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Extending Peano derivatives: necessary and sufficient conditions

by

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Abstract. The paper treats functions which are defined on closed subsets of $[0,1]$ and which are k times Peano differentiable. A necessary and sufficient condition is given for the existence of a k times Peano differentiable extension of such a function to $[0,1]$. Several applications of the result are presented. In particular, functions defined on symmetric perfect sets are studied.

1. Introduction. Let P be a closed subset of $[0,1]$, and let $f : P \to \mathbb{R}$ be a given real-valued function defined on P. Let k be a positive integer. We say that f is *k times Peano differentiable at $x \in P$ relative to P* with Peano derivatives $f_{(1)}(x), \ldots, f_{(k)}(x)$ if we can write ($f_{(0)} := f$)

$$f(x+h) = \sum_{j=0}^{k} f_{(j)}(x)\frac{h^j}{j!} + \varepsilon(x,h)\frac{h^k}{k!}$$

with

$$\varepsilon(x,h) \to 0 \quad \text{as } 0 \neq h \to 0,\ x+h \in P.$$

This condition is empty if x is an isolated point of P. At an isolated point the Peano derivatives $f_{(1)}(x), \ldots, f_{(k)}(x)$ are arbitrarily assigned. If f is k times Peano differentiable at every point $x \in P$, then we say that f is *k times Peano differentiable on P relative to P*. If P is perfect, this definition is due to Denjoy [4, p. 280]. The extension to closed sets was given by Fejzić, Mařík and Weil [7].

Let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P with Peano derivatives $f_{(1)}, \ldots, f_{(k)}$. In this paper we deal with the following question: does there exist a function $F : [0,1] \to \mathbb{R}$ which is k times Peano differentiable on $[0,1]$ and has the property that $F(x) = f(x)$ and $F_{(j)}(x) =$

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$f_{(j)}(x)$ for all $x \in P$ and all $j = 1, \ldots, k$? We will call such a function F a *k-extension* of f for short.

This question was raised in the very interesting papers [2, 7] which inspired the present paper. It was shown in [7] that 1-extensions always exist but examples of Buczolich [1] and Denjoy [4] show that, for every $k \geq 2$, there are k times Peano differentiable functions which do not admit a k-extension. A more general class of such examples is presented in Section 4 of the present paper.

The main result of this paper is Corollary 3.10 of Theorem 3.2 which gives a necessary and sufficient condition for the existence of k-extensions. The necessity of the condition is known from [7, Cor. 4.8]. We recall this important theorem in Section 2. In Corollary 3.8 we prove that a k times Peano differentiable function $f : P \to \mathbb{R}$ admits a k-extension if and only if its restriction to the perfect kernel of the boundary of P admits a k-extension.

As in [7] we say that a closed subset P of $[0, 1]$ belongs to the class $\mathbf{P}_k$ if every k times Peano differentiable function $f : P \to \mathbb{R}$ admits a k-extension. Corollary 3.9 establishes that every closed set with countable boundary belongs to $\mathbf{P}_k$.

In Section 4 we investigate the problem whether a given symmetric perfect set specified by a sequence $\{\varepsilon_n\}$ belongs to $\mathbf{P}_k$. For many sequences we solve the problem but one case is still open.

2. A property of Peano derivatives. Let H be a perfect subset of $[0, 1]$. We say that H is of *finite Denjoy index* [3, p. 138], [7, p. 392] if there exist two constants $\theta > 0$ and $\beta > 1$ such that, for every $x \in H$, there is a real sequence h_n, $n \in \mathbb{N}$, such that $0 \neq h_n \to 0$ as $n \to \infty$, $x + h_n \in H$ for $n \in \mathbb{N}$, $|h_1| \geq \theta$, and

$$1 < |h_n|/|h_{n+1}| \leq \beta \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

The following theorem will be used in Section 3.

THEOREM 2.1. *Let H be a perfect subset of $[0, 1]$ of finite Denjoy index. Let $f : H \to \mathbb{R}$ be k times Peano differentiable on H relative to H with Peano derivatives $f_{(1)}, \ldots, f_{(k)}$. Let P be a perfect subset of H. Then there is a dense open subset E of P such that, for each $x \in E$ and $p = 1, \ldots, k-1$, $f_{(p)}$ is $k-p$ times Peano differentiable at x relative to P with Peano derivatives $f_{(p+1)}(x), \ldots, f_{(k)}(x)$.*

Theorem 2.1 is related to a result of Denjoy [4, p. 293] (which is given without proof), namely that the set E is only residual (complement of a set of first category). Theorem 2.1 is proved for $H = [0, 1]$ in [6, Thm. 1.1.20]. In [7, Cor. 4.8] an extension theorem is used to generalize it to the case where H is of finite Denjoy index. The author has found a more direct proof of Theorem 2.1 that is omitted here. It is of interest to have such a proof because we will show that Theorem 2.1 can be used to prove the extension theorem (Corollary 3.11).

Theorem 2.1 with $H = P = [0,1]$ shows that every function $f : [0,1] \to \mathbb{R}$ which is k times Peano differentiable on $[0,1]$ is k times differentiable on a dense open subset of $[0,1]$. This was proved by Oliver [8] in a different way.

3. A necessary and sufficient condition. The following lemma shows that we can assume without loss of generality that P is nowhere dense when we study the extension problem.

LEMMA 3.1. *Let P be a closed subset of* $[0,1]$, *and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. If f restricted to the topological boundary ∂P of P has a k-extension*, *then so does f.*

Proof. Let G be a k-extension of $f|\partial P$. The function $h := f - G$ is k times Peano differentiable on P relative to P, and it vanishes together with its first k Peano derivatives on ∂P. Define $H : [0,1] \to \mathbb{R}$ by $H(x) = h(x)$ for $x \in P$ and $H(x) = 0$ for $x \notin P$. Then H is a k-extension of h. Now $G + H$ is a k-extension of f. ∎

Let P be a nowhere dense closed subset of $[0,1]$, and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. Let $R(f,P)$ be the set of all $x \in P$ for which there exists an open interval (a,b) with $a < x < b$ and $a, b \notin P$ such that $f|(a,b) \cap P$ has a k-extension. Note that $R(f,P)$ is open relative to P and contains every isolated point of P. We also set $Q(f,P) := P - R(f,P)$. This is a closed subset of P.

Our goal is to prove the following theorem.

THEOREM 3.2. *Let P be a closed nowhere dense subset of* $[0,1]$, *and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. If f satisfies the condition*:

(3.1) *for every nonempty closed subset P_0 of P*, *$R(f,P_0)$ is nonempty*,

then f admits a k-extension.

For the proof a series of lemmas will be needed.

LEMMA 3.3. *Let P be a closed subset of* $[0,1]$. *Let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. Suppose there is a k-extension $F : [0,1] \to \mathbb{R}$ of f. For every open interval I containing P and every $\varepsilon > 0$, there is another k-extension $H : [0,1] \to \mathbb{R}$ of f such that*

$$\max_{x\in[0,1]} |H(x)| \le \max_{x\in P} |f(x)| + \varepsilon \tag{3.2}$$

and

$$H(x) = 0 \quad \text{for all } x \text{ outside } I.$$

Proof. Let $A := \max_{x\in P} |f(x)|$. Define a function $G : [0,1] \to \mathbb{R}$ by $G(x) = F(x)$ if $|F(x)| \le A + \varepsilon$, $G(x) = A + \varepsilon$ if $F(x) > A + \varepsilon$ and $G(x) = -A - \varepsilon$ if $F(x) < -A - \varepsilon$. Then G might not be k times Peano differentiable on $[0,1]$ any more but G agrees with F in a neighborhood of each $x \in P$. Inspection of the proof of Lemma 4.6 of [7] shows that G can be "smoothened" to a function H in such a way that it becomes a k-extension of f and still $|H(x)| \le A + \varepsilon$ for all $x \in [0,1]$. It is clear that we can change H so that H vanishes outside I without destroying condition (3.2). ∎

LEMMA 3.4. *Let P be a closed nowhere dense subset of* $[0,1]$, *and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. Let A be a compact subset of $R(f,P)$. Then $f|A$ admits a k-extension.*

Proof. For every $x \in A$, there is an open interval I containing x whose endpoints are not in P such that $f|I \cap A$ admits a k-extension. By compactness of A, finitely many of these intervals, say $I_1, \ldots, I_n$, cover A. We can also assume that these intervals are pairwise disjoint. By Lemma 3.3, for every $j = 1, \ldots, n$, there is a k-extension F_j of $f|I_j \cap A$ which vanishes outside I_j. Then $F_1 + \ldots + F_n$ is a k-extension of $f|A$. ∎

LEMMA 3.5. *Let P be a closed nowhere dense subset of* $[0,1]$. *Let Q be a nonempty closed subset of P. Then there exists a countable collection of open intervals I_n which has the following properties*:

(i) *the I_n are pairwise disjoint*, *disjoint from Q and $P_n := I_n \cap P$ is nonempty*;

(ii) *the length $|I_n|$ of I_n is less than the distance* $\operatorname{dist}(I_n, Q)$ *from I_n to Q*;

(iii) *the endpoints of I_n are not in P so that P_n is closed*;

(iv) $P - Q = \bigcup_n P_n$.

Proof. Consider a complementary interval (a,b) of Q. Since P is nowhere dense, it is easy to find points c_n, $n \in \mathbb{Z}$, which are not in P such that $a < \ldots < c_{-1} < c_0 < c_1 < \ldots < b$, $c_n \to a$ as $n \to -\infty$, $c_n \to b$ as $n \to \infty$ and $\operatorname{dist}((c_n, c_{n+1}), Q) > |c_n - c_{n+1}|$ for all n. Then let $I_n = (c_n, c_{n+1})$. If we do this for every complementary interval, the collection of all the I_n that meet P has the desired properties. ∎

LEMMA 3.6. *Let P be a closed nowhere dense subset of* $[0,1]$, *and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. Suppose that*, *for all $x \in Q(f,P)$*,

$$f(x) = f_{(1)}(x) = \ldots = f_{(k)}(x) = 0. \tag{3.3}$$

Then f admits a k-extension.

Proof. If $Q := Q(f, P)$ is empty, then the conclusion follows from Lemma 3.4 with $A = P$. So let Q be nonempty. By Lemma 3.5, there are countably many open intervals I_n having the properties (i) through (iv) as given in the lemma. Let $P_n := P \cap I_n$. Since $P_n \cap Q = \emptyset$, Lemmas 3.3 and 3.4 tell us that, for every n, there is $F_n : [0,1] \to \mathbb{R}$ such that

(a) F_n is k times Peano differentiable on $[0,1]$;
(b) $(F_n)_{(j)}(x) = f_{(j)}(x)$ for all $x \in P_n$ and all $j = 0, \ldots, k$;
(c) F_n has support in I_n;
(d) $|F_n(x)| \le \max_{y \in P_n} |f(y)| + \operatorname{dist}(I_n, Q)^{k+1}$ for all x.

Define $F : [0,1] \to \mathbb{R}$ by

$$F(x) := \sum_n F_n(x).$$

This is a well-defined function because the supports of the F_n are pairwise disjoint. We now show that F is a k-extension of f. Each $x \in [0,1] - Q$ has a neighborhood which meets only finitely many supports of the F_n. This proves that F is k times Peano differentiable at each $x \in [0,1] - Q$. If $x \in P - Q$, then there is n such that $x \in P_n$ and F agrees with F_n in I_n. Thus $F_{(j)}(x) = f_{(j)}(x)$ for all $j = 0, \ldots, k$.

By (3.3), all what is left to show is that $F(x)/(x-b)^k \to 0$ as $x \to b$ for every $b \in Q$. Let $b \in Q$, $\varepsilon > 0$. By assumption, there is $0 < \delta < \varepsilon$ such that

$$|y - b| < \delta, y \in P \Rightarrow |f(y)| \le \varepsilon |y - b|^k. \tag{3.4}$$

Let $x \in [0,1]$ with $|x - b| < \delta/2$. Since there is nothing to prove if $F(x) = 0$, let $x \in I_n$ for some n. So

$$|I_n| \le \operatorname{dist}(I_n, Q) \le |x - b|. \tag{3.5}$$

If $y \in P_n$, then

$$|y - b| \le |y - x| + |x - b| \le |I_n| + |x - b| \le 2|x - b| < \delta.$$

By (3.4), $|f(y)| \le \varepsilon |y - b|^k \le \varepsilon 2^k |x - b|^k$. By (d) and (3.5),

$$|F(x)| \le \varepsilon 2^k |x - b|^k + |x - b|^{k+1} \le \varepsilon (2^k + 1) |x - b|^k.$$

Since this is true for all x with $|x - b| < \delta/2$, the conclusion follows. ∎

Let P be a closed nowhere dense subset of $[0,1]$, and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. By transfinite induction, for every ordinal α, we define a closed subset $T_\alpha = T_\alpha(f, P)$ of P as follows:

(i) if $\alpha = 0$, then $T_0 := P$;
(ii) if $\alpha = \beta + 1$, then $T_\alpha := Q(f, T_\beta)$;
(iii) if α is a limit number, then $T_\alpha := \cap_{\beta < \alpha} T_\beta$. ∎

Clearly, we have $T_\beta \subset T_\alpha$ (with equality allowed) whenever $\alpha < \beta$. Under condition (3.1), T_β is a proper subset of T_α whenever $\alpha < \beta$ and T_α

is nonempty. In this case the Cantor–Baire stationary principle implies that there is a smallest ordinal $\mu = \mu(f, P)$ in the first or second number class for which $T_\mu = \emptyset$. We will use transfinite induction on μ in order to construct a k-extension of f. Let us first use an ordinary induction.

LEMMA 3.7. *Let P be a closed nowhere dense subset of* $[0,1]$, *and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P. Assume that there is a positive integer n such that $T_n = \emptyset$. Then f admits a k-extension.*

Proof. The proof is by induction on n. If $n = 1$, then we are done by Lemma 3.4. Assume that the statement of the lemma is true for $n-1$ in place of n, and let P and f be given with $T_n(f, P) = \emptyset$. Define $Q := Q(f, P)$. Then Q is a closed subset of $[0, 1]$ with $T_{n-1}(f, Q) = \emptyset$. By induction hypothesis, there is a function $G : [0, 1] \to \mathbb{R}$ which is k times Peano differentiable on $[0, 1]$ and $G_{(j)}(x) = f_{(j)}(x)$ for all $x \in Q$ and $j = 0, \ldots, k$. The function $f - G$ is k times Peano differentiable on P relative to P. This function together with its first k Peano derivatives vanishes on Q. Note that $Q(f, P) = Q(f - G, P)$. By Lemma 3.6, there is a function $H : [0, 1] \to \mathbb{R}$ which is k times Peano differentiable on $[0, 1]$ and $H_{(j)}(x) = f_{(j)}(x) - G_{(j)}(x)$ for all $x \in P$ and $j = 0, \ldots, k$. Now $F := G + H$ is a k-extension of f. ∎

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\mu = \mu(f, P)$ be the smallest ordinal (of the first or second number class) such that $T_\mu(f, P) = \emptyset$. We prove the theorem by transfinite induction on $\mu(f, P)$. We have already shown in Lemma 3.7 that the theorem is true if $\mu(f, P)$ is finite. Assume now that the theorem is true if $\mu(f, P) < \gamma$ where γ is a given ordinal in the second number class. Let P be a closed nowhere dense subset of $[0, 1]$, and let $f : P \to \mathbb{R}$ be k times Peano differentiable on P relative to P with $\mu(f, P) = \gamma$. We have to show that f admits a k-extension. The ordinal γ cannot be a limit number. So γ is of the form $\gamma = \beta + m$, where β is a limit number and m is a positive integer. Let

$$S := T_\beta = \bigcap_{\alpha<\beta} T_\alpha.$$

Since $T_m(f, S) = \emptyset$ we know from Lemma 3.7 that $f|S$ has a k-extension G. Define $h(x) := f(x) - G(x)$ for $x \in P$. Note that $T_\alpha(f, P) = T_\alpha(h, P)$ for all ordinals α, and

$$h(x) = h_{(1)}(x) = \ldots = h_{(k)}(x) = 0 \quad \text{for all } x \in S. \tag{3.6}$$

Let x be in $P - S$. Then there is an ordinal $\alpha < \beta$ such that $x \notin T_\alpha$. Choose an open interval (a, b) disjoint from T_α containing x and such that $a, b \notin P$. Then $P_0 := P \cap (a, b)$ is disjoint from T_α. Since $T_\alpha(h, P_0)$ is a subset of both $T_\alpha = T_\alpha(h, P)$ and P_0, $T_\alpha(h, P_0)$ is empty. By induction hypothesis, $h|P_0$

admits a k-extension which implies $x \in R(h, P)$. Since x was arbitrary in $P - S$, we see that $P - S$ is contained in $R(h, P)$ and so $Q(h, P)$ is a subset of S. By Lemma 3.6 and (3.6), h admits a k-extension H. Then $G + H$ is a k-extension of f. ■

We now draw some conclusions from Theorem 3.2.

COROLLARY 3.8. *Let P be a closed subset of* $[0, 1]$, *and let* $f : P \to \mathbb{R}$ *be k times Peano differentiable on P relative to P. Let $\partial P = A \cup B$ be the* (*unique*) *decomposition of ∂P into a perfect* (*or empty*) *set A and an at most countable set B. If $f|A$ admits a k-extension*, *then so does f.*

P r o o f. We verify that $f|\partial P$ satisfies condition (3.1). Let P_0 be a closed nonempty subset of ∂P. If P_0 has an isolated point, then this point is in $R(f, P_0)$ and $R(f, P_0)$ is nonempty. If P_0 does not have an isolated point, then P_0 is perfect and it is a subset of A. Since $f|A$ has a k-extension, this implies $R(f, P_0) = P_0$. So condition (3.1) is satisfied, and the conclusion follows from Lemma 3.1 and Theorem 3.2. ■

Corollary 3.8 shows that it is sufficient to consider nowhere dense perfect sets P when we investigate the extension problem.

COROLLARY 3.9. *Let P be a closed subset of* $[0, 1]$ *with the property that ∂P is countable. Then P belongs to the class $\mathbf{P}_k$.*

We now obtain a necessary and sufficient condition for the existence of k-extensions.

COROLLARY 3.10. *Let P be a closed subset of* $[0, 1]$, *and let* $f : P \to \mathbb{R}$ *be k times Peano differentiable on P relative to P with Peano derivatives $f_{(1)}, \ldots, f_{(k)}$. Then there exists a k-extension of f if and only if the following condition holds*: *in every perfect subset P_0 of ∂P there exists a point x such that*, *for all y in a neighborhood I of x relative to P_0 and all $p = 1, \ldots, k - 1$, $f_{(p)}$ is $k - p$ times Peano differentiable at y relative to P_0 with Peano derivatives $f_{(p+1)}(y), \ldots, f_{(k)}(y)$.*

P r o o f. By Theorem 2.1 with $H = [0, 1]$, the condition is necessary for the existence of a k-extension of f. Now let the condition be satisfied. In order to show that f admits a k-extension it is enough to verify condition (3.1) for $f|\partial P$ (by Lemma 3.1 and Theorem 3.2). Let P_0 be a perfect subset of ∂P. By assumption, there is $x \in P_0$ and an open interval I containing x whose endpoints do not lie in P such that for all $y \in I \cap P_0$ and all $p = 1, \ldots, k - 1$, $f_{(p)}$ is $k - p$ times Peano differentiable at y relative to P_0 with Peano derivatives $f_{(p+1)}(y), \ldots, f_{(k)}(y)$. By [7, Theorem 3.3], this implies that $f|I \cap P_0$ admits a k-extension. ■

By combining Theorem 2.1 with Corollary 3.10 we obtain a new proof of the main result of [7].

Corollary 3.11. *Every perfect subset of* $[0,1]$ *which has finite Denjoy index belongs to* $\mathbf{P}_k$.

4. Extension of functions defined on symmetric perfect sets. Let λ_n, $n \in \mathbb{N}$, be a given sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. We assume that

$$\mu_n := \sum_{m=n+1}^{\infty} \lambda_m < \lambda_n \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Let P be the set of all finite or infinite subsums of the series $\sum_n \lambda_n$:

$$P := \Big\{ \sum_{n \in A} \lambda_n : A \in \mathbf{P}(\mathbb{N}) \Big\}, \tag{4.2}$$

where $\mathbf{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$. The empty sum is defined as 0.

Let $T : \mathbf{P}(\mathbb{N}) \to P$ be the map defined by $T(A) := \sum_{n \in A} \lambda_n$. Then T is a measure on $\mathbf{P}(\mathbb{N})$ and P is the range of T. Condition (4.1) implies that T is one-to-one. We turn $\mathbf{P}(\mathbb{N})$ into a metric space by defining

$$d(A,B) := \sum_{n \in A \triangle B} 2^{-n}.$$

It is easy to see that T is continuous from $\mathbf{P}(\mathbb{N})$ onto P. Since $\mathbf{P}(\mathbb{N})$ is compact, this shows that P is compact and T is a topological map. It is also easy to see that P has no isolated points and so is a perfect set. The set P is called a *symmetric perfect set*.

The right end-points of complementary intervals of P are exactly the points $T(A)$ with A finite. The left end-points of complementary intervals of P are exactly the points $T(A)$ with $\mathbb{N} - A$ finite.

We define $\eta_n := \mu_n/\lambda_n \in (0,1)$ and $\varepsilon_n := (1-\eta_n)/(1+\eta_n)$. It is easy to see that P can be obtained by successively removing middle intervals from $[0,1]$ of proportion ε_n in the nth step as described in [9, p. 205] and [5, p. 116]. The symmetric perfect set P is completely determined by the numbers η_n (or ε_n) which can be arbitrarily chosen in $(0,1)$. For example, in the Cantor set we have $\varepsilon_n = 1/3$, $\eta_n = 1/2$, $\lambda_n = 2 \cdot 3^{-n}$ and $\mu_n = 3^{-n}$.

We pose the problem: for which choices of sequences η_n does P belong to the class $\mathbf{P}_k$?

We present two results.

Theorem 4.1. *If* $\liminf \eta_n > 0$, *then the symmetric perfect set* P *is of finite Denjoy index. Thus it belongs to* $\mathbf{P}_k$.

Proof. By assumption, there is $a > 0$ such that $\eta_n \ge a$ for all $n \in \mathbb{N}$. We claim that P has finite Denjoy index with corresponding constants $\theta = \lambda_1$ and $\beta = 2/a$. Let $x = T(A) \in P$. We define $h_n := \lambda_n$ if $n \notin A$ and $h_n := -\lambda_n$

if $n \in A$. Then $x + h_n \in P$ for all n. Since $0 < \lambda_n \to 0$, we have $0 \neq h_n \to 0$. Also, $|h_1| = \lambda_1 = \theta$. Since

$$\frac{\lambda_n}{\lambda_{n+1}} = \frac{\mu_n}{\eta_n \lambda_{n+1}} = \frac{\lambda_{n+1} + \mu_{n+1}}{\eta_n \lambda_{n+1}} = \frac{1 + \eta_{n+1}}{\eta_n},$$

we obtain

$$1 < \frac{|h_n|}{|h_{n+1}|} < \frac{2}{a} = \beta$$

for all n. So P has finite Denjoy index. By Corollary 3.11, P belongs to $\mathbf{P}_k$. ■

THEOREM 4.2. *Assume that* $\liminf \eta_n = 0$ *and* $\limsup \eta_n < 1$. *Let* $k \geq 2$. *Then the symmetric perfect set* P *does not belong to* $\mathbf{P}_k$.

Proof. We will construct a function $f : P \to \mathbb{R}$ which is k times Peano differentiable on P relative to P but does not admit a k-extension. By assumption, there is $\delta > 0$ such that $1 - \eta_n \geq \delta$ for all n. Moreover, there are positive integers $n_1 < n_2 < n_3 < \ldots$ converging to infinity such that $\eta_{n_i} \to 0$. We decompose $\mathbb{N}$ into blocks $D_i := \{n_{i-1} + 1, \ldots, n_i\}$, $i \in \mathbb{N}$, where $n_0 := 0$. For each subset A of $\mathbb{N}$ and every $i \in \mathbb{N}$, we define $j(A, i)$ as the number of $q \in \{1, \ldots, i - 1\}$ for which $A \cap D_q$ is nonempty. We define $f : P \to \mathbb{R}$ as follows:

$$f(x) := \sum_{i=1}^{\infty} 2^{-j(A,i)} \Big(\sum_{n \in A \cap D_i} \lambda_n \Big)^k \quad \text{for } x = T(A).$$

We now show that f is k times Peano differentiable at a given $x \in P$ relative to P. We distinguish two cases:

FIRST CASE: $x = T(A)$ and A is an infinite set. Let $y = T(B) \in P$, $A \neq B$. Let p be the minimal element in $A \triangle B$. Define m by $p \in D_m$. We have

$$|y - x| \geq \lambda_p - \mu_p = (1 - \eta_p)\lambda_p \geq \delta \lambda_p. \tag{4.3}$$

Also,

$$|f(y) - f(x)| \leq 2^{-j(A,m)} 2 \mu_{p-1}^k \leq 2^{-j(A,m)} 2^{k+1} \lambda_p^k. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$|f(y) - f(x)| \leq 2^{-j(A,m)} 2^{k+1} \delta^{-k} |y - x|^k.$$

Now $y \to x$ implies $m \to \infty$. Since A is infinite, this in turn implies $j(A, m) \to \infty$. Hence f is k times Peano differentiable at x relative to P with the first k Peano derivatives equal to 0.

SECOND CASE: $x = T(A)$ and A is a finite set. Let $y = T(B)$, $A \neq B$. Since we are only interested in y close to x and A is finite, we can assume that $B \supset A$ so that $y > x$. Let again p be the minimal element in $A \triangle B = B - A$

and $p \in D_m$. Of course, we can assume that $A \cap D_m = \emptyset$. Write $y-x = w+z$ with

$$w := \sum_{n \in B \cap D_m} \lambda_n \ge \lambda_p \ge \lambda_{n_m}$$

and

$$z := \sum_{q>m} \sum_{n \in B \cap D_q} \lambda_n \le \mu_{n_m} = \eta_{n_m} \lambda_{n_m} \le \eta_{n_m} w.$$

Then we have

$$w \le y - x = w + z \le (1 + \eta_{n_m}) w.$$

This implies that

$$0 \le (y-x)^k - w^k \le ((1+\eta_{n_m})^k - 1) w^k \le ((1+\eta_{n_m})^k - 1)(y-x)^k.$$

Setting $j := j(A,m) = j(B,m)$ we obtain

$$\begin{aligned} |f(y) - f(x) - 2^{-j} w^k| &\le \sum_{q>m} \Big(\sum_{n \in B \cap D_q} \lambda_n \Big)^k \\ &\le \mu_{n_m}^k \le \eta_{n_m}^k w^k \le \eta_{n_m}^k (y-x)^k. \end{aligned}$$

Thus

$$\begin{aligned} |f(y) - f(x) - 2^{-j}(y-x)^k| &\le |f(y) - f(x) - 2^{-j} w^k| + |w^k - (y-x)^k| \\ &\le \{\eta_{n_m}^k + (1+\eta_{n_m})^k - 1\}(y-x)^k. \end{aligned}$$

As $y \to x$, j stays fixed but $m \to \infty$. Since $\eta_{n_m} \to 0$ as $m \to \infty$, we see that f is k times Peano differentiable at x relative to P. The first $k-1$ derivatives are zero but the kth equals $k! 2^{-j}$.

Since the set of all $T(A)$ with finite A is dense in P, Corollary 3.10 shows that f does not admit a k-extension. So P does not belong to $\mathbf{P}_k$. ∎

Theorems 4.1 and 4.2 solve our problem except in the case of

$$\liminf \eta_n = 0 \quad \text{and} \quad \limsup \eta_n = 1. \tag{4.5}$$

This leads us to asking the question: can a symmetric perfect set P whose corresponding sequence η_n satisfies (4.5) belong to $\mathbf{P}_k$?

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