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On z°-ideals in $C(X)$

by

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Abstract. An ideal I in a commutative ring R is called a z°-ideal if I consists of zero divisors and for each $a \in I$ the intersection of all minimal prime ideals containing a is contained in I. We characterize topological spaces X for which z-ideals and z°-ideals coincide in $C(X)$, or equivalently, the sum of any two ideals consisting entirely of zero divisors consists entirely of zero divisors. Basically disconnected spaces, extremally disconnected and P-spaces are characterized in terms of z°-ideals. Finally, we construct two topological almost P-spaces X and Y which are not P-spaces and such that in $C(X)$ every prime z°-ideal is either a minimal prime ideal or a maximal ideal and in $C(Y)$ there exists a prime z°-ideal which is neither a minimal prime ideal nor a maximal ideal.

1. Introduction. An ideal I of a commutative ring R is called a *z-ideal* if whenever any two elements of R are contained in the same set of maximal ideals and I contains one of them, then it also contains the other one (see [5], 4A.5, for an equivalent definition). These ideals which are both algebraic and topological objects were first introduced by Kohls (see [5]) and play a fundamental role in studying the ideal structure of $C(X)$, the ring of real-valued continuous functions on a completely regular Hausdorff space X. Maximal ideals, minimal prime ideals and most of the important ideals in $C(X)$ are z-ideals.

In this article we investigate ideals in $C(X)$ which we call z°-ideals. It turns out that the concept of z°-ideals is very useful when dealing with ideals in $C(X)$ consisting of zero divisors.

This article consists of three sections. In Section 2, z°-ideals are studied in $C(X)$, and it is also shown that every ideal in $C(X)$ consisting of zero divisors is contained in a prime z°-ideal. This immediately shows that every maximal ideal in $C(X)$ consisting of zero divisors is a z°-ideal. We

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characterize topological spaces X such that z-ideals and z°-ideals coincide in $C(X)$. We also investigate topological spaces X such that the sum of any two z°-ideals in $C(X)$ is either $C(X)$ or a z°-ideal. Characterizations of basically disconnected, extremally disconnected and P-spaces are given in terms of z°-ideals. Finally, we present two natural questions concerning z°-ideals in $C(X)$ which are answered in Section 3.

We first recall some general information from [5]. If $f \in C(X)$, then $Z(f) = \{x \in X : f(x) = 0\}$ is the *zero set* of f and $\mathrm{Coz}(f) = X - Z(f)$ its *cozero set*. A subspace Y of X is said to be *C-embedded* in X if the map that sends each $f \in C(X)$ to its restriction to Y is onto. An ideal I of $C(X)$ is called a z-*ideal* if $Z(f) = Z(g)$ and $f \in I$ imply that $g \in I$. X is called *extremally* (*basically*) *disconnected* if each open (cozero) set has an open closure, or equivalently, if the interior of each closed set (zero set) is closed. If $A \subseteq X$, then $O_A = \{f \in C(X) : A \subseteq \operatorname{int} Z(f)\}$, and if $A \subseteq \beta X$, then $O^A = \{f \in C(X) : A \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)\}$, where βX is the Stone–Čech compactification of X. We also recall that every maximal ideal M of $C(X)$ is of the form $M = M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$, where $p \in \beta X$, and if $p \in X$, then $M^p = M_p = \{f \in C(X) : f(p) = 0\}$. For each $S \subseteq C(X)$, by the *annihilator* of S we mean $\mathrm{Ann}(S) = \{f \in C(X) : Sf = 0\}$. For undefined terms and notations, the readers are referred to [5].

2. z°-ideals and $C(X)$. For each $f \in R$ let P_f be the intersection of all minimal prime ideals containing f; by convention, the intersection of an empty set of ideals is $C(X)$. Next we give the definition of z°-ideals.

DEFINITION. A proper ideal I in $C(X)$ is called a z°-*ideal* if for each $f \in I$ we have $P_f \subseteq I$. Clearly, P_f is a z°-ideal which is called a *basic* z°-*ideal*.

We begin with the following lemma.

LEMMA 2.1. *If* $f, g \in C(X)$, *then* $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$ *if and only if* $\mathrm{Ann}(f) \subseteq \mathrm{Ann}(g)$.

Proof. Let $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$ and $h \in \mathrm{Ann}(f)$; then $hf = 0$ implies that $X - Z(h) \subseteq \operatorname{int} Z(f) \subseteq Z(g)$. This means that $gh = 0$ and therefore $h \in \mathrm{Ann}(g)$. Conversely, let $\mathrm{Ann}(f) \subseteq \mathrm{Ann}(g)$. To prove that $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$, it suffices to show that $\operatorname{int} Z(f) \subseteq Z(g)$. Suppose $x \in \operatorname{int} Z(f)$ and $x \notin Z(g)$. Since $x \notin X - \operatorname{int} Z(f)$, there is $0 \neq h \in C(X)$ with $h(X - \operatorname{int} Z(f)) = \{0\}$ and $h(x) = 1$. Clearly $hf = 0$ and $hg \neq 0$, which is impossible.

The following propositions are now immediate.

PROPOSITION 2.2. *If* I *is a proper ideal in* $C(X)$, *then the following statements are equivalent*:

(1) *I is a z°-ideal in R.*
(2) *$P_f = P_g$ and $g \in I$ imply that $f \in I$.*
(3) *$\mathrm{Ann}(f) = \mathrm{Ann}(g)$ and $f \in I$ imply that $g \in I$.*
(4) *$f \in I$ implies that $\mathrm{Ann}(\mathrm{Ann}(f)) \subseteq I$.*
(5) *$\mathrm{int}\, Z(f) = \mathrm{int}\, Z(g)$ and $f \in I$ imply that $g \in I$.*
(6) *If $\mathrm{Ann}(S) \subseteq \mathrm{Ann}(g)$ and $S \subset I$ is a finite set, then $g \in I$.*

PROPOSITION 2.3. *For every $f \in C(X)$ we have*

$$P_f = \{g \in C(X) : \mathrm{Ann}(f) \subseteq \mathrm{Ann}(g)\}.$$

It would be interesting to characterize reduced rings such that conditions (1) and (6) in Proposition 2.2 are equivalent.

Examples of z°*-ideals in* $C(X)$. (1) If S is a regular closed set in X, i.e., $\mathrm{cl}(\mathrm{int}\, S) = S$, then the ideal $M_S = \{f \in C(X) : S \subseteq Z(f)\}$ is a z°-ideal.

(2) O_x for $x \in X$, and more generally, O^A for $A \subseteq \beta X$, are z°-ideals in $C(X)$.

(3) If X is a noncompact space, then the ideal $C_{\mathrm{K}}(X)$ of functions with compact support is a z°-ideal.

(4) Every maximal ideal of $C(X)$ consisting of zero divisors is a z°-ideal (see Corollary 2.6).

(5) Every minimal prime ideal in $C(X)$ is a z°-ideal. More generally, one can prove that if I is a z°-ideal in $C(X)$ and P is a prime ideal in $C(X)$ minimal over I, then P is a prime z°-ideal.

(6) Every intersection of z°-ideals in $C(X)$ is a z°-ideal.

REMARK 2.4. Clearly, every z°-ideal in $C(X)$ is a z-ideal but the converse is not true. To see this, let $I = \{f \in C(X) : [0,1] \cup \{2\} \subseteq Z(f)\}$.

THEOREM 2.5. *If I is an ideal in $C(X)$ consisting of zero divisors, then I is contained in a z°-ideal.*

Proof. We define $I_0 = I$ and $I_1 = \sum_{f \in I_0} \mathrm{Ann}(\mathrm{Ann}(f))$. If α is a limit ordinal we define $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$, where β is an ordinal, and if $\alpha = \beta + 1$, we set $I_\alpha = \sum_{f \in I_\beta} \mathrm{Ann}(\mathrm{Ann}(f))$. Thus we get an ascending chain $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_\alpha \subseteq I_{\alpha+1} \subseteq \ldots$ and since $C(X)$ is a set, there exists the smallest ordinal α such that $I_\alpha = I_\gamma$ for all $\gamma \geq \alpha$. We claim that I_α is a proper ideal which is also a z°-ideal. If I_α is a proper ideal, it certainly is a z°-ideal, for $I_\alpha = I_{\alpha+1} = \sum_{f \in I_\alpha} \mathrm{Ann}(\mathrm{Ann}(f))$. This means that $\mathrm{Ann}(\mathrm{Ann}(f)) \subseteq I_\alpha$ for all $f \in I_\alpha$ and therefore, by Proposition 2.2, we are through.

Thus, it remains to be shown that I_α is a proper ideal. To see this, it suffices to prove that for each α, I_α consists entirely of zero divisors. We proceed by transfinite induction on α. For $\alpha = 0$, the result is evident. Let us assume it is true for all ordinals $\beta < \alpha$ and prove it for α. If α is a limit ordinal, then $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$ and therefore I_α consists of zero divisors. Now let $\alpha = \beta+1$ be a nonlimit ordinal; then $I_\alpha = \sum_{f\in I_\beta} \mathrm{Ann}(\mathrm{Ann}(f))$. We must show that each element g of I_α is a zero divisor. Put $g = g_1 + \ldots + g_n$, where $g_i \in \mathrm{Ann}(\mathrm{Ann}(f_i))$, $f_i \in I_\beta$, $i = 1, \ldots, n$. But by the induction hypothesis each element of I_β is a zero divisor. Now since every finitely generated ideal in $C(X)$ consisting of zero divisors has a nonzero annihilator, there exists $0 \neq h \in \mathrm{Ann}(f_1C(X) + \ldots + f_nC(X))$, i.e., $gh = 0$.

Corollary 2.6. *Every maximal ideal in $C(X)$ consisting only of zero divisors is a* z°*-ideal.*

Corollary 2.7. *If I is an ideal in $C(X)$ consisting of zero divisors, then there is the smallest* z°*-ideal containing I and also there is a maximal* z°*-ideal containing I which is also a prime* z°*-ideal.*

The following shows that certain z-ideals in $C(X)$ are z°-ideals.

Proposition 2.8. (i) *Every finitely generated z-ideal* (*even a semiprime ideal*) *in $C(X)$ is a basic* z°*-ideal generated by an idempotent.*

(ii) *If X is compact*, *then every countably generated z-ideal in $C(X)$ is a* z°*-ideal.*

Proof. (i) is clear.

(ii) If I is a countably generated z-ideal in $C(X)$, where X is compact, then by the Corollary of the main Theorem in [4], we have $I = \bigcap_{p\in A} O_p$, where A is a zero set of X. But we have seen that each O_p is a z°-ideal, i.e., I is a z°-ideal.

Remark 2.9. In [4], De Marco has given a direct proof that every f.g. semiprime ideal in $C(X)$ is generated by an idempotent.

Next we give an algebraic characterization of basically and extremally disconnected spaces in terms of z°-ideals.

Theorem 2.10. (i) *Every basic* z°*-ideal in $C(X)$ is principal if and only if X is basically disconnected.*

(ii) *Every intersection of basic* z°*-ideals in $C(X)$ is principal if and only if X is extremally disconnected.*

Proof. (i) Suppose every basic z°-ideal is principal. We are to show that $\mathrm{int}\, Z(f)$ is closed for $f \in C(X)$. It suffices to prove this for $f \in C(X)$ which is a zero divisor, for if $\mathrm{Ann}(f) = (0)$, then $\mathrm{int}\, Z(f) = \emptyset$. Now let $P_f = (g)$ and $\mathrm{Ann}(f) \neq (0)$. Then by Proposition 2.8, we have $P_f = (e)$, where $e = e^2$. Hence $f \in (e)$ implies that $Z(e) \subseteq Z(f)$ and $e \in P_f$ implies that $\mathrm{int}\, Z(f) \subseteq \mathrm{int}\, Z(e) = Z(e)$. Hence $Z(e) = \mathrm{int}\, Z(f)$ is closed.

Conversely, let X be a basically disconnected space and $f \in C(X)$ with $\mathrm{Ann}(f) \neq (0)$. Then $F = \mathrm{int}\, Z(f) \neq \emptyset$ is a closed set. Now we may define $e \in C(X)$ with $e(F) = \{0\}$ and $e(X-F) = \{1\}$. Clearly $e = e^2$ and $P_f = (e)$, for we recall that $P_f = \{g \in C(X) : \mathrm{int}\, Z(f) \subseteq \mathrm{int}\, Z(g)\}$.

(ii) Suppose every intersection of basic z°-ideals is principal and G is an open set in X. Then there is $S \subseteq C(X)$ such that $G = \bigcup_{f\in S} \operatorname{int} Z(f)$ (see [5], p. 38). But by our hypothesis, there is $g \in C(X)$ such that $\bigcap_{f\in S} P_f = (g)$. Then (g) is a z°-ideal, i.e., (g) is a z-ideal and therefore by Proposition 2.8, $(g) = (e)$, where $e = e^2$. This shows that $Z(g) = Z(e)$ is open.

We now claim that $\operatorname{cl} G = Z(g)$. To see this, we note that $g \in P_f$ for all $f \in S$, i.e., $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g) \subseteq Z(g)$ for all $f \in S$. Hence $G \subseteq Z(g)$ implies that $\operatorname{cl} G \subseteq Z(g)$. Now suppose for contradiction that $x \in Z(g)$ and $x \notin \operatorname{cl} G$. Define $h \in C(X)$ with $h(\operatorname{cl} G) = \{0\}$, $h(x) = 1$, i.e., $\operatorname{int} Z(f) \subseteq Z(h)$ for all $f \in S$. Hence by the definition of P_f, we have $h \in P_f$ for all $f \in S$. This shows that $h \in \bigcap_{f\in S} P_f = (g)$. But $x \in Z(g)$ and $h(x) = 1$ imply that $Z(g) \nsubseteq Z(h)$, i.e., $h \notin (g)$, which is our desired contradiction.

Conversely, let X be an extremally disconnected space and let $I = \bigcap_{f\in S} P_f, S \subseteq C(X)$. Since $G = \operatorname{cl}(\bigcup_{f\in S} \operatorname{int} Z(f))$ is an open set, there exists an idempotent $e \in C(X)$ with $e(G) = \{0\}$ and $e(X - G) = \{1\}$. Clearly $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(e)$ for all $f \in S$, which means that $e \in P_f$ for all $f \in S$. Hence $(e) \subseteq I$ and we also claim that $I \subseteq (e)$. To show this, let $g \in I$; then $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$ for all $f \in S$, which means that $G \subseteq Z(g)$. Hence $g = ge$, i.e., $I \subseteq (e)$ and therefore $I = (e)$.

Remark 2.11. The previous result immediately shows that every prime z°-ideal in $C(X)$, where X is a basically disconnected space, is a minimal prime ideal. To see this, let $P \subseteq C(X)$ be a prime z°-ideal and $Q \subset P$ be any prime ideal. Then there exists $f \in P - Q$. By the proof of the previous result, $f \in P_f = (e) \subseteq P$, where $e = e^2$. Hence $e \notin Q$ implies that $1 - e \in Q \subseteq P$, which is impossible.

We recall that an element $f \in C(X)$ is a non-zero divisor if and only if $\operatorname{int} Z(f) = \emptyset$. We know that an ideal consisting entirely of zero divisors may not be a z°-ideal. For example, if $f \in C(\mathbb{R})$, where $f(x) = x$ for $x \leq 0$ and $f(x) = 0$ for $x > 0$, then the principal ideal (f) is not a z°-ideal (not even a z-ideal), but clearly $\operatorname{int} Z(f) \neq \emptyset$ or $\operatorname{Ann}(f) \neq (0)$, i.e., every member of (f) is a zero divisor.

The next result shows that in any space which is not a P-space there exists an example similar to the previous one. Before stating the result, we recall a definition and some well-known facts.

Definition. A completely regular space X is called an *almost P-space* if every non-empty zero set of X has a non-empty interior. This is equivalent to saying that every element of $C(X)$ is either a unit or a zero divisor (i.e., $C(X)$ is its own classical ring of quotients), or equivalently, for every $f \in C(X)$, $Z(f)$ is a regular closed set (see [10], [11]). Almost P-spaces were first introduced by A. I. Veksler in [11] as a generalization of P-spaces and were further studied by R. Levy in [10].

PROPOSITION 2.12. *The following statements are equivalent*:

(1) *X is a P-space.*

(2) *Every ideal in $C(X)$ consisting of zero divisors is a* z°*-ideal.*

(3) *Every nonunit element of $C(X)$ is a zero divisor and P_f is a principal ideal in $C(X)$ for all $f \in C(X)$.*

Proof. (1)⇒(2) is clear.

(2)⇒(3). Let f be a nonunit element in $C(X)$. First we show that f is a zero divisor. We may assume that $f \geq 0$ for otherwise we consider $|f|$ (note that f is a zero divisor if and only if $|f|$ is). If $f(X)$ is finite set, then it is clear that f is a zero divisor. Hence let f take the values $0 < a < b$. Now put $\{y \in X : f(y) \geq a\} = Z(g)$ and $\{z \in X : f(z) \leq a\} = Z(h)$ for some $g, h \in C(X)$. Clearly $g \neq 0 \neq h$ and $fgh = 0$ imply that $\mathrm{Ann}(fg) \neq (0)$. Hence by (2), (fg) is a z°-ideal and by Proposition 2.8, $(fg) = (e)$, where $e^2 = e$. Thus $Z(fg) = Z(e)$ is an open set. But $Z(fg) = Z(f) \cup Z(g)$ and $Z(f) \cap Z(g) = \emptyset$ imply that $Z(f) = Z(fg) - Z(g)$ is open, i.e., $\mathrm{Ann}(f) \neq (0)$, for we recall that $\mathrm{Ann}(f) \neq (0)$ if and only if $\operatorname{int} Z(f) \neq \emptyset$. Now by (2) again, (f) is a z°-ideal. But by the definition of z°-ideals, we have $f \in P_f \subseteq (f)$ and clearly $(f) \subseteq P_f$, i.e., $P_f = (f)$.

(3)⇒(1). Let $f \in C(X)$ be a nonunit element and $P_f = (g) \neq C(X)$ for some $g \in C(X)$. Then by Proposition 2.8, $P_f = (e)$, where $e = e^2$. But $f \in (e)$ implies that $Z(e) \subseteq Z(f)$ and $e \in P_f$ implies that $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(e) = Z(e)$, i.e., $Z(e) = \operatorname{int} Z(f)$. This shows that $\operatorname{int} Z(f) \neq \emptyset$ whenever $Z(f) \neq \emptyset$, i.e., X is an almost P-space. But in an almost P-space, we have $\mathrm{cl}(\operatorname{int} Z(f)) = Z(f)$, i.e., $Z(f) = Z(e)$ is an open set, which means that X is a P-space.

We recall that the sum of two z-ideals in $C(X)$ is either a z-ideal or $C(X)$ (see [5], p. 198). But the sum of two z°-ideals in $C(X)$ may be a proper ideal which is not a z°-ideal, for if $S = [0, \infty)$ and $T = (-\infty, 0]$, then M_S and M_T are z°-ideals in $C(\mathbb{R})$ (see Example (1) earlier in this section). But $M_S + M_T$ is not a z°-ideal, since the function $i \in C(\mathbb{R})$, where $i(x) = x$ for $x \in \mathbb{R}$, is in $M_S + M_T$, but clearly $\mathrm{Ann}(i) = (0)$. We also note that $M_S + M_T \subseteq M_0$, i.e., $M_S + M_T \neq C(X)$.

Next we are going to investigate topological spaces X such that the sum of two z°-ideals in $C(X)$ either is a z°-ideal or equals $C(X)$. For a similar result, see Theorem 4.4 in [7]. We have not been able to characterize all topological spaces such that the sum of any two z°-ideals is either a z°-ideal or $C(X)$.

PROPOSITION 2.13. *If X is a basically disconnected space*, *then the sum of two* z°*-ideals is either a* z°*-ideal or $C(X)$.*

Proof. Let I and J be two z°-ideals in $C(X)$ and suppose that $I + J \neq C(X)$. Let $f \in I + J$ and $\operatorname{int} Z(f) = \operatorname{int} Z(g)$ for some $g \in C(X)$. We are to show that $g \in I + J$. We have $f = k + h$, where $k \in I$ and $h \in J$. We may assume that $k \neq 0 \neq h$, for otherwise we clearly have $g \in I + J$. Now since X is a basically disconnected space, $\operatorname{int} Z(k)$ and $\operatorname{int} Z(h)$ are closed sets and since I and J are z°-ideal, we have $\operatorname{int} Z(k) \neq \emptyset \neq \operatorname{int} Z(h)$. Then we put $A = X - \operatorname{int} Z(k)$ and note that A and $\operatorname{int} Z(k)$ are two disjoint open and closed sets. Thus there exists $k' \in C(X)$ with $k'(A) = \{1\}$ and $k'(\operatorname{int} Z(k)) = \{0\}$. Therefore $Z(k') = \operatorname{int} Z(k)$. Similarly, there exists $h' \in C(X)$ with $Z(h') = \operatorname{int} Z(h)$. Since I and J are z°-ideals, we infer that $k' \in I$ and $h' \in J$. But $Z(k'^2 + h'^2) = \operatorname{int} Z(k) \cap \operatorname{int} Z(h)$ implies that $Z(k'^2 + h'^2) \subseteq \operatorname{int} Z(f) = \operatorname{int} Z(g)$. Now it is clear that g is a multiple of $k'^2 + h'^2$ (see [5], 1D), i.e., $g \in I + J$.

The next result, which is an algebraic characterization of almost P-spaces, immediately shows that the sum of z°-ideals in $C(X)$, where X is an almost P-space, is either a z°-ideal or $C(X)$.

THEOREM 2.14. *The following statements are equivalent*:

(1) *X is an almost P-space.*

(2) *Every* z*-ideal in $C(X)$ is a* z°*-ideal.*

(3) *Every maximal ideal* (*prime* z*-ideal*) *in $C(X)$ is a* z°*-ideal.*

(4) *Every maximal ideal in $C(X)$ consists entirely of zero divisors.*

(5) *The sum of any two ideals consisting of zero divisors either is $C(X)$ or consists of zero divisors.*

(6) *For each nonunit element $f \in C(X)$, there exists a nonzero $g \in C(X)$ with $P_f \subseteq \operatorname{Ann}(g)$.*

Proof. (1)⇒(2). Let I be a z-ideal and $\operatorname{int} Z(f) = \operatorname{int} Z(g)$, $f \in I$. Since X is an almost P-space, $Z(f) = \operatorname{cl}(\operatorname{int} Z(f)) = \operatorname{cl}(\operatorname{int} Z(g)) = Z(g)$, i.e., $f \in I$ implies that $g \in I$.

(2)⇒(3)⇒(4)⇒(5) are evident.

(5)⇒(1)⇒(6). Let $f \in C(X)$ be a nonunit element; we show that $\operatorname{int} Z(f) \neq \emptyset$. We may assume that $x, y \notin Z(f)$ with $x \neq y$. Now we define $g, h \in C(X)$ with $g, h \geq 0$ and $Z(g) \cap Z(h) = \emptyset$, where $g \in O_x$ and $h \in O_y$ (see [5], Theorem 1.15 and statement (b) on page 38). Hence (fg) and (fh) consist only of zero divisors and since $(fg) + (fh) \neq C(X)$, by (5) we have $\emptyset \neq \operatorname{int} Z(fg + fh) = \operatorname{int}(Z(f) \cup Z(g + h)) = \operatorname{int} Z(f)$. Next we observe that (1) clearly implies (6): let $0 \neq g \in \operatorname{Ann}(f)$, i.e., $f \in \operatorname{Ann}(g)$, which means that $P_f \subseteq \operatorname{Ann}(g)$.

(6)⇒(1). Let $P_f \subseteq \operatorname{Ann}(g)$, where f is a nonunit element of $C(X)$ and $0 \neq g \in C(X)$. Now $fg = 0$ implies that $X - Z(g) \subseteq \operatorname{int} Z(f) \neq \emptyset$. This means that X is an almost P-space.

It is easy to see that if every prime z°-ideal in $C(X)$ is maximal, then X is a P-space. This shows that for a non-P-space which is an almost P-space, there exists a prime z°-ideal in $C(X)$ which is not a maximal ideal. We also note that by Theorem 2.14, for an almost P-space X which is not a P-space, there exists a prime z°-ideal which is not a minimal prime ideal. Thus, the following questions are now in order.

Question 1. Does there exist an almost P-space X which is not a P-space and has the property that every prime z°-ideal in $C(X)$ is either a minimal prime ideal or a maximal ideal?

Question 2. Does there exist an almost P-space X with a prime z°-ideal in $C(X)$ which is neither a minimal prime ideal nor a maximal ideal?

It seems that the spaces we are after are rare animals indeed, but we are going to catch them in the next section.

3. Some peculiar almost P-spaces which are not P-spaces. We conclude this article with catching the rare animals we are after.

First we recall some well-known facts. We observe that if F and E are two distinct maximal chains of prime ideals in $C(\mathbb{R})$ such that $E \cap F$ is an infinite set (see [5], 14I.8), then $P = \bigcap F$ and $Q = \bigcap E$ are minimal prime ideals and $P \in F$ and $Q \in E$. This shows that $P + Q \in F \cap E$. To see this, we first observe that $P \neq Q$, for otherwise $E \cup F$ becomes a chain, which is impossible. Now if $P \supseteq A$ for some $A \in E \cap F$, then $P = A = Q$, which is absurd. Hence we must have $P \subseteq A$ for all $A \in E \cap F$, i.e., $P + Q \subseteq A$ for all $A \in E \cap F$. This means that $P + Q \neq C(X)$ is also a z-ideal (see [5], p. 198). But a z-ideal containing a prime ideal is a prime ideal, i.e., $P + Q$ is a prime ideal. This shows that $C(\mathbb{R})$ contains a prime z-ideal (namely, $P + Q$) which is neither a maximal nor a minimal prime ideal.

Now we construct an almost P-space Y which is not a P-space but there exists an epimorphism $\phi : C(Y) \to C(\mathbb{R})$. Then $\phi^{-1}(P+Q)$ is a prime z-ideal (i.e., a prime z°-ideal, by Theorem 2.14) in $C(Y)$ which is neither a maximal nor a minimal prime ideal. Of course, since ϕ is onto, the contractions of z-ideals in $C(\mathbb{R})$ are z-ideals in $C(Y)$. Thus, once we construct Y, we will have an affirmative answer to our second question. In what follows, we construct the space Y.

Let D be an uncountable discrete space and let $X = D \cup \{a\}$, $a \notin D$, be the one-point compactification of D. Clearly X is an almost P-space. Now put $Y = X \times \mathbb{R}$, and define a topology on Y as follows. Every basic neighborhood of $(a, r) \in Y$, $r \in \mathbb{R}$, is of the form $G \times H$, where G is an open set in X containing a and $H \subseteq \mathbb{R}$ is an open set containing r, and let the other points of Y be isolated, i.e., each (x, r) with $x \neq a$ and $r \in \mathbb{R}$ is isolated. Clearly Y is a locally compact space, i.e., Y is completely regular Hausdorff. We also observe that Y is an almost P-space, for if $\emptyset \neq Z \in Z(Y)$ and $(a, r) \notin Z$ for all $r \in \mathbb{R}$, then Z is an open set. Hence let $(a, r) \in Z$ for some $r \in \mathbb{R}$. Now if we put $X' = X \times \{r\}$, then X' is homeomorphic to X, i.e., X' is an almost P-space, and therefore $\mathrm{int}_{X'}(Z \cap X') \neq \emptyset$. In other words there is $x \neq a$ with $(x, r) \in Z \cap X'$, which means that $(x, r) \in \mathrm{int}\, Z \neq \emptyset$. Thus, Y is an almost P-space and it is clear that Y is not a P-space. We also note that $A = \{a\} \times \mathbb{R}$, which is homeomorphic to $\mathbb{R}$, is C-embedded in Y, for if $f : \{a\} \times \mathbb{R} \to \mathbb{R}$ is a continuous map we define $\bar{f} : Y \to \mathbb{R}$ by $\bar{f}(x, r) = f(a, r)$ for $x \in X$. Hence $\mathbb{R}$ is C-embedded in Y and the map $\phi : C(Y) \to C(\mathbb{R})$ defined by $\phi(g) = g|_{\mathbb{R}}$ is onto, and now we are through.

Finally, we are going to give our affirmative answer to the first question. Again let us recall some well-known facts. Let Y and Z be arbitrary topological spaces and let $D = Y + Z$ denote the disjoint union of Y and Z. Suppose that A is the subspace of D consisting of two points $y_0 \in Y$ and $z_0 \in Z$. Then the quotient space obtained from D by collapsing $A = \{y_0, z_0\}$ to a point is called the *one-point union* of Y and Z with base points $y_0 \in Y$ and $z_0 \in Z$, denoted by $Y \vee Z$. We can consider Y and Z as subspaces of $Y \vee Z$ in the obvious way, and $Y \vee Z$ can be considered as a subspace of $Y \times Z$ with the product topology by means of the embedding $i : Y \vee Z \to Y \times Z$ defined by $i(y) = (y, z_0)$ if $y \in Y$ and $i(z) = (y_0, z)$ if $z \in Z$. Note that $i(Y) \cap i(Z) = \{y_0, z_0\}$. Hence without losing generality, we may assume that $Y \vee Z = Y \cup Z$, where $Y \cap Z = \{a\}$ and $Y - \{a\}$ and $Z - \{a\}$ are open sets in $Y \vee Z$. Clearly if Y and Z are completely regular Hausdorff spaces, then so is $X = Y \vee Z$. We also note that Y and Z are C-embedded in X, for if $f \in C(Y)$, then we define $\bar{f} \in C(X)$ by $\bar{f}|_Y = f$ and $\bar{f}(x) = f(a)$ for $x \in Z$, and similarly for Z.

In what follows we always have $X = Y \vee Z$, i.e., $X = Y \cup Z$, $Y \cap Z = \{a\}$. We now have the following facts.

(1) If $X = Y \cup Z$ and $Y \cap Z = \{a\}$, then we define $\phi_1 : C(X) \to C(Y)$ by $\phi_1(f) = f|_Y$ and $\phi_2 : C(X) \to C(Z)$ by $\phi_2(f) = f|_Z$. It is easy to see that

$$O_a(X) = \phi_1^{-1}(O_a(Y)) \cap \phi_2^{-1}(O_a(Z)),$$

where for any space W and $a \in W$, $O_a(W) = \{f \in C(W) : a \in \mathrm{int}_W\, Z(f)\}$. When we write ϕ_i, $i = 1, 2$, we always mean these maps.

(2) $\beta X = \beta Y \cup \beta Z$ and $\beta Y \cap \beta Z = \{a\}$.

(3) If $a \neq p \in \beta X$, then either

$$C(X)/O^p(X) \cong C(Y)/O^p(Y) \quad \text{or} \quad C(X)/O^p(X) \cong C(Z)/O^p(Z).$$

To see this, note that $p \in \beta X$ implies $p \in \beta Y$ or $p \in \beta Z$. Let $p \in \beta Y$, and define $\theta : C(X) \to C(Y)/O^p(Y)$ by $\theta(f) = \phi_1(f) + O^p(Y)$. Clearly θ is onto and $\ker \theta = O^p(X)$.

(4) As we have noted in the previous fact, if $a \neq p \in \beta Y$, then there is a one-one correspondence between prime ideals in $C(X)$ containing $O^p(X)$,

and prime ideals in $C(Y)$ containing $O^p(Y)$, and there is a similar correspondence if $p \in \beta Z$. We also observe that if P is an ideal in $C(X)$ containing $O_a(X)$, then P is a prime ideal if and only if $P = \phi_1^{-1}(Q_1)$ or $P = \phi_2^{-1}(Q_2)$, where Q_1 and Q_2 are prime ideals in $C(Y)$ and $C(Z)$ containing $O_a(Y)$ and $O_a(Z)$ respectively. To see this, we recall that $O_a(X) = \phi_1^{-1}(O_a(Y)) \cap \phi_2^{-1}(O_a(Z)) \subseteq P$ and if P is a prime ideal, then either $\phi_1^{-1}(O_a(Y)) \subseteq P$ or $\phi_2^{-1}(O_a(Z)) \subseteq P$. Let $\phi_1^{-1}(O_a(Y)) \subseteq P$, i.e., $\phi_1(P) = Q_1$ is a prime ideal containing $O_a(Y)$, for $\ker \phi_1 \subseteq P$ and ϕ_1 is onto. It is clear that if P is a minimal prime ideal which is not maximal, then so is Q_1.

(5) Let X be any topological space, $p \in \beta X$ and $O^p(X) \subseteq I \neq M^p(X)$, where I is an ideal which is not a z°-ideal. Then there are $f \in I$ and $g \in M^p(X) - I$ with $\operatorname{int} Z(f) \subseteq \operatorname{int} Z(g)$. To see this, we use the fact that there are $f_1 \in I$ and $h \in C(X)$ with $\operatorname{int} Z(f_1) \subseteq \operatorname{int} Z(h)$ and $h \notin I$. If $h \in M^p(X)$, then we are through. Hence let $p \notin \operatorname{cl}_{\beta X} Z(h)$; then there is $k \in O^p(X)$ with $Z(k) \cap Z(h) = \emptyset$. Clearly $k^2 + f_1^2 \in I$ and $\operatorname{int} Z(k^2 + f_1^2) = \operatorname{int} Z(k) \cap \operatorname{int} Z(f_1) \subseteq \operatorname{int}(Z(h) \cap Z(k)) = \emptyset$. Hence it suffices to take $f = k^2 + f_1^2$ and let g be any element of $M^p(X) - I$.

Now we are ready to give our promised example.

Let Y be a nondiscrete P-space and y_0 be a nonisolated point in Y. Take $Z = \Sigma = \mathbb{N} \cup \{\sigma\}$, where $\sigma \notin \mathbb{N}$, and let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$. All points of $\mathbb{N}$ are isolated and the neighborhoods of σ are sets $G \cup \{\sigma\}$ for $G \in \mathcal{F}$ (see [5], 4M for some interesting properties of Z). Since Z is extremally disconnected, by Remark 2.11 every prime z°-ideal in $C(Z)$ is a minimal prime ideal. We also note that for each $p \neq \sigma$ in βZ, $M^p(Z)$ is a minimal prime ideal and $M^p(Z) = O^p(Z)$. Now let $X = Y \vee Z$ be the one-point union of Y and Z with base points $y_0 \in Y$ and $z_0 = \sigma \in Z$. Hence we may assume that $X = Y \cup Z$, $Y \cap Z = \{a\}$.

Now we claim that X is an almost P-space such that every prime z°-ideal is either a minimal prime ideal or a maximal ideal. Hence let $f \in C(X)$ be nonunit. Then since $a \in Y$ is a nonisolated point and Y is a P-space, we have $\{a\} \neq Z(f)$, i.e., there is $x \in X$ with $a \neq x \in Z(f)$. Now if $x \in Z$, then $x \neq \sigma$ and therefore $\{x\}$ is open in Z, i.e., $\{x\}$ is open in X. If $x \in Y$, then $x \neq y_0$ and $x \in \operatorname{int} Z(f')$, where $f' = f|_Y$. Clearly $\operatorname{int} Z(f') \cap (Y - \{a\}) \subseteq \operatorname{int} Z(f)$ and therefore in any case we have $\operatorname{int} Z(f) \neq \emptyset$, i.e., X is an almost P-space and clearly not a P-space, for $Z = \Sigma$ is not a P-space.

Finally, assume that P is a nonmaximal prime z°-ideal in $C(X)$. We then claim that P is a minimal prime ideal. By what we have already said in facts (1)–(5), $P = {\phi_1}^{-1}(Q_1)$ or $P = {\phi_2}^{-1}(Q_2)$, where Q_1 and Q_2 are prime ideals in $C(Y)$ and $C(Z)$ respectively. Since P is not maximal, $P \neq {\phi_1}^{-1}(Q_1)$, for otherwise Q_1 is maximal (note that Y is a P-space), and therefore P is a maximal ideal, which is absurd. Hence $P = {\phi_2}^{-1}(Q_2)$, i.e., Q_2 is a prime ideal which is not maximal. Now since Z is extremally disconnected, it is basically disconnected and therefore it suffices to show that $\phi_2(P)$ is a z°-ideal, for then, by Remark 2.11, $\phi_2(P)$ is a minimal prime ideal and therefore P is a minimal prime ideal.

Assume for contradiction that $\phi_2(P)$ is not a z°-ideal. It is clear that $O_x = M_x$ in $C(Z)$ for $x \neq \sigma$ and $O_a \subseteq \phi_2(P) \neq M_a$. But by fact (5), there are $f \in \phi_2(P)$ and $g \in M_a(Z) - \phi_2(P)$ with $\mathrm{int}_Z\, Z(f) \subseteq \mathrm{int}_Z\, Z(g)$. Now we have $\mathrm{int}_X\, Z(\bar{f}) \subseteq \mathrm{int}_X\, Z(\overline{g})$, where $\bar{f}|_Z = f$ and $\overline{g}|_Z = g$ and also $\bar{f}|_Y = 0 = \overline{g}|_Y$. But $\bar{f} \in \phi_2^{-1}(\phi_2(P)) = P$, which is impossible, for P is a z°-ideal, and $\mathrm{int}\, Z(\bar{f}) \subseteq \mathrm{int}\, Z(\overline{g})$ and $\bar{f} \in P$ imply that $\overline{g} \in P$, i.e., $g \in \phi_2(P)$, which is absurd.

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Mass., 1969.

[2] F. Azarpanah, *Essential ideals in* $C(X)$, Period. Math. Hungar. 3 (1995), no. 12, 105–112.

[3] —, *Intersection of essential ideals in* $C(X)$, Proc. Amer. Math. Soc. 125 (1997), 2149–2154.

[4] G. De Marco, *On the countably generated z-ideals of* $C(X)$, ibid. 31 (1972), 574–576.

[5] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer, 1976.

[6] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. 115 (1965), 110–130.

[7] C. B. Huijsmans and B. de Pagter, *On z-ideals and d-ideals in Riesz spaces. I*, Indag. Math. 42 (Proc. Netherl. Acad. Sci. A 83) (1980), 183–195.

[8] O. A. S. Karamzadeh, *On a question of Matlis*, Comm. Algebra 25 (1997), 2717–2726.

[9] O. A. S. Karamzadeh and M. Rostami, *On the intrinsic topology and some related ideals of* $C(X)$, Proc. Amer. Math. Soc. 93 (1985), 179–184.

[10] R. Levy, *Almost P-spaces*, Canad. J. Math. 2 (1977), 284–288.

[11] A. I. Veksler, *p'-points, p'-sets, p'-spaces. A new class of order-continuous measures and functions*, Soviet Math. Dokl. 14 (1973), 1445–1450.

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