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Partition properties of subsets of $\mathcal{P}_\kappa\lambda$

by

Masahiro Shioya (Tsukuba)

Abstract. Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. The following partition property is shown to be consistent relative to a supercompact cardinal: For any $f : \bigcup_{n<\omega}[X]^n_\subset \to \gamma$ with $X \subset \mathcal{P}_\kappa\lambda$ unbounded and $1 < \gamma < \kappa$ there is an unbounded $Y \subset X$ with $|f``[Y]^n_\subset| = 1$ for any $n < \omega$.

Let κ be a regular cardinal $> \omega$, λ a cardinal $\geq \kappa$ and F a filter on $\mathcal{P}_\kappa\lambda$. Partition properties of the form $\mathcal{P}_\kappa\lambda \to (F^+)^2_2$ (see below for the definition) were introduced by Jech [6]. The case where F is the club filter $\mathcal{C}_{\kappa\lambda}$ was particularly studied in connection with a supercompact cardinal: Menas [14] proved $\mathcal{P}_\kappa\lambda \to (\mathcal{C}^+_{\kappa\lambda})^2_2$ for a $2^{\lambda^{<\kappa}}$-supercompact κ via a normal ultrafilter U with $\mathcal{P}_\kappa\lambda \to (U^+)^2_2$. As noted by Kamo [9], Menas' argument can be modified to give the partition property of $\mathcal{P}_\kappa\lambda$ for κ just λ-supercompact. For the converse direction Di Prisco and Zwicker [4] and others refined the global result of Magidor [12]: The partition property of $\mathcal{P}_\kappa 2^{\lambda^{<\kappa}}$ implies that κ is λ-supercompact.

In [8] Johnson introduced properties of the form $X \to (F^+)^2_2$ for $X \in F^+$, which means that for any $f : [X]^2_\subset \to 2$ there is $Y \in F^+$ with $Y \subset X$ and $|f``[Y]^2_\subset| = 1$, as well as $F^+ \to (F^+)^2_2$, which means $X \to (F^+)^2_2$ for any $X \in F^+$. Abe [1] asked whether $\mathcal{F}^+_{\kappa\lambda} \to (\mathcal{F}^+_{\kappa\lambda})^2_2$ would fail in ZFC, where $\mathcal{F}_{\kappa\lambda}$ denotes the minimal fine filter on $\mathcal{P}_\kappa\lambda$.

In this note we answer the question of Abe:

Theorem. *Let κ be a supercompact cardinal and λ a cardinal $> \kappa$. Then there is a κ^+-c.c. poset forcing that κ is supercompact and $\mathcal{F}^+_{\kappa\lambda} \to (\mathcal{F}^+_{\kappa\lambda})^{<\omega}_\gamma$ for any $1 < \gamma < \kappa$.*

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Here $F^+ \to (F^+)^{<\omega}_\gamma$ means that for any $f : \bigcup_{n<\omega}[X]^n_\subset \to \gamma$ with $X \in F^+$ there is $Y \in F^+$ with $Y \subset X$ and $|f``[Y]^n_\subset| = 1$ for any $n < \omega$. Note that κ is Ramsey iff $\mathcal{F}^+_{\kappa\kappa} \to (\mathcal{F}^+_{\kappa\kappa})^{<\omega}_\gamma$ for any $1 < \gamma < \kappa$.

We generally follow the terminology of Kanamori [10] with the following exception: For a cardinal $\mu \ge \omega$ we set $[X]^\mu = \{x \subset X : |x| = \mu\}$, $[X]^{<\mu} = \{x \subset X : |x| < \mu\}$ and $\lim A = \{\alpha < \mu : \sup(A \cap \alpha) = \alpha > 0\}$ for $A \subset \mu$. We understand $\bigcup a \subsetneq \bigcap b$ whenever the union $a \cup b$ of $a \in [\mathcal{P}_\kappa\lambda]^m_\subset$ and $b \in [\mathcal{P}_\kappa\lambda]^n_\subset$ with $m, n < \omega$ is formed.

We first give two negative partition results, which motivated Abe's question. In [1] Abe proved $\mathcal{F}^+_{\kappa\lambda} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$ under $\lambda^{<\kappa} = 2^\lambda$. On the other hand, Matet [13], extending a result of Laver (see [7]), got the same conclusion from the opposite assumption:

Proposition 1. *Assume* $\lambda^\kappa = \lambda$. *Then* $\mathcal{F}^+_{\kappa\lambda} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$.

Proof. First set $\mathcal{P}_\kappa\lambda = \{x_\xi : \xi < \lambda\}$ and $[\mathcal{P}_\kappa\lambda]^\kappa = \{Y_\alpha : \alpha < \lambda\}$. By induction on $\xi < \lambda$ we construct $z_\xi \in \mathcal{P}_\kappa\lambda$ and $\{y^{\alpha i}_\xi : \alpha \in z_\xi \wedge i < 2\}$ so that $x_\xi \subset z_\xi$, $z_\xi \ne z_\zeta$, $y^{\alpha i}_\xi \in Y_\alpha$, $y^{\alpha i}_\xi \subsetneq z_\xi$ and $y^{\alpha 0}_\xi \ne y^{\beta 1}_\xi$ for any $\zeta < \xi$, $i < 2$ and $\alpha, \beta \in z_\xi$ as follows: At stage $\xi < \lambda$ by induction on $n < \omega$ build $z_{\xi n} \in \mathcal{P}_\kappa\lambda$ and $\{y^{\alpha i}_\xi : \alpha \in z_{\xi n} \wedge i < 2\}$ so that $x_\xi \subset z_{\xi 0} \not\subset \bigcup_{\zeta<\xi} z_\zeta$, $y^{\alpha i}_\xi \in Y_\alpha$, $y^{\alpha 0}_\xi \ne y^{\beta 1}_\xi$ and $z_{\xi n} \cup \bigcup\{y^{\alpha i}_\xi : \alpha \in z_{\xi n} \wedge i < 2\} \subsetneq z_{\xi n+1}$. Finally set $z_\xi = \bigcup_{n<\omega} z_{\xi n}$. We claim that f defined by $f(\{y^{\alpha i}_\xi, z_\xi\}) = i$ witnesses $\{z_\xi : \xi < \lambda\} \not\to (\mathcal{F}^+_{\kappa\lambda})^2_2$.

Fix an unbounded set $X \subset \{z_\xi : \xi < \lambda\}$. We show $f``[X]^2_\subset = 2$. Take $\alpha < \lambda$ with $Y_\alpha \in [X]^\kappa$, and $\xi < \lambda$ with $\alpha \in z_\xi \in X$. Then $f(\{y^{\alpha i}_\xi, z_\xi\}) = i$ for $i < 2$ by definition, as desired. ∎

The above proof yields in fact for any $\gamma < \kappa$ an unbounded set $X \subset \mathcal{P}_\kappa\lambda$ and $f : [X]^2_\subset \to \gamma$ such that $f``[Y]^2_\subset = \gamma$ for any unbounded $Y \subset X$.

The analogous problem for the club filter has been solved by Abe [2] via an extension of Magidor's theorem [12]: $\mathcal{C}^+_{\kappa\lambda} \not\to (\mathcal{C}^+_{\kappa\lambda})^2_2$. Let us give a canonical witness to his observation by appealing to Magidor's idea more directly:

Proposition 2. *Let* $\mu < \kappa$ *be regular. Then* $\{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(x \cap \kappa) = \mu\} \not\to (\mathcal{C}^+_{\kappa\lambda})^2_2$.

Proof. Set $S = \{x \in \mathcal{P}_\kappa\lambda : \mathrm{cf}(x \cap \kappa) = \mu\}$ and for $x \in S$ fix an unbounded set $c_x \subset x \cap \kappa$ of order type μ. For $\{x, y\} \in [S]^2_\subset$ let $f(\{x, y\})$ be 0 when $\min(c_x \Delta c_y) \in c_x$, and 1 otherwise. Fix a stationary set $T \subset S$. We show $f``[T]^2_\subset = 2$.

First, we have $\gamma < \kappa$ such that for any $w \in \mathcal{P}_\kappa\lambda$ there are $w \subset x, y \in T$ with $\gamma \in c_x - c_y$: Let $g : \kappa \to \mathcal{P}_\kappa\lambda$ witness the contrary, i.e. $\gamma \in c_x$ iff $\gamma \in c_y$ for any $\gamma < \kappa$ and $g(\gamma) \subset x, y \in T$. Take $x, y \in C(g) \cap T$ with $x \cap \kappa < y \cap \kappa$

by the stationarity of $\{z \cap \kappa : z \in C(g) \cap T\}$ in κ. Then $c_x = c_y \cap x \cap \kappa$ has order type μ, contradicting the choice of c_y.

Now, let $\gamma < \kappa$ be minimal as above. Then for $\alpha < \gamma$ we have $w_\alpha \in \mathcal{P}_\kappa\lambda$ such that $\alpha \in c_x$ iff $\alpha \in c_y$ for any $w_\alpha \subset x, y \in T$. Set $w = \bigcup_{\alpha<\gamma} w_\alpha \in \mathcal{P}_\kappa\lambda$. Take $w \subset x \subset y \subset z$ from T with $\gamma \in c_x \cap c_z - c_y$. Then $\min(c_x \Delta c_y) = \min(c_y \Delta c_z) = \gamma$ by $w_\alpha \subset x \subset y \subset z$ for any $\alpha < \gamma$, and hence $f(\{x, y\}) = 0$ and $f(\{y, z\}) = 1$ by definition, as desired. ∎

The rest of the paper is devoted to establishing our Theorem. We refer to Baumgartner's expository paper [3] for the rudiments of iterated forcings. We call a poset κ-*centered closed* when any centered subset of size $< \kappa$ has a lower bound.

Assume for the moment that κ is a compact cardinal and $\lambda \le 2^\kappa$. Fix a coloring $f : \bigcup_{n<\omega}[S]^n_\subset \to \gamma$ with $S \subset \mathcal{P}_\kappa\lambda$ unbounded and $1 < \gamma < \kappa$. Our definition of the poset Q_f below owes much to Galvin (see [7]), who proved under $\mathrm{MA}(\lambda)$ that for any $f : [X]^2_\subset \to 2$ with $X \subset [\lambda]^{<\omega}$ cofinal there is a cofinal $Y \subset X$ with $|f``[Y]^2_\subset| = 1$.

Fix a fine ultrafilter U on S and define inductively a κ-complete ultrafilter U_n on $[S]^n_\subset$ by $U_0 = \{\{\emptyset\}\}$ and $U_{n+1} = \{X : \{x : \{a : \{x\} \cup a \in X\} \in U_n\} \in U\}$. For $n < \omega$ let β_n be the unique $\beta < \gamma$ with $\{a \in [S]^n_\subset : f(a) = \beta\} \in U_n$. Let $Q_f = \{p \in [S]^{<\kappa} : \forall m, n < \omega \forall a \in [p]^m_\subset(\{b \in [S]^n_\subset : f(a \cup b) = \beta_{m+n}\} \in U_n)\}$, and $q \le p$ iff $q \supset p$ and $y \not\subset x$ for any $x \in p$ and $y \in q - p$. Let us observe some basic properties of Q_f.

First, for a generic filter $G \subset Q_f$, $\bigcup G$ is unbounded in $\mathcal{P}_\kappa\lambda$ by the density of $\{q \in Q_f : \exists y \in q(x \subset y)\}$ for any $x \in \mathcal{P}_\kappa\lambda$, and homogeneous for f: $f``[\bigcup G]^n_\subset = \{\beta_n\}$ for any $n < \omega$.

Next, we have the κ-centered closure of Q_f: $\bigcup D$ is a lower bound of a centered set $D \in [Q_f]^{<\kappa}$.

Finally, we invoke an argument of Engelking and Karłowicz [5] to show that Q_f is κ-linked. Fix an injection $\pi : \mathcal{P}_\kappa\lambda \to {}^\kappa 2$. For $A \subset {}^\alpha 2$ with $\alpha < \kappa$ set $Q_{f,A} = \{p \in Q_f : \{\pi(x)|\alpha : x \in p\} = A \wedge \langle \pi(z)|\alpha : z \in \bigcup_{x\in p} \mathcal{P}x\rangle$ is injective$\}$. Then $Q_f = \bigcup\{Q_{f,A} : \exists \alpha < \kappa(A \subset {}^\alpha 2)\}$ by the inaccessibility of κ. To see that $Q_{f,A}$ is linked, fix $p, q \in Q_{f,A}$. Then $x \not\subset y$ for any $x \in p - q$ and $y \in q$: Otherwise we would have $x = z$ for some $x \in p - q$, $y \in q$ with $x \subset y$ and $z \in q$ with $\pi(x)|\alpha = \pi(z)|\alpha$. Similarly, $y \not\subset x$ for any $x \in p$ and $y \in q - p$. Thus $p \cup q \le p, q$, as desired.

Before starting the proof of our Theorem, we need to generalize a result of Baumgartner [3]:

LEMMA. *Assume $2^{<\kappa} = \kappa$. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \beta\rangle$ be a $<\kappa$-support iteration such that $\Vdash_\alpha$ "$\dot{Q}_\alpha$ is κ-centered closed and κ-linked" for any $\alpha < \beta$. Then P_β is κ-directed closed and κ^+-c.c.*

Proof. It is easily seen that the κ-centered closure implies the κ-directed closure, which is preserved by $<\kappa$-support iterations.

To see the κ^+-c.c., fix $X \in [P_\beta]^{\kappa^+}$. For $\alpha < \beta$ let $\Vdash_\alpha$ "$\dot{Q}_\alpha = \bigcup_{\gamma<\kappa} \dot{Q}_{\alpha\gamma}$ with $\dot{Q}_{\alpha\gamma}$ linked for any $\gamma < \kappa$". For $p \in X$ by induction on $\xi < \kappa$ build $p_\xi \le p$, $\alpha^p_\xi \in \mathrm{supp}(p_\xi)$ and $\gamma^p_\xi < \kappa$ so that $p_\xi \le p_\zeta$ for any $\zeta < \xi$, $p_{\xi+1}|\alpha^p_\xi \Vdash_{\alpha^p_\xi}$ "$p_\xi(\alpha^p_\xi) \in \dot{Q}_{\alpha^p_\xi\gamma^p_\xi}$", and $\{\xi < \kappa : \alpha^p_\xi = \alpha\}$ is unbounded for any $\alpha \in \bigcup_{\zeta<\kappa} \mathrm{supp}(p_\zeta)$. Take $Y \in [X]^{\kappa^+}$ and $\delta < \kappa$ so that $\delta \in \Delta_{\zeta<\kappa} \bigcap\{\lim\{\xi < \kappa : \alpha^p_\xi = \alpha\} : \alpha \in \mathrm{supp}(p_\zeta)\}$ for any $p \in Y$. Note that $\{\alpha^p_\xi : \xi < \delta\} = \bigcup_{\zeta<\delta} \mathrm{supp}(p_\zeta)$ for any $p \in Y$. Next take $Z \in [Y]^{\kappa^+}$ so that $\{\{\alpha^p_\xi : \xi < \delta\} : p \in Z\}$ forms a Δ-system with root $d \in [\beta]^{<\kappa}$. Finally, take $W \in [Z]^{\kappa^+}$ and $H \in [\delta \times d \times \kappa]^{<\kappa}$ so that $\{(\xi, \alpha^p_\xi, \gamma^p_\xi) : \xi < \delta \wedge \alpha^p_\xi \in d\} = H$ for any $p \in W$. We show that W is linked, as desired.

Fix $p, q \in W$. Inductively we build a lower bound $r \in P_\beta$ of $\{p_\xi : \xi < \delta\} \cup \{q_\xi : \xi < \delta\}$ with support $\bigcup_{\zeta<\delta} \mathrm{supp}(p_\zeta) \cup \bigcup_{\zeta<\delta} \mathrm{supp}(q_\zeta)$. At stage $\alpha < \beta$ we claim that $\{\xi < \delta : r|\alpha \Vdash_\alpha$ "$p_\xi(\alpha) \parallel q_\xi(\alpha)$"$\}$ is unbounded, which implies $r|\alpha \Vdash_\alpha$ "$\{p_\xi(\alpha) : \xi < \delta\} \cup \{q_\xi(\alpha) : \xi < \delta\}$ is centered", as desired, since $r|\alpha \Vdash_\alpha$ "$\{p_\xi(\alpha) : \xi < \delta\}$ and $\{q_\xi(\alpha) : \xi < \delta\}$ are descending". Let us concentrate on the nontrivial case where $\alpha \in d = \bigcup_{\zeta<\delta} \mathrm{supp}(p_\zeta) \cap \bigcup_{\zeta<\delta} \mathrm{supp}(q_\zeta)$.

Fix $\xi < \delta$ with $\alpha^p_\xi = \alpha$. Then $r|\alpha \le p_{\xi+1}|\alpha, q_{\xi+1}|\alpha$ forces "$p_\xi(\alpha), q_\xi(\alpha) \in \dot{Q}_{\alpha\gamma}$", where $(\xi, \alpha, \gamma) \in H$. Now the claim follows, since $\{\xi < \delta : \alpha^p_\xi = \alpha\}$ is unbounded by the choice of δ. ■

Proof of Theorem. First, we force with the Laver poset [11] for κ and then add λ Cohen subsets of κ to ensure that κ is supercompact and $\lambda \le 2^\kappa$ in the further extensions. Next, we perform a $<\kappa$-support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < 2^{\lambda^{<\kappa}}\rangle$ with $\Vdash_\alpha$ "$\dot{Q}_\alpha = Q_{\dot{f}}$" for some canonical P_α-name $\dot{f}$ for a coloring. The standard inductive argument, together with the κ-closure and the κ^+-c.c. of P_α, shows that for any $\alpha < 2^{\lambda^{<\kappa}}$, P_α is of size $\le 2^{\lambda^{<\kappa}}$, and so is the set of canonical P_α-names for colorings, whose union can be identified with that of canonical $P_{2^{\lambda^{<\kappa}}}$-names for colorings. Thus the iteration can be arranged so that a homogeneous set for a coloring in the final model by $P_{2^{\lambda^{<\kappa}}}$ appears in an intermediate model, which, by absoluteness of $\mathcal{P}_\kappa\lambda$, remains unbounded, as desired. ■

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Institute of Mathematics
University of Tsukuba
Tsukuba, 305-8571 Japan
E-mail: shioya@math.tsukuba.ac.jp

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