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ON SUBRINGS OF AMALGAMATED FREE PRODUCTS OF RINGS

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Abstract. The aim of this paper is to develop the homological machinery needed to study *amalgams of subrings*. We follow Cohn [1] and describe an amalgam of subrings in terms of reduced iterated tensor products of the rings forming the amalgam and prove a result on embeddability of amalgamated free products. Finally we characterise the commutative *perfect amalgamation bases*.

1. Introduction. An *amalgam* of rings $[R;S_i]$ consists of a family of rings S_i together with a common subring R, called the *core* of the amalgam. The amalgam is said to be (*weakly*) *embeddable* in a ring W if there are monomorphisms $\theta_i : S_i \to W$ such that $\theta_i|_R = \theta_j|_R$ for all $i \neq j$. If in addition $\operatorname{im}\theta_i \cap \operatorname{im}\theta_j \cong R$, then we say that the embedding is *strong*. It is easy to establish that an amalgam of rings $[R;S_i]$ is embeddable if and only if it is embeddable in its *amalgamated free product* $\prod_R^* S_i$.

It is well known that not every amalgam of rings is embeddable and P. M. Cohn [1] gave some conditions under which an amalgam is embeddable. About the same time, Howie [2] studied the case for semigroup amalgams. The author extended this work in both the semigroup and ring cases [4], [5].

In [3], Howie studied the idea of subsemigroups of amalgamated free-products and again this work was extended by the author [6], [7]. We wish now to study the case for rings. In more detail, suppose that $[R;T_i]$ and $[R;S_i]$ are amalgams with $R \subseteq T_i \subseteq S_i$. We shall call the amalgam $[R;T_i]$ an *amalgam of subrings* of the amalgam $[R;S_i]$. We wish to ask the question: is it true that $\prod_R^* T_i$ is embeddable in $\prod_R^* S_i$? In fact we need only consider amalgams with a *finite* index set, because of the following easily proved result (see [6] for the semigroup case):

THEOREM 1.1. *Let $[R;T_i : i \in I]$ be an amalgam of subrings of the amalgam $[R;S_i : i \in I]$. Then $\prod_R^* T_i$ is embeddable in $\prod_R^* S_i$ if and only if $\Pi_R^*\{T_i : i \in F\}$ is embeddable in $\Pi_R^*\{S_i : i \in F\}$ for all finite subsets F of I.*

We shall have occasion to use the following theorem.

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Theorem 1.2 (cf. [8, Theorem 2.18]. *Let I be a directed quasi-ordered set. Suppose that (A_i, ϕ^i_j) and (B_i, θ^i_j) are direct systems in the category of R-modules (sharing the same index set) with direct limits (A, α_i) and (B, β_i) respectively. Suppose also there exist maps $f_i : A_i \to B_i$ such that $f_j\theta^i_j = \phi^i_j f_i$ for all $i \geq j$. Then there exists a unique map $f : A \to B$ such that $\beta_i f_i = f\alpha_i$ for all i and if each f_i is one-to-one then f is one-to-one also.*

Conversely, if f and each ϕ^i_j are one-to-one then each f_i is also one-to-one.

We begin in Section 2 by recalling some definitions from [4] and proving a technical result on *free extensions of R-modules*. Given an amalgam of subrings $[R; T_1, T_2]$ of an amalgam $[R; S_1, S_2]$, we describe, in Section 3, the canonical map $T_1 *_R T_2 \to S_1 *_R S_2$, in terms of maps between two directed systems of R-modules. This construction is then used to prove the main results. All rings are assumed to be unitary rings and all tensor products, unless otherwise stated, are assumed to be over R.

2. Purity and free extensions. Let R be a subring of a ring S. Let $A \in$ MOD-S, $B \in$ MOD-R and suppose that $f : A \to B$ is an R-map. The *free S-extension of A and B* is a right S-module $F(S; A, B)$ together with an S-map $h : A \to F(S; A, B)$ and a right R-map $g : B \to F(S; A, B)$ such that

1. $g \circ f = h$, and
2. whenever there is an S-module C, an S-map $\beta : A \to C$ and an R-map $\alpha : B \to C$ with $\alpha \circ f = \beta$, then there exists a unique S-map $\psi : F(S; A, B) \to C$ such that $\psi \circ g = \alpha$ and $\psi \circ h = \beta$.

Recall that a right R-monomorphism $f : X \to Y$ is called (*right*) *pure* if for all $A \in R$-MOD, the induced map $f \otimes 1 : X \otimes A \to Y \otimes A$ is one-to-one. If $X, Y \in R$-MOD-R and if $f : X \to Y$ is an (R, R)-monomorphism then f is called *pure* if for all $A \in$ MOD-R and $B \in R$-MOD the induced map $1 \otimes f \otimes 1 : A \otimes X \otimes B \to A \otimes Y \otimes B$ is one-to-one.

Let $f : X \to Y$ be a right R-map and $\lambda : A \to B$ a left R-map and consider the commutative diagram

$$\begin{array}{ccc} X \otimes A & \xrightarrow{1_X \otimes \lambda} & X \otimes B \\ {\scriptstyle f\otimes 1_A}\downarrow & & \downarrow{\scriptstyle f\otimes 1_B} \\ Y \otimes A & \xrightarrow[1_Y \otimes \lambda]{} & Y \otimes B \end{array} \tag{1}$$

We say that the pair (f, λ) is *stable* if

$$\mathrm{im}(f \otimes 1_B) \cap \mathrm{im}(1_Y \otimes \lambda) = \mathrm{im}(f \otimes \lambda).$$

In other words, (f, λ) is stable if whenever $\sum y \otimes \lambda(a) = \sum f(x) \otimes b$ in $Y \otimes B$, then there exists $\sum x' \otimes a'$ in $X \otimes A$ such that $\sum y \otimes \lambda(a) = \sum f(x') \otimes \lambda(a')$. It follows that if all the maps in the diagram (1) are one-to-one then (f, λ) is stable if and only if (1) is a pullback.

We say that a right R-monomorphism $f : X \to Y$ is (*right*) *stable* if for all $A, B \in$ MOD-R and all left R-monomorphisms $\lambda : A \to B$, the pair (f, λ) is stable. The following is an easy consequence of [4, Theorem 3.11].

LEMMA 2.1. *If $f : X \to Y$ is right pure and $\lambda : A \to B$ is left pure then the diagram* (1) *is a pullback.*

Suppose now that $R \subseteq T \subseteq S$ are rings. We show that under certain conditions, if we have a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & F(T; A, B) \arrow[d] \\ C \arrow[r] & D \arrow[r] & F(S; C, D) \end{tikzcd}$$

and if the first square satisfies a suitable property P, say, then so does the second square. This will form the basis for an inductive process in the next section.

THEOREM 2.2. *Let $R \subseteq T \subseteq S$ be rings, with $R \to S$ and $T \to S$ both pure as R-monomorphisms. Whenever $A \in R$-MOD-T, $B, D \in R$-MOD-R, $C \in R$-MOD-S and $\alpha_1 : A \to B$, $\alpha_2 : C \to D$ are pure R-monomorphisms and whenever there exist "connecting" pure R-monomorphisms $\delta : A \to C$ and $\varepsilon : B \to D$ such that for all $X \in$ MOD-R and all $Y \in R$-MOD the diagram*

$$\begin{tikzcd} X \otimes A \otimes Y \arrow[r, "1\otimes\alpha_1\otimes 1"] \arrow[d, "1\otimes\delta\otimes 1"'] & X \otimes B \otimes Y \arrow[d, "1\otimes\varepsilon\otimes 1"] \\ X \otimes C \otimes Y \arrow[r, "1\otimes\alpha_2\otimes 1"'] & X \otimes D \otimes Y \end{tikzcd}$$

is a pullback, then there exists a unique pure R-monomorphism $\psi : F(T; A, B) \to F(S; C, D)$ such that $\psi \circ \beta_1 = \beta_2 \circ \varepsilon$ (where the maps β_i are the canonical maps). Moreover, when these conditions hold, then for all $X \in$ MOD-R, $Y \in R$-MOD the diagram

$$\begin{tikzcd} X \otimes B \otimes Y \arrow[r, "1\otimes\beta_1\otimes 1"] \arrow[d, "1\otimes\varepsilon\otimes 1"'] & X \otimes F(T; A, B) \otimes Y \arrow[d, "1\otimes\psi\otimes 1"] \\ X \otimes D \otimes Y \arrow[r, "1\otimes\beta_2\otimes 1"'] & X \otimes F(S; C, D) \otimes Y \end{tikzcd}$$

is also a pullback.

Proof. For the sake of brevity, let us denote $F(T;A,B)$ by P and $F(S;C,D)$ by Q. We see from [4, Theorem 3.15] that the maps $B \to P$ and $D \to Q$ are pure monomorphisms. Consider the following commutative diagram:

$$\begin{tikzcd}
X \otimes A \otimes T \otimes Y \arrow[rr] \arrow[dr] \arrow[dd] & & X \otimes B \otimes T \otimes Y \arrow[dr] \arrow[dd] & \\
& X \otimes C \otimes S \otimes Y \arrow[rr] \arrow[dd] & & X \otimes D \otimes S \otimes Y \arrow[dd] \\
X \otimes A \otimes Y \arrow[rr] \arrow[dr] & & X \otimes P \otimes Y \arrow[dr] & \\
& X \otimes C \otimes Y \arrow[rr] & & X \otimes Q \otimes Y
\end{tikzcd}$$

If we can show that the top square

$$\begin{tikzcd}
X \otimes A \otimes T \otimes Y \arrow[r] \arrow[d] & X \otimes B \otimes T \otimes Y \arrow[d] \\
X \otimes C \otimes S \otimes Y \arrow[r] & X \otimes D \otimes S \otimes Y
\end{tikzcd}$$

is a pullback then it will follow from [4, Theorem 2.9] that the map $X \otimes P \otimes Y \to X \otimes Q \otimes Y$ is one-to-one and so $P \to Q$ will be pure as required.

Consider then the commutative diagram

$$\begin{tikzcd}
X \otimes A \otimes T \otimes Y \arrow[r] \arrow[d] & X \otimes B \otimes T \otimes Y \arrow[d] \\
X \otimes C \otimes T \otimes Y \arrow[r] \arrow[d] & X \otimes D \otimes T \otimes Y \arrow[d] \\
X \otimes C \otimes S \otimes Y \arrow[r] & X \otimes D \otimes S \otimes Y
\end{tikzcd}$$

The top square is a pullback, by assumption, and the bottom is a pullback by Lemma 2.1. Hence the "outer" rectangle is also a pullback. ∎

3. Free products of rings. Recall the following construction from Cohn [1] (see also [4]). Let $[R;S_1,S_2]$ be an amalgam of rings. Let $W_1 = S_1$, $W_2 = S_1 \otimes S_2$ and define $f_1 : W_1 \to W_2$ by $f_1(s_1) = s_1 \otimes 1$. Now define, inductively, a sequence of (S_1,S_i)-bimodules W_n and (S_1,R)-maps $f_n : W_n \to W_{n+1}$ $(i \equiv n \pmod 2)$ by $W_{n+1} = F(S_i, W_{n-1}, W_n)$ and f_n the canonical map.

It was proved in Cohn [1] that $S_1 *_R S_2$, the free product of the amalgam, is the direct limit in the category of R-modules of the direct system (W_n, f_n).

The direct system comes equipped with maps $\phi_n : W_n \to S_1 *_R S_2$ such that

$$\phi_{n+1} \circ f_n = \phi_n, \qquad n = 1, 2, \ldots$$

It is clear that if $[R; T_1, T_2]$ is an amalgam of subrings of the amalgam $[R; S_1, S_2]$ then a similar construction, say (Z_n, g_n), can be made. Hence we can construct a commutative diagram

(2)
$$\begin{tikzcd}
 & & T_1 *_R T_2 & & \\
 & Z_1 \arrow[r, "g_1"'] \arrow[ur, "\theta_1"] \arrow[dd, "\varepsilon_1"] & Z_2 \arrow[r, "g_2"'] \arrow[u, "\theta_2"] \arrow[dd, "\varepsilon_2"] & Z_3 \arrow[r, "g_3"'] \arrow[ul, "\theta_3"'] \arrow[dd, "\varepsilon_3"] & {} \\
U \arrow[ur] \arrow[dr] & & & & \\
 & W_1 \arrow[r, "f_1"] \arrow[dr, "\phi_1"'] & W_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & W_3 \arrow[r, "f_3"] \arrow[dl, "\phi_3"] & {} \\
 & & S_1 *_R S_2 & &
\end{tikzcd}$$

where $\varepsilon_1 : Z_1 \to W_1$ is the inclusion, $\varepsilon_2 : Z_2 \to W_2$ is given by $\varepsilon_2(t_1 \otimes t_2) = t_1 \otimes t_2$, and in general, $\varepsilon_n : Z_n \to W_n$ is the unique T_i-map ($i \equiv n \pmod 2$) which makes the diagram

$$\begin{tikzcd}
Z_{n-2} \arrow[rr, "g_{n-2}"] \arrow[dr] \arrow[ddr, "f_{n-1}f_{n-2}\varepsilon_{n-2}"'] & & Z_{n-1} \arrow[dl, "g_{n-1}"'] \arrow[ddl, "f_{n-1}\varepsilon_{n-1}"] \\
 & Z_n \arrow[d, "\varepsilon_n"'] & \\
 & W_n &
\end{tikzcd}$$

commute. We see from Theorem 1.2 that if each ε_i is one-to-one, then so is the canonical map $\psi : T_1 *_R T_2 \to S_1 *_R S_2$. In fact, since "tensor products preserve direct limits" [8, Corollary 2.20], if $X \in$ MOD-R and $Y \in R$-MOD then we can apply the functors $X \otimes -$ and $- \otimes Y$ to the diagram (2) and deduce that if each ε_i is pure then so is ψ. Our aim therefore is to consider when each ε_i is a pure R-monomorphism.

THEOREM 3.1 ([4, Theorem 5.3]). *Let $[R; S_1, S_2]$ be an amalgam of rings such that $R \to S_i$ is pure. Then the amalgam is strongly embeddable and $R \to S_1 *_R S_2$ is pure. Moreover, the maps $\phi_n : W_n \to S_1 *_R S_2$ in* (2) *are all pure monomorphisms.*

We extend this result to amalgams of subrings as follows. The idea is to apply Theorem 2.2 to the first square in (2) and then use induction to prove that each ε_i is pure.

THEOREM 3.2. *Let* $[R;T_1,T_2]$ *be an amalgam of subrings of the amalgam* $[R;S_1,S_2]$ *and suppose that the maps* $R\to T_i$ *and* $T_i\to S_i$ *are pure* R-*monomorphisms* $(i=1,2)$. *Then the canonical map* $\psi: T_1 *_R T_2 \to S_1 *_R S_2$ *is a pure* R-*monomorphism.*

Proof. It is easy to establish that f_1, g_1, ε_1 and ε_2 in (2) are all pure R-monomorphisms. Let $X\in$ MOD-R and $Y\in R$-MOD and consider the commutative diagram

$$\begin{tikzcd} X\otimes T_1\otimes Y \arrow[r, "1\otimes g_1\otimes 1"] \arrow[d, "\varepsilon_1"'] & X\otimes T_1\otimes T_2\otimes Y \arrow[d, "\varepsilon_2"] \\ X\otimes S_1\otimes Y \arrow[r, "1\otimes f_1\otimes 1"'] & X\otimes S_1\otimes S_2\otimes Y \end{tikzcd} \tag{3}$$

Now since the map $X\otimes T_1\to X\otimes S_1$ is right pure and $Y\to T_2\otimes Y$ is left pure, it follows from Lemma 2.1 that the diagram

$$\begin{tikzcd} X\otimes T_1\otimes Y \arrow[r, "1\otimes g_1\otimes 1"] \arrow[d, "\varepsilon_1"'] & X\otimes T_1\otimes T_2\otimes Y \arrow[d, "\varepsilon_2"] \\ X\otimes S_1\otimes Y \arrow[r, "1\otimes f_1\otimes 1"'] & X\otimes S_1\otimes T_2\otimes Y \end{tikzcd}$$

is a pullback. But the map $X\otimes S_1\otimes T_2\otimes Y\to X\otimes S_1\otimes S_2\otimes Y$ is one-to-one since $T_2\to S_2$ is pure and so (3) is also a pullback as required. Hence, by induction and by Theorem 2.2, we can deduce that each ε_i in (2) is pure and so $\psi: T_1 *_R T_2\to S_1 *_R S_2$ is pure. ■

Using the fact that amalgamated free products are associative, it is easy to extend the above theorem to amalgams with *finite* index sets. The general case then follows from Theorem 1.1.

THEOREM 3.3. *If* $[R;T_i]$ *is an amalgam of subrings of the amalgam* $[R;S_i]$ *and if the maps* $R\to T_i$ *and* $T_i\to S_i$ *are all pure* R-*monomorphisms*, *then the canonical map* $\prod_R^* T_i\to\prod_R^* S_i$ *is also a pure* R-*monomorphism.*

Using techniques of the same kind, it is also possible to prove a similar result for flatness in place of purity.

THEOREM 3.4. *If* $[R;T_i]$ *is an amalgam of subrings of the amalgam* $[R;S_i]$ *and if* S_i/T_i *and* T_i/R *are all right flat* R-*modules*, *then the canonical map* $\prod_R^* T_i\to\prod_R^* S_i$ *is one-to-one and* $\prod_R^* S_i/\prod_R^* T_i$ *and* $\prod_R^* T_i/R$ *are again right flat.*

We say that a ring R is a (*weak*, *strong*) *amalgamation base* if every amalgam with R as core can be (weakly, strongly) embedded. It was shown in [4, Theorem 5.9] that R is an amalgamation base if and only if for every ring S containing R as a subring, the inclusion $R \to S$ is a pure R-monomorphism. We call such rings R *absolutely extendable*. In particular (von Neumann) regular rings are amalgamation bases ([1, Theorem 4.7], [4, Theorem 3.4]).

Let us now define a ring R to be a *perfect amalgamation base* if

1. R is an amalgamation base, and

2. whenever $[R; T_i]$ is an amalgam of subrings of the amalgam $[R; S_i]$ then $\prod_R^* T_i \to \prod_R^* S_i$ is one-to-one.

It is clear from the above remarks and from the above theorem that if R is a regular ring then R is a perfect amalgamation base. We aim to prove that when R is commutative the converse is also true. First, if R is a subring of a ring S, we say that (R, S) is a *perfect amalgamation pair* if

1. every amalgam of the form $[R; S; S']$ is embeddable (i.e. (R, S) is an *amalgamation pair*), and

2. whenever $[R; T, T']$ is an amalgam of subrings of the amalgam $[R; S, S']$ then the map $T *_R T' \to S *_R S'$ is one-to-one.

It was proved in [1, Theorem 5.1] that if condition 1 holds, then R must be absolutely extendable.

THEOREM 3.5. *If R is commutative and (R, S) is a perfect amalgamation pair, then S is flat.*

Proof. Let $f : X \to Y$ be a left R-monomorphism and let T' and S' be the tensor algebras over X and Y respectively. We can clearly consider T' as a subring of S' and so we have an amalgam $[R; S', S]$ with an amalgam of subrings $[R; T', S]$. By assumption then, $T' *_R S \to S' *_R S$ is one-to-one. Now $R \to S$ and $R \to T'$ are both pure, by the above remarks, and so by Theorems 1.2 and 3.1, it follows that $T' \otimes S \to S' \otimes S$ is one-to-one. Since $X \otimes S$ is a direct summand of $T' \otimes S$ and $Y \otimes S$ is a direct summand of $S' \otimes S$, it is then straightforward to deduce that $X \otimes S \to Y \otimes S$ is one-to-one as required. ∎

We can now deduce, from [4, Lemma 3.3],

THEOREM 3.6. *A commutative ring is a perfect amalgamation base if and only if it is regular.*

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