge LM/K + t + \vartheta \ge (HN)^{\varepsilon}.$$

Removing the smooth coefficient $g(r)$ by partial summation and using the exponent pair (κ, λ) yield the inequality $S_{d,n_1,n_2,l}(\vartheta) \ll (W/R)^{\kappa}R^{\lambda}$ uniformly for $\vartheta \in [0, 1]$. Thus by (3.17), we find

$$S_{3,1} \ll JL(W/R)^{\kappa}R^{\lambda}D\mathcal{L}_0^3 \ll X^{1+\kappa}H^{-1-\kappa}M^{-1}N^{1-\kappa}K^{1+\lambda}\mathcal{L}_0^3,$$

which implies, via (3.18),

$$\begin{aligned} MQ_1|S_3| \ll{}& (X^{1/2+\kappa}H^{\lambda-\kappa}MN^{2-\kappa+\lambda}Q_1^{-\lambda+1/2} \\ &+ HM^2NQ_1 + X^{-1}HM^2N^3Q_1)x^{\varepsilon}, \end{aligned}$$

where we have used the fact that

$$X^{1/2}HMNQ_1^{-1/2} + HMN^3 \ll X^{1/2+\kappa}H^{\lambda-\kappa}MN^{2-\kappa+\lambda}Q_1^{-\lambda+1/2}.$$

The same estimate holds also for $MQ_1|S_j|$ $(j = 1, 2, 4)$. Thus we obtain, for any $Q_1 \in [100, HN]$,

$$|S_{II}|^2 \ll (X^{1/2+\kappa}H^{\lambda-\kappa}MN^{2-\kappa+\lambda}Q_1^{-\lambda+1/2} + HM^2NQ_1 + X^{-1}HM^2N^3Q_1)x^{\varepsilon}.$$

This implies (3.2). The proof of Theorem 3 is finished. ■

4. Rosser–Iwaniec's sieve and bilinear forms. Let

$$\mathcal{A}_d := \{n \in \mathcal{A} : d \mid n\}, \quad r(\mathcal{A}, d) := |\mathcal{A}_d| - y/d \quad \text{and} \quad P^*(z) := \prod_{p<z} p.$$

We recall the formula of the Rosser–Iwaniec linear sieve [15] in the form stated in [1], Lemma 10.

LEMMA 4.1. *Let $0 < \varepsilon < 1/8$ and $2 \le z \le D^{1/2}$. Then*

$$S(\mathcal{A}, z) \le yV(z)\{F(\log D/\log z) + E\} + \mathcal{R}(\mathcal{A}, D),$$

where $V(z) := \prod_{p<z}(1 - 1/p)$, $E = C\varepsilon + O(\log^{-1/3} D)$ with an absolute constant C and $F(t) := 2e^\gamma/t$ for $1 \le t \le 3$ (γ is the Euler constant). Here

$$\mathcal{R}(\mathcal{A}, D) := \sum_{(D)} \sum_{\substack{\nu < D^\varepsilon \\ \nu \mid P^*(D^{\varepsilon^2})}} c_{(D)}(\nu, \varepsilon) \sum_{\substack{D_i \le p_i < D_i^{1+\varepsilon^7} \\ p_i \mid P^*(z)}} r(\mathcal{A}, \nu p_1 \ldots p_t),$$

where $|c_{(D)}(\nu, \varepsilon)| \le 1$ and $\sum_{(D)}$ runs over all subsequences $D_1 \ge \ldots \ge D_t$ (including the empty subsequence) of $\{D^{\varepsilon^2(1+\varepsilon^7)^n} : n \ge 0\}$ for which $D_1 \ldots D_{2l} D_{2l+1}^3 \le D$ $(0 \le l \le (t-1)/2)$.

Let $r_0(\mathcal{A}, d) := \psi((x+y)/d) - \psi(x/d)$, where $\psi(t)$ is defined as in Section 3. Then

$$|\mathcal{A}_d| = \sum_{x^\theta < dk \le ex^\theta} \{y/(dk) + r_0(\mathcal{A}, dk)\} = y/d + O(y/x^\theta) + \sum_{x^\theta < dk \le ex^\theta} r_0(\mathcal{A}, dk).$$

Thus $r(\mathcal{A}, d) = O(y/x^\theta) + \sum_{x^\theta < dk \le ex^\theta} r_0(\mathcal{A}, dk)$ and

$$\begin{aligned}\mathcal{R}(\mathcal{A}, D) \\ = \sum_{(D)} \sum_{\substack{\nu < D^\varepsilon \\ \nu \mid P^*(D^{\varepsilon^2})}} c_{(D)}(\nu, \varepsilon) \sum_{D_i \le p_i < \min\{z, D_i^{1+\varepsilon^7}\}} \sum_{x^\theta < \nu k p_1 \ldots p_t \le ex^\theta} r_0(\mathcal{A}, \nu k p_1 \ldots p_t) \\ + O(Dy/x^\theta).\end{aligned}$$

We would like to find $D = D(\theta)$, *as large as possible*, such that $\mathcal{R}(\mathcal{A}, D) \ll_\varepsilon y/\mathcal{L}^2$. For this, it suffices to impose $D \le x^{\theta - \varepsilon'}$ and to prove

$$\mathcal{R}^*(\mathcal{A}, D) := \sum_{A_1 \le p_1 < B_1} \cdots \sum_{A_t \le p_t < B_t} \sum_{x^\theta < \nu k p_1 \ldots p_t \le ex^\theta} r_0(\mathcal{A}, \nu k p_1 \ldots p_t) \ll yx^{-\varepsilon} \tag{4.1}$$

for

$$\begin{cases} 1 \le \nu \le D^{\varepsilon},\ t \ll 1,\ A_i \ge 1,\ B_i \le 2A_i,\ A_1 \ge \ldots \ge A_t, \\ A_1 \ldots A_{2l} A_{2l+1}^3 \le D^{1+\varepsilon}\ (0 \le l \le (t-1)/2). \end{cases}$$

In order to prove (4.1), we need to treat the following bilinear forms:

$$\mathcal{R}_I(M,N;x^\theta) := \sum_{\substack{m\sim M \\ x^\theta < mn \le ex^\theta}} \sum_{n\sim N} b_n r_0(\mathcal{A}, mn),$$

$$\mathcal{R}_{II}(M,N;x^\theta) := \sum_{\substack{m\sim M \\ x^\theta < mn \le ex^\theta}} \sum_{n\sim N} a_m b_n r_0(\mathcal{A}, mn),$$

where $|a_m| \le 1$, $|b_n| \le 1$. Using the Fourier expansion of $\psi(t)$, we reduce the estimation for $\mathcal{R}_I$, $\mathcal{R}_{II}$ to the estimation for the exponential sums S_I, S_{II} (cf. [7], Lemma 9). Applying Corollaries 2 and 3 to these sums, we can immediately get the desired results on $\mathcal{R}_I$ and $\mathcal{R}_{II}$.

Before stating our results, it is necessary to introduce some notation. Let $\phi_1 := 3/5 = 0.6$, $\phi_2 := 11/18 \approx 0.611$, $\phi_3 := 35/54 \approx 0.648$, $\phi_4 := 2/3 \approx 0.666$, $\phi_5 := 90/131 \approx 0.687$, $\phi_6 := 226/323 \approx 0.699$, $\phi_7 := 546/771 \approx 0.708$, $\phi_8 := 23/32 \approx 0.718$ and $\phi_9 := 0.738$. For $\phi_1 \le \theta \le \phi_8$, we define $I = I(\theta) := [ax^{\varepsilon'}, bx^{-\varepsilon'}]$ with $a = a(\theta) := x^{\theta-1/2}$, $b = b(\theta) := x^{\tau(\theta)}$ and

$$\tau(\theta) := \begin{cases} 2-3\theta & \text{if } \phi_1 \le \theta \le \phi_2, \\ 1/6 & \text{if } \phi_2 \le \theta \le \phi_3, \\ (9\theta-3)/17 & \text{if } \phi_3 \le \theta \le \phi_4, \\ (12\theta-5)/17 & \text{if } \phi_4 \le \theta \le \phi_5, \\ (55\theta-25)/67 & \text{if } \phi_5 \le \theta \le \phi_6, \\ (59\theta-28)/66 & \text{if } \phi_6 \le \theta \le \phi_7, \\ (245\theta-119)/261 & \text{if } \phi_7 \le \theta \le \phi_8. \end{cases}$$

For $\mathcal{R}_I$, we have the following result, which improves Corollary 1 of [2].

Lemma 4.2. *Let $1/2 < \theta < 3/4$ and $N \le x^{2/5-\varepsilon'}$. Then $\mathcal{R}_I(M,N;x^\theta) \ll_\varepsilon yx^{-3\eta}$.*

For $\mathcal{R}_{II}$, we have the following result, which improves Lemmas 2 and 3 of [2].

Lemma 4.3. *Let $1/2 < \theta < \phi_8$ and $N \in I(\theta)$. Then $\mathcal{R}_{II}(M,N;x^\theta) \ll_\varepsilon yx^{-3\eta}$.*

Let $D = D(\theta) := (b/a)x^{2/5-\varepsilon'}$ for $\phi_1 \le \theta \le \phi_8$ and $D := x^{2/5-\varepsilon'}$ for $\phi_8 \le \theta \le \phi_9$. We define $\varrho(\theta)$ by $D = x^{\varrho(\theta)-\varepsilon'}$, i.e.

$$\varrho(\theta)=\begin{cases}(29-40\theta)/10 & \text{if } \phi_1\le\theta\le\phi_2,\\ (16-15\theta)/15 & \text{if } \phi_2\le\theta\le\phi_3,\\ (123-80\theta)/170 & \text{if } \phi_3\le\theta\le\phi_4,\\ (103-50\theta)/170 & \text{if } \phi_4\le\theta\le\phi_5,\\ (353-120\theta)/670 & \text{if } \phi_5\le\theta\le\phi_6,\\ (157-35\theta)/330 & \text{if } \phi_6\le\theta\le\phi_7,\\ (1159-160\theta)/2610 & \text{if } \phi_7\le\theta\le\phi_8,\\ 2/5 & \text{if } \phi_8\le\theta\le\phi_9.\end{cases}$$

For our choice of D, it is easy to verify $D\le x^{\theta-\varepsilon'}$. Next we prove (4.1).

LEMMA 4.4. *Let $\phi_1\le\theta\le\phi_9$ and let D be defined as before. Then* (4.1) *holds.*

Proof. If $\phi_8\le\theta\le\phi_9$, then $A_1\ldots A_t\ll D^{1+\varepsilon}\ll x^{2/5-\varepsilon'}$. Thus Lemma 4.2 gives (4.1). When $\phi_1\le\theta\le\phi_8$, we have $D=(b/a)x^{2/5-\varepsilon'}$. If there exists $\mathcal{J}\subset\{1,\ldots,t\}$ satisfying $\prod_{j\in\mathcal{J}}A_j\in I(\theta)$, we can apply Lemma 4.3 with a suitable choice of a_m,b_n to get (4.1). Otherwise Lemma 5 of [6] implies $A_1\ldots A_t\le D^{1+2\varepsilon}a/b<x^{2/5-\varepsilon'}$. Thus Lemma 4.2 is applicable to give (4.1). ■

Combining Lemmas 4.1 and 4.4, we immediately obtain the following result.

LEMMA 4.5. *Let $D^{1/3}\le z\le D^{1/2}$. Then $S(\mathcal{A},z)\le\{1+O(\varepsilon)\}2y/(\varrho(\theta)\mathcal{L})$.*

5. An alternative sieve. In this section, we insert our new results on bilinear forms $\mathcal{R}_I$ and $\mathcal{R}_{II}$ into the alternative sieve of Baker and Harman ([2], Section 5). This allows us to improve all results there. Since the proof is very similar, we just state our results and omit the details.

Let $\omega(t)$ be the Buchstab function, in particular,

$$t\omega(t)=\begin{cases}1 & \text{if } 1\le t\le 2,\\ 1+\log(t-1) & \text{if } 2\le t\le 3,\\ 1+\log(t-1)+\int_2^{t-1}s^{-1}\log(s-1)\,ds & \text{if } 3\le t\le 4.\end{cases}$$

Let $\mathcal{B}=\mathcal{B}(\theta):=\{n:x^\theta<n\le ex^\theta\}$. For $\mathcal{E}=\mathcal{A}$ or $\mathcal{B}$, we write $\mathcal{E}_m=\{n:mn\in\mathcal{E}\}$. Define

$$S(\mathcal{B}_m,z):=\sum_{mn\in\mathcal{B},\,P^-(n)\ge z}y/(mn).$$

Corresponding to Lemma 9 of [2], we have the following sharper result.

LEMMA 5.1. *Let $|b_n|\le 1$. For $N\le x^{2/5-\varepsilon'}$, we have*

$$\sum_{n\le N}b_n|\mathcal{A}_n|=y\sum_{n\le N}b_n/n+O_\varepsilon(yx^{-3\eta}).$$

Proof. In the proof of Lemma 9 of [2], replace Corollary 1 there by our Lemma 4.2. ∎

The next lemma is an improvement of Lemma 10 of [2].

LEMMA 5.2. *Let* $N \le x^{2/5-\varepsilon'}$, $0 \le b_n \le 1$, $b_n = 0$ *unless* $P^-(n) \ge x^\eta$ $(1 \le n \le N)$. *Then*

$$\sum_{n\le N} b_n S(\mathcal{A}_n, x^\eta) = \{1 + O(G(\varepsilon/\eta))\} \sum_{n\le N} b_n S(\mathcal{B}_n, x^\eta) + O_\varepsilon(yx^{-3\eta}),$$

where $G(t) := \exp\{1 + (\log t)/t\}$ $(t > 0)$.

Proof. In the proof of Lemma 10 of [2], replace Lemma 9 there by Lemma 5.1 above. ∎

We can improve Lemma 11 of [2] as follows.

LEMMA 5.3. *Let* $|a_m| \le 1$ *and* $|b_n| \le 1$. *For* $\phi_1 \le \theta \le \phi_8$ *and* $N \in I(\theta)$, *we have*

$$\sum_{\substack{mn\in\mathcal{A}\\ m\sim M,\, n\sim N}}\sum a_m b_n = y \sum_{\substack{mn\in\mathcal{B}\\ m\sim M,\, n\sim N}}\sum a_m b_n/(mn) + O_\varepsilon(yx^{-5\eta}).$$

Proof. In the proof of Lemma 11 of [2], replace (4.1) of [2] by our Lemma 4.3. ∎

Finally, similar to Lemmas 12, 13 and 15 of [2], we have the following results.

LEMMA 5.4. *Let* $h \ge 1$ *be given and suppose that* $\mathcal{J} \subset \{1, \ldots, h\}$. *For* $\phi_1 \le \theta \le \phi_8$, $N \in I(\theta)$ *and* $N_1 < 2N$, *we have*

$$\sum_{p_1} \ldots \sum_{p_h}{}^{*} S(\mathcal{A}_{p_1\ldots p_h}, p_1) = \sum_{p_1} \ldots \sum_{p_h}{}^{*} S(\mathcal{B}_{p_1\ldots p_h}, p_1) + O_\varepsilon(yx^{-5\eta}).$$

Here $*$ *indicates that* $p_1, \ldots, p_h$ *satisfy* $x^\eta \le p_1 < \ldots < p_h$ *and*

$$N \le \prod_{j\in\mathcal{J}} p_j < N_1 \tag{5.1}$$

together with no more than ε^{-1} *further conditions of the form*

$$R \le \prod_{j\in\mathcal{J}'} p_j \le S. \tag{5.2}$$

LEMMA 5.5. *Let* $M \le a$ *and* $N \le x^{2/5-\varepsilon'}/(2a)$. *Let* $M \le M_1 \le 2M$ *and* $N \le N_1 \le 2N$. *Let* $x^\eta \le z \le b/a$. *Suppose that* $\{1, \ldots, h\}$ *partitions into two sets* $\mathcal{J}$ *and* $\mathcal{K}$. *Then*

$$\sum_{p_1} \ldots \sum_{p_h}{}^{*} S(\mathcal{A}_{p_1\ldots p_h}, z) = \{1 + O(\varepsilon)\} \sum_{p_1} \ldots \sum_{p_h}{}^{*} S(\mathcal{B}_{p_1\ldots p_h}, z).$$

Here $*$ *indicates that* $p_1, \ldots, p_h$ *satisfy* $z \le p_1 < \ldots < p_h$ *and*

$$M \le \prod_{j\in\mathcal{J}} p_j < M_1, \qquad N \le \prod_{j\in\mathcal{K}} p_j < N_1 \tag{5.3}$$

together with no more than ε^{-1} *further conditions of the form* (5.2). *The case* $h = 0$, $\mathcal{J}$ *and* $\mathcal{K}$ *empty is permitted.*

LEMMA 5.6. *Let* $\phi_1 \le \theta \le \phi_2$, $ev/b^2 < P \le x^{-\varepsilon'} v/a^3$ *and* $b/a < Q \le b$. *Then*

$$\sum_{p\sim P}\sum_{q\sim Q} S(\mathcal{A}_{pq}, q) = \{1 + O(\varepsilon)\} \sum_{p\sim P}\sum_{q\sim Q} S(\mathcal{B}_{pq}, q).$$

Proof. In view of Lemma 5.4, we can suppose $Q < a$. By the Buchstab identity, we write

$$\begin{aligned}\sum_{p\sim P}\sum_{q\sim Q} S(\mathcal{A}_{pq}, q) \\ = \sum_{p\sim P}\sum_{q\sim Q} S(\mathcal{A}_{pq}, b/a) - \sum_{p\sim P}\sum_{q\sim Q}\sum_{b/a\le r<q} S(\mathcal{A}_{pqr}, r).\end{aligned} \tag{5.4}$$

Since $P \le x^{-\varepsilon'} v/a^3 \le x^{2/5-\varepsilon'}/(2a)$ and $Q \le a$, Lemma 5.5 can be applied to the first sum on the right-hand side of (5.4). When $\phi_1 \le \theta \le \phi_2$, we have $(b/a)^2 \ge a$. Thus the parts of the second sum with $qr \le b$ may be evaluated asymptotically via Lemma 5.4. For the remaining portion of the sum we note that it counts numbers $pqrs \in \mathcal{A}$ where $s < ev/(Pqr) \le ev/((ev/b^2)b) = b$ and $s > v/(8PQ^2) \ge v/(8(x^{-\varepsilon'} v/a^3)a^2) = x^{\varepsilon'} a/8 \ge a$. Hence Lemma 5.4 is again applicable and this completes the proof. ∎

6. The proof of (1.3). We establish (1.3) by three different methods according to the size of θ. Our function $u(\theta)$ is better than that of Baker and Harman [2]. We begin with the simplest case. Applying directly Lemma 4.5 with $z = D^{1/3}$, we have the following result.

LEMMA 6.1. *If* $\phi_1 \le \theta \le \phi_9$, *then* (1.3) *holds with* $u(\theta) = 5\theta$.

This result is very rough. In fact $S(\mathcal{A}, D^{1/3})$ counts many numbers not counted by $S(\theta)$. For some of these we can apply Lemma 4.3 and so obtain an improved bound by removing the "deductible" terms. Similarly to Lemma 17 of [2], we have the following sharper result.

LEMMA 6.2. *Let* $\theta_0 := \varrho(\theta)/(3\theta)$, $\theta_1 := (\theta - 1/2)/\theta$ *and* $\theta_2 := \tau(\theta)/\theta$. *If* $189/290 \le \theta \le \phi_8$, *then* (1.3) *holds with*

$$u(\theta)=\frac{2}{3\theta_0}-\int_{\theta_1}^{\theta_2}\omega\Big(\frac{1-\alpha}{\alpha}\Big)\frac{d\alpha}{\alpha^2}-\int_{\theta_0}^{\theta_1}\frac{d\alpha_1}{\alpha_1}\int_{\theta_1}^{\theta_2}\omega\Big(\frac{1-\alpha_1-\alpha_2}{\alpha_2}\Big)\frac{d\alpha_2}{\alpha_2^2}$$
$$-\int_{\theta_0}^{\theta_1}\frac{d\alpha_1}{\alpha_1}\int_{\alpha_1}^{\theta_1}\frac{d\alpha_2}{\alpha_2}\int_{\theta_1}^{\theta_2}\omega\Big(\frac{1-\alpha_1-\alpha_2-\alpha_3}{\alpha_3}\Big)\frac{d\alpha_3}{\alpha_3^2}.$$

REMARK. We have $\theta_1\ge\theta_0$ for $\theta\ge 189/290$. Therefore the last two integrals are positive.

Proof (of Lemma 6.2). By using the Buchstab identity, we write, with $z=D^{1/3}$,

$$S(\mathcal{A},(ev)^{1/2}) = S(\mathcal{A},z)-\sum_{z\le p<a}S(\mathcal{A}_p,p)-\sum_{a\le p<b}S(\mathcal{A}_p,p)-\sum_{b\le p<(ev)^{1/2}}S(\mathcal{A}_p,p). \tag{6.1}$$

Applying again the Buchstab identity yields

$$\sum_{z\le p<a}S(\mathcal{A}_p,p)=\sum_{z\le p<a}S(\mathcal{A}_p,b)+\sum_{z\le p\le q<a}\sum S(\mathcal{A}_{pq},q)+\sum_{z\le p<a\le q<b}\sum S(\mathcal{A}_{pq},q), \tag{6.2}$$

$$\sum_{z\le p\le q<a}\sum S(\mathcal{A}_{pq},q)=\sum_{z\le p\le q<a}\sum S(\mathcal{A}_{pq},b)+\sum_{z\le p\le q\le r<a}\sum\sum S(\mathcal{A}_{pqr},r)+\sum_{z\le p\le q<a\le r<b}\sum\sum S(\mathcal{A}_{pqr},r). \tag{6.3}$$

Inserting (6.2) and (6.3) into (6.1), we find

$$\begin{aligned} S(\mathcal{A},(ev)^{1/2}) &= S(\mathcal{A},z)-\sum_{a\le p<b}S(\mathcal{A}_p,p)-\sum_{z\le p<a\le q<b}\sum S(\mathcal{A}_{pq},q)\\ &\quad-\sum_{z\le p\le q<a\le r<b}\sum\sum S(\mathcal{A}_{pqr},r)\\ &\quad-\sum_{z\le p<a}S(\mathcal{A}_p,b)-\sum_{z\le p\le q<a}\sum S(\mathcal{A}_{pq},b)\\ &\quad-\sum_{z\le p\le q\le r<a}\sum\sum S(\mathcal{A}_{pqr},r)-\sum_{b\le p<(ev)^{1/2}}S(\mathcal{A}_p,p)\\ &=: R_1-R_2-R_3-R_4-\ldots-R_8\\ &\le R_1-R_2-R_3-R_4. \end{aligned} \tag{6.4}$$

By Lemma 4.5, we have

$$R_1\le\{1+O(\varepsilon)\}\frac{2y}{\varrho(\theta)\mathcal{L}}. \tag{6.5}$$

We may evaluate asymptotically R_2, R_3, R_4 via Lemma 5.4. Applying Lemma 8 of [2] and using the standard procedure for replacing sums over primes by integrals, we can prove

$$R_2 = \{1+O(\varepsilon)\}\frac{y}{\theta\mathcal{L}}\int_{\theta_1}^{\theta_2}\omega\Big(\frac{1-\alpha}{\alpha}\Big)\frac{d\alpha}{\alpha^2}, \tag{6.6}$$

$$R_3 = \{1+O(\varepsilon)\}\frac{y}{\theta\mathcal{L}}\int_{\theta_0}^{\theta_1}\frac{d\alpha_1}{\alpha_1}\int_{\theta_1}^{\theta_2}\omega\Big(\frac{1-\alpha_1-\alpha_2}{\alpha_2}\Big)\frac{d\alpha_2}{\alpha_2^2}, \tag{6.7}$$

$$R_4 = \{1+O(\varepsilon)\}\frac{y}{\theta\mathcal{L}}\int_{\theta_0}^{\theta_1}\frac{d\alpha_1}{\alpha_1}\int_{\alpha_1}^{\theta_1}\frac{d\alpha_2}{\alpha_2}\int_{\theta_1}^{\theta_2}\omega\Big(\frac{1-\alpha_1-\alpha_2-\alpha_3}{\alpha_3}\Big)\frac{d\alpha_3}{\alpha_3^2}. \tag{6.8}$$

Inserting (6.5)–(6.8) into (6.4), we obtain the required result. ■

Finally, we apply the alternative sieve of Baker and Harman to deduce the desired upper bound $u(\theta)$ for $\phi_1 \le \theta < 7/10$. By the Buchstab identity, we can write

$$\begin{aligned} S(\mathcal{A},(ev)^{1/2}) = S(\mathcal{A},b/a) &- \sum_{b/a\le p<a} S(\mathcal{A}_p,p) \\ &- \sum_{a\le p\le b} S(\mathcal{A}_p,p) - \sum_{b<p<(ev)^{1/2}} S(\mathcal{A}_p,p). \end{aligned} \tag{6.9}$$

For the second term on the right-hand side, we apply again two times the Buchstab identity

$$\begin{aligned} \sum_{b/a\le p<a} S(\mathcal{A}_p,p) = \sum_{b/a\le p<a} S(\mathcal{A}_p,b/a) &- \sum\sum_{b/a\le q<p<a} S(\mathcal{A}_{pq},b/a) \\ &+ \sum\sum\sum_{b/a\le r<q<p<a} S(\mathcal{A}_{pqr},r). \end{aligned} \tag{6.10}$$

Inserting (6.10) into (6.9) yields

$$\begin{aligned} S(\mathcal{A},(ev)^{1/2}) &= S(\mathcal{A},b/a) - \sum_{b/a\le p<a} S(\mathcal{A}_p,b/a) \\ &\quad + \sum\sum_{b/a\le q<p<a} S(\mathcal{A}_{pq},b/a) - \sum\sum\sum_{b/a\le r<q<p<a} S(\mathcal{A}_{pqr},r) \\ &\quad - \sum_{a\le p\le b} S(\mathcal{A}_p,p) - \sum_{b<p<(ev)^{1/2}} S(\mathcal{A}_p,p) \\ &=: S_1 - S_2 + S_3 - S_4 - S_5 - S_6. \end{aligned} \tag{6.11}$$

Noticing $a \le x^{2/5-\varepsilon'}/(2a)$ for $\theta < 7/10$, Lemma 5.5 allows us to get the asymptotic formulae for S_j $(1 \le j \le 3)$. In addition, by Lemma 5.4 we also obtain the asymptotic formula for S_5.

In order to treat S_4, it is necessary to introduce some notation. We write $p = v^{\alpha_1}$, $q = v^{\alpha_2}$, $r = v^{\alpha_3}$, $s = v^{\alpha_4}$, $t = v^{\alpha_5}$ and $\overline{\alpha} := (\alpha_1, \ldots, \alpha_n)$. Let $\theta_3 := \theta_2 - \theta_1$ and

$$\mathbb{E}_n := \{(\alpha_1, \ldots, \alpha_n) : \theta_3 \le \alpha_n < \ldots < \alpha_1 < \theta_1, \\ \alpha_1 + \ldots + \alpha_{n-1} + 2\alpha_n \le 1 + 1/(\theta\mathcal{L})\}.$$

A point $\overline{\alpha}$ of $\mathbb{E}_n$ is said to be *bad* if no sum $\sum_{j\in\mathcal{J}} \alpha_j$ lies in $[\theta_1 + \varepsilon', \theta_2 - \varepsilon']$ where $\mathcal{J} \subset \{1, \ldots, n\}$. The set of all bad points is denoted by $\mathbb{B}_n$. The points of $\mathbb{G}_n := \mathbb{E}_n \setminus \mathbb{B}_n$ are called *good*. Let $\theta_4 := (9/10 - \theta)/\theta$, $\mathbb{U} := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{B}_3 : \alpha_2 + 2\alpha_3 \ge \theta_4 - \varepsilon'\}$, $\mathbb{V} := \mathbb{B}_3 \setminus \mathbb{U}$ and $\mathbb{W} := \mathbb{G}_3$. We see that $\mathbb{E}_3$ partitions into $\mathbb{U}, \mathbb{V}, \mathbb{W}$. Thus

$$S_4 = \sum_{\overline{\alpha}\in\mathbb{U}} S(\mathcal{A}_{pqr}, r) + \sum_{\overline{\alpha}\in\mathbb{V}} S(\mathcal{A}_{pqr}, r) + \sum_{\overline{\alpha}\in\mathbb{W}} S(\mathcal{A}_{pqr}, r) =: S_7 + S_8 + S_9.$$

According to the definition of $\mathbb{W}$, S_9 can be evaluated asymptotically. For S_8, we use the Buchstab identity to write

$$\begin{aligned} S_8 &= \sum_{\overline{\alpha}\in\mathbb{V}} S(\mathcal{A}_{pqr}, b/a) - \sum_{\overline{\alpha}\in\mathbb{X}_1} S(\mathcal{A}_{pqrs}, s) - \sum_{\overline{\alpha}\in\mathbb{X}_2} S(\mathcal{A}_{pqrs}, s) \\ &=: S_{10} - S_{11} - S_{12}, \end{aligned}$$

with $\mathbb{X}_1 := \{(\alpha_1, \ldots, \alpha_4) \in \mathbb{G}_4 : (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{V}\}$, $\mathbb{X}_2 := \{(\alpha_1, \ldots, \alpha_4) \in \mathbb{B}_4 : (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{V}\}$.

If $\overline{\alpha} \in \mathbb{V}$, then $qr = v^{\alpha_2+\alpha_3} < v^{\theta_4-\varepsilon'} \le x^{2/5-\varepsilon'}/(2a)$. Hence Lemma 5.5 allows us to get the desired asymptotic formula for S_{10}. In addition, the definition of $\mathbb{X}_1$ shows that S_{11} may be evaluated asymptotically. For S_{12}, we again apply the Buchstab identity to write

$$\begin{aligned} S_{12} &= \sum_{\overline{\alpha}\in\mathbb{X}_2} S(\mathcal{A}_{pqrs}, b/a) - \sum_{\overline{\alpha}\in\mathbb{Y}_1} S(\mathcal{A}_{pqrst}, t) - \sum_{\overline{\alpha}\in\mathbb{Y}_2} S(\mathcal{A}_{pqrst}, t) \\ &=: S_{13} - S_{14} - S_{15}, \end{aligned}$$

where

$$\mathbb{Y}_1 := \{(\alpha_1, \ldots, \alpha_5) \in \mathbb{G}_5 : (\alpha_1, \ldots, \alpha_4) \in \mathbb{X}_2\},$$
$$\mathbb{Y}_2 := \{(\alpha_1, \ldots, \alpha_5) \in \mathbb{B}_5 : (\alpha_1, \ldots, \alpha_4) \in \mathbb{X}_2\}.$$

When $\overline{\alpha} \in \mathbb{X}_2$, we find that $qrs = v^{\alpha_2+\alpha_3+\alpha_4} \le v^{\alpha_2+2\alpha_3} \le v^{\theta_4-\varepsilon'} \le x^{2/5-\varepsilon'}/(2a)$. Thus we have the desired asymptotic formula for S_{13} by Lemma 5.5.

Inserting these into (6.11), we obtain

$$\begin{aligned} S(\mathcal{A}, (ev)^{1/2}) = S_1 - S_2 + S_3 - S_5 - S_6 - S_7 - S_9 - S_{10} \\ + S_{11} + S_{13} + S_{14} - S_{15}. \end{aligned}$$

We have the desired asymptotic formulae for S_j, except for $j = 6, 7, 15$.

Obviously the same decomposition also holds for $S(\mathcal{B},(ev)^{1/2})$, i.e.

$$S(\mathcal{B},(ev)^{1/2}) = S_1' - S_2' + S_3' - S_5' - S_6' - S_7' - S_9' - S_{10}' + S_{11}' + S_{13}' + S_{14}' - S_{15}',$$

where S_j' is defined similarly to S_j with the only difference that $\mathcal{A}$ is replaced by $\mathcal{B}$. Since $S_j = \{1+O(\varepsilon)\}S_j'$ except for $j = 6, 7, 15$, we can obtain

$$S(\mathcal{A},(ev)^{1/2}) = \{1+O(\varepsilon)\}\{S(\mathcal{B},(ev)^{1/2}) + S_6' + S_7' + S_{15}'\} - S_6 - S_7 - S_{15}. \tag{6.12}$$

By Lemma 8 of [2] and by using the standard procedure for replacing sums over primes by integrals, we can deduce

$$S(\mathcal{B},(ev)^{1/2}) + S_6' = \{1+O(\varepsilon)\}\frac{1}{\theta_2}\omega\left(\frac{1}{\theta_2}\right)\frac{y}{\theta\mathcal{L}}, \tag{6.13}$$

$$S_7' = \{1+O(\varepsilon)\}\frac{K(\theta)y}{\theta\mathcal{L}}, \qquad S_{15}' = \{1+O(\varepsilon)\}\frac{R(\theta)y}{\theta\mathcal{L}}, \tag{6.14}$$

where

$$\begin{cases} K(\theta) := \displaystyle\int_{\mathbb{U}} \omega\left(\frac{1-\alpha_1-\alpha_2-\alpha_3}{\alpha_3}\right)\frac{d\alpha_1\,d\alpha_2\,d\alpha_3}{\alpha_1\alpha_2\alpha_3^2}, \\ R(\theta) := \displaystyle\int_{\mathbb{Y}_2} \omega\left(\frac{1-\alpha_1-\ldots-\alpha_5}{\alpha_5}\right)\frac{d\alpha_1\ldots d\alpha_5}{\alpha_1\ldots\alpha_4\alpha_5^2}. \end{cases} \tag{6.15}$$

Finally, we give a non-trivial lower bound for S_6 when $\phi_1 \le \theta \le \phi_2$. In this case, we have $b \le ev/b^2 < x^{-\varepsilon'}v/a^3 \le (ev)^{1/2}$. Thus by the Buchstab identity, we can write

$$\begin{aligned} S_6 &\ge \sum_{ev/b^2<p<x^{-\varepsilon'}v/a^3} S(\mathcal{A}_p, p) \\ &= \sum_{ev/b^2<p<x^{-\varepsilon'}v/a^3} S(\mathcal{A}_p, b/a) - \sum_{\substack{ev/b^2<p<x^{-\varepsilon'}v/a^3 \\ b/a\le q<\min\{p,(ev/p)^{1/2}\}}} S(\mathcal{A}_{pq}, q). \end{aligned}$$

Since $x^{-\varepsilon'}v/a^3 \le x^{2/5-\varepsilon'}/(2a)$, we have an asymptotic formula for the first term on the right-hand side from Lemma 5.5. In addition, we note that $p > ev/b^2$ implies $(ev/p)^{1/2} \le b$. Thus the second term may be evaluated asymptotically via Lemma 5.6. Hence

$$\begin{aligned} S_6 &\ge \{1+O(\varepsilon)\}\sum_{ev/b^2<p<x^{-\varepsilon'}v/a^3} S(\mathcal{B}_p, p) \\ &= \{1+O(\varepsilon)\}\frac{y}{\theta\mathcal{L}}\log\left(\frac{3-4\theta}{6\theta-3}\cdot\frac{4-6\theta}{7\theta-4}\right). \end{aligned} \tag{6.16}$$

Inserting (6.13), (6.14) and (6.16) into (6.12) and using $S_7, S_{15} \ge 0$, we get the following result.

LEMMA 6.3. *For* $\phi_1 \le \theta < 7/10$, *we have* (1.3) *with* $u(\theta) = M(\theta) + K(\theta) + R(\theta)$, *where* $K(\theta)$ *and* $R(\theta)$ *are defined as in* (6.15) *and*

$$M(\theta) = \begin{cases} \dfrac{1}{\theta_2}\omega\left(\dfrac{1}{\theta_2}\right) - \log\left(\dfrac{3-4\theta}{6\theta-3}\cdot\dfrac{4-6\theta}{7\theta-4}\right) & \text{if } \phi_1 \le \theta < \phi_2, \\ \dfrac{1}{\theta_2}\omega\left(\dfrac{1}{\theta_2}\right) & \text{if } \phi_2 \le \theta < 7/10. \end{cases}$$

REMARK. The functions $M(\theta)$, $K(\theta)$ and $R(\theta)$ are each θ times the corresponding functions in Baker and Harman [2].

7. The proof of (1.4). We recall the notation: $\theta_0 := \varrho(\theta)/(3\theta)$, $\theta_1 := (\theta - 1/2)/\theta$ and $\theta_2 := \tau(\theta)/\theta$.

A. *The interval* $\phi_1 \le \theta \le 0.661$. In this case we use Lemma 6.3. Noticing $3 \le 1/\theta_2 \le 4$, we have

$$\frac{1}{\theta_2}\omega\left(\frac{1}{\theta_2}\right) = 1 + \log 2 + \int_2^{1/\theta_2 - 1} \frac{1 + \log(t-1)}{t}\,dt$$

and $\int_{\phi_1}^{0.661} M(\theta)\,d\theta < 0.123182$. Clearly (7.3) of [2] implies $\int_{\phi_1}^{0.661} \{K(\theta) + R(\theta)\}\,d\theta < 0.0125$ (see the final remark). Hence

$$\int_{\phi_1}^{0.661} u(\theta)\,d\theta < 0.135682. \tag{7.1}$$

B. *The interval* $0.661 \le \theta \le \phi_8$. In this case we apply Lemma 6.2. We have $2 \le (1-\alpha)/\alpha \le 4$ for $\theta_1 \le \alpha \le \theta_2$. By using $t\omega(t) \ge 1 + \log(t-1)$ for $2 \le t \le 4$, we can deduce

$$\int_{\theta_1}^{\theta_2} \omega\left(\frac{1-\alpha}{\alpha}\right)\frac{d\alpha}{\alpha^2} \ge \log\frac{1/\theta_1 - 1}{1/\theta_2 - 1} + \int_{1/\theta_2 - 1}^{1/\theta_1 - 1} \frac{\log(\alpha - 1)}{\alpha}\,d\alpha.$$

Similarly noticing $1 \le (1 - \alpha_1 - \alpha_2)/\alpha_2 \le 3$ for $\theta_0 \le \alpha_1 \le \theta_1 \le \alpha_2 \le \theta_2$ and $t\omega(t) \ge 1$ for $1 \le t \le 3$, we see that

$$\int_{\theta_0}^{\theta_1} \frac{d\alpha_1}{\alpha_1} \int_{\theta_1}^{\theta_2} \omega\left(\frac{1-\alpha_1-\alpha_2}{\alpha_2}\right)\frac{d\alpha_2}{\alpha_2^2} \ge \int_{\theta_0}^{\theta_1} \log\left(\frac{1-\theta_1-\alpha}{1-\theta_2-\alpha}\cdot\frac{\theta_2}{\theta_1}\right)\frac{d\alpha}{\alpha(1-\alpha)}.$$

Finally, using $\omega(t) \ge 1/2$ for $t \ge 1$ ([16, IV], p. 437), we deduce

$$\int_{\theta_0}^{\theta_1} \frac{d\alpha_1}{\alpha_1} \int_{\alpha_1}^{\theta_1} \frac{d\alpha_2}{\alpha_2} \int_{\theta_1}^{\theta_2} \omega\left(\frac{1-\alpha_1-\alpha_2-\alpha_3}{\alpha_3}\right)\frac{d\alpha_3}{\alpha_3^2} \ge \frac{1}{4}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\log^2\frac{\theta_1}{\theta_0}.$$

Hence we have

$$u(\theta) \le f(\theta) - g(\theta),$$

where

$$f(\theta) := \frac{2}{3\theta_0} - \log\frac{1/\theta_1 - 1}{1/\theta_2 - 1} - \frac{1}{4}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\log^2\frac{\theta_1}{\theta_0},$$

$$g(\theta) := \int\limits_{1/\theta_2-1}^{1/\theta_1-1} \frac{\log(\alpha-1)}{\alpha}\,d\alpha + \int\limits_{\theta_0}^{\theta_1} \log\left(\frac{1-\theta_1-\alpha}{1-\theta_2-\alpha}\cdot\frac{\theta_2}{\theta_1}\right)\frac{d\alpha}{\alpha(1-\alpha)}.$$

A numerical computation gives us

$[\alpha,\beta]$	$[0.661,\phi_4]$	$[\phi_4,\phi_5]$	$[\phi_5,\phi_6]$	$[\phi_6,\phi_7]$	$[\phi_7,\phi_8]$
$\int_\alpha^\beta f(\theta)\,d\theta <$	0.0177872	0.0666379	0.0433597	0.0296966	0.0376814
$\int_\alpha^\beta g(\theta)\,d\theta >$	0.0004544	0.0009964	0.0002399	0.0000643	0.0000231

$$\int\limits_{0.661}^{\phi_8} u(\theta)\,d\theta < 0.193385. \tag{7.2}$$

C. *The interval* $\phi_8 \le \theta \le \phi_9$. From Lemma 6.1, we have

$$\int\limits_{\phi_8}^{\phi_9} u(\theta)\,d\theta = 2.5(\phi_9^2 - \phi_8^2) < 0.070107. \tag{7.3}$$

Now (1.4) follows from (7.1)–(7.3), completing the proof of Theorem 1. ■

FINAL REMARK. Since our estimates for exponential sums are better than those of Baker and Harman [2], our $\mathbb{U}$, $\mathbb{Y}_2$ are smaller than their corresponding $\mathbb{U}$, $\mathbb{Y}_2$. Therefore we can certainly obtain a smaller value in place of 0.0125. This leads to a better exponent than 0.738. It seems that we could not have arrived at 0.74 by computing precisely $\int_{\phi_1}^{0.661}\{K(\theta)+R(\theta)\}\,d\theta$.

References

[1] R. C. Baker, *The greatest prime factor of the integers in an interval*, Acta Arith. 47 (1986), 193–231.

[2] R. C. Baker and G. Harman, *Numbers with a large prime factor*, ibid. 73 (1995), 119–145.

[3] R. C. Baker, G. Harman and J. Rivat, *Primes of the form* $[n^c]$, J. Number Theory 50 (1995), 261–277.

[4] A. Balog, *Numbers with a large prime factor I*, Studia Sci. Math. Hungar. 15 (1980), 139–146; *II*, in: Topics in Classical Number Theory, Colloq. Math. Soc. János Bolyai 34, North-Holland, 1984, 49–67.

[5] A. Balog, G. Harman and J. Pintz, *Numbers with a large prime factor IV*, J. London Math. Soc. (2) 28 (1983), 218–226.
[6] E. Fouvry, *Sur le théorème de Brun–Titchmarsh*, Acta Arith. 43 (1984), 417–424.
[7] E. Fouvry and H. Iwaniec, *Exponential sums with monomials*, J. Number Theory 33 (1989), 311–333.
[8] S. W. Graham, *The greatest prime factor of the integers in an interval*, J. London Math. Soc. (2) 24 (1981), 427–440.
[9] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge Univ. Press, 1991.
[10] G. Harman, *On the distribution of αp modulo one*, J. London Math. Soc. (2) 27 (1983), 9–13.
[11] D. R. Heath-Brown, *The Pjateckiĭ–Šapiro prime number theorem*, J. Number Theory 16 (1983), 242–266.
[12] —, *The largest prime factor of the integers in an interval*, Sci. China Ser. A 39 (1996), 449–476.
[13] D. R. Heath-Brown and C. H. Jia, *The largest prime factor of the integers in an interval II*, J. Reine Angew. Math. 498 (1998), 35–59.
[14] M. N. Huxley, *Area, Lattice Points and Exponential Sums*, Oxford Sci. Publ., Clarendon Press, Oxford, 1996.
[15] H. Iwaniec, *A new form of the error term in the linear sieve*, Acta Arith. 37 (1980), 307–320.
[16] C. H. Jia, *The greatest prime factor of the integers in an interval I*, Acta Math. Sinica 29 (1986), 815–825; *II*, ibid. 32 (1989), 188–199; *III*, ibid. (N.S.) 9 (1993), 321–336; *IV*, ibid. 12 (1996), 433–445.
[17] M. Jutila, *On numbers with a large prime factor IV*, J. Indian Math. Soc. (N.S.) 37 (1973), 43–53.
[18] H.-Q. Liu, *The greatest prime factor of the integers in an interval*, Acta Arith. 65 (1993), 301–328.
[19] —, *A special triple exponential sum*, Mathematika 42 (1995), 137–143.
[20] K. Ramachandra, *A note on numbers with a large prime factor I*, J. London Math. Soc. (2) 1 (1969), 303–306; *II*, J. Indian Math. Soc. 34 (1970), 39–48.
[21] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.
[22] J. Wu, *Nombres $\mathcal{B}$-libres dans les petits intervalles*, Acta Arith. 65 (1993), 97–116.

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