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FANS ARE NOT C-DETERMINED

BY

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Abstract. A *continuum* is a compact connected metric space. For a continuum X, let $C(X)$ denote the hyperspace of subcontinua of X. In this paper we construct two nonhomeomorphic fans (dendroids with only one ramification point) X and Y such that $C(X)$ and $C(Y)$ are homeomorphic. This answers a question by Sam B. Nadler, Jr.

1. Introduction. A *continuum* is a compact connected metric space. For a continuum X, let $C(X)$ denote the space of all the subcontinua of X, with the Hausdorff metric H. A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(\{x\}) = 0$ for each $x \in X$ and if $A \subsetneq B$, then $\mu(A) < \mu(B)$. For the existence of Whitney maps see [9, 0.50.1]. A *dendroid* is an arcwise connected hereditarily unicoherent continuum. Given points p and q in a dendroid X, pq denotes the unique arc joining p and q if $p \neq q$, and $pq = \{p\}$ if $p = q$. A *fan* is a dendroid with only one ramification point. Let X be a fan with ramification point v; it is said to be a *smooth fan* provided that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to a point $x \in X$, then $vx_n \to vx$.

A class Λ of continua is said to be *C-determined* ([9, Definition 0.61)]) provided that if $X, Y \in \Lambda$ and $C(X) \cong C(Y)$ ($C(X)$ is homeomorphic to $C(Y)$), then $X \cong Y$. The following classes of continua are known to be C-determined:

(a) finite graphs different from an arc ([3, 9.1]),

(b) hereditarily indecomposable continua ([9, 0.60]),

(c) smooth fans ([4, Corollary 3.3]),

(d) indecomposable continua such that all their proper nondegenerate subcontinua are arcs ([7]), and

(e) metric compactifications of the half-ray $[0,\infty)$ ([1]).

Recently, answering a question by Nadler, the author showed that the class of chainable continua is not C-determined ([5]). In [9, Questions 0.62]

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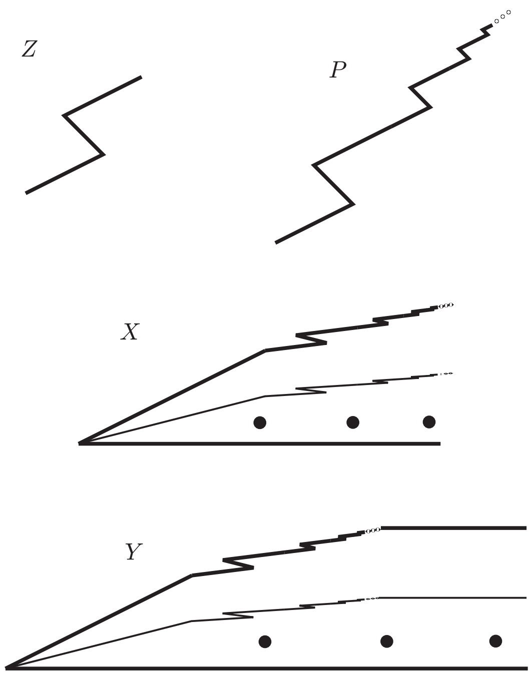
Nadler asked if the class of fans is C-determined. Here, we answer this question in the negative.

Description of the examples. Given two points p, q in the Euclidean plane $\mathbb{R}^2$, pq denotes the convex segment which joins them. Given points $p_1, \ldots, p_n$ in $\mathbb{R}^2$, let $\langle p_1, \ldots, p_n \rangle = p_1p_2 \cup p_2p_3 \cup \ldots \cup p_{n-1}p_n$. Given a point $p \in \mathbb{R}^2$ and a subset A of $\mathbb{R}^2$, let $p + A = \{p + a : a \in A\}$. The set of positive integers is denoted by $\mathbb{N}$. Let $\theta = (0,0) \in \mathbb{R}^2$, $B_0 = \theta(2,0)$ and $C_0 = (2,0)(3,0)$.



Let

$$Z = \langle \theta, (2,1), (1,2), (3,3) \rangle.$$

Notice that $Z \subset \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2x\}$.

For each $n \in \mathbb{N}$, let

$$P_n = \left(1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^{n-1}}\right) + \left\{\frac{1}{3 \cdot 2^n} p : p \in Z\right\}$$

Let

$$P = \left[\bigcup \{P_n : n \in \mathbb{N}\}\right] \cup \{(1,1)\}.$$

Notice that $P \subset \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2x\}$.

Given $m \in \mathbb{N}$, let

$$B_m = \theta\left(1, \frac{1}{2^{m-1}}\right) \cup \left\{\left(1+x, \frac{1}{2^{m-1}} + \frac{y}{2^{m+1}}\right) : (x,y) \in P\right\},$$

$$C_m = \left(2, \frac{1}{2^{m-1}} + \frac{1}{2^{m+1}}\right)\left(3, \frac{1}{2^{m-1}} + \frac{1}{2^{m+1}}\right).$$

Notice that $B_m \subset \{(x,y) \in \mathbb{R}^2 : y \leq x/2^{m-1}$ and $y \leq 1/2^{m-1} + 1/2^{m+1}\}$.

Finally, let

$$X = \bigcup\{B_m : m = 0,1,\ldots\}, \qquad Y = \bigcup\{B_m \cup C_m : m = 0,1,\ldots\}.$$

Clearly, X and Y are fans and X is not homeomorphic to Y.

$C(X)$ **is homeomorphic to** $C(Y)$**.** Fix a Whitney map $\mu : C(X) \to [0,1]$. By the main result of [10], we may assume that $\mu(B_m) = 1/2$ for every $m = 0,1,\ldots$ Let $\pi_i : \mathbb{R}^2 \to \mathbb{R}^1$ be the projection on the ith coordinate, $i = 1,2$.

We denote the Hilbert cube by $\mathbb{Q}$. Let $C(\{\theta\}, X) = \{A \in C(X) : \theta \in A\}$ and $C(\{\theta\}, Y) = \{A \in C(Y) : \theta \in A\}$. In [4], Eberhart and Nadler constructed geometric models for the hyperspace of subcontinua of a smooth fan. We will use some of the ideas and results from that paper.

As a consequence of Theorem 2.3 of [4], we know that $C(\{\theta\}, X)$ and $C(\{\theta\}, Y)$ are homeomorphic to $\mathbb{Q}$.

Let $N(X) = \{\theta p \in C(X) : p \in X\}$, $N(Y) = \{\theta p \in C(Y) : p \in Y\}$, $T(X) = \bigcup\{C(B_m) : m = 0,1,\ldots\}$, and $T(Y) = \bigcup\{C(B_m \cup C_m) : m = 0,1,\ldots\}$. Clearly, $T(X)$ and $T(Y)$ are compact, $C(\{\theta\}, X) \cap T(X) = N(X)$, $C(\{\theta\}, Y) \cap T(Y) = N(Y)$, $C(X) = C(\{\theta\}, X) \cup T(X)$ and $C(Y) = C(\{\theta\}, Y) \cup T(Y)$.

Claim 1. *$N(X)$ (respectively, $N(Y)$) is a Z-set in $C(\{\theta\}, X)$ (respectively, $C(\{\theta\}, Y)$).*

Recall that, by definition, $N(X)$ is a Z-set in $C(\{\theta\}, X)$ if and only for each $\varepsilon > 0$, there exists a continuous function

$$g_\varepsilon : C(\{\theta\}, X) \to C(\{\theta\}, X) - N(X)$$

such that $H(g_\varepsilon(A), A) < \varepsilon$ for every $A \in C(\{\theta\}, X)$.

In order to prove Claim 1, let $\varepsilon > 0$. Suppose that $\varepsilon < 1$. Let $D_\varepsilon = \{p \in X : \|p - \theta\| \leq \varepsilon/2\}$. Then define $g_\varepsilon : C(\{\theta\}, X) \to C(\{\theta\}, X) - N(X)$ by

$$g_\varepsilon(A) = A \cup D_\varepsilon.$$

Clearly, g_ε has the required properties. Therefore, $N(X)$ is a Z-set in $C(\{\theta\}, X)$. Similarly, $N(Y)$ is a Z-set in $C(\{\theta\}, Y)$.

Notice that, for each $m \in \mathbb{N}$, $\pi_2|B_m : B_m \to [0, 1/2^{m-1} + 1/2^{m+1}]$ is one-to-one. Let $\mathcal{B} = \{A \in C(X) : \pi_1(A) \subset [1,2]\}$.

For $i = 1, 2$, let $M_i, m_i : C(X) \to \mathbb{R}$ be the maps defined by $M_i(A) = \max \pi_i(A)$ and $m_i(A) = \min \pi_i(A)$. Let $\omega : N(X) \cup \mathcal{B} \to [0,1]$ be given by

$$\omega(A) = \begin{cases} \dfrac{M_1(A) + M_2(A)}{2(2 + 1/2^{m-1} + 1/2^{m+1})} \\ \qquad\qquad \text{if } A \in C(B_m) \cap N(X) \text{ for some } m \in \mathbb{N}, \\ M_1(A)/4 \quad \text{if } A \subset B_0 \text{ and } A \in N(X), \\ (M_1(A) - m_1(A) + M_2(A) - m_2(A))/4 \quad \text{if } A \in \mathcal{B}. \end{cases}$$

Claim 2. *The set $\mathcal{B}$ and the function ω have the following properties*:

(a) *$\mathcal{B}$ is closed in $C(X)$, $\mathcal{B} \cap N(X) = \emptyset$,*

(b) *ω is continuous,*

(c) *if $A \subsetneq B$, then $\omega(A) < \omega(B)$,*

(d) *$\omega(\{p\}) = 0$ for each $\{p\} \in N(X) \cup \mathcal{B}$ and $\omega(B_m) = 1/2$ for each $m = 0, 1, \ldots$*

Statements (a), (b) and (d) are easy to prove. In order to prove (c), let $A, B \in N(X)$ be such that $A \subsetneq B \subset B_m$ for some $m \in \mathbb{N}$. Since A and B are arcs, θ is an end point of A and of B and $\pi_2|B_m$ is one-to-one, we conclude that $M_2(A) < M_2(B)$. Notice that $M_1(A) \le M_1(B)$. Thus $\omega(A) < \omega(B)$. The case $A, B \subset B_0$ is easier. The case $A, B \in \mathcal{B}$ follows from the fact that $\pi_2|B_m$ is one-to-one for every $m \in \mathbb{N}$. Finally, the case $A \in \mathcal{B}$ and $B \in N(X)$ is easy to check. This completes the proof of Claim 2.

Clearly, $N(X) \cup \mathcal{B}$ is a compact subset of $C(X)$. Thus we may apply the main result of [10]. In this way we may assume that the Whitney map μ also satisfies $\mu|(N(X) \cup \mathcal{B}) = \omega$.

Let $g : T(X) \to \mathbb{R}^3$ be given by

$$g(A) = \begin{cases} \left(\dfrac{M_1(A) + M_2(A)}{2(2 + 1/2^{m-1} + 1/2^{m+1})}, M_2(A), \mu(A) \right) \\ \qquad\qquad\qquad\qquad \text{if } A \subset B_m \text{ for some } m \in \mathbb{N}, \\ (M_1(A)/4, 0, \mu(A)) \quad \text{if } A \subset B_0, \end{cases}$$

Clearly, g is a continuous function.

Claim 3. *g is one-to-one.*

In order to prove Claim 3, suppose that $A, B \in T(X)$ and $g(A) = g(B)$.

If $A \subset B_0$, then $0 = M_2(A) = M_2(B)$. Thus $B \subset B_0$. Since $M_1(A) = M_1(B)$, it follows that A and B are (possibly degenerate) subarcs of B_0 with the same right end point. Thus, $A \subset B$ or $B \subset A$. But $\mu(A) = \mu(B)$. Therefore, $A = B$.

Therefore we may assume that $A \not\subset B_0$ and $B \not\subset B_0$. Thus $M_2(A) = M_2(B) > 0$.

If $A, B \subset B_m$ for some $m \in \mathbb{N}$, then since $\pi_2|B_m$ is one-to-one and $M_2(A) = M_2(B)$, we conclude that A and B are (possibly degenerate) subarcs of B_m with a common end point. Then $A \subset B$ or $B \subset A$. Since $\mu(A) = \mu(B)$, we conclude that $A = B$.

Finally, we consider the case when $A \subset B_n$ and $B \subset B_m$ with $0 < n < m$. We know that $B_m \subset \{(x, y) \in \mathbb{R}^2 : y \leq x/2^{m-1}$ and $y \leq 1/2^{m-1} + 1/2^{m+1}\}$. This implies $M_2(B) \leq M_1(B)/2^{m-1}$ and $M_2(B) = M_2(A) \leq 1/2^{m-1} + 1/2^{m+1}$. The way that B_n and B_m were constructed implies $A \subset \theta(1, 1/2^{n-1})$ and $M_2(A) = M_1(A)/2^{n-1}$.

Since
$$\frac{M_1(A) + M_2(A)}{2(2 + 1/2^{n-1} + 1/2^{n+1})} = \frac{M_1(B) + M_2(B)}{2(2 + 1/2^{m-1} + 1/2^{m+1})}$$
and $n < m$, we obtain
$$2(2 + 1/2^{m-1} + 1/2^{m+1})M_1(A) > 2(2 + 1/2^{n-1} + 1/2^{n+1})M_1(B).$$
Then
$$2(2 + 1/2^{m-1} + 1/2^{m+1})2^{n-1}M_2(A) > 2(2 + 1/2^{n-1} + 1/2^{n+1})2^{m-1}M_2(B).$$
Thus
$$\frac{1}{2^{m-1}}\left(2 + \frac{1}{2^{m-1}} + \frac{1}{2^{m+1}}\right) > \frac{1}{2^{n-1}}\left(2 + \frac{1}{2^{n-1}} + \frac{1}{2^{n+1}}\right).$$
This is a contradiction since $n < m$. Thus the proof of Claim 3 is complete, i.e. g is one-to-one.

By Claim 3, the map g is a homeomorphism from $T(X)$ onto $g(T(X)) \subset \mathbb{R}^3$. Thus we have obtained a model for $T(X)$.

Let $S = \{(x, z) \in \mathbb{R}^2 : 0 \leq x \leq 1/2,\ z \geq 2 - 4x$ and $0 \leq z \leq x\}$ and $R = (2/5, 2/5)(1/2, 1/2)$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection defined as $\pi(x, y, z) = (x, z)$.

Claim 4. *For each* $m = 0, 1, \ldots$, *let* $\mathcal{A}_m = (\pi \circ g)^{-1}(S) \cap C(B_m)$, $\mathcal{C}_m = N(X) \cap C(B_m) = \{A \in C(B_m) : \theta \in A\}$, $e_m = (2, 1/2^{m-1} + 1/2^{m+1})$ *for* $m \geq 1$, $e_0 = (2, 0)$ *and* $\mathcal{D}_m = \{A \in C(B_m) : e_m \in A\}$. *Then*:

(i) $\pi \circ g|\mathcal{C}_m : \mathcal{C}_m \to \theta(1/2, 1/2)$, $\pi \circ g|\mathcal{D}_m : \mathcal{D}_m \to (1/2, 0)(1/2, 1/2)$ *and* $\pi \circ g|\mathcal{A}_m : \mathcal{A}_m \to S$ *are homeomorphisms*,

(ii) $\pi(g(\mathcal{A}_m \cap \mathcal{C}_m)) = R$,

(iii) $\mathrm{Bd}_{C(B_m)}(\mathcal{A}_m) = (\pi \circ g|C(B_m))^{-1}((1/2, 0)(2/5, 2/5))$, *and*

(iv) $\mathrm{Bd}_{T(X)}(\bigcup\{\mathcal{A}_m : m = 0, 1, \ldots\}) = (\pi \circ g)^{-1}((1/2, 0)(2/5, 2/5))$.

Since B_m is an arc with end points θ and e_m, there is a homeomorphism from $C(B_m)$ into a triangle such that the following sets are sent to the respective sides of the triangle: the set of singletons of B_m, $\mathcal{C}_m$ and $\mathcal{D}_m$. In particular, $\mathcal{C}_m$ and $\mathcal{D}_m$ are arcs. The end points of $\mathcal{C}_m$ are $\{\theta\}$ and B_m and the end points of $\mathcal{D}_m$ are $\{e_m\}$ and B_m. Given $A \neq B$ in $\mathcal{C}_m$ (respectively,

$\mathcal{D}_m$), we have $A \subsetneq B$ or $B \subsetneq A$. This implies that $\mu(A) < \mu(B)$ or vice versa. Thus, $\mu|\mathcal{C}_m$ (respectively, $\mu|\mathcal{D}_m$) is one-to-one.

Let $A \in \mathcal{C}_m$. Then $A \subset B_m$. Thus $\mu(A) \leq \mu(B_m) = 1/2$ and $\pi(g(A)) = (\omega(A), \mu(A)) \in \theta(1/2, 1/2)$. Since $\pi(g(\{\theta\})) = \theta$ and $\pi(g(B_m)) = (1/2, 1/2)$, we conclude that $\pi \circ g|\mathcal{C}_m : \mathcal{C}_m \to \theta(1/2, 1/2)$ is a homeomorphism.

Given $A \in \mathcal{D}_m$, we have $e_m \in A \subset B_m$. Then $M_1(A) = 2$ and $M_2(A) = 1/2^{m-1} + 1/2^{m+1}$ if $m \geq 1$, and $M_1(A) = 2$ if $m = 0$. Thus $\pi(g(A)) = (1/2, \mu(A)) \in (1/2, 0)(1/2, 1/2)$. Since $\mu(\{e_m\}) = 0$ and $\mu(B_m) = 1/2$, we conclude that $\pi \circ g|\mathcal{D}_m : \mathcal{D}_m \to (1/2, 0)(1/2, 1/2)$ is a homeomorphism.

Now we show that $\pi \circ g|\mathcal{A}_m$ is one-to-one.

Let $D_1 = \bigcup\{\alpha : \alpha$ is a straight line segment contained in B_1 and the slope of α is negative$\}$. Let $D_0 = \pi_1(D_1)$.

Let $m \geq 0$. Let $A \in C(B_m)$ be such that $A \subset \pi_1^{-1}(D_0)$. Let $g(A) = (x', y', z')$. We prove that $z' < 2 - 4x'$ (and then $A \notin \mathcal{A}_m$). We only consider the case $m \geq 1$, the case $m = 0$ is easier. Notice that $\pi_1(A) \subset [1, 2]$. Then $A \in \mathcal{B}$ and $\mu(A) = (M_1(A) - m_1(A) + M_2(A) - m_2(A))/4$. Notice that, from the way B_m was constructed, $M_1(A) - m_1(A) \leq (2 - M_1(A))/4$ and $M_2(A) - m_2(A) \leq (2 - M_1(A))/8$. Thus $\mu(A) < (2 - M_1(A))/2$.

Notice that

$$x' = \frac{M_1(A) + M_2(A)}{2(2 + 1/2^{m-1} + 1/2^{m+1})} < \frac{M_1(A) + 1/2^{m-1} + 1/2^{m+1}}{2(2 + 1/2^{m-1} + 1/2^{m+1})}.$$

Then

$$1 - 2x' > \frac{2 - M_1(A)}{2 + 1/2^{m-1} + 1/2^{m+1}} \geq \frac{2 - M_1(A)}{4} \geq \frac{\mu(A)}{2}.$$

Thus $z' = \mu(A) < 2 - 4x'$.

Now, we are ready to prove that $\pi \circ g|\mathcal{A}_m$ is one-to-one. It is easy to show that $\pi \circ g|C(B_0)$ is one-to-one. So, we only consider the case of $m \geq 1$. Suppose that $A, B \in \mathcal{A}_m$ and $\pi(g(A)) = \pi(g(B))$. If $M_2(A) = M_2(B)$, then since $\pi_2|B_m$ is one-to-one, A and B are (possibly degenerate) arcs with a common end point. Thus $A \subset B$ or $B \subset A$. Since $\mu(A) = \mu(B)$, we conclude that $A = B$. Hence, we may assume that $M_2(A) < M_2(B)$. Since $\pi(g(A)) = \pi(g(B))$, $M_1(A) > M_1(B)$. From the way B_m was constructed, it follows that $B \subset \pi_1^{-1}(D_0)$. By the paragraph above $B \notin \mathcal{A}_m$. This contradiction proves that $\pi \circ g|\mathcal{A}_m$ is one-to-one.

Now, we show that $\pi \circ g|\mathcal{A}_m : \mathcal{A}_m \to S$ is onto. Let $(x, z) \in S$. Then $0 \leq x \leq 1/2$, $z \geq 2 - 4x$ and $0 \leq z \leq x$. Since $0 \leq z \leq 1/2$, we have $(z, z) \in \theta(1/2, 1/2)$ and $(1/2, z) \in (1/2, 0)(1/2, 1/2)$. Thus there exist $C \in \mathcal{C}_m$ and $D \in \mathcal{D}_m$ such that $(\mu(C), \mu(C)) = \pi(g(C)) = (z, z)$ and $(1/2, \mu(D)) = \pi(g(D)) = (1/2, z)$. Let $\mathcal{E} = (\mu|C(B_m))^{-1}(z)$. Since B_m is an arc, by [6, 6.4(a)], $\mathcal{E}$ is an arc with end points C and D. Notice that for every $E \in \mathcal{E}$, $\pi(g(E)) \in \mathbb{R} \times \{z\}$. Since $z \leq x \leq 1/2$, the Intermediate Value Theorem implies that there exists $E_0 \in \mathcal{E}$ such that $\pi(g(E_0)) = (x, z)$. Notice that $E_0 \in \mathcal{A}_m$. Therefore, $\pi \circ g|\mathcal{A}_m : \mathcal{A}_m \to S$ is bijective.

Clearly, $\mathcal{A}_m$ is compact. Hence, $\pi \circ g|\mathcal{A}_m : \mathcal{A}_m \to S$ is a homeomorphism.

The equality $\pi(g(\mathcal{A}_m \cap \mathcal{C}_m)) = R$ is easy to prove. Now, we check that $\mathrm{Bd}_{C(B_m)}(\mathcal{A}_m) = (\pi \circ g|C(B_m))^{-1}((1/2, 0)(2/5, 2/5))$. Let $A \in C(B_m)$ and let $\pi(g(A)) = (x', z')$. We analyze two possibilities for A.

If $A \subset \pi_1^{-1}(D_0)$, then as we saw before, $z' < 2 - 4x'$. Thus $A \notin \mathcal{A}_m$ and $A \notin (\pi \circ g)^{-1}((1/2, 0)(2/5, 2/5))$.

If $A \not\subset \pi_1^{-1}(D_0)$, let $p, q \in A$ be such that $M_1(A) = \pi_1(p)$ and $M_2(A) = \pi_2(q)$ (if $m=0$, we can take $q=p$; then $M_1(A)=\pi_1(q)$). Let J (respectively, K and L) be the (possibly degenerate) subarc of B_m which joins θ and q (respectively, p and q and θ and p). Notice that $K \subset A$, and K is a one-point set or $K \subset \pi_1^{-1}(D_0)$. Then $M_1(K) = M_1(A)$, $M_2(K) = M_2(A)$ and $\pi(g(K)) = (x', \mu(K))$. Thus $\mu(K) = 0$ or $\mu(K) < 2 - 4x'$.

Since $A \not\subset \pi_1^{-1}(D_0)$, it is easy to prove that $M_1(A) = M_1(J)$ and $M_2(A) = M_2(J)$. Let $\alpha : [0, 1] \to J$ be a continuous function such that $\alpha(0) = q$ and $\alpha(1) = \theta$. Let $\beta : [0, 1] \to C(B_m)$ be given by $\beta(t) = K \cup \alpha([0, t])$. Then β is continuous, $\beta(0) = K$, $\beta(1) = J$, there exists $t_0 \in [0, 1]$ such that $\beta(t_0) = A$, $M_1(\beta(t)) = M_1(A)$, $M_2(\beta(t)) = M_2(A)$ for every $t \in [0, 1]$ and if $s \le t$, then $\mu(\beta(s)) \le \mu(\beta(t))$. Thus $\pi(g(\beta(t))) = (x', \mu(\beta(t)))$ for each $t \in [0, 1]$.

Since $J = \beta(1) \in \mathcal{C}_m$, we have $x' = \mu(J) = \mu(\beta(1)) \ge \mu(\beta(t_0)) = \mu(A)$. Moreover, since $x' \le 1/2$, $\pi(g(A))$ is in the triangle in $\mathbb{R}^2$ which has vertices θ, $(1/2, 1/2)$ and $(1/2, 0)$.

Combining the conclusions of the two cases $A \subset \pi_1^{-1}(D_0)$ and $A \not\subset \pi_1^{-1}(D_0)$, we find that $\mathcal{A}_m = (\pi \circ g)^{-1}(S) \cap C(B_m) = (\pi \circ g)^{-1}(\{(x, z) \in \mathbb{R}^2 : 2 - 4x \le z \text{ and } 0 \le z\}) \cap C(B_m)$. Thus

$$\mathrm{Bd}_{C(B_m)}(\mathcal{A}_m) \subset (\pi \circ g|C(B_m))^{-1}((1/2, 0)(2/5, 2/5)).$$

Now, take $A \in (\pi \circ g|C(B_m))^{-1}((1/2, 0)(2/5, 2/5) - \{(1/2, 0)\})$. Then $A \not\subset \pi_1^{-1}(D_0)$. Let β and t_0 be as before. Since $\mu(A) > 0$, we have $A \neq K$ and β can be chosen to be one-to-one. Thus $0 < t_0$ and for each $t \in [0, t_0)$, $\mu(\beta(t)) < \mu(\beta(t_0))$. This implies that $\beta(t) \notin \mathcal{A}_m$ for every $t < t_0$. Hence, $A \in \mathrm{Bd}_{C(B_m)}(\mathcal{A}_m)$. Since $\mathrm{Bd}_{C(B_m)}(\mathcal{A}_m)$ is closed, we conclude that

$$\mathrm{Bd}_{C(B_m)}(\mathcal{A}_m) = (\pi \circ g|C(B_m))^{-1}((1/2, 0)(2/5, 2/5)).$$

Finally, the equality

$$\mathrm{Bd}_{T(X)}\Big(\bigcup\{\mathcal{A}_m : m = 0, 1, \ldots\}\Big) = (\pi \circ g)^{-1}((1/2, 0)(2/5, 2/5))$$

easily follows. This completes the proof of Claim 4.

Claim 5. *There is a homeomorphism $F : T(Y) \to T(X)$ such that $F(N(Y)) = N(X)$.*

For each $m = 1, 2, \dots$, let $t_m = 1/2^{m-1} + 1/2^{m+1}$ and $E_m = (2, t_m)(3, t_m) = e_m(3, t_m)$. Let $t_0 = 0$, $E_0 = (2, t_0)(3, t_0) = e_0(3, 0)$ and $\mathcal{A} = \bigcup\{\mathcal{A}_m : m = 0, 1, \dots\}$.

Let $G : T(Y) \to \mathbb{R}^3$ be given by

$$G(A) = \begin{cases} g(A) \quad \text{if } A \in T(X), \\ (1/2 + M_1(A) - 2, t_m, \mu(A \cap B_m)) \\ \qquad\qquad\qquad\qquad \text{if } e_m \in A \subset B_m \cup E_m \text{ for some } m \geq 0, \\ (1/2 + M_1(A) - 2, t_m, 2 - m_1(A)) \\ \qquad\qquad\qquad\qquad \text{if } A \subset E_m \text{ for some } m = 0, 1, \dots \end{cases}$$

If $A \in T(X) \cap C(B_m \cup E_m)$ and $e_m \in A$, then $A \subset B_m$. So $\mu(A \cap B_m) = \mu(A)$, $M_1(A) = 2$ and $M_2(A) = 1/2^{m-1} + 1/2^{m+1} = t_m$. It follows that $(1/2 + M_1(A) - 2, t_m, \mu(A \cap B_m)) = (1/2, t_m, \mu(A)) = g(A)$.

If $A \subset E_m$ and $e_m \in A$, then $m_1(A) = 2$. Thus, $\mu(A \cap B_m) = \mu(\{e_m\}) = 0 = 2 - m_1(A)$.

This proves that G is well defined. It is easy to show that G is continuous and one-to-one. Therefore, $G : T(Y) \to G(T(Y))$ is a homeomorphism.

Let $S_1 = S \cup ([1/2, 3/2] \times [0, 1/2]) \cup \{(x, z) \in \mathbb{R}^2 : 1/2 \leq x \leq 3/2$ and $1/2 - x \leq z \leq 0\}$, $R_1 = R \cup ([1/2, 3/2] \times \{1/2\})$ and R_2 be the triangle with vertices θ, $(1/2, 0)$ and $(2/5, 2/5)$.

Clearly, there is a homeomorphism $h : S_1 \to S$ such that $h(R_1) = R$ and $h|R_2$ is the identity on R_2. Suppose that $h = (h_1, h_3)$.

For each $m = 0, 1, \dots$, let $\mathcal{F}_m = \mathcal{A}_m \cup \{A \in C(B_m \cup E_m) : e_m \in A\} \cup C(E_m)$. It is easy to check that $\pi \circ G|\mathcal{F}_m : \mathcal{F}_m \to S_1$ is a homeomorphism. Let $\mathcal{F} = \bigcup\{\mathcal{F}_m : m = 0, 1, \dots\}$. Notice that $\mathrm{Bd}_{T(Y)}(\mathcal{F}) = \mathrm{Bd}_{T(X)}(\mathcal{A}) = (\pi \circ g)^{-1}((1/2, 0), (2/5, 2/5))$.

Define $F : T(Y) \to T(X)$ by

$$F(A) = \begin{cases} ((\pi \circ g)|\mathcal{A}_m)^{-1}(h(\pi(G(A)))) & \text{if } A \in \mathcal{F}_m \text{ for some } m = 0, 1, \dots, \\ A & \text{if } A \notin \mathcal{F}. \end{cases}$$

If $A \in \mathcal{F}_m$ for some $m = 0, 1, \dots$ and $A \in \mathrm{Cl}_{T(Y)}(T(Y) - \mathcal{F})$, then $A \in \mathrm{Bd}_{T(Y)}(\mathcal{F})$. This implies that $A \in T(X)$ and $\pi(g(A)) \in (1/2, 0)(2/5, 2/5) \subset R_2$. Thus $((\pi \circ g)|\mathcal{A}_m)^{-1}(h(\pi(G(A)))) = ((\pi \circ g)|\mathcal{A}_m)^{-1}(\pi(g(A))) = A$.

It is easy to show that F is a homeomorphism.

If $A \in N(Y)$, $A \in \mathcal{F}_m$ and $A \subset B_m$, then $A \in \mathcal{A}_m \cap \mathcal{C}_m$ and $\pi(G(A)) \in R$. Thus $h(\pi(G(A))) \in R$. Hence, $F(A) \in ((\pi \circ g)|\mathcal{A}_m)^{-1}(h(\pi(G(A)))) \subset \mathcal{C}_m \subset N(X)$.

If $A \in N(Y)$, $A \in \mathcal{F}_m$ and $A \notin C(B_m)$, then $\mu(A \cap B_m) = \mu(B_m) = 1/2$ and $2 \leq M_1(A) \leq 3$. Thus $\pi(G(A)) \in R_1$. Therefore, $F(A) \in N(X) \cap \mathcal{C}_m \subset N(X)$.

This implies that $F(N(Y)) \subset N(X)$.

Now, let $B \in N(X)$ be such that $B \in \mathcal{A}_m$. Then $B \in \mathcal{C}_m$ and $\pi(g(B)) \in R$. Thus $h^{-1}(\pi(g(B))) \in R_1 = R \cup ([1/2, 3/2] \times \{1/2\})$.

If $h^{-1}(\pi(g(B)) \in R$, then by Claim 4, there exists $A \in \mathcal{A}_m \cap \mathcal{C}_m \subset N(X)$ such that $\pi(g(A)) = h^{-1}(\pi(g(B)))$. Hence $B = F(A)$.

If $h^{-1}(\pi(g(B)) \in [1/2, 3/2] \times \{1/2\}$, then $h^{-1}(\pi(g(B))) = (1/2 + t - 2, 1/2)$ for some $t \in [2, 3]$. Let $A = B_m \cup ([2, t] \times \{t_m\})$. Then $\pi(G(A)) = h^{-1}(\pi(g(B)))$ and $A \in N(Y)$.

This completes the proof that $F(N(Y)) = N(X)$.

Claim 6. *$C(X)$ is homeomorphic to $C(Y)$.*

Since $N(X)$ (respectively, $N(Y)$) is a Z-set in $C(\{\theta\}, X)$ (respectively, $C(\{\theta\}, Y)$) and $C(\{\theta\}, X)$ and $C(\{\theta\}, Y)$ are homeomorphic to Hilbert cubes (see Theorem 2.3 of [4]), by [2] (see also 1.3 of [4]), there exists a homeomorphism $F_1 : C(\{\theta\}, Y) \to C(\{\theta\}, X)$ such that $F_1|N(Y) = F|N(Y)$. Define $F_2 : C(Y) \to C(X)$ by

$$F_2(A) = \begin{cases} F(A) & \text{if } A \in T(Y), \\ F_1(A) & \text{if } A \in C(\{\theta\}, Y). \end{cases}$$

Then F_2 is a homeomorphism.

Final remarks. Recently, Acosta ([1]) has introduced the following notion: A continuum X is said to *have unique hyperspace* $C(X)$ provided that if Y is a continuum such that $C(X) \cong C(Y)$, then $X \cong Y$. He has showed that if X is a continuum in one of the following classes, then X has unique hyperspace $C(X)$:

(a) finite graphs different from an arc and from a circle,

(b) hereditarily indecomposable continua,

(c) indecomposable continua such that all their proper nondegenerate subcontinua are arcs, and

(d) metric compactifications of the half-ray $[0, \infty)$.

Macías in [8] has defined the corresponding notion with 2^X in place of $C(X)$; namely, X is said to *have unique hyperspace* 2^X provided that if Y is a continuum such that $2^X \cong 2^Y$, then $X \cong Y$. He has showed that the hereditarily indecomposable continua have unique hyperspace 2^X.

The following question remains open.

Question [9, Questions 0.62]. Is the class of circle-like continua C-determined?

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