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On decompositions of Banach spaces into a sum of operator ranges

by

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Dedicated to M. I. Kadets on the occasion of his seventy-fifth birthday

Abstract. It is proved that a separable Banach space X admits a representation $X = X_1 + X_2$ as a sum (not necessarily direct) of two infinite-codimensional closed subspaces X_1 and X_2 if and only if it admits a representation $X = A_1(Y_1) + A_2(Y_2)$ as a sum (not necessarily direct) of two infinite-codimensional operator ranges. Suppose that a separable Banach space X admits a representation as above. Then it admits a representation $X = T_1(Z_1) + T_2(Z_2)$ such that neither of the operator ranges $T_1(Z_1)$, $T_2(Z_2)$ contains an infinite-dimensional closed subspace if and only if X does not contain an isomorphic copy of l_1.

In this paper we consider a decomposition of a separable Banach space X into a sum (*not necessarily direct*) of two operator ranges. By an *operator range* $A(Y) \subset X$ we mean the image of some Banach space Y under a bounded linear operator $A : Y \to X$. Since every operator range is the range of some one-to-one operator (by passing to the quotient space) we may assume without loss of generality that A is a one-to-one (injective) operator.

There are two cases (known to the authors) where a decomposition into a sum of operator ranges actually gives a decomposition into a sum of subspaces (by a subspace we mean a *closed* linear manifold). First if a Banach space is a *direct* sum of two operator ranges then, by using the Closed Graph Theorem, it is easy to show that both operator ranges are subspaces. The second case is the following. Let $H = A_1(H) + A_2(H)$ be a decomposition of a Hilbert space H into a sum (not necessarily direct) of two ranges of operators on the same Hilbert space. Then there exist subspaces N and M with $N \subset A_1(H), M \subset A_2(H)$ and $N \cap M = 0$ such that $H = N + M$ (see [1]).

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In this paper we are interested only in decompositions of the kind

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(1)\qquad \begin{aligned} &X = A_1(Y_1) + A_2(Y_2),\\ &\dim A_1(Y_1) = \operatorname{codim} A_1(Y_1) = \dim A_2(Y_2) = \operatorname{codim} A_2(Y_2) = \infty, \end{aligned}
$$

of a separable Banach space X into a sum of two operator ranges $A_1(Y_1)$, $A_2(Y_2)$ without the assumption that $A_1(Y_1) \cap A_2(Y_2) = 0$. It turns out that a decomposition as in (1) exists for a given separable Banach space if and only if it is possible to decompose the space into a sum of two infinite-dimensional and infinite-codimensional subspaces (Theorem 2.1). Let us note that not every separable Banach space has a decomposition (1). For instance, the dual to the Gowers–Maurey space [4] has no decomposition (1) (see Example 2.2 below).

Now assume that a given separable Banach space admits a decomposition into a sum of two operator ranges. Is it necessary that (at least) one of them contains an infinite-dimensional subspace? The answer (Theorem 3.1) depends on whether the space contains an isomorphic copy of l_1.

The paper is organized as follows. In Section 1 we show that every infinite-codimensional operator range can be covered by the linear span of the union of an infinite-codimensional subspace and a compact set.

Section 2 is devoted to the proof of Theorem 2.1. The proof is based on the results of Section 1. By using Theorem 2.1 we show that the dual space Z^* to the Gowers–Maurey space cannot be decomposed into a sum (1).

In Section 3 we introduce the notion of a "thin" operator range $A(Y)$, by which we mean that $A(Y)$ does not contain any infinite-dimensional subspace, and we prove that a separable Banach space X with the decomposition property (1) can be decomposed into a sum of two thin operator ranges if and only if X does not contain an isomorphic copy of the space l_1.

Section 4 contains some results on complemented operator ranges.

We use standard Banach space theory notation [6]. By $U(E)$ we denote the unit ball of a normed space E. For a subset $A \subset E$ of a normed space E we denote by $[A]$ the closed linear hull of A, by $\operatorname{co} A$ the convex hull, by $\operatorname{lin} A$ the linear hull of A and by $\operatorname{cl} A$ the closure of A in the norm topology. Most results of this paper were announced in [3].

1. Covering of an operator range. We begin with two auxiliary lemmas. The proofs are standard and we omit them.

LEMMA 1.1. *Let X be a Banach space, $A(Y) \subset X$ be an operator range of infinite-codimension in X and K be a compact set in X. Then*

$$\operatorname{codim}(\operatorname{lin}\{K \cup A(Y)\}) = \infty.$$

LEMMA 1.2. *Let $A : Y \to X$ be an injective operator with* $\operatorname{codim} A(Y) = \infty$. *Then* $\operatorname{codim}(\operatorname{lin} \operatorname{cl} AU(Y)) = \infty$.

PROPOSITION 1.3. *Let $A : Y \to X$ be a dense injection of a Banach space Y into a separable Banach space X such that* $\operatorname{codim} A(Y) = \infty$. *Then there exist a subspace $X_1 \subset X$ with* $\operatorname{codim} X_1 = \infty$ *and a complete minimal sequence $\{x_i\}$ in X such that $\|x_i\| < 2^{-i}$ and*

$$A(Y) \subset \operatorname{lin}\{X \cup \operatorname{clco}\{\pm x_i\}\}.$$

Proof. Put $V = \operatorname{cl} AU(Y)$ and let Z be the Banach space $\operatorname{lin} V$ with the norm generated by the set V as the unit ball. Denote by $T : Z \to X$ the natural embedding. By Lemma 1.2, $\operatorname{codim} T(Z) = \infty$. Since $T(U(Z))$ is closed, the linear manifold $T^*(X^*)$ is 1-norming and by [2, Proposition, p. 279], there exists a fundamental minimal system $\{x_i\} \subset T(Z)$ with a conjugate system $\{h_i\} \subset X^*$ such that $\|x_i\| \le 2^{-i}$ and $\|T^* h_i\| \le 2^{-i}, i = 1, 2, \ldots$ Put $X_1 = [h_i]^{\top}$. Suppose $y \in U(Z)$, $x = \sum h_i(Ty)x_i$ and $K = \operatorname{clco}\{\pm x_i\}$. It is obvious that $x \in K$ and $Ty - x \in X_1$. So $T(Z) \subset \operatorname{lin}(K \cup X_1)$. Since $A(Y) \subset T(Z)$, we get $A(Y) \subset \operatorname{lin}(K \cup X_1)$. The proof is complete.

Recall that two infinite-dimensional subspaces N, M of a Banach space X are said to be *quasicomplemented* if $N \cap M = 0$ and $N + M$ is a non-closed dense (in X) linear manifold.

Let H be a Hilbert space. It is known [1] that every dense operator range $A(H) \subset H$ which contains an infinite-dimensional subspace can be represented as a sum

$$A(H) = N + M,$$

where N, M are quasicomplemented.

For a separable Banach space the following statement is true.

PROPOSITION 1.4. *Every operator range $A(Y) \subset X$ in a separable Banach space X is contained in a sum of two quasicomplemented subspaces.*

Proof. We take up the approach and notation of the proof of Proposition 1.3. Let $A : Y \to X$ be an injection and $\{z_n\}$, with $\|z_n\| = 1$, be a basic sequence in X_1 with basic constant less than 2. Put

$$u_i = z_i + 0.2x_i, \quad (i = 1, 2, \ldots).$$

By the stability theorem, $\{u_i\}$ is also a basic sequence. Put $X_2 = [u_i]$. Since $\{x_i\}$ is complete in X, the linear manifold $X_1 + X_2$ is dense in X. Let us verify that $X_1 \cap X_2 = 0$. Indeed, let $x \in X_1 \cap X_2$. Then $x = \sum \xi_i(z_i + 0.2x_i) \in X_1$. Thus $\xi_i = 0$ and $x = 0$. Standard considerations show that $X_1 + X_2$ is not closed. Hence, X_1 and X_2 are quasicomplemented and it follows from Proposition 1.3 that $X_1 + X_2 \supset A(Y)$. The proof is complete.

2. Decomposition into a sum of subspaces. The simplest example of an operator range is a subspace. In this section we show that a Banach space is decomposable into a sum of two operator ranges if and only if it may be decomposed into a sum of two subspaces.

THEOREM 2.1. *Let X be a separable Banach space. The following assertions are equivalent*:

(i) *There exist infinite-dimensional and infinite-codimensional subspaces X_1 and X_2 in X such that $X = X_1 + X_2$.*

(ii) *There exist operator ranges $A_1(Y_1)$ and $A_2(Y_2)$ such that the representation* (1) *is valid.*

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i). By Proposition 1.3 there exist subspaces X_1, X_2 of X with $\operatorname{codim} X_1 = \operatorname{codim} X_2 = \infty$ and compact subsets K_1, K_2 of X such that

$$A_1(Y_1) \subset \operatorname{lin}\{K_1 \cup X_1\}, \qquad A_2(Y_2) \subset \operatorname{lin}\{K_2 \cup X_2\}.$$

Put

$$K = K_1 \cup K_2, \qquad Y = X_1 \oplus X_2$$

and define an operator $T : Y \to X$ by the formula

$$T(x_1, x_2) = x_1 + x_2, \qquad (x_1, x_2) \in Y.$$

Then $X_1 + X_2 = T(Y)$ and $X = \operatorname{lin}\{K \cup T(Y)\}$. By Lemma 1.1, we have $\operatorname{codim}(X_1 + X_2) < \infty$, and so there exists a finite-dimensional subspace $X_3 \subset X$ such that

$$X = X_3 \dotplus (X_1 + X_2).$$

Set $N = X_3 + X_1$ and $M = X_2$ to obtain $X = M + N$, where $\operatorname{codim} N = \operatorname{codim} M = \infty$. The proof is complete.

EXAMPLE 2.2. There exists a separable reflexive Banach space X which does not admit any representation (1). Indeed, let Z be the reflexive hereditarily indecomposable separable Banach space constructed by Gowers and Maurey [4], that is, Z has the following property: there is no pair of infinite-dimensional subspaces $Z_1, Z_2 \subset Z$ such that $Z_1 \cap Z_2 = 0$ and $Z_1 + Z_2$ is closed. Put $X = Z^*$. So $X^* = Z^{**} = Z$. Assume that a representation (1) for X is valid. By Theorem 2.1 there exist infinite-dimensional and infinite-codimensional subspaces N, M of X such that $X = N + M$. Let $Y = N \oplus M$ and $T : Y \to X$ be the natural onto mapping (that is, $T(N) = N$ and $T(M) = M$). Then $T^* : Z \to Y^* = N^* \oplus M^*$ is an isomorphic embedding. Since $T^*(N^\perp) \subset M^*$ and $T^*(M^\perp) \subset N^*$ we have $\dim T^*(Z) \cap N^* = \dim T^*(Z) \cap M^* = \infty$. But $T^*(Z) \cap N^* + T^*(Z) \cap M^*$ is closed and we have a contradiction to the main property of the space Z.

3. Decomposition of a Banach space into a sum of thin operator ranges

DEFINITION 3.1. An operator range $A(Y)$ in a Banach space X is said to be *thin* if it does not contain any infinite-dimensional subspace.

THEOREM 3.2. *Let X be a separable Banach space with a decomposition* (1). *Then the following assertions are equivalent*:

(i) *There exists a decomposition of X into a sum of operator ranges*

$$X = T_1(Y_1) + T_2(Y_2)$$

such that both $T_1(Y_1)$ and $T_2(Y_2)$ are thin.

(ii) *X does not contain an isomorphic copy of l_1.*

Proof. (ii)$\Rightarrow$(i). By Theorem 2.1 we may write $X = E_1 + E_2$, where E_1 and E_2 are infinite-codimensional subspaces of X. Let us denote by l_1' and l_1'' two isometric copies of the space l_1. It is well known ([6, p. 108]) that there exist surjective operators

$$A_1 : l_1' \to E_1, \qquad A_2 : l_1'' \to E_2.$$

Denote by $\{e_i'\}$ and $\{e_i''\}$ the canonical bases of l_1' and l_2'' respectively. Put $x_i = A_1 e_i' \in E_1$ and $y_i = A_2 e_i'' \in E_2$ $(i = 1, 2, \ldots)$. Let $\{u_i\}$, $\|u_i\| = 1$, $i = 1, 2, \ldots$, be a basic sequence in E_1 with $[u_i]_{i=1}^\infty \cap E_2 = 0$ and $\{v_i\}$, $\|v_i\| = 1$, $i = 1, 2, \ldots$, be a basic sequence in E_2 such that $[v_i]_{i=1}^\infty \cap E_1 = 0$. Set $Z = (l_1' \oplus l_1'')_1$ and define the operators $T_1 : Z \to X$ and $T_2 : Z \to X$ by

$$T_1(a, b) = \sum \xi_i(x_i + 2^{-i} v_i) + \sum \mu_i 2^{-i} u_i,$$
$$T_2(a, b) = \sum \xi_i(y_i + 2^{-i} u_i) + \sum \mu_i 2^{-i} v_i,$$

where $(a, b) \in Z$, $a = \sum \xi_i e_i'$, $b = \sum \mu_i e_i''$.

We claim that T_1 and T_2 are injective. Indeed, if

$$T_1(a, b) = \sum \xi_i(x_i + 2^{-i} v_i) + \sum \mu_i 2^{-i} u_i = 0,$$

then

$$\sum (\xi_i x_i + \mu_i 2^{-i} u_i) + \sum \xi_i 2^{-i} v_i = 0.$$

But $\sum(\xi_i x_i + \mu_i 2^{-i} u_i) \in E_1$ and $E_1 \cap [v_i] = \{0\}$. Thus $\sum(\xi_i x_i + \mu_i 2^{-i} u_i) = 0$ and $\sum \xi_i 2^{-i} v_i = 0$. Since $\{u_i\}$ and $\{v_i\}$ are basic sequences, it follows that $\xi_i = \mu_i = 0$, $i = 1, 2, \ldots$, which proves that T_1 is injective. The same considerations show that T_2 is also injective. Since X does not contain a copy of l_1, neither $T_1(Z) \subset X$ nor $T_2(Z) \subset X$ contains an infinite-dimensional subspace of X.

Now we show that
$$T_1(Z)+T_2(Z)=X.$$
Take $u\in X$, $u=x+y$, where $x\in E_1$ and $y\in E_2$. Since A_1, A_2 are surjective operators, we have
$$x=\sum \xi_i x_i \quad\text{and}\quad y=\sum \nu_i y_i.$$
Put
$$w_1=\Big(\sum \xi_i e_i',-\sum \nu_i e_i''\Big),\qquad w_2=\Big(\sum \nu_i e_i',-\sum \xi_i e_i''\Big).$$
Then $u=T_1w_1+T_2w_2$ and this concludes the proof (ii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii). Suppose the contrary, i.e. X contains a copy of l_1. Let $X=T_1(Y_1)+T_2(Y_2)$, where $T_i:Y\to X$ $(i=1,2)$ are injective operators and both $T_1(Y_1)$ and $T_2(Y_2)$ are thin. Put $Y=(Y_1\oplus Y_2)_{l_1}$ and let $T:Y\to X$ be defined by
$$T(x_1,x_2)=T_1x_1+T_2x_2,$$
where $(x_1,x_2)\in Y$.

It is clear that $T(Y)=X$. Let $\{e_i\}\subset S(X)$ be the canonical basis of $l_1\subset X$. Put $Z=T^{-1}(l_1)$. Then $T|_Z:Z\to l_1$ is an onto mapping, and so there exists a bounded sequence $\{z_i\}\subset Z$ with $\|z_i\|\le C$, $i=1,2,\ldots$, such that $Tz_i=e_i$, $i=1,2,\ldots$ It is clear that

(2) $$z_i=x_i+y_i,\quad x_i\in Y_1,\ y_i\in Y_2,\ \|x_i\|,\|y_i\|\le C,\ i=1,2\ldots$$

Let us consider two cases.

Case 1: For all $\varepsilon>0$ and n there are m and $\{\xi_i\}_{i=n+1}^{n+m}$ such that

(3) $$\sum_{i=n+1}^{n+m}|\xi_i|=1,\qquad \Big\|T\Big(\sum_{i=n+1}^{n+m}\xi_i x_i\Big)\Big\|<\varepsilon.$$

Let $0<\varepsilon_n<1/2$, $n=1,2,\ldots$ By using condition (3) for $n=1,2,\ldots$, we construct a sequence of integers $m_1<m_2<\ldots$ and a sequence of numbers $\{\xi_i\}$ such that

(4) $$\sum_{i=m_n+1}^{m_{n+1}}|\xi_i|=1,\qquad \Big\|T\Big(\sum_{i=m_n+1}^{m_{n+1}}\xi_i x_i\Big)\Big\|<\varepsilon_n.$$

Put
$$u_n=T\Big(\sum_{i=m_n+1}^{m_{n+1}}\xi_i(x_i+y_i)\Big)=\sum_{i=m_n+1}^{m_{n+1}}\xi_i e_i,\qquad n=1,2,\ldots$$

It is clear that the sequence $\{u_n\}$ is equivalent to the canonical basis of l_1. By using the inequality

$$\Big\|u_n-T\Big(\sum_{i=m_n+1}^{m_{n+1}}\xi_i y_i\Big)\Big\|=\Big\|T\Big(\sum_{i=m_n+1}^{m_{n+1}}\xi_i x_i\Big)\Big\|<\varepsilon_n.$$
and the stability property of the canonical basis of l_1, we conclude that the sequence $\{T(\sum_{i=m_n+1}^{m_{n+1}}\xi_i y_i)\}$ is γ-equivalent to the canonical basis of l_1 (for some $\gamma>0$). Taking into account the inequality $\|\sum_{i=m_n+1}^{m_{n+1}}\xi_i y_i\|\le C$, $n=1,2,\ldots$ (see (2) and (4)), we have
$$\gamma\|T\|^{-1}\sum|a_n|\le\|T\|^{-1}\Big\|\sum a_n\Big(\sum_{i=m_n+1}^{m_{n+1}}\xi_i Ty_i\Big)\Big\|$$
$$\le\Big\|\sum a_n\Big(\sum_{i=m_n+1}^{m_{n+1}}\xi_i y_i\Big)\Big\|\le C\sum|a_n|.$$

Thus, the sequence $\{\sum_{i=m_n+1}^{m_{n+1}}\xi_i y_i\}$ is equivalent to the canonical basis of l_1. Put $E=[\sum_{i=m_n+1}^{m_{n+1}}\xi_i y_i]_{n=1}^{\infty}$. Hence $T|_E$ is an isomorphism, and so $T_2(Y_2)$ is not thin ($T_2(Y_2)\supset l_1$).

Case 2: There are $\varepsilon>0$ and n_0 such that for all m and $\{\xi_i\}_{i=n_0+1}^{n_0+m}$ with $\sum_{i=n_0+1}^{n_0+m}|\xi_i|=1$ we have
$$\Big\|T\Big(\sum_{i=n_0+1}^{n_0+m}\xi_i x_i\Big)\Big\|>\varepsilon.$$

In this case $\{Tx_i\}_{i=n_0+1}^{\infty}$ is equivalent to the canonical basis of l_1:
$$\varepsilon\sum|\xi_i|\le\Big\|\sum\xi_i Tx_i\Big\|\le C\|T\|\sum|\xi_i|.$$
Since $\|x_i\|<C$, $i=1,2,\ldots$, we have
$$\varepsilon\|T\|^{-1}\sum|\xi_i|\le\|T\|^{-1}\Big\|\sum\xi_i Tx_i\Big\|\le\Big\|\sum\xi_i x_i\Big\|\le C\sum|\xi_i|.$$

Thus, $\{x_i\}_{i=n_0+1}^{\infty}$ is also equivalent to the canonical basis of l_1. Therefore $T|_{[x_i]}$ is an isomorphism and $T_1(Y_1)\supset l_1$.

So, in both cases one of the ranges $T_1(Y_1)$ or $T_2(Y_2)$ contains a copy of l_1, and so it is not thin. This contradiction completes the proof.

Example 3.3. The following examples show that Theorem 3.2(ii)$\Rightarrow$(i) is not true if we restrict ourselves to *own* operator ranges, that is, if $Y_1=Y_2=X$.

(i) As mentioned above, if the space l_2 is represented as a sum of two *own* operator ranges
$$l_2=A_1(l_2)+A_2(l_2),$$
then there exist two infinite-dimensional subspaces $N\subset A_1(l_2)$ and $M\subset A_2(l_2)$ such that $N\cap M=\{0\}$ and $l_2=N+M$.

(ii) Let $X = l_p$, $1 < p < \infty$, and let $A_i : l_p \to l_p$ $(i = 1, 2)$ be two linear bounded one-to-one operators such that

$$(5)\qquad \begin{aligned} &l_p = A_1(l_p) + A_2(l_p),\\ &\dim A_1(l_p) = \operatorname{codim} A_1(l_p) = \dim A_2(l_p) = \operatorname{codim} A_2(l_p) = \infty. \end{aligned}$$

We claim that both $A_1(l_p)$ and $A_2(l_p)$ contain infinite-dimensional subspaces. Indeed, suppose that, say, $A_1(l_p)$ does not contain any infinite-dimensional subspace. Then $A_1 : l_p \to l_p$ is strictly singular, and so compact. Now, taking into account (5), we have a contradiction with Lemma 1.1.

QUESTION. Suppose the space l_p, $1 < p < \infty$, is represented as in (5). Are there infinite-dimensional subspaces $N \subset A_1(l_p)$ and $M \subset A_2(l_p)$ such that $N \cap M = \{0\}$ and $N + M = l_p$?

4. Complemented operator ranges. Let X be a Banach space and $A(Y) \subset X$ be an operator range with $\dim A(Y) = \operatorname{codim} A(Y) = \infty$. We say that $A(Y)$ is *complemented* if there exists an operator range $A_1(Y_1) \subset X$ with $\dim A_1(Y_1) = \operatorname{codim} A_1(Y_1) = \infty$ such that

$$A(Y) + A_1(Y_1) = X.$$

We recall some definitions (see [7], [8, pp. 49–50]). An operator $T : Y \to X$ is said to be *strictly singular* if for every infinite-dimensional subspace $M \subset Y$ the restriction $T|_M$ is not an isomorphism. An operator $T : Y \to X$ is said to be *strictly cosingular* if for every infinite-codimensional subspace $N \subset X$ the operator $\tau_N \circ T$, where $\tau_N : X \to X/N$ is the quotient map, is not surjective.

PROPOSITION 4.1. *Let X be a separable Banach space and $A(Y) \subset X$ be an operator range such that* $\dim A(Y) = \operatorname{codim} A(Y) = \infty$. *The following assertions are equivalent*:

(i) *$A(Y)$ is complemented.*

(ii) *There exists a subspace $N \subset X$ such that* $\dim N = \operatorname{codim} N = \infty$ *and $A(Y) + N = X$.*

(iii) *The operator $A : Y \to X$ is not strictly cosingular.*

If the dual space X^* is separable then (i)–(iii) *are equivalent to*

(iv) *The operator $A^* : X^* \to Y^*$ is not strictly singular.*

Proof. The equivalence (ii)⇔(iii) follows directly from the definitions, while (ii)⇒(i) is trivial.

(i)⇒(ii). Let $X = A(Y) + A_1(Y_1)$, where $A_1(Y_1)$ is of infinite codimension. By Proposition 1.3 there exist a subspace $M \subset X$ with $\operatorname{codim} M = \infty$ and a compact set $K \subset X$ such that $A_1(Y_1) \subset \operatorname{lin}(M \cup K)$. Put $Z = Y \oplus M$ and define an operator $T : Z \to X$ by

$$T(y_1, y_2) = A_1 y_1 + y_2.$$

From Lemma 1.1 it follows that $\operatorname{codim}(A(Y) + M) < \infty$. Hence, there exists a finite-dimensional subspace $M_1 \subset X$ such that $X = A(Y) + M + M_1$. Put $N = M + M_1$.

(ii)⇒(iv). Let $\tau : X \to X/N$ be the quotient map and $\psi = \tau \circ A$. Then $\psi : Y \to X/N$ is a surjection, and so $\psi^* : (X/N)^* \to Y^*$ is an isomorphic embedding. But this means that $A^*|_{N^\perp}$ is an isomorphism. Hence the operator A^* is not strictly singular.

Now we suppose that X^* is separable and prove (iv)⇒(ii).

Since $A^* : X^* \to Y^*$ is not strictly singular, there exists an infinite-dimensional subspace $M \subset X^*$ such that $A^*|_M$ is an isomorphism. From separability of X^* it follows that there exists an infinite-dimensional w^*-closed subspace $L \subset M$ (see [5]). Set $N = L^\top$ and let $\tau : X \to X/N$ be the quotient map. Consider the operator $\psi = \tau \circ A$. Since $N^\perp = L$ and $A^*|_L$ is an isomorphism, it follows that ψ^* is an isomorphic embedding, and so ψ is a surjection. Hence $A(Y) + N = X$, which concludes the proof.

EXAMPLE 4.2. Let $X = L_p[0,1]$, $1 < p < \infty$, $Y = C[0,1]$, and let $A : Y \to X$ be the natural embedding. Denote by L the subspace of $L_q[0,1]$, $1/p + 1/q = 1$, generated by the Rademacher functions. By using Khinchin's inequalities, it is not difficult to show that $A^*|_L$ is an isomorphism. Since X is reflexive, the subspace L is w^*-closed. Now from the proof of Proposition 4.1(iv)⇒(ii) it follows that

$$L_p[0,1] = A(C[0,1]) + L^\top.$$

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[6] D. Beck, *Introduction to Dynamical Systems*, Vol. 2, Progr. Math. 54, Birkhäuser, Basel, 1978.

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