Contents of Volume 133, Number 1

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1999): Vols. 132–137 (18 issues); $33.50 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman

PRINTED IN POLAND

ISSN 0039-3223

STUDIA MATHEMATICA 133 (1) (1999)

An oscillatory singular integral operator with polynomial phase

by

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Abstract. We prove the continuity of an oscillatory singular integral operator T with polynomial phase $P(x,y)$ on an atomic space H^1_P related to the phase P. Moreover, we show that the cancellation condition to be imposed on T holds under more general conditions. To that purpose, we obtain a van der Corput type lemma with integrability at infinity.

1. Introduction. In this paper, we consider a continuous linear operator $T : C_0^\infty \to D'$ defined as

$$(T(f), g) = (k(x,y), e^{iP(x,y)} f(y)g(x))$$

where $k(x,y)$ is the distribution kernel of a Calderón–Zygmund operator, $P(x,y)$ is a real polynomial in $x, y \in \mathbb{R}^n$, $f, g \in C_0^\infty$, and $(\,,)$ denotes the (D', C_0^∞) duality.

These operators appear, for instance, in connection with singular integrals on lower dimensional varieties, on the Heisenberg group in relation to twisted convolution, and as the model for operators occurring in the theory of the singular Radon transforms (cf. [11]). Many authors have studied the behavior of these and related operators on various function spaces. For example, D. Geller and E. M. Stein [6] and D. H. Phong and E. M. Stein [11] showed that if P is a bilinear form, the operator T is of strong type (p,p), $1 < p < \infty$; F. Ricci and E. M. Stein [12] showed that T is of strong type (p,p), $1 < p < \infty$; S. Chanillo and M. Christ [2] proved that T is of weak type $(1,1)$; Y. Pan [10] showed that T maps continuously an appropriate atomic space H^1_P into L^1. He also showed that the result is false in general, even when P is a bilinear form, if the exponent 1 is replaced by $0 < p < 1$. Y. Hu and Y. Pan [9] proved that if $P = P(x-y)$, $k = k(x-y)$, and $\nabla P(0) = 0$, the operator T maps continuously the weighted Hardy space H^1_w into itself for $w \in A_1$. They also proved that the hypothesis $\nabla P(0) = 0$

1991 *Mathematics Subject Classification*: 42B20, 42B30.

Research of J. Hounie supported in part by CNPq.

is necessary. There are also weighted results on L^p and H^1 due to Y. Hu (cf. [7], [8]).

In this paper, we state an appropriate cancellation condition to impose on T, so that T will map continuously the space H_P^1 into itself. When the phase P is a constant, it is known (cf. [1]) that a Calderón–Zygmund operator T maps the space H^1 into itself if and only if $T^*(1) = 0$ in the BMO sense. The standard proof of this result involves showing that T maps H^1-atoms into H^1-molecules. Following this model case, we first define in H_P^1 a suitable notion of molecule and then we show that the operator under consideration maps H_P^1-atoms into H_P^1-molecules. As usual, a molecule satisfies the same cancellation condition as an atom, but instead of having compact support, it satisfies an additional size condition at infinity. When verifying that the image of an atom satisfies the cancellation condition, the model case suggests that the appropriate cancellation condition on T should be $T^*(e^{iP(z,y)}) = 0$ for each $z \in \mathbb{R}^n$, in some sense. More precisely, the (H_P^1, L^1) continuity result proved by Y. Pan [10] implies that the operator T^* maps continuously L^∞ into a BMO like space. It is in this sense that the above cancellation condition on T can be understood. The (H_P^1, L^1) continuity requires assuming that the distribution kernel $k(x,y)$ defines an operator bounded on L^q, for some $1 < q < \infty$.

It is interesting to observe, however, that the action of T^* on the function $e^{iP(z,x)}$, $z \in \mathbb{R}^n$ fixed, can be made precise without assuming any continuity condition. This is already true in the case of a constant phase. Indeed, it is known that the pointwise condition

$$|\nabla_{x,y} k(x,y)| \leq \frac{C}{|x-y|^{n+\varepsilon}}$$

for some $\varepsilon > 0$, or the Hörmander condition

$$\int_{|x-z|>2|y-z|} (|k(x,y) - k(y,z)| + |k(y,x) - k(z,x)|)\, dx < \infty, \tag{1.1}$$

are enough to define the action of the operator on $L^\infty \cap C^\infty$. This observation lies at the heart of the celebrated $T(1)$ Theorem of G. David and J. L. Journé [4].

It turns out that in the case of oscillatory singular integrals, to obtain the integrability needed to make precise the action of T^* on $e^{iP(z,x)}$, a quite more involved argument is required. We discuss these matters in the last part of the paper. In particular, we obtain a van der Corput type lemma with integrability at infinity. We believe that this result may be of interest on its own.

We view these results as the main contribution of our paper.

More precisely, following the model case we are led to consider the function

$$\Psi(y) = \frac{1}{|y-z|^n} \int e^{iP(y,x)} g(x)\, dx$$

where g is a C_0^∞ function. Then we prove that Ψ is integrable at infinity, possibly imposing a cancellation condition on g. This is the result we refer to as a van der Corput type lemma.

The notation we use in this paper is the standard one in the subject. The symbols C_0^∞, S', D', L^p, etc. indicate the usual spaces of distributions or functions defined on $\mathbb{R}^n$, with complex values. Moreover, $\|f\|_p$ denotes the L^p norm of the function f, and $\|T\|_{p,p}$ denotes the operator norm of an operator bounded on L^p. We denote by χ_A the characteristic function of a set A, by $|A|$ the Lebesgue measure of a measurable set A, and $B(z,r)$ is the ball centered at z with radius r. As usual, the letter C indicates an absolute constant, probably different at different occurrences. Other notations will be introduced at the appropriate time.

2. Definition of the space H_P^1. Y. Pan [10] introduced the atomic space H_P^1 following the definition given by D. H. Phong and E. M. Stein [11] in the case when the phase P is a bilinear form.

DEFINITION 2.1. Given $1 < q \leq \infty$ a $(1,q)$ *atom* is a q-integrable function a satisfying the conditions:

(i) a is supported on a ball $B = B(z,r)$,
(ii) $\|a\|_q \leq |B|^{-1+1/q}$,
(iii) $\int e^{iP(z,y)} a(y)\, dy = 0$.

DEFINITION 2.2. We denote by $H_P^{1,q}$ the family of tempered distributions f that can be written as

$$f = \sum_{j\geq 1} \lambda_j a_j \quad \text{in the sense of } S'$$

where a_j is a $(1,q)$ atom, $\lambda_j \in \mathbb{C}$, and $\sum_{j\geq 1} |\lambda_j| < \infty$.

Clearly, $H_P^{1,q}$ is a complex vector space. It becomes a complete quasi-normed space upon defining

$$\|f\| = \inf_{\sum \lambda_j a_j = f} \sum_{j\geq 1} |\lambda_j|.$$

The proof starting on page 266 of J. García-Cuerva and J. L. Rubio de Francia's book [5], appropriately modified, shows that the definition of $H_P^{1,q}$ does not depend on q, for $1 < q \leq \infty$. So, from now on we will denote the space as H_P^1.

As noted by Y. Pan [10] there is a BMO like space associated with H^1_P. The definition of this space BMO_P follows the one given by D. H. Phong and E. M. Stein [11] for the operator with a bilinear phase. More precisely,

DEFINITION 2.3. Given a locally integrable function f we say that $f \in \mathrm{BMO}_P$ if

$$\|f\|_{\mathrm{BMO}_P} = \sup_{\substack{z\in\mathbb{R}^n\\ r>0}} \frac{1}{|B(z,r)|} \int\limits_{B(z,r)} |f(x) - f^P_B(x)|\,dx < \infty$$

where $f^P_B(x) = \frac{e^{iP(z,x)}}{|B(z,r)|}\int_{B(z,r)} e^{-iP(z,y)} f(y)\,dy$.

It follows from Definitions 2.1 and 2.3 that given $f \in \mathrm{BMO}_P$ and given a $(1,\infty)$ atom a, we have

$$\Big|\int f(x)a(x)\,dx\Big| \le \|f\|_{\mathrm{BMO}_P}.$$

This shows that the spaces H^1_P and BMO_P are in duality.

3. A notion of molecule in the space H^1_P

DEFINITION 3.1. Let $1 < q < \infty$ and $\alpha > n(q-1)$. A q-integrable function M is a $(1,q,\alpha)$ *molecule* if there exists a ball $B(z,r)$ and a positive constant C such that the following conditions are satisfied:

(i) $\|M|y-z|^{\alpha/q}\|_q \le C|B|^{\alpha/(nq)+1/q-1}$,
(ii) $\|M\|_q \le C|B|^{-1+1/q}$,
(iii) $\int e^{iP(z,y)}M(y)\,dy = 0$.

REMARK 3.2. A function M satisfying the first two conditions above is integrable, thus giving a meaning to the third condition. Indeed,

$$\int\limits_{B(z,r)} |M(x)|\,dx = \Big(\int |M(x)|^q\,dx\Big)^{1/q}|B|^{1-1/q} \le C|B|^{1/q-1}|B|^{1-1/q} = C.$$

Moreover,

$$\begin{aligned}\int\limits_{\mathbb{R}^n\setminus B(z,r)} |M(x)|\,dx &= \int\limits_{\mathbb{R}^n\setminus B(z,r)} |M(x)||x-z|^{\alpha/q}|x-z|^{-\alpha/q}\,dx\\ &\le \Big(\int |M(x)|^q|x-z|^{\alpha}\,dx\Big)^{1/q}\Big(\int\limits_{\mathbb{R}^n\setminus B(z,r)} |x-z|^{-\alpha q'/q}\,dx\Big)^{1/q'}\\ &\le C|B|^{\alpha/(nq)+1/q-1+1/q'-\alpha/(nq)} = C.\end{aligned}$$

M. Taibleson has pointed out to us that by defining a new parameter $\beta = \alpha/q$ and rewriting the above conditions in terms of β one can allow $q = \infty$.

PROPOSITION 3.3. *Let M be a $(1,q,\alpha)$ molecule. Then M belongs to the space H^1_P.*

Proof. Given a molecule M associated with a ball $B = B(z,r)$, we want to decompose it as an appropriate linear combination of $(1,q)$ atoms. Let

$$C_j = \begin{cases} B = B_0, & j = 0,\\ B(z,2^jr)\setminus B(z,2^{j-1}r) = B_j\setminus B_{j-1}, & j = 1,2,\ldots\end{cases}$$

Consider the complex vector space generated by $e^{-iP(z,x)}$. It becomes a Hilbert space V_j with the scalar product

$$(f,g)_j = \int f\bar{g}\frac{\chi_{C_j}}{|C_j|}\,dy.$$

For this space V_j the function $e^{-iP(z,x)}$ is an ON basis. Let $P_j(M)$ be the projection of M onto V_j. That is,

$$P_j(M)(x) = e^{-iP(z,x)}(M, e^{-iP(z,y)})_j = \frac{e^{-iP(z,x)}}{|C_j|}\int\limits_{C_j} M(y)e^{iP(z,y)}\,dy.$$

For $m = 1,2,\ldots$ fixed, consider

$$\sum_{j=0}^{m}(M - P_j(M))\chi_{C_j} = M\chi_{B_m} - \sum_{j=0}^{m} P_j(M)\chi_{C_j}.$$

Let us write the sum on the right hand side differently:

$$\begin{aligned}\sum_{j=0}^{m} P_j(M)\chi_{C_j} &= \sum_{j=0}^{m}\frac{e^{-iP(z,x)}}{|C_j|}\Big(\int\limits_{B_j\setminus B_{j-1}} M(y)e^{iP(z,y)}\,dy\Big)\chi_{C_j}\\ &= \sum_{j=0}^{m}\frac{e^{-iP(z,x)}}{|C_j|}\Big(\int\limits_{\mathbb{R}^n\setminus B_{j-1}} M(y)e^{iP(z,y)}\,dy\Big)\chi_{C_j}\\ &\quad - \sum_{j=0}^{m}\frac{e^{-iP(z,x)}}{|C_j|}\Big(\int\limits_{\mathbb{R}^n\setminus B_j} M(y)e^{iP(z,y)}\,dy\Big)\chi_{C_j}\\ &= \sum_{j=0}^{m-1} e^{-iP(z,x)}\frac{\chi_{C_{j+1}}}{|C_{j+1}|}\int\limits_{\mathbb{R}^n\setminus B_j} M(y)e^{iP(z,y)}\,dy\\ &\quad - \sum_{j=0}^{m} e^{-iP(z,x)}\frac{\chi_{C_j}}{|C_j|}\int\limits_{\mathbb{R}^n\setminus B_j} M(y)e^{iP(z,y)}\,dy\\ &= \sum_{j=0}^{m-1} e^{-iP(z,x)}\Big(\frac{\chi_{C_{j+1}}}{|C_{j+1}|} - \frac{\chi_{C_j}}{|C_j|}\Big)\int\limits_{\mathbb{R}^n\setminus B_j} M(y)e^{iP(z,y)}\,dy\\ &\quad - e^{-iP(z,x)}\frac{\chi_{C_m}}{|C_m|}\int\limits_{\mathbb{R}^n\setminus B_m} M(y)e^{iP(z,y)}\,dy.\end{aligned}$$

It is understood that $B_{-1} = \emptyset$.

Since $M\in L^1$, we see that $M\chi_{B_m}\to M$ in L^1 as $m\to\infty$. Likewise,

$$e^{-iP(z,x)}\frac{\chi_{C_m}}{|C_m|}\int_{\mathbb{R}^n\setminus B_m} M(y)e^{iP(z,y)}\,dy \underset{m\to\infty}{\longrightarrow} 0$$

also in L^1.

Let us now estimate the general term in each of the sums

$$\sum_{j=0}^{m}(M-P_j(M))\chi_{C_j}$$

and

$$\sum_{j=0}^{m-1} e^{-iP(z,x)}\left(\frac{\chi_{C_{j+1}}}{|C_{j+1}|}-\frac{\chi_{C_j}}{|C_j|}\right)\int_{\mathbb{R}^n\setminus B_j} M(y)e^{iP(z,y)}\,dy.$$

First, let

$$\alpha_j=\left(M-\frac{e^{-iP(z,x)}}{|C_j|}\int_{C_j}M(y)e^{iP(z,y)}\,dy\right)\chi_{C_j}.$$

Then

$$\begin{aligned}
\text{(3.1)}\quad \|\alpha_j\|_q^q &\le \int_{C_j}\left|M-\frac{e^{-iP(z,x)}}{|C_j|}\int_{C_j}M(y)e^{iP(z,y)}\,dy\right|^q dx\\
&\le 2^q\left(\int_{C_j}|M(x)|^q\,dx+\frac{|C_j|}{|C_j|^q}\Big(\int_{C_j}|M(y)|\,dy\Big)^q\right)\\
&\le 2^q\left(\int_{C_j}|M(y)|^q\,dy+|C_j|^{1-q}\int_{C_j}|M(y)|^q\,dy\;|C_j|^{q/q'}\right).
\end{aligned}$$

Since $q/q'=q-1$, we can estimate (3.1) as

$$\begin{aligned}
2^{q+1}\int_{C_j}|M(y)|^q\,dy &\le 2^{q+1}\int_{\mathbb{R}^n\setminus B_{j-1}}|M(y)|^q|y-z|^{\alpha}|y-z|^{-\alpha}\,dy\\
&\le C(2^jr)^{-\alpha}r^{\alpha+n(1-q)}=C2^{-j\alpha}|B|^{1-q}.
\end{aligned}$$

Thus,

$$\|\alpha_j\|_q\le C2^{-j\alpha/q}|B|^{1/q-1}=C2^{-j(\alpha/q-n+n/q)}|B_j|^{1/q-1}.$$

This means that we can write

$$\alpha_j=C2^{-j(\alpha/q-n+n/q)}\frac{2^{j(\alpha/q-n+n/q)}}{C}\alpha_j=\lambda_j a_j$$

where a_j is a $(1,q)$ atom supported in the ball $B(z,2^jr)$ and $\sum_{j\ge1}|\lambda_j|<\infty$.

Moreover,

$$\|\alpha_j\|_1\le\Big(\int_{C_j}|\alpha_j|^q\,dx\Big)^{1/q}|B_j|^{1-1/q}\le C2^{-j(\alpha/q-n+n/q)}.$$

Thus, $\sum_{j\ge1}\alpha_j$ converges in L^1, by the condition $\alpha>n(q-1)$.

Let us now consider the general term of the second sum. Let

$$\beta_j=e^{-iP(z,x)}\left(\frac{\chi_{C_{j+1}}}{|C_{j+1}|}-\frac{\chi_{C_j}}{|C_j|}\right)\int_{\mathbb{R}^n\setminus B_j}M(y)e^{iP(z,y)}\,dy.$$

We have

$$\begin{aligned}
\|\beta_j\|_q^q&\le 2^q(|C_j|^{1-q}+|C_{j+1}|^{1-q})\\
&\quad\times\Big(\int_{\mathbb{R}^n\setminus B_j}|M(y)||y-z|^{\alpha/q}|y-z|^{-\alpha/q}\,dy\Big)^q\\
&\le 2^q(|C_j|^{1-q}+|C_{j+1}|^{1-q})\Big(\int|M(y)|^q|y-z|^{\alpha}\,dy\Big)\\
&\quad\times\Big(\int_{\mathbb{R}^n\setminus B_j}|y-z|^{-\alpha q'/q}\,dy\Big)^{q/q'}
\end{aligned}$$

The condition $\alpha>n(q-1)$ is precisely the condition $-\alpha q'/q<-n$ so the last integral converges. We then have the estimate

$$\begin{aligned}
\|\beta_j\|_q^q&\le C(C^{1-q}((2^jr)^n-(2^{j-1}r)^n)^{1-q}+C^{1-q}((2^{j+1}r)^n-(2^jr)^n)^{1-q})\\
&\quad\times|B|^{\alpha/n+1-q}\Big(\int_{2^jr}^{\infty}t^{n-1-\alpha q'/q}\,dt\Big)^{q/q'}\\
&=C(2^jr)^{n(1-q)}|B|^{\alpha/n+1-q}(2^jr)^{(n-\alpha q'/q)q/q'}\\
&=C(2^j)^{n(1-q)+nq/q'-\alpha}|B|^{\alpha/n+1-q-\alpha/n+q-1+1-q}=C2^{-j\alpha}|B|^{1-q}\\
&=C2^{-j(\alpha-n/q+n)}|B_j|^{1-q}.
\end{aligned}$$

As before, this estimate shows that β_j can be written as

$$\beta_j=C2^{-j(\alpha/q-n+n/q)}\frac{2^{j(\alpha/q-n+n/q)}}{C}\beta_j=\mu_jb_j$$

where b_j is a $(1,q)$ atom supported in the ball $B(z,2^jr)$ and $\sum_{j\ge1}|\mu_j|<\infty$. Likewise,

$$\|\beta_j\|_1\le\|\beta_j\|_q|B_{j+1}|^{1-1/q}\le C2^{-(j/q)(\alpha-n(q-1))}.$$

Thus, $\sum_{j\ge1}\beta_j$ converges in L^1, due to the condition $\alpha>n(q-1)$.

Finally, we have shown that

$$M = \sum_{j\geq 1} \lambda_j a_j + \sum_{j\geq 1} \mu_j b_j$$

in the sense of L^1, with a_j and b_j $(1,q)$ atoms in H_P^1 and

$$\sum_{j\geq 1} |\lambda_j| < \infty, \quad \sum_{j\geq 1} |\mu_j| < \infty.$$

Thus, $M \in H_P^1$. This completes the proof of Proposition 3.3.

REMARK 3.4. The proof above shows that given a $(1,q,\alpha)$ molecule M, we have $\|M\| \leq cC$, where c is a positive constant not depending on M, and C is the constant that appears in Definition 3.1.

4. H_p^1-continuity of an oscillatory singular integral operator. We first make precise the definition of the operator under consideration. For completeness, we start with the definition of a Calderón–Zygmund operator (cf. [3]).

DEFINITION 4.1. Let $T : C_0^\infty \to D'$ be a continuous linear operator. Following [3], we say that T is a *Calderón–Zygmund operator* if

(i) the distribution kernel $k(x,y)$ of T is a locally integrable function outside the diagonal, satisfying the conditions

$$|k(x,y)| \leq \frac{C}{|x-y|^n},$$

$$|k(x,y) - k(x,z)| + |k(y,x) - k(z,x)| \leq C\frac{|y-z|^\delta}{|x-z|^{n+\delta}},$$

for some $0 < \delta \leq 1$, if $2|y-z| < |x-z|$,

(ii) the operator T extends to a continuous operator from L^{p_0} to itself, for some $1 < p_0 < \infty$,

(iii) given $f, g \in C_0^\infty$ with disjoint supports,

$$(T(f), g) = \int k(x,y) f(y) g(x)\, dy\, dx.$$

In this paper, we consider a continuous linear operator $T : C_0^\infty \to D'$ defined as

$$(T(f), g) = (k(x,y), e^{iP(x,y)} f(y) g(x)) \tag{4.1}$$

where $k(x,y)$ is the distribution kernel of a Calderón–Zygmund operator, $P(x,y)$ is a real polynomial in $x, y \in \mathbb{R}^n$, $f, g \in C_0^\infty$, and $(\,,\,)$ denotes the (D', C_0^∞) duality.

Under the above conditions, Y. Pan [10] proved that T maps continuously H_P^1 into L^1. He also showed by means of a counterexample that there is no p-version of this result for $0 < p < 1$, where H_P^p is defined as the natural extension of H_P^1 at least for p small enough.

As we mentioned in the introduction, the (H_P^1, L^1) continuity result proved by Y. Pan [10] implies that the operator T^* maps continuously L^∞ into the dual of H_P^1. This observation will allow us to state an appropriate cancellation condition, which will extend the condition $T^*(1) = 0$ in the BMO sense to the class of operators given by (4.1).

DEFINITION 4.2. Let T be an operator as in (4.1). We say that T^* satisfies the *cancellation condition* $T^*(e^{iP(z,x)}) = 0$, $z \in \mathbb{R}^n$ fixed, if

$$\int e^{iP(z,x)} T(a)(x)\, dx = 0$$

for each $(1,\infty)$ atom supported on some ball centered at z.

In a somewhat imprecise way, we then say that $T^*(e^{iP(z,x)}) = 0$ for each $z \in \mathbb{R}^n$ *in the* BMO$_P$ *sense*. In fact, we are using a subset of BMO$_P$, given as a centralized version for each $z \in \mathbb{R}^n$.

PROPOSITION 4.3. *Let a be a $(1,\infty)$ atom in H_P^1 supported in the ball $B = B(z,r)$ and let T be an oscillatory singular integral operator as* (4.1) *such that $T^*(e^{iP(z,x)}) = 0$ for each $z \in \mathbb{R}^n$ in the* BMO$_P$ *sense. Given $\alpha > 0$ and $1 < q < \infty$ such that $n(q-1) < \alpha < q(n+\delta) - n$, there exists a positive constant c not depending on a or T such that $T(a)$ satisfies conditions* (i)–(iii) *in Definition* 3.1 *with constant $c(\|T\|_{q,q} + C)$, where C is the constant that appears in Definition* 4.1(i).

Proof. It is known (cf. [12]) that the operator T is continuous on L^q. Thus,

$$\|T(a)\|_q \leq \|T\|_{q,q} \|a\|_q \leq \|T\|_{q,q} |B|^{-1+1/q}.$$

This shows that condition (ii) is satisfied. Consider now

$$\|T(a)|x-z|^{\alpha/q}\|_q^q = \int_{B(z,2r)} |T(a)|^q |x-z|^\alpha\, dx + \int_{\mathbb{R}^n \setminus B(z,2r)} |T(a)|^q |x-z|^\alpha\, dx. \tag{4.2}$$

We estimate the first term in (4.2) as

$$c|B|^{\alpha/n} \int |T(a)|^q\, dx \leq c\|T\|_{q,q} |B|^{1-q+\alpha/n}.$$

For the second term we write

$$\int_{\mathbb{R}^n \setminus B(z,2r)} |x-z|^\alpha \left| \int e^{iP(x,y)} [k(x,y) - k(x,z)] a(y)\, dy \right|^q dx \tag{4.3}$$

$$+ \int_{\mathbb{R}^n \setminus B(z,2r)} |x-z|^\alpha \left| \int e^{iP(x,y)} k(x,z) a(y)\, dy \right|^q dx. \tag{4.4}$$

Using condition (i) in Definition 3.1 we can estimate (4.3) as

$$(4.5)\qquad \le C \int_{\mathbb{R}^n\setminus B(z,2r)} |x-z|^{\alpha-q(n+\delta)}\Big(\int |y-z|^{\delta}|a(y)|\,dy\Big)^q dx$$

$$\le cC|B|^{\delta q/n-q+q}\int_{2r}^{\infty} t^{\alpha-q(n+\delta)+n-1}\,dt.$$

Under the conditions imposed on α and q, (4.5) converges and we obtain the estimate

$$cC|B|^{\delta q/n-q+q+\alpha/n-q(1+\delta/n)+1} = cC|B|^{\alpha/n-q+1}.$$

Finally, we estimate (4.4) as

$$\le C\int_{\mathbb{R}^n\setminus B(z,2r)} |x-z|^{\alpha-qn}\Big(\int |a(y)|\,dy\Big)^q dx \le cC|B|^{\alpha/n-q+1}.$$

Thus, condition (i) is also satisfied.

In order to show that $T(a)$ is a $(1,q,\alpha)$ molecule, it remains to prove that $T(a)$ satisfies the cancellation condition (iii). That is to say,

$$(4.6)\qquad \int e^{iP(z,x)}T(a)(x)\,dx = 0.$$

As observed by Y. Pan [10], by duality from his result on (H^1_P, L^1) continuity, one deduces that T^* is continuous from L^∞ to BMO_P. But this is exactly the condition $T^*(e^{iP(z,x)}) = 0$ in the BMO_P sense, as stated in Definition 4.2. This completes the proof of Proposition 4.3.

COROLLARY 4.4. *Let T be an oscillatory singular integral operator and assume that $T^*(e^{iP(z,x)}) = 0$ for each $z\in\mathbb{R}^n$ in the* BMO_P *sense. Then T maps continuously H^1_P into itself.*

Proof. Given $f\in H^1_P$ and $\varepsilon>0$ consider an atomic decomposition $f=\sum_{j\ge1}\lambda_j a_j$ such that $\sum_{j\ge1}|\lambda_j| < \|f\|+\varepsilon$. We can write $T(f_N) = \sum_{j=1}^N \lambda_j T(a_j)$ where $f_N = \sum_{j=1}^N \lambda_j a_j$. According to Proposition 4.3, we have the estimate

$$\|T(f_N)\| \le \sum_{j=1}^N |\lambda_j|\,\|T(a_j)\| \le c(\|T\|_{q,q}+C)\sum_{j=1}^N|\lambda_j|$$
$$\le c(\|T\|_{q,q}+C)(\|f\|+\varepsilon)$$

uniformly in N. This estimate implies that the sequence $\{T(f_N)\}$ is a Cauchy sequence in H^1_P. Thus, we can take the limit as $N\to\infty$ to obtain

$$\|T(f)\| \le c(\|T\|_{q,q}+C)\|f\|.$$

This completes the proof of Corollary 4.4.

REMARK 4.5. The cancellation condition imposed on T is necessary, in general. Indeed, given $\psi\in C_0^\infty(\mathbb{R})$ and the real polynomial $P(x,y)=(x-y)^2y(1-y)$, $x,y\in\mathbb{R}$, consider the operator T defined pointwise as

$$T(f)(x)=\psi(x)f(x),\qquad f\in C_0^\infty(\mathbb{R}).$$

Then T is an oscillatory singular integral operator defined by the phase $P(x,y)$ and the distribution kernel $\psi(x)\delta(x-y)$, as

$$(\psi(x)\delta(x-y), e^{iP(x,y)}f(y)g(x)) = \int \psi(x)f(x)g(x)\,dx.$$

Given the $(1,\infty)$ atom $a=\chi_{(0,1/2)}-\chi_{(1/2,1)}$, $T(a)$ does not satisfy in general the cancellation condition.

Let us point out that when the polynomial phase reduces to a constant, the continuity of T on H^1 implies the condition $T^*(1)=0$. Indeed, if a is a $(1,\infty)$ atom, we can write an atomic decomposition for $T(a)$ in the L^1 sense, namely, $T(a)=\sum_{j\ge1}\lambda_j a_j$. Thus,

$$\int T(a)(x)\,dx = \sum_{j\ge1}\lambda_j\int a_j(x)\,dx = 0,$$

or $T^*(1)=0$ in the BMO sense.

5. A van der Corput type lemma. As explained in the introduction, the purpose of this section is to show that the cancellation condition imposed on T in Corollary 4.4 still makes sense without assuming that the operator associated with the kernel $k(x,y)$ is bounded on some L^p space. Let us first specify our hypothesis on T.

DEFINITION 5.1. We will consider a continuous linear operator $T: C_0^\infty\to D'$ given by

$$(T(f),g) = (k(x,y), e^{iP(x,y)}f(y)g(x))$$

where $P(x,y)$ is a real polynomial on $\mathbb{R}^n\times\mathbb{R}^n$, and the distribution kernel $k(x,y)$ is a locally integrable function outside the diagonal, satisfying the conditions (cf. Definition 4.1(i), (iii))

(i) $|k(x,y)|\le C/|x-y|^n$,

(ii) $|k(x,y)-k(x,z)|+|k(y,x)-k(z,x)|\le C|y-z|^\delta/|x-z|^{n+\delta}$ for some $0<\delta\le1$, if $2|y-z|<|x-z|$,

(iii) given $f,g\in C_0^\infty$ with disjoint supports,

$$(T(f),g)=\int e^{iP(x,y)}k(x,y)f(y)g(x)\,dy\,dx.$$

Following the definition of $T^*(1)$ in the model case (cf. [4]), we write, for $z\in\mathbb{R}^n$ fixed,

$$(5.1)\qquad (T^*(e^{iP(z,x)}),\psi) := (e^{iP(z,x)}\varphi(x), T(\psi)) + (e^{iP(z,x)}(1-\varphi(x)), T(\psi))$$

where, in principle, $\varphi,\psi \in C_0^\infty$ and $\varphi = 1$ on a suitable neighborhood of $\operatorname{supp}(\psi)$.

Since $e^{iP(z,x)}\varphi(x) \in C_0^\infty$ and $T(\psi) \in D'$, the first term in (5.1) is well defined in the sense of distributions. Now, if $x \notin \operatorname{supp}(\psi)$, we can write

$$\begin{aligned} T(\psi)(x) &= \int e^{iP(x,y)} k(x,y)\psi(y)\,dy \\ &= \int e^{iP(x,y)}[k(x,y)-k(x,z)]\psi(y)\,dy + \int e^{iP(x,y)}k(x,z)\psi(y)\,dy. \end{aligned}$$

If $2|y-z| < |x-z|$, the integrand in the first term is bounded by $C/|x-z|^{n+\delta}$ which is integrable at infinity as a function of x.

In fact, to see that the double integral

$$\iint\limits_{2|y-z|<|x-z|} e^{iP(z,x)}(1-\varphi(x))e^{iP(x,y)}[k(x,y)-k(x,z)]\psi(y)\,dy\,dx$$

exists, it suffices to assume that the kernel $k(x,y)$ satisfies the first half of the Hörmander condition (1.1).

The second term is bounded by

$$\frac{C}{|x-z|^n}\Big|\int e^{iP(x,y)}\psi(y)\,dy\Big|.$$

We will prove that this function of x is integrable at infinity, possibly assuming that ψ satisfies a cancellation condition. Whether it is necessary to impose some condition on ψ will have to do with how the polynomial P depends on the variables x and y. More precisely,

PROPOSITION 5.2 (van der Corput type lemma). *In the notation above, let*

$$\Psi(x) = \frac{1}{|x-z|^n}\int e^{iP(x,y)}\psi(y)\,dy.$$

Then

(i) *if $P(x,y) = P_1(x) + P_2(y)$, the function $\Psi(x)$ is integrable on $\mathbb{R}^n$ only when $\int e^{iP_2(y)}\psi(y)\,dy = 0$,*

(ii) *if $P(x,y) \neq P_1(x) + P_2(y)$, and if $\partial_{x_j}P$ is not identically zero for $j = 1,\ldots,n$, the function Ψ is integrable at infinity for any ψ,*

(iii) *if $P(x,y) \neq P_1(x) + P_2(y)$, and if $\partial_{x_j}P$ is identically zero for some j's, the function Ψ is integrable on $\mathbb{R}^n$ when $\int e^{iP(z,y)}\psi(y)\,dy = 0$.*

Let us point out that the condition on the phase used in Proposition 5.2 already appears in [12], p. 192, in connection with the L^p boundedness of an oscillatory singular integral operator. Part (i) in Proposition 5.2 is obvious, as well as part (ii) in the case when the phase $P(x,y)$ is a non-degenerate bilinear form. The other cases will follow immediately from Corollary 5.7 and Lemma 5.8 below. Before going into these results, let us state our conclusion, which is a clear consequence of Proposition 5.2.

COROLLARY 5.3. *Let T be an operator as in Definition 5.1. Then*

(i) *if $P(x,y) = P_1(x) + P_2(y)$, $T^*(e^{iP(z,x)})$ is well defined for every $z \in \mathbb{R}^n$ as a continuous linear functional on*

$$D_0 = \Big\{\psi \in C_0^\infty : \int e^{iP_2(y)}\psi(y)\,dy = 0\Big\},$$

(ii) *if $P(x,y) \neq P_1(x) + P_2(y)$, and if $\partial_{x_j}P$ is not identically zero for $j = 1,\ldots,n$, $T^*(e^{iP(z,x)})$ is well defined for every $z \in \mathbb{R}^n$ in the sense of distributions,*

(iii) *if $P(x,y) \neq P_1(x) + P_2(y)$, and if $\partial_{x_j}P$ is identically zero for some j's, $T^*(e^{iP(z,x)})$ is well defined for every $z \in \mathbb{R}^n$ as a continuous linear functional on*

$$D_0(z) = \Big\{\psi \in C_0^\infty : \int e^{iP(z,y)}\psi(y)\,dy = 0\Big\}.$$

We will now state a version with parameters of the van der Corput lemma (cf. [13], p. 317).

LEMMA 5.4. *Suppose $\psi(y) \in C_0^\infty$, $\varphi(y,\lambda)$ is real-valued and depends smoothly on y, and for some multi-index α with $k = |\alpha| > 0$, and positive numbers λ_0 and K, $|D_y^\alpha\varphi(y,\lambda)| \ge 1$ for $y \in \operatorname{supp}(\psi)$, $\lambda \ge \lambda_0$. Moreover, suppose also that*

(i) *if $|\alpha| = 1$, i.e., if $D_y^\alpha = D_j$, then $D_jD_j\varphi$ changes sign $\le K$ times as a function of y_j,*

(ii) *if $|\alpha| > 1$, then $|\nabla_y D_y^\alpha\varphi(y,\lambda)| \le K$ for $y \in \operatorname{supp}(\psi)$ and $\lambda \ge \lambda_0$.*

Then

$$\Big|\int e^{i\lambda\varphi(y,\lambda)}\psi(y)\,dy\Big| \le C\lambda^{-1/k}(\|\psi\|_\infty + \|\nabla\psi\|_1)$$

for $\lambda \ge \lambda_0$, with positive constant $C = C(K,k,\varphi)$ independent of λ and ψ.

We will also need the following result (cf. [13], p. 183).

LEMMA 5.5. *Let $P(x)$ be a non-identically zero polynomial of $x \in \mathbb{R}^n$. Then the function $\log|P(x)|$ belongs to the space* BMO.

We are now ready to state and prove our main results.

LEMMA 5.6. *Let $P(x,y)$, $x,y \in \mathbb{R}^n$, be a real polynomial, let $\psi \in C_0^\infty$, and consider the function*

$$g(x) = \int e^{iP(x,y)}\psi(y)\,dy.$$

Let $x = (x_1, x')$, $x' = (x_2, \ldots, x_n)$, and let Γ_1 be the truncated cone defined by the inequality $\max(|x_2|, \ldots, |x_n|) \le |x_1|$ *for* $x_1 \ge 1$. *Finally, assume that $\partial_{x_1} P$ is not identically zero. Then there exists a constant $C = C(P) > 0$ such that*

$$\int_{\Gamma_1} \frac{|g(x)|}{|x^n|}\,dx \le C(\|\psi\|_\infty + \|\psi\|_1 + \|\nabla\psi\|_1).$$

Proof. We first parameterize the cone Γ_1 using projective coordinates,

$$z_1 = x_1, \quad z_j = x_j/x_1, \quad j = 2, \ldots, n.$$

Thus, Γ_1 can be described as $\{z = (z_1, z') : z_1 \ge 1,\ z' \in B'\}$, where B' is the closed cube of side length 2 centered at the origin of $\mathbb{R}^{n-1}$. Since the Jacobian determinant $|(\partial x/\partial z)|$ is z_1^{n-1} and $|x|$ is comparable to $x_1 = z_1$ on Γ_1, we get

$$\int_{\Gamma_1} \frac{|g(x)|}{|x^n|}\,dx \le C \int_{B'} \int_1^\infty |g(z_1, z_1 z')| z_1^{-1}\,dz_1\,dz'. \tag{5.2}$$

To estimate the integral on the right hand side of (5.2), we fix $z' \in B'$ and integrate first in z_1. Let $P(x, y)$ in the coordinates $z = (z_1, z')$ be $\widetilde{P}(z, y) = P(z_1, z_1 z', y)$. The assumption on P implies that $\widetilde{P}$ truly depends on z_1. Let α be an n-tuple of maximal length so that $D_y^\alpha \widetilde{P}$ truly depends on z_1. Thus, we can write

$$D_y^\alpha \widetilde{P}(z, y) = a(z_1, z') + b(z', y)$$

for some polynomials a, b with $\partial_{z_1} a$ not identically zero. We have, for some positive integer r,

$$a(z_1, z') = A_r(z') z_1^r + \ldots + A_0(z'). \tag{5.3}$$

First assume that $A_r(z') \ne 0$ for our fixed $z' \in B'$. This is indeed the case outside a null subset of B' because the polynomial A_r does not vanish identically. Thus, it follows from (5.3) that

$$a(z_1, z') = z_1^r(A_r(z') + O(z_1^{-1}))$$

uniformly in $z' \in B'$, which easily implies that

$$|a(z_1, z')| \ge z_1^r \frac{|A_r(z')|}{2}$$

for $z_1 \ge C/|A_r(z')|$ with C independent of z'. Since $b(z', y)$ remains bounded for $z' \in B'$ and $y \in \operatorname{supp}(\psi)$, we can conclude that there exists $M > 0$, independent of z', such that

$$|D_y^\alpha \widetilde{P}(z, y)| \ge |a(z_1, z')| - |b(z', y)| \ge z_1^r \frac{|A_r(z')|}{4} \tag{5.4}$$

provided that $z_1 \ge M/|A_r(z')|$ and $y \in \operatorname{supp}(\psi)$. Furthermore,

$$\nabla_y D_y^\alpha \widetilde{P}(z, y) = \nabla_y b(z', y)$$

and we obtain

$$|\nabla_y D_y^\alpha \widetilde{P}(z, y)| \le C \le K z_1^r \frac{|A_r(z')|}{4} \tag{5.5}$$

for $z_1 \ge M/|A_r(z')|$ and $y \in \operatorname{supp}(\psi)$. Estimates (5.4) and (5.5) show that we can apply Lemma 5.4 to the integral

$$\int e^{i(z_1^r A_r(z')/4)[4A_r(z')^{-1} z_1^{-r} P(z_1, z_1 z', y)]}\psi(y)\,dy$$

with $\lambda = z_1^r A_r(z')/4$, $\lambda_0 = M/|A_r(z')|$, and

$$\varphi(y, \lambda) = 4P(z_1, z_1 z', y)/(z_1^r A_r(z')).$$

It follows that

$$|g(z_1, z_1 z')| \le C z_1^{-r\gamma} |A_r(z')|^{-\gamma}(\|\psi\|_\infty + \|\nabla\psi\|_1)$$

for $z_1 \ge \lambda_0$, with $\gamma = 1/|\alpha|$.

On the other hand, it is clear that $\|g(z)\|_\infty \le \|\psi\|_1$. Thus, writing

$$N(\psi) = \|\psi\|_\infty + \|\psi\|_1 + \|\nabla\psi\|_1$$

we obtain

$$\begin{aligned}\int_1^\infty |g(z_1, z_1 z')| z_1^{-1} dz_1 &\le N(\psi) \int_1^{\lambda_0} \frac{dz_1}{z_1} + CN(\psi)|A_r(z')|^{-\gamma} \int_{\lambda_0}^\infty z_1^{-(1+r\gamma)}\,dz \\ &\le CN(\psi)(|\ln|A_r(z')|| + |A_r(z')|^{(r-1)\gamma}).\end{aligned} \tag{5.6}$$

Using Lemma 5.5 and the fact that r is a positive integer, we conclude that (5.6) is integrable on B'. According to (5.2), this completes the proof of Lemma 5.6.

COROLLARY 5.7. *Let $g(x)$ be as in Lemma 5.6 and assume that $\partial_{x_j} P$ is not identically zero for $j = 1, \ldots, n$. Then there exists a positive constant $C = C(P)$ such that*

$$\int_{\mathbb{R}^n \setminus Q} \frac{|g(x)|}{|x^n|}\,dx \le C(\|\psi\|_\infty + \|\psi\|_1 + \|\nabla\psi\|_1) \tag{5.7}$$

where Q is the cube centered at the origin with side length 2.

Proof. It is enough to decompose $\mathbb{R}^n \setminus Q$ as the union of the n truncated cones

$$\Gamma_j = \{\max(|x_1|, \ldots, |x_{j-1}|, |x_{j+1}|, \ldots, |x_n|) \le |x_j|,\ x_j \ge 1 \text{ or } x_j \le -1\}$$

for $j = 1, \ldots, n$. On each of these cones Γ_j we may reason as in Lemma 5.6.

This completes the proof of Corollary 5.7.

We now consider an integral analogous to (5.7) but involving an additional set of variables $z = (z_1, \ldots, z_m)$ on which the function $g(x)$ does not depend.

LEMMA 5.8. *Let $g(x)$ be as in Lemma 5.6 and assume that $\partial_{x_j} P$ is not identically zero for $j = 1, \ldots, n$. If $g(x)$ vanishes at a point $x_0 \in \mathbb{R}^n$, then there exists a positive constant $C = C(P)$ such that*

$$\int_{\mathbb{R}^{n+m}} \frac{|g(x)|}{|(x-x_0, z)|^{n+m}}\, dx\, dz \le C\Big(\sup_{\substack{|x|\le 1\\ y\in \operatorname{supp}(\psi)}} |\nabla_x P(x,y)|\, \|\psi\|_\infty + \|\psi\|_1 + \|\nabla\psi\|_1\Big).$$

Proof. Consider polar coordinates (r, θ) in $\mathbb{R}^n$ centered at x_0 so that $g(0) = 0$, and also polar coordinates (ϱ, α) in $\mathbb{R}^m$. The proof of Lemma 5.4 shows that there exists a function $0 \le A(\theta) \le 1$, $\theta \in S^{n-1}$, two positive constants γ and C and a positive integer j such that

$$\text{(a)}\qquad \int_{S^{n-1}} \ln A(\theta)\, d\theta < \infty,$$

$$\text{(b)}\qquad |g(x)| \le N(\psi) r^{-j\gamma} A(\theta)^{-1} \quad \text{for } r \ge A(\theta)^{-1},$$

where

$$N(\psi) = \sup_{\substack{|x|\le 1\\ y\in \operatorname{supp}(\psi)}} |\nabla_x P(x,y)|\, \|\psi\|_\infty + \|\psi\|_1 + \|\nabla\psi\|_1.$$

We also have $|g(x)| \le C\|\psi\|_1$. Furthermore, since g is smooth and vanishes at the origin, we obtain, for $r = |x| \le 1$,

$$\text{(c)}\qquad |g(x)| \le \sup_{|u|\le 1} |\nabla g(u)| r \le \sup_{\substack{|x|\le 1\\ y\in \operatorname{supp}(\psi)}} |\nabla_x P(x,y)|\, \|\psi\|_\infty r.$$

Thus, we can estimate

$$\text{(5.8)}\qquad \int_{\mathbb{R}^{n+m}} \frac{|g(x)|}{|(x-x_0, z)|^{n+m}}\, dx\, dz \le \int_{S^{n-1}} d\theta \int_0^\infty \int_0^\infty |g| r^{n-1} \varrho^{m-1} (r^2+\varrho^2)^{-(n+m)/2}\, dr\, d\varrho.$$

Introducing polar coordinates $\mu = (r^2+\varrho^2)^{1/2}$ and $\beta = \arctan(\varrho/r)$ in the first quadrant of the (r,ϱ)-plane, and taking into account estimates (b) and (c) above, we see that the inner integral with respect to (r,ϱ) in (5.8) is dominated by $N(\psi)$ times the integral

$$\text{(5.9)}\qquad \int_0^{\pi/2} d\beta \Big[\int_0^{1/\cos\beta} \cos\beta\, d\mu + \int_{1/\cos\beta}^{1/(A(\theta)\cos\beta)} \frac{d\mu}{\mu} + A(\theta)^{-\gamma} \int_{1/(A(\theta)\cos\beta)}^{\infty} (\mu\cos\beta)^{-j\gamma} \frac{d\mu}{\mu}\Big]$$
$$= \frac{\pi}{2}(1 + |\ln A(\theta)| + A(\theta)^{\gamma(j-1)}).$$

This expression (5.9) is an integrable function of $\theta \in S^{n-1}$, so integrating this with respect to θ we obtain the desired conclusion. This completes the proof of Lemma 5.8.

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Received May 12, 1997
Revised version July 15, 1998 (3884)

STUDIA MATHEMATICA 133 (1) (1999)

On Sobolev spaces of fractional order and ε-families of operators on spaces of homogeneous type

by

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Dedicated to Professor Carlos Segovia Fernández

Abstract. We introduce Sobolev spaces L^p_α for $1 < p < \infty$ and small positive α on spaces of homogeneous type as the classes of functions f in L^p with fractional derivative of order α, $D^\alpha f$, as introduced in [2], in L^p. We show that for small α, L^p_α coincides with the continuous version of the Triebel–Lizorkin space $F_p^{\alpha,2}$ as defined by Y. S. Han and E. T. Sawyer in [4]. To prove this result we give a more general definition of ε-families of operators on spaces of homogeneous type, in which the identity operator is replaced by an invertible operator. Then we show that the family $t^\alpha D^\alpha q(x,y,t)$ is an ε-family of operators in this new sense, where $q(x,y,t) = t\frac{\partial}{\partial t}s(x,y,t)$, and $s(x,y,t)$ is a Coifman type approximation to the identity.

1. Definitions and statement of results. Let (X,δ,μ) be a space of homogeneous type of infinite measure and such that $\mu(\{x\}) = 0$ for every x in X. Without loss of generality it can be assumed that (X,δ,μ) is a normal space of order γ, $0 < \gamma \le 1$. For $0 < \alpha < 1$ let

$$\delta_{-\alpha}(x,y) = \left(\int_0^\infty t^{-\alpha} s(x,y,t)\,\frac{dt}{t}\right)^{1/(-\alpha-1)}$$

where $s(x,y,t)$ is a Coifman type approximation to the identity. In [2] it is shown that $\delta_{-\alpha}(x,y)$ is a quasidistance equivalent to $\delta(x,y)$.

Let C_0^η, $0 < \eta \le \gamma$, be the space of Lipschitz functions of order η with bounded support. The fractional derivative of order α of a function f belonging to C_0^η, $0 < \alpha < \eta$, was defined in [2] by the formula

1991 *Mathematics Subject Classification*: Primary 46E15; Secondary 26A33.
The first author supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences of DePaul University. It is also a pleasure for this author to thank Y. S. Han for a helpful discussion of the present subject.