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TIGHTNESS AND π-CHARACTER IN CENTERED SPACES

BY

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Abstract. We continue an investigation into centered spaces, a generalization of dyadic spaces. The presence of large Cantor cubes in centered spaces is deduced from tightness considerations. It follows that for centered spaces X, $\pi\chi(X) = t(X)$, and if X has uncountable tightness, then $t(X) = \sup\{\kappa : 2^\kappa \subset X\}$. The relationships between 9 popular cardinal functions for the class of centered spaces are justified. An example is constructed which shows, unlike the dyadic and polyadic properties, that the centered property is not preserved by passage to a zeroset.

1. Introduction. For S a non-empty collection of non-empty subsets of a set X, put $\mathrm{Cen}(S) = \{T \subset S : T$ is centered, i.e., whenever $F \subset T$ is finite, then $\bigcap F \neq \emptyset\}$. Give $\mathrm{Cen}(S)$ the topology that uses $\{s^+, s^- : s \in S\}$ as a subbase, where $s^+ = \{T \in \mathrm{Cen}(S) : s \in T\}$ and $s^- = \{T \in \mathrm{Cen}(S) : s \notin T\}$. For $A \subset X$, put $A^+ = \{T \in \mathrm{Cen}(S) : A \subset T\}$ and $A^- = \{T \in \mathrm{Cen}(S) : A \cap T = \emptyset\}$. Then $\{A^+ \cap B^- : A, B$ are finite subsets of X and $A \cap B = \emptyset\}$ forms a clopen base for $\mathrm{Cen}(S)$. The boolean spaces $\mathrm{Cen}(S)$ have served topologists well as a rich source of examples. Although isolated uses of these kind of spaces existed, they were formally introduced (by an equivalent definition) by Talagrand [14] who called them adequate compact spaces (at the time of [4], I was not aware of this reference). If we consider the two extreme examples: the Cantor cube 2^κ (where S is a centered collection of cardinality κ) and the 1-point compactification of a discrete space $\alpha\kappa$ (where S is a disjoint collection of cardinality κ), then their topological sum $2^\kappa + \alpha\kappa$, for $\kappa > \omega$, is not of the form $\mathrm{Cen}(S)$. By studying Hausdorff continuous images of $\mathrm{Cen}(S)$ (these are *centered* spaces [4]), we include these simple combinations. Independently of the author, Plebanek [11] began an investigation into centered spaces which he called AD-compact spaces.

In this paper, we expand our investigations into centered spaces to include tightness and π-character considerations. In Section 2 we look at a

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third way of viewing the spaces $\mathrm{Cen}(S)$; this way, compact c-sets, is presented in the fashion of the *weakly dyadic* spaces (a generalization of centered spaces) introduced by Kulpa and Turzański [10]; this way is more akin to the dyadic approach and will be expedient for this paper. We acknowledge a debt to the paper of Gerlits [7] and collect relevant results in Section 3. In Section 4 our main theorems show that if X is a centered space of uncountable tightness, then $\pi\chi(X)$ and $t(X)$ are equal to $\sup\{\kappa : 2^\kappa \subset X\}$. In particular, $\pi\chi(X) = t(X)$ for centered spaces. This generalizes the same theorem proven for *polyadic* spaces (Hausdorff continuous images of $\alpha\kappa^\lambda$) by Gerlits [7]. In Section 5 we look at the partial order of 9 popular cardinal functions and show that unlike the polyadic case, $d\chi(X)$ can be less than $t(X)$ (Example 5.2) and $d\pi\chi(X)$ can be less than $d\chi(X)$ (Example 5.3). In Section 6, we clear up a loose end (a basic structural question) by presenting an example of a centered space which has a zeroset Z that is not centered.

We reserve κ, λ, μ, ν and τ for cardinals and α, β, γ, δ and σ for ordinals. Cardinal functions used are:

$$\begin{aligned}
w(X) &= \min\{|\mathcal{P}| : \mathcal{P} \text{ is a base for } X\},\\
\chi(x,X) &= \min\{|\mathcal{P}| : \mathcal{P} \text{ is a local base at } x\},\\
\chi(X) &= \sup\{\chi(x,X) : x \in X\},\\
\pi(X) &= \min\{|\mathcal{P}| : \mathcal{P} \text{ is a } \pi\text{-base for } X\},\\
\pi\chi(x,X) &= \min\{|\mathcal{P}| : \mathcal{P} \text{ is a local } \pi\text{-base at } x\},\\
\pi\chi(X) &= \sup\{\pi\chi(x,X) : x \in X\}.
\end{aligned}$$

If $A \subset X$ and $x \in \overline{A}$ (the closure of A in X), then

$$\begin{aligned}
a(x,A) &= \min\{|B| : x \in \overline{B} \text{ and } B \subset A\},\\
t(x,X) &= \sup\{a(x,A) : A \subset X \text{ and } x \in \overline{A}\},\\
t(X) &= \sup\{t(x,X) : x \in X\},\\
c(X) &= \sup\{|\mathcal{P}| : \mathcal{P} \text{ is a disjoint collection of open subsets of } X\},\\
d(X) &= \min\{|D| : D \text{ is dense in } X\},\\
d\chi(X) &= \min\{\kappa : \{x \in X : \chi(x,X) \le \kappa\} \text{ is dense in } X\},\\
d\pi\chi(X) &= \min\{\kappa : \{x \in X : \pi\chi(x,X) \le \kappa\} \text{ is dense in } X\}.
\end{aligned}$$

Some convenient notations are:

- $X \approx Y$ means that X is homeomorphic to Y.
- $\phi : X \twoheadrightarrow Y$ means that ϕ is a continuous surjection.
- $X \twoheadrightarrow Y$ means that $\phi : X \twoheadrightarrow Y$ for some ϕ.
- $p \in 2^\kappa \subset X$ means that there exists $L \subset X$ with $L \approx 2^\kappa$ and $p \in L$.

Sets and properties defined with a parameter κ will be of the $< \kappa$ version and not the $\le \kappa$ version. For example, a G_κ-set Z is a set which is the intersection of less than κ open sets. All spaces appearing in this paper are Hausdorff.

2. Compact c-set, Cen(S) **or Adequate Compact.** Fix $\tau \geq \omega$ and let 2^τ be the Cantor cube of weight τ, where $2 = \{0,1\}$.

For $x \in 2^\tau$ put $G(x) = \{y \in 2^\tau : y^{-1}(1) \subset x^{-1}(1)\}$. We call $W \subset 2^\tau$ a *c-set* if whenever $x \in W$, then $G(x) \subset W$. Compact c-sets, Cen(S)'s and adequate compact spaces are three slightly different ways of looking at the same class of spaces (see Turzański [17]); each has its own advantages. So, a centered space is a continuous image of a compact c-set.

We now give the basic notations that will be used in the rest of the paper. For $\kappa \leq \tau$, put $J_\kappa = \{s : s$ is a function from A to 2 where $A \subset \tau$ and $|A| < \kappa\}$. If $x \in 2^\tau$, put $J_\kappa(x) = \{x{\restriction}A : A \subset \tau$ and $|A| < \kappa\}$. For $\kappa \leq \tau$ and $s \in J_\kappa$, put $H^s = \{x \in 2^\tau : x$ extends $s\}$. If s and t are partial functions, $s \subset t$ means that t extends s; if s and t are compatible functions, then $s \cup t$ is the unique minimal extension of both s and t. Put $\Sigma_\kappa = \{x \in 2^\tau : |x^{-1}(1)| < \kappa\}$.

Let W be a compact c-set in 2^τ. Put $W_\kappa = W \cap \Sigma_\kappa$. Then W_ω is dense in W, W_κ is closed in Σ_κ and $\lambda < \kappa$ implies that $W_\lambda \subset W_\kappa$. If $x \in W$ and $s \in J_\kappa(x)$, then put $G^s(x) = H^s \cap G(x)$. If $s \in J_\kappa$, then $\widehat{s}$ represents the point in 2^τ defined by $\widehat{s}(\alpha) = s(\alpha)$ for $\alpha \in \operatorname{dom}(s)$ and $\widehat{s}(\alpha) = 0$ for $\alpha \notin \operatorname{dom}(s)$. Note that if $s \in J_\kappa(x)$, then $\widehat{s} \in G(x)$.

For a boolean space X, put Cen$(X) =$ Cen(CO(X)) where CO(X) is the set of all clopen subsets of X. Two of our examples will be derived in this way. The remaining three examples will be spaces of complete subgraphs of a graph. A *graph* G on a set X is a $G \subset [X]^2 = \{D \subset X : |D| = 2\}$. A subset $Y \subset X$ is *complete* if $\{y_1, y_2\} \in G$ for every $y_1 \neq y_2$ in Y. Put $G^* = \{Y \subset X : Y$ is complete$\}$. For $A \subset X$, put $A^+ = \{Y \in G^* : A \subset Y\}$ and $A^- = \{Y \in G^* : A \cap Y = \emptyset\}$. Then $\{A^+ \cap B^- : A, B$ are finite subsets of X and $A \cap B = \emptyset\}$ forms a clopen base for G^*.

3. Basic results required. We devote this section to collecting results which we need but which quickly follow from the results of Gerlits [7].

THEOREM 3.1 (Gerlits [6]). *Let $\phi : 2^\tau \twoheadrightarrow X$, $\tau \geq \kappa > \omega$ and $x \in 2^\tau$. If for every $A \subset \tau$ with $|A| < \kappa$ there exists $y \in 2^\tau$ such that $y{\restriction}A = x{\restriction}A$ and $\phi(y) \neq \phi(x)$, then $\phi(x) \in 2^\kappa \subset X$.*

In fact, Gerlits' Theorem will be applied in this paper in the following way.

COROLLARY 3.2. *Let W be a compact c-set in 2^τ, $\phi : W \twoheadrightarrow X$, $\kappa > \omega$, $p \in X$ and $x, y \in W$ with $x \in \phi^{-1}(p) \cap G(y)$. If $G^s(y) \not\subset \phi^{-1}(p)$ for every $s \in J_\kappa(x)$ then $p \in 2^\kappa \subset X$. Alternatively stated, if there does not exist*

$L \subset X$ with $L \approx 2^\kappa$ and $p \in L$, then there exists $s \in J_\kappa(x)$ such that $G^s(y) \subset \phi^{-1}(p)$.

Proof. Consider $\phi\restriction G(y) : G(y) \twoheadrightarrow \phi(G(y)) \subset X$. If $R = \{\alpha < \tau : y(\alpha) = 1\}$, then $|R| \geq \kappa$ because $x \in \phi^{-1}(p)$ and $G^s(y) \not\subset \phi^{-1}(p)$ for every $s \in J_\kappa(x)$. Now apply Theorem 3.1 with $G(y) \approx 2^R$ in the role of 2^τ to get $p \in 2^\kappa \subset \phi(G(y))$. ■

We will require two notions (due to Arkhangel'skiĭ [1], [2]): a space X is *κ-monolithic* if whenever $Y \subset X$ and $|Y| < \kappa$, then $w(\overline{Y}) < \kappa$. If $Y \subset X$, then Y is *κ-closed* in X if whenever $A \subset Y$ and $|A| < \kappa$, then $\overline{A}^X \subset Y$.

FACT 3.3. *Let W be a compact c-set in 2^τ and $\kappa > \omega$.*

(1) *If κ is regular, then W_κ is κ-closed in W and W_κ is κ-monolithic.*
(2) *$W_\kappa \not\twoheadrightarrow I^\kappa$.*

Proof. (1) This follows because W_κ is a closed subset of Σ_κ and, for κ regular, Σ_κ is κ-monolithic and is κ-closed in 2^τ.

(2) The same argument that Gerlits uses to prove that $\Sigma_\kappa \not\twoheadrightarrow K$, where K is a compact space with $K \supset 2^\kappa$ (a⇒j of Theorem 9 in [7]), also shows that no closed subset L of Σ_κ can continuously map onto K where K is a compact space with $K \supset 2^\kappa$. ■

FACT 3.4. *Let W be a compact c-set in 2^τ, $\phi : W \twoheadrightarrow X$ and $\kappa > \omega$.*

(1) *If $p \in X \setminus \phi(W_\kappa)$, then $p \in 2^\kappa \subset X$ and $\pi\chi(p, X) \geq \kappa$.*
(2) *If $X = \phi(W_\kappa)$ and κ is regular, then X is κ-monolithic.*

Proof. (1) Choose $x \in \phi^{-1}(p)$. For every $s \in J_\kappa(x)$, $\widehat{s} \in G^s(x) \cap W_\kappa$ and so $\phi(\widehat{s}) \neq p$; by Corollary 3.2 (with $y = x$), $p \in 2^\kappa \subset X$. Striving for a contradiction, assume that $\pi\chi(p, X) = \lambda < \kappa$. Let $\mathcal{P}$ be a local π-base at p such that $|\mathcal{P}| = \lambda$. Since $\phi(W_\omega)$ is dense in X, for every $P \in \mathcal{P}$ choose $x_P \in W_\omega \subset W_{\lambda^+}$ such that $\phi(x_P) \in P$. Since W_{λ^+} is λ^+-closed in W, $\overline{\{x_P : P \in \mathcal{P}\}} \subset W_{\lambda^+} \subset W_\kappa$. But $p \in \phi(\overline{\{x_P : P \in \mathcal{P}\}})$, therefore $p \in \phi(W_\kappa)$; a contradiction. Hence, $\pi\chi(p, X) \geq \kappa$.

(2) This follows from compactness of W and Fact 3.3(1). ■

FACT 3.5. *If $A \subset 2^\tau$, Z is a closed G_κ-set of 2^τ and $\overline{A} \cap Z \neq \emptyset$, then there exists $D \subset A$ with $|D| < \kappa$ and $\overline{D} \cap Z \neq \emptyset$.*

4. Tightness and π-character of centered spaces. We want to extend Gerlits' results [7] on tightness and π-character in polyadic spaces to centered spaces. We point out 2 obstacles to this. The first is that with a product space preimage, there are many dense Σ-products for each $\kappa \geq \omega$. As an example, for each $x \in 2^\tau$, if we put $\Sigma_\kappa(x) = \{y \in 2^\tau : |\{\alpha \in \tau : y(\alpha) \neq x(\alpha)\}| < \kappa\}$, then $\Sigma_\kappa(x)$ is dense in 2^τ. With a compact c-set preimage, we are guaranteed only one dense Σ-product for each $\kappa \geq \omega$. Let $\overline{0} \in 2^\tau$ be the constantly 0 function. For each $x \in W$, if we put $W_\kappa(x) = W \cap \Sigma_\kappa(x)$, then, unless $x = \overline{0}$, $W_\kappa(x)$ may not be dense in W and so we lose the tool (so useful in [7]) of many dense Σ-products. Secondly, Gerlits proves that for polyadic spaces X, if κ is regular uncountable, then $\{x \in X : \pi\chi(x, X) < \kappa\}$ is closed in X. Examples 5.2, 5.3 or 5.6 below, for $\kappa = \omega_1$, show that this is not true for centered spaces.

THEOREM 4.1. *Let X be a centered space*, $\mathrm{cf}(\kappa) > \omega$ *and* $p \in X$. *There exists $A \subset X$ with $p \in \overline{A}$ and $a(p, A) \geq \kappa$ if and only if $p \in 2^\kappa \subset X$. Consequently*, $t(p, X) = \sup\{\kappa : p \in 2^\kappa \subset X\}$ *for all points p in a centered space X with $t(p, X) > \omega$.*

Proof. Let W be a compact c-set in 2^τ and let $\phi : W \twoheadrightarrow X$ be a continuous surjection. The final consequence follows from the equivalence, whose sufficiency is clear. For necessity, let $A \subset X$ with $p \in \overline{A}$ and $a(p, A) \geq \kappa$. As ϕ is a closed map, choose $x \in \phi^{-1}(p) \cap \overline{\phi^{-1}(A)}$. Striving for a contradiction, we assume that there does not exist $L \subset X$ with $L \approx 2^\kappa$ and $p \in L$.

Using Corollary 3.2 (with $y = x$), choose $t_0 \in J_\kappa(x)$ such that $G^{t_0}(x) \subset \phi^{-1}(p)$. As H^{t_0} is a closed G_κ-set containing x, choose $D_0 \subset \phi^{-1}(A)$ with $|D_0| < \kappa$ and $x_0 \in \overline{D}_0 \cap H^{t_0}$. Again by Corollary 3.2 (with $y = x_0$), as $\widehat{t}_0 \in \phi^{-1}(p) \cap G(x_0)$, choose $s_0 \in J_\kappa$ such that $t_0 \subset s_0 \subset \widehat{t}_0$ and $G^{s_0}(x_0) \subset \phi^{-1}(p)$. By recursion on $n < \omega$, we construct $t_n \in J_\kappa(x)$, $s_n \in J_\kappa$, $D_n \subset \phi^{-1}(A)$ with $|D_n| < \kappa$ and $x_n \in \overline{D}_n \cap H^{t_n}$ such that $t_n \subset s_n \subset \widehat{t}_n$, $G^{s_n}(x_n) \subset \phi^{-1}(p)$ and for $n > 0$, $t_{n-1} \subset t_n$ and $\mathrm{dom}(t_n) = \mathrm{dom}(s_{n-1})$. At stage $n+1$, put $t_{n+1} = x \restriction \mathrm{dom}(s_n)$. As H^{t_n} is a closed G_κ-set containing x, choose $D_{n+1} \subset \phi^{-1}(A)$, $|D_{n+1}| < \kappa$ and $x_{n+1} \in \overline{D}_{n+1} \cap H^{t_{n+1}}$. By Corollary 3.2 (with $y = x_{n+1}$), as $\widehat{t}_{n+1} \in \phi^{-1}(p) \cap G(x_{n+1})$, choose $s_{n+1} \in J_\kappa$ such that $t_{n+1} \subset s_{n+1} \subset \widehat{t}_{n+1}$ and $G^{s_{n+1}}(x_{n+1}) \subset \phi^{-1}(p)$.

We now show that every cluster point of the sequence $\{x_n : n < \omega\}$ is in $\phi^{-1}(p)$. If not, get $s \in J_\omega$ and an infinite $R \subset \omega$ such that $\{x_n : n \in R\} \subset H^s \subset W \setminus \phi^{-1}(p)$. Put $t = \bigcup_{n \in R} t_n$. Since $x_n \in H^s \cap H^{t_n}$ for $n \in R$, compactness of W implies that if $r = t \cup s$, then $\widehat{r} \in W$. As $\widehat{r} \in H^s$, $\widehat{r} \notin \phi^{-1}(p)$. Put $T = \bigcup_{n<\omega} \mathrm{dom}(s_n) = \bigcup_{n<\omega} \mathrm{dom}(t_n)$. Get $m \in R$ such that $\mathrm{dom}(s) \cap T = \mathrm{dom}(s) \cap \mathrm{dom}(t_m)$. For $k \in R$, $k \geq m$, put $r_k = s_k \cup s$. As $\widehat{r}_k \in G^{s_k}(x_k)$, we have $\widehat{r}_k \in \phi^{-1}(p)$. But $\widehat{r}_k \to \widehat{r}$ and $\phi^{-1}(p)$ is closed, therefore $\widehat{r} \in \phi^{-1}(p)$; a contradiction.

If we put $D = \bigcup_{n<\omega} D_n$, then as $x_n \in \overline{D}_n$, we deduce that $\overline{D} \cap \phi^{-1}(p) \neq \emptyset$. Thus, $p \in \overline{\phi(D)}$. As $\mathrm{cf}(\kappa) > \omega$, we have $|\phi(D)| < \kappa$; this contradicts $a(p, A) \geq \kappa$. ∎

QUESTION 4.2. Can $\mathrm{cf}(\kappa) > \omega$ be replaced by just $\kappa > \omega$? That is, if X is a centered space, $\kappa > \omega$, $p \in X$ and there exists $A \subset X$ with $p \in \overline{A}$

and $a(p,A) \geq \kappa$, then is $p \in 2^\kappa \subset X$? Gerlits has shown this to be true for polyadic spaces.

LEMMA 4.3. *Let W be a compact c-set in 2^τ and $\kappa > \omega$ be regular. If $A \subset W_\kappa$, $x \in \overline{A}$ and $s \in J_\kappa(x)$, then there exists $D \subset A$ with $|D| < \kappa$ and $\overline{D} \cap G^s(x) \neq \emptyset$.*

Proof. Put $s_0 = s$. As H^{s_0} is a closed G_κ-set containing x, choose $D_0 \subset A$ and $x_0 \in W$ such that $|D_0| < \kappa$ and $x_0 \in \overline{D}_0 \cap H^{s_0}$. By recursion on $n < \omega$, we construct $s_n \in J_\kappa(x)$, $D_n \subset A$ with $|D_n| < \kappa$ and $x_n \in \overline{D}_n \cap H^{s_n}$ such that for $n > 0$, $s_{n-1} \subset s_n$ and $x_{n-1}^{-1}(1) \subset \mathrm{dom}(s_n)$. At stage $n+1$, as $x_n \in \overline{D}_n$, $D_n \subset W_\kappa$, $|D_n| < \kappa$ and κ is regular, we have $x_n \in W_\kappa$ from Fact 3.3(1). Put $s_{n+1} = x \restriction (\mathrm{dom}(s_n) \cup x_n^{-1}(1))$. Choose $D_{n+1} \subset A$ and $x_{n+1} \in W$ with $|D_{n+1}| < \kappa$ and $x_{n+1} \in \overline{D}_{n+1} \cap H^{s_{n+1}}$.

Put $D = \bigcup_{n<\omega} D_n$ and $t = \bigcup_{n<\omega} s_n$. Then $|D| < \kappa$ and $\widehat{t} \in \overline{D}$ because $x_n \to \widehat{t}$. Hence $\widehat{t} \in \overline{D} \cap G^s(x)$.

THEOREM 4.4. *Let X be a centered space and $\kappa > \omega$. There exists $p \in X$ with $\pi\chi(p,X) \geq \kappa$ if and only if $2^\kappa \subset X$. Consequently, for centered spaces X with uncountable π-character, $\pi\chi(X) = \sup\{\kappa : 2^\kappa \subset X\}$.*

Proof. Let W be a compact c-set in 2^τ and let $\phi : W \twoheadrightarrow X$ be a continuous surjection. The final consequence follows from the equivalence. For necessity, assume that $p \in X$ with $\pi\chi(p,X) = \mu \geq \kappa$. If μ is regular, then by a result of Shapirovskiĭ (cf. p. 54 of [8]) we can get $A \subset X$ such that $p \in \overline{A}$ and $a(p,A) = \mu$. Theorem 4.1 gives us $p \in 2^\mu \subset X$. So, assume $\mathrm{cf}(\mu) = \lambda < \mu$ and $\mu = \sum_{\alpha<\lambda} \mu_\alpha$ where μ_α are regular cardinals $< \mu$ for every $\alpha < \lambda$. For $\alpha < \lambda$, put $A_\alpha = \{x \in X : \pi\chi(x,X) < \mu_\alpha\}$.

CASE 1: $\exists \beta < \lambda$ with $p \in \overline{A}_\beta$. Then Fact 3.4(1) implies $A_\beta \subset \phi(W_{\mu_\beta})$. Choose $x \in \phi^{-1}(p) \cap \overline{\phi^{-1}(A_\beta) \cap W_{\mu_\beta}}$. Either $p \in 2^\mu \subset X$, or by Corollary 3.2 we can choose $s \in J_\mu(x)$ with $G^s(x) \subset \phi^{-1}(p)$. In the latter case, by taking the maximum of μ_β and $|\mathrm{dom}(s)|$ we may as well assume that $|\mathrm{dom}(s)| \leq \mu_\beta$; then, by invoking Lemma 4.3 (with $\kappa = \mu_\beta$), we choose $D \subset \phi^{-1}(A_\beta) \cap W_{\mu_\beta}$ with $|D| < \mu_\beta$ such that $\overline{D} \cap G^s(x) \neq \emptyset$. Therefore, $p \in \overline{\phi(D)}$, $\phi(D) \subset A_\beta$ and $|\phi(D)| < \mu_\beta$. It follows that $\pi\chi(p,X) < \mu_\beta < \mu$; a contradiction. So, $p \in 2^\mu \subset X$.

CASE 2: $p \notin \bigcup_{\alpha<\lambda} \overline{A}_\alpha$. Choose a closed G_{λ^+}-set $Z \ni p$ such that $\pi\chi(x,X) \geq \mu$ for every $x \in Z$. Since $\lambda < \mu$, it follows that $\pi\chi(x,Z) \geq \mu$ for every $x \in Z$. By a theorem of Shapirovskiĭ [13], we have $X \twoheadrightarrow I^\mu$. Fact 3.3(2) tells us that $W_\mu \not\twoheadrightarrow X$. By Fact 3.4(1), if $q \in X \setminus \phi(W_\mu)$, then $q \in 2^\mu \subset X$.

For sufficiency, assume that $2^\kappa \subset X$. As above, $W_\kappa \not\twoheadrightarrow X$. By Fact 3.4(1), if $q \in X \setminus \phi(W_\kappa)$, then $\pi\chi(q,X) \geq \kappa$. ∎

Remark. Certainly, a major simplification of Theorem 4.4 can be obtained if it were known that for each point x in a compact space X, there is $Y \subset X$ with $\pi\chi(x, X) = a(x, Y)$. However, to our knowledge, this is still not fully resolved. The statement does appear in [9] but there is a simple gap in the sketch of proof supplied. One of the authors of [9] has acknowledged this gap and has indicated that a repair has not been found.

Theorems 4.1 and 4.4 give us

Corollary 4.5. *If X is a centered space of uncountable tightness, then* $\pi\chi(X) = t(X) = \sup\{\kappa : 2^\kappa \subset X\}$.

Question 4.6. If X is a centered space and $\pi\chi(p, X) = \mu > \omega$, then must $p \in 2^\mu \subset X$?

Looking at the proof of Theorem 4.4, it is seen that we have a positive answer to Question 4.6 except for the case when μ is singular and Case 2 of that proof ensues. However, if X is simply a compact c-set, then the singular cardinal difficulties can be overcome.

Corollary 4.7. *Let W be a compact c-set in 2^τ, $\kappa > \omega$ and $p \in W$. If $\pi\chi(p, W) = \kappa$, then $p \in 2^\kappa \subset W$. Consequently, $\pi\chi(p, W) \le t(p, W)$ for all points p in a compact c-set W.*

Proof. Assume that $\pi\chi(p, W) = \kappa$. In the proof of Theorem 4.4, it is only Case 2 (with μ replaced by κ) where we do not achieve our goal. Assume that $\mathrm{cf}(\kappa) = \lambda < \kappa$ and that Z is a closed G_{λ^+}-set of W such that $p \in Z$ and $\pi\chi(x, W) \ge \kappa$ for every $x \in Z$. Get $s \in J_{\lambda^+}$ such that $p \in W^s = H^s \cap W \subset Z$. Put $R = \{x \in W^s : p \in G(x)\}$. As R is closed in W, get $q \in R$ such that $q^{-1}(1)$ is maximal under inclusion among all points x in R. Then $\{q\} = W^s \cap \bigcap_{q(\alpha)=1} \{x \in W : x(\alpha) = 1\}$, so $\chi(q, W) \le \lambda + |q^{-1}(1)|$. Since $\pi\chi(q, W) \ge \kappa$, we have $|q^{-1}(1)| \ge \kappa$. So, $p \in G(q) \approx 2^\nu$ where $\kappa \le \nu$. ■

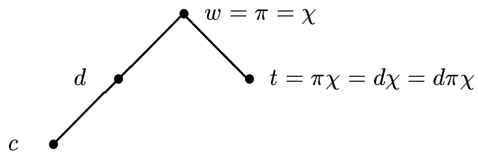
5. Cardinal functions of centered spaces. We consider relations between the 9 cardinal functions w, π, χ, t, $\pi\chi$, $d\chi$, $d\pi\chi$, d, and c.

The classical theory of dyadic spaces gives rise to the following partial order. Of course, all dyadic spaces satisfy $c = \omega$.

$$\text{Dyadic} \qquad \begin{array}{l} \bullet\; w = \pi = \chi = t = \pi\chi = d\chi = d\pi\chi \\ | \\ \bullet\; d \end{array}$$

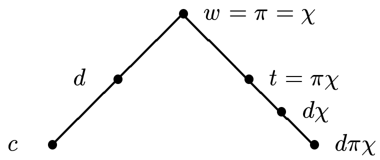
Gerlits' investigations [7] into polyadic spaces give rise to the following partial order. The fact that he proved that $\max\{c, t\} = w$ renders the join semi-lattice structure correct.

Polyadic



In this section we show that centered spaces satisfy the following partial order. The join semi-lattice structure (between the 2 branches) is more complicated than in the polyadic case.

Centered



That $w = \pi = \chi$ was proved in Bell [4] and $t = \pi\chi$ is the content of Corollary 4.5.

Proposition 5.1. *If X is a centered space, then $d\chi(X) \le t(X)$.*

Proof. Let W be a compact c-set in 2^τ, $\phi : W \twoheadrightarrow X$ be a continuous surjection and $t(X) = \kappa$. Since $X \not\supset 2^{\kappa^+}$, by Fact 3.4(1) we get $X = \phi(W_{\kappa^+})$ and therefore Fact 3.4(2) implies that X is κ^+-monolithic. It remains to apply Theorem 2.2.4 of Arkhangel'skiĭ [3] which implies that if X is compact, $t(X) = \kappa$ and X is κ^+-monolithic, then $d\chi(X) \le \kappa$. ■

Example 5.2 (A centered X with $d\chi(X) < t(X)$). Define a graph G on the set $\omega_1 \times \{0,1\}$ as follows: $\{(\alpha,i),(\beta,j)\} \in G$ if and only if $(i = j = 0)$ or $(i = 0,\ j = 1$ and $\alpha < \beta)$ or $(i = 1,\ j = 0$ and $\beta < \alpha)$. Put $X = G^*$. For $\alpha < \omega_1$, $B_\alpha = (\alpha,1)^+$ is a clopen and second countable subspace of X. As $\bigcup_{\alpha<\omega_1} B_\alpha$ is dense in X, we have $d\chi(X) = \omega$. As $\{A \times \{0\} : A \subset \omega_1\}$ is a subspace of X which is homeomorphic to 2^{ω_1}, we obtain $t(X) = \omega_1$.

Example 5.3 (A centered X with $d\pi\chi(X) < d\chi(X)$). For a tree $(T,<)$ and $t \in T$, $\mathrm{ht}(T)$ denotes the height of T, $\mathrm{ht}(t)$ denotes the height of t in T, $\mathrm{succ}(t) = \{s \in T : t < s \text{ and } \mathrm{ht}(s) = \mathrm{ht}(t)+1\}$, $L(t) = \{s \in T : s \le t\}$, and $\mathrm{Fin}(t) = \{F \subset L(t) : F \text{ is finite}\}$.

Let T be a tree of height ω_1, with no countable maximal chains and such that $|\mathrm{succ}(t)| = \omega_1$ for every $t \in T$. For each $t \in T$, choose a countably infinite $C_t \subset \mathrm{succ}(t)$ and let ϕ_t be an infinite-to-one surjection $\phi_t : C_t \to \mathrm{Fin}(t)$. Define a graph G on T as follows: $\{s,t\} \in G$ if and only if (either $s < t$ or $t < s$) and (if $\max\{s,t\} \in C_r$ for some $r \in T$, then $\min\{s,t\} \notin \phi_r(\max\{s,t\})$). Put $X = G^*$. It is easily checked that every maximal complete subset of T has cardinality ω_1; so, if $x \in X$ and m is a maximal complete subset of T with $x \subset m$, then $x \in \{y : y \subset m\} \approx 2^{\omega_1} \subset X$. Thus X has uniform character ω_1 and $d\chi(X) = \omega_1$.

Put $D = \{x \in X : x \text{ is countable}\}$. Then D is dense in X. Let $x \in D$. Choose $r \in T$ such that $t < r$ for every $t \in x$. Let $\mathcal{P}$ be the set of all $t^+ \cap A^+ \cap \phi_r(t)^-$ where $t \in C_r$, A is a finite subset of x and $\phi_r(t) \cap A = \emptyset$. Then $\mathcal{P}$ is a countable π-base for x. To see this, let $x \in A^+ \cap B^-$ where A is a finite subset of x, B is a finite subset of T and $A \cap B = \emptyset$. Put $E = B \cap L(r)$ and $F = B \setminus L(r)$. Since C_r is infinite and ϕ_r is infinite-to-one, we can choose $t \in C_r$ such that $\phi_r(t) = E$, and $\{s, t\} \notin G$ for every $s \in F$. Then $\emptyset \neq t^+ \cap A^+ \cap \phi_r(t)^- \subset A^+ \cap B^-$. Thus, $d\pi\chi(X) = \omega$.

As for the join semi-lattice structure for centered spaces, we have:

THEOREM 5.4. *If X is a centered space, then* $\max\{d(X), t(X)\} = w(X)$.

Proof. In every space X, we have $\max\{d(X), \pi\chi(X)\} = \pi(X)$. The theorem follows because for centered spaces X, we have $w(X) = \pi(X)$ and $t(X) = \pi\chi(X)$. ∎

In [4] and [11] consistent examples of centered spaces X are presented which satisfy $\max\{c(X), t(X)\} < w(X)$. Here is an "honest" example.

EXAMPLE 5.5 (A centered X with $\max\{c(X), t(X)\} < w(X)$). We begin with a boolean space K and a cardinal $\kappa \geq \omega$ such that $c(K) = \kappa$ and K does not have precaliber κ^+, i.e. there exists $\mathcal{P} \subset \mathrm{CO}(K)$ such that $|\mathcal{P}| = \kappa^+$ and $\mathcal{P}$ does not have a centered subcollection of cardinality κ^+. An example of such a space K and cardinal κ appears in Corollary 3 of Todorčević [15]. Our example is $X = \mathrm{Cen}(\mathcal{P})$. Since $w(\mathrm{Cen}(S)) = |S|$, we have $w(X) = |\mathcal{P}| = \kappa^+$. Since $c(\mathrm{Cen}(Y)) = c(Y)$ for any boolean space Y, we see that $c(\mathrm{Cen}(K)) = \kappa$. Since the intersection map is a retraction of $\mathrm{Cen}(K)$ onto X, we find that $c(X) \leq \kappa$. Since $\mathcal{P}$ does not have a centered subcollection of cardinality κ^+, $\mathrm{Cen}(\mathcal{P})$ can be embedded into $\Sigma_{\kappa^+} \subset 2^{\mathcal{P}}$ as characteristic functions. As $t(\Sigma_{\kappa^+}) = \kappa$, we have $t(\mathrm{Cen}(\mathcal{P})) \leq \kappa$.

EXAMPLE 5.6 (A centered X with $\max\{d(X), d\chi(X)\} < w(X)$). We begin with a separable boolean space K which has no isolated points and which has an uncountable π-base $\mathcal{P}$ of clopen sets and a dense set D such that $\{b \in \mathcal{P} : d \in b\}$ is a countable base at d in K for every $d \in D$. Note that D cannot be countable. Such a space K appears in Example 3.6 of Bell [5]. Our present example is $X = \mathrm{Cen}(\mathcal{P})$. We have $w(X) = |\mathcal{P}| > \omega$. Turzański [16] has shown that $d(\mathrm{Cen}(Y)) = d(Y)$ for any boolean space Y; it follows that $d(\mathrm{Cen}(K)) = \omega$. Since the intersection map is a retraction of $\mathrm{Cen}(K)$ onto X, we have $d(X) = \omega$. For each $d \in D$, put $x(d) = \{b \in \mathcal{P} : d \in b\} \in X$. We observe that if y is a co-finite subset of $x(d)$, then $\chi(y, X) = \omega$. To see this, let F be a finite subset of $x(d)$ with $y = x(d) \setminus F$. Then, because d is a non-isolated point of K and $x(d)$ is a countable base at d in K, we have

$\{y\} = \bigcap_{b \in y} b^+ \cap \bigcap_{b \in F} b^-$; so y is a point of first countability in X. We now show that these points are dense in X, so that $d\chi(X) = \omega$. Let F and G be disjoint finite subsets of $\mathcal{P}$ such that $F^+ \cap G^- \neq \emptyset$. Therefore, $\bigcap F \neq \emptyset$. As $\mathcal{P}$ is a π-base for K, choose a non-empty $b \in \mathcal{P}$ such that $b \subset \bigcap F$. As D is dense in K, choose $d \in D \cap b$. Then $x(d) \setminus G \in F^+ \cap G^-$.

6. Zerosets in centered spaces. As both the dyadic and polyadic properties are preserved by zerosets (i.e., closed G_δ-sets in compact spaces), a natural question (see [4], [11], [12] and [16]) is whether the same is true for the centered property. Our next example shows that this is not the case.

EXAMPLE 6.1 (A centered X with a zeroset $Z \subset X$ such that Z is not centered). Let T be the tree $\bigcup_{\alpha \leq \omega} 2^\alpha$ ordered by $s \leq t$ if and only if t extends s. Define a graph G on T as follows: $\{s, t\} \in G$ if and only if $s < t$ or $t < s$. Put $X = G^*$. For each $n < \omega$, put $B_n = \{x \in X : x \cap 2^n \neq \emptyset\}$. Each B_n is a clopen subset of X and so $Z = \bigcap_{n<\omega} B_n$ is a zeroset of X. Further, Z consists of x_α's and y_α's for $\alpha \in 2^\omega$, where $x_\alpha = \{\alpha \restriction n : n < \omega\}$ and $y_\alpha = x_\alpha \cup \{\alpha\}$. The y_α's form a discrete subspace and Z is homeomorphic to the Alexandrov duplicate of the Cantor set 2^ω, a space of uncountable weight and countable character. But centered spaces Z satisfy $w(Z) = \chi(Z)$, hence Z is not centered.

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