(see [9]), one easily checks that ϕ is in $H^2(\partial B^2)$. Proposition 4.2 thus shows that $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B^2)\ominus N, H^2(\partial B^2))$ is zero. The proof is complete.

For a proper Hardy submodule N of finite codimension in $H^2(\partial B^2)$, since $\mathrm{Ext}_{\mathcal{N}}(H^2(\partial B^2)\ominus N, N)$ is never zero, it follows that N is never similar to $H^2(\partial B^2)$ by Proposition 4.4. We refer the reader to [3] for a further consideration of the rigidity of Hardy submodules over the ball algebra.

REMARK 4.5. The main results of the present paper are also valid for strongly pseudoconvex domains with smooth boundary by [5, 10].

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Canonical functional extensions on von Neumann algebras

by

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Abstract. The topology and the structure of the set of the canonical extensions of positive, weakly continuous functionals from a von Neumann subalgebra M_0 to a von Neumann algebra M are described.

1. Introduction. The aim of this paper is to give some results about the structure of the set $R(M, M_0)$ of the canonical extensions of positive, weakly continuous functionals (called canonical functional extensions, c.f.e.) from a von Neumann subalgebra M_0 to a von Neumann algebra M (cf. [3]–[5]). After the necessary preliminaries in Section 2, Section 3 is devoted to the introduction of a set $V(\omega_0)$ of vectors in the Hilbert space of the standard representation for M, canonically associated with $R(M, M_0)$ in the framework of the modular theory of von Neumann algebras. Section 4 contains some topological results on $R(M, M_0)$ and $V(\omega_0)$. In Section 5 structural properties for different c.f.e. are compared and the possibility of defining Radon–Nikodym derivatives for c.f.e. in the spirit of Connes' type cocycles for conditional expectations on von Neumann algebras (cf. [6]) is considered. In Section 6 we consider the special situation in which a c.f.e. is dominated (i.e. majorized by some multiple of another) in order to obtain some further comparison results, and we conclude by giving a sufficient condition for a given c.f.e. to dominate no other c.f.e.

2. Preliminaries and notations. Let M be a von Neumann algebra acting on a Hilbert space H. We denote by $S(M)$ (resp. $S_{\mathrm{f}}(M)$) the set of normal (resp. normal faithful) states on M. For ξ in H and a in M we set $\omega_\xi(a) = \langle \xi, a\xi\rangle$. Let ϕ and ω be in $(M_*)^+$. We say that ϕ is *dominated* by ω (and denote by $m(\omega)$ the set of such functionals) if it is majorized by some positive multiple of ω. If M_0 is a von Neumann subalgebra of M we set $\omega_0 = \omega|M_0$ for all ω in M_*. For ω in $S(M)$ we denote by $[\varepsilon(\omega)]$ the ω conditional expectation from M to M_0 introduced in [1] (see also [4], [5]).

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For ω in $(M_*)^+$, $[\sigma(\omega)]^t$ denotes the modular automorphism group for ω on M, and if also ϕ is in $(M_*)^+$ and its support is contained in the support of ω then $(D\phi : D\omega)_t$ will denote the Connes cocycle for ω and ϕ in M. This cocycle admits ([7]) a continuous extension $(D\phi : D\omega)_{i/2}$ in $i/2$ as a linear, densely defined, in general not closable operator which commutes with M' and whose domain coincides with $D(H,\omega) = \{\xi \in H : \omega_\xi \in m(\omega)\}$. We denote the linear operator which coincides with $(D\phi : D\omega)_{i/2}$ on $\operatorname{supp}\omega$ and is zero on its orthogonal space by $\{\phi/\omega\}$. If in particular ϕ itself is in $m(\omega)$, then $\{\phi/\omega\}$ admits a bounded closure $[\phi/\omega]$ in M.

Let now ξ be a vector in H, $H(M,\xi)$ the closure of $\{a\xi : a \in M\}$ and $E(M,\xi)$ the orthogonal projection from H to $H(M,\xi)$. A *selfdual positive cone* for M in H is any cone V in H containing a separating vector ξ for M in H such that V is the selfdual positive cone for $E(M,\xi)M$ in $H(M,\xi)$ in the sense of the modular theory of von Neumann algebras. We denote the associated isometric involution by $J(M,V)$ (or, briefly, by $J(V)$ when there is no danger of confusion); in the above situation we also write $E(V)$ for $E(M,\xi)$. Each ϕ in $(M_*)^+$ is implemented by the unique vector $\{\phi/\omega_\xi\}\xi$ in V.

Let V_1, V_2 be two selfdual positive cones for M in H, and for all ω in $S(M)$ let $\xi_i(\omega) \in V_i$ $(i = 1,2)$. There is then a unique partial isometry $u'(V_1,V_2)$ in M' with initial projection $E(V_1)$ and final projection $E(V_2)$ such that $u'(V_1,V_2)\xi_1(\omega) = \xi_2(\omega)$ for all ω in $S(M)$.

We use the following standard fact from the modular theory of von Neumann algebras [2]:

2.1. THEOREM. *Let ξ and η be in V. Then*

$$\|\xi-\eta\|^2 \le \|\omega_\xi - \omega_\eta\| \le \|\xi-\eta\|\cdot\|\xi+\eta\|.$$

We now reformulate our key definition introduced in [3]–[5]:

2.2. DEFINITION. Let M be a von Neumann algebra with a von Neumann subalgebra M_0. A mapping $\varrho : ((M_0)_*)^+ \to (M_*)^+$ is called a (V_0-*implemented*) *canonical functional extension* (c.f.e.) if there is a faithful representation $\pi(M)$ of M on a Hilbert space H and a selfdual positive cone V_0 for $\pi(M_0)$ in H such that $\varrho((\omega_\xi\circ\pi)_0) = \omega_\xi\circ\pi$ for all vectors ξ in V_0.

We denote the set of all c.f.e. for the couple (M,M_0) by $R(M,M_0)$.

If ε is a norm one projection from M to M_0 then the mapping $\phi_0 \mapsto \phi_0\circ\varepsilon$ (ϕ_0 in $((M_0)_*)^+$) is in $R(M,M_0)$. A V_0-implemented canonical functional extension is of this form iff V_0 is contained in some selfdual positive cone V for M in H (cf. [3]), and will be called *regular*.

In the following we simplify our notation by assuming, since this is not restrictive for our purposes, M to act standardly on the Hilbert space H. We also fix a reference selfdual positive cone V for M in H together with a reference ϕ_0 in $S_{\mathrm{f}}(M_0)$, denote by $\xi(\phi)$ the unique vector in V implementing ϕ in $(M_*)^+$ and shorten $J(V)$ to J. All the above recalled objects from the modular theory of von Neumann algebras will be endowed with a subscript "0" when referring to the von Neumann subalgebra M_0.

3. The set $V(\omega_0)$. Any ϕ in $(M_*)^+$ can be written (uniquely if its restriction ϕ_0 to M_0 is faithful) as $\phi = \varrho(\phi_0)$ with ϱ in $R(M,M_0)$ (cf. [4], [5]); conversely, for all ϱ in $R(M,M_0)$ and ϕ_0 in $((M_0)_*)^+$ there is a unique ϕ in $(M_*)^+$ satisfying $\phi = \varrho(\phi_0)$. A natural object to be associated with the resulting parametrization of $(M_*)^+$ through $R(M,M_0)$ and $((M_0)_*)^+$ in the framework of the modular theory of von Neumann algebras is the set $V(\omega_0)$ defined below.

3.1. DEFINITION. For all ϱ in $R(M,M_0)$ we let $V(\omega_0,\varrho)$ be the selfdual positive cone for M_0 in H containing $\xi(\varrho(\omega_0))$ (which therefore implements ϱ), and $V(\omega_0) = \bigcup_{\varrho\in R(M,M_0)} V(\omega_0,\varrho)$.

It is an immediate consequence of our construction that for any ϕ in $(M_*)^+$ there are representative vectors $[\xi(V(\omega_0))](\phi)$ (or, briefly, $\xi_0(\phi)$ when no confusion arises) in $V(\omega_0)$. The uniqueness is guaranteed if ϕ_0 is faithful. However, if $\phi = \varrho(\phi_0)$ we denote by $\xi_0(\varrho(\phi_0))$ the representative vector of ϕ in $V(\omega_0,\varrho)$, which is unique. If V_1 and V_2 are selfdual positive cones for M in H, we have $V_2(\omega_0) = u'(V_1,V_2)V_1(\omega_0)$. We also remark (cf. Prop. 4.1 in the following) that $V(\omega_0)$ does depend on our choice of ω_0.

3.2. EXAMPLE (the abelian case). Let $M \cong L^\infty(\Omega,\Sigma,\mu)$ and $M_0 \cong L^\infty(\Omega,\Sigma_0,\mu_0)$ with Σ_0 a subsigma algebra of Σ, $\mu(\Omega) = 1$ and $\mu_0 = \mu|\Sigma_0$. Then $V(\omega)$ is the same as $V(\omega_0)$ and corresponds to the positive functions in $L^2(\Omega,\Sigma,\mu)$, there is a bijection between $R(M,M_0)$ and the set of positive functions in $L^1(\Omega,\Sigma,\mu)$ with conditional expectation in $L^1(\Omega,\Sigma_0,\mu_0)$ the identity function, and $V(\omega_0,\varrho)$ is the set of the positive functions in $L^2(\Omega,\Sigma,\mu)$ which can be written as the product of the square root of the function in $L^1(\Omega,\Sigma,\mu)$ which corresponds under the above mentioned bijection to ϱ and a positive function in $L^2(\Omega,\Sigma_0,\mu_0)$.

We can associate ([4], [5]) with each couple (ϱ,ϕ_0) in $R(M,M_0)\times((M_0)_*)^+$ the unique partial isometry $u(\varrho,\phi_0)$ in M satisfying $u(\varrho,\phi_0) = Ju(V,W)J\operatorname{supp}\varrho(\phi_0)$, where W is any selfdual positive cone for M in H containing $\xi_0(\varrho(\phi_0))$.

The proof of the following proposition is straightforward.

3.3. PROPOSITION. *Let ψ_0 be in $S_{\mathrm{f}}(M_0)$ and for all ϕ_0 in $((M_0)_*)^+$ denote by $v(\varrho,\phi_0)$ the partial isometry obtained by taking in our construction ψ_0 instead of ω_0 as our reference state in $S_{\mathrm{f}}(M_0)$. Then $v(\varrho,\omega_0) = u(\varrho,\psi_0)^+$, and, for all ϕ_0 in $((M_0)_*)^+$, $v(\varrho,\phi_0) = v(\varrho,\omega_0)u(\varrho,\phi_0)$.*

It is immediate to check that $u(\varrho,\phi_0)$ does not depend on the cone V we start with. For a fixed ϱ in $R(M,M_0)$ the mapping $\phi_0 \mapsto u(\varrho,\phi_0)$ (ϕ_0 in $((M_0)_*)^+$) tells us how far ϱ is from a regular canonical functional extension. In this case (and only then) $u(\varrho,\phi_0)$ coincides with $\operatorname{supp}\varrho(\phi_0)$ for all ϕ_0 in $((M_0)_*)^+$ (cf. [3]). The partial isometry $u(\varrho,\phi_0)$ is the one occurring in [3]–[5] in the comparison of the ω-conditional expectations $[\varepsilon(\varrho(\phi_0))]$ and $[\varepsilon(\varrho(\omega_0))]$ from M to M_0.

4. Topological results

4.1. DEFINITION. We say a sequence ϱ_n converges to ϱ in $R(M,M_0)$ when $\varrho_n(\phi_0)\to\varrho(\phi_0)$ (in norm) for all ϕ_0 in $((M_0)_*)^+$.

We recall that by [5] a sufficient condition for ϱ_n to converge to ϱ in $R(M,M_0)$ is that $\varrho_n(\omega_0)\to\varrho(\omega_0)$ in norm.

4.2. THEOREM. *Let ϱ_n and ϱ be in $R(M,M_0)$, $(\phi_0)_n$ be in $((M_0)_*)^+$ and ϕ_0 be in $((M_0)_*)^+$ and faithful. Then $\varrho_n\to\varrho$ in $R(M,M_0)$ and $(\phi_0)_n\to\phi_0$ in norm iff $\varrho_n(\phi_0)\to\varrho(\phi_0)$ in norm.*

Proof. With no loss of generality we can assume $\phi_0=\omega_0$. If $\varrho_n(\phi_0)\to\varrho(\omega_0)$ in norm then trivially $(\phi_0)_n\to\omega_0$ in norm. For all natural n let $(u_0')_n$ be a partial isometry in M_0' such that $\xi(\varrho_n(\omega_0))=(u_0')_n\xi(\varrho(\omega_0))$; then also

$$\begin{aligned}\xi_0(\varrho_n((\phi_0)_n)) &= \{(\phi_0)_n/\omega_0\}\xi(\varrho_n(\omega_0)) = (u_0')_n\{(\phi_0)_n/\omega_0\}\xi(\varrho(\omega_0))\\ &= (u_0')_n\xi_0(\varrho((\phi_0)_n)),\end{aligned}$$

and we get

$$\begin{aligned}\|\xi(\varrho_n(\omega_0))-\xi_0(\varrho_n((\phi_0)_n))\| &= \|(u_0')_n\xi(\varrho(\omega_0))-(u_0')_n\xi_0(\varrho((\phi_0)_n))\|\\ &\le \|\xi(\varrho(\omega_0))-\xi_0(\varrho((\phi_0)_n))\|\to 0\end{aligned}$$

by Theorem 2.1, since $\xi(\varrho(\omega_0))=\xi_0(\varrho(\omega_0))$ (resp. $\xi_0(\varrho((\phi_0)_n))$) is the representative vector of ω_0 (resp. $(\phi_0)_n$) in $V(\omega_0,\varrho)$ and $(\phi_0)_n\to\omega_0$ (in norm).

Again by Theorem 2.1,

$$\begin{aligned}&\|\varrho_n(\omega_0)-\varrho_n((\phi_0)_n)\|\\ &\quad\le \|\xi(\varrho_n(\omega_0))-\xi_0(\varrho_n((\phi_0)_n))\|\cdot\|\xi(\varrho_n(\omega_0))+\xi_0(\varrho_n((\phi_0)_n))\|,\end{aligned}$$

which implies $\|\varrho_n(\omega_0)-\varrho_n((\phi_0)_n)\|\to 0$.

The proof of our first implication is now completed by noticing that $\|\varrho_n(\omega_0)-\varrho(\omega_0)\|\le\|\varrho_n(\omega_0)-\varrho_n((\phi_0)_n)\|+\|\varrho_n((\phi_0)_n)-\varrho(\omega_0)\|$, and that both the latter summands converge to zero.

The converse implication has a similar proof:

$$\begin{aligned}\|\varrho_n((\phi_0)_n)-\varrho(\omega_0)\| &\le \|\varrho_n((\phi_0)_n)-\varrho_n(\omega_0)\|+\|\varrho_n(\omega_0)-\varrho(\omega_0)\|\\ &\le \|\xi_0(\varrho_n(\omega_0))-\xi_0(\varrho_n((\phi_0)_n))\|\\ &\quad\times\|\xi(\varrho_n(\omega_0))+\xi_0(\varrho_n((\phi_0)_n))\|+\|\varrho_n(\omega_0)-\varrho(\omega_0)\|;\end{aligned}$$

the first summand tends to zero by Theorem 2.1 since $(\phi_0)_n\to\omega_0$ as earlier, and the last because $\varrho_n\to\varrho$ in $R(M,M_0)$ implies $\varrho_n(\omega_0)\to\varrho(\omega_0)$ in norm.

A slight generalization of the preceding proof shows that the "only if" part of the above statement also holds when ϕ_0 is no longer faithful.

4.3. THEOREM. *Let ϱ_n and ϱ be in $R(M,M_0)$. Then $\varrho_n\to\varrho$ in $R(M,M_0)$ iff $V(\omega_0,\varrho_n)\to V(\omega_0,\varrho)$ pointwise in norm.*

Proof. Let $\varrho_n\to\varrho$ in $R(M,M_0)$. We must prove that $\xi_0(\varrho_n(\phi_0))\to\xi_0(\varrho(\phi_0))$ for all ϕ_0 in $((M_0)_*)^+$. Let $\varepsilon>0$. As $m(\omega_0)$ is norm dense in $((M_0)_*)^+$ there is a ψ_0 in $m(\omega_0)$ such that $\|\phi_0-\psi_0\|\le\varepsilon$. By Theorem 2.1 we have

$$\|\xi_0(\varrho(\psi_0))-\xi_0(\varrho(\phi_0))\|\le\varepsilon^{1/2},$$

and, for all n,

$$\|\xi_0(\varrho_n(\psi_0))-\xi_0(\varrho_n(\phi_0))\|\le\varepsilon^{1/2},$$

since $\xi_0(\varrho(\psi_0))$ (resp. $\xi_0(\varrho_n(\psi_0))$) and $\xi_0(\varrho(\phi_0))$ (resp. $\xi_0(\varrho_n(\phi_0))$) are the representative vectors of ψ_0 and ϕ_0 in the same selfdual positive cone $V(\omega_0,\varrho)$ (resp. $V(\omega_0,\varrho_n)$) for M_0 in H. Similarly,

$$\|\xi(\varrho_n(\omega_0))-\xi(\varrho(\omega_0))\|\le\|\varrho_n(\omega_0)-\varrho(\omega_0)\|^{1/2}.$$

Then

$$\begin{aligned}&\|\xi_0(\varrho_n(\phi_0))-\xi_0(\varrho(\phi_0))\|\\ &\quad\le\|\xi_0(\varrho_n(\phi_0))-\xi_0(\varrho_n(\psi_0))\|+\|\xi_0(\varrho_n(\psi_0))-\xi_0(\varrho(\psi_0))\|\\ &\qquad+\|\xi_0(\varrho(\psi_0))-\xi_0(\varrho(\phi_0))\|\\ &\quad\le 2\varepsilon^{1/2}+\|[\psi_0/\omega_0]\|\cdot\|\xi(\varrho_n(\omega_0))-\xi(\varrho(\omega_0))\|\\ &\quad\le 2\varepsilon^{1/2}+\|[\psi_0/\omega_0]\|\cdot\|\varrho_n(\omega_0)-\varrho(\omega_0)\|^{1/2},\end{aligned}$$

and our claim is proved, since $\|\varrho_n(\omega_0)-\varrho(\omega_0)\|\to 0$.

Conversely, $\|\xi(\varrho_n(\omega_0))-\xi(\varrho(\omega_0))\|\to 0$ implies, by Theorem 2.1 again, $\varrho_n(\omega_0)\to\varrho(\omega_0)$ in norm and therefore $\varrho_n\to\varrho$ in $R(M,M_0)$.

4.4. COROLLARY. *Let ϱ_n and ϱ be in $R(M,M_0)$, ϕ_0 and $(\phi_0)_n$ be in $((M_0)_*)^+$, and let ϕ_0 be faithful. Then $\varrho_n((\phi_0)_n)\to\varrho(\phi_0)$ in norm iff $\xi_0(\varrho_n((\phi_0)_n))\to\xi_0(\varrho(\phi_0))$ in norm.*

Proof. By Theorem 4.2, $\varrho_n((\phi_0)_n)\to\varrho(\phi_0)$ in norm iff $\varrho_n\to\varrho$ in $R(M,M_0)$ and $(\phi_0)_n\to\phi_0$ in norm. If this is the case, then

$$\begin{aligned}\|\xi_0(\varrho_n((\phi_0)_n))-\xi_0(\varrho(\phi_0))\| &\le\|\xi_0(\varrho_n((\phi_0)_n))-\xi_0(\varrho((\phi_0)_n))\|\\ &\quad+\|\xi_0(\varrho((\phi_0)_n))-\xi_0(\varrho(\phi_0))\|;\end{aligned}$$

the first summand tends to zero by Theorem 4.3 and the second by Theorem 2.1.

The converse is trivial (and true even when ϕ_0 is no longer faithful).

4.5. THEOREM. *The mapping $(\varrho, \phi_0) \mapsto u(\varrho, \phi_0)$ is continuous from the product of the norm topology on $((M_0)_*)^+$ and the $R(M, M_0)$ topology to the strong operator topology.*

Proof. Let $\varrho_n \to \varrho$ in $R(M, M_0)$ and $(\phi_0)_n \to \phi_0$ in norm, and let us shorten $u(\varrho_n, (\phi_0)_n)$ to u_n and $u(\varrho, \phi_0)$ to u. For a in M, using once more Theorem 2.1, we have

$$\begin{aligned}\|J(u_n - u)Ja\xi(\varrho(\phi_0))\| &\le \|a\| \cdot \|Ju_n\xi(\varrho(\phi_0)) - \{\phi_0/\omega_0\}\xi(\varrho(\omega_0))\| \\ &\le \|a\|(\|Ju_n\xi(\varrho(\phi_0)) - Ju_n\xi(\varrho_n((\phi_0)_n))\| \\ &\quad + \|Ju_n\xi(\varrho_n((\phi_0)_n)) - \{\phi_0/\omega_0\}\xi(\varrho(\omega_0))\|) \\ &\le \|a\|(\|\xi(\varrho(\phi_0)) - \xi(\varrho_n((\phi_0)_n))\| \\ &\quad + \|\xi_0(\varrho((\phi_0)_n)) - \xi_0(\varrho(\phi_0))\|) \\ &\le \|a\|(\|\varrho(\phi_0) - \varrho_n((\phi_0)_n)\|^{1/2} + \|(\phi_0)_n - \phi_0\|^{1/2}),\end{aligned}$$

and by our hypothesis and Theorem 4.4 both summands in parenthesis tend to zero. This implies $\|(u_n - u)\xi\| \to 0$ for ξ in $\operatorname{supp} \varrho(\phi_0)$, and therefore our claim, since $\operatorname{supp} \varrho((\phi_0)_n) \to \operatorname{supp} \varrho(\phi_0)$ strongly.

5. Comparing the structure of canonical functional extensions

5.1. LEMMA. *Let ϕ_0 be in $((M_0)_*)^+$ and R be a subset of $R(M, M_0)$. There is a selfdual positive cone U for M in H containing $\xi_0(\varrho(\phi_0))$ for all ϱ in R iff there is a partial isometry u in M with initial projection the identity such that $u(\varrho, \phi_0) = u \operatorname{supp} \varrho(\phi_0)$ for all ϱ in R.*

Proof. Let u be as above. Then $U = \{Ju(\phi_0)[\xi(V)](\psi) : \psi \in (M_*)^+\}$ is a selfdual positive cone for M in H which clearly satisfies our requirements.

Conversely, if U is as above then the partial isometry $u(\phi_0)$ in M defined by setting $Ju(\phi_0)[\xi(V)](\psi) = [\xi(U)](\psi)$ for a fixed faithful ψ in $(M_*)^+$ is unique and, for all ϱ in R, $u(\varrho, \phi_0) = u \operatorname{supp} \varrho(\phi_0)$.

5.2. REMARK. If $R = R(M, M_0)$ our first condition above is the same as requiring the ϕ_0 section of $V(\omega_0)$ to be contained in some selfdual positive cone for M in H.

5.3. PROPOSITION. *Let ϕ_0 be in $((M_0)_*)^+$, ϱ_1, ϱ_2 in $R(M, M_0)$ and $\operatorname{supp} \varrho_1(\omega_0) \le \operatorname{supp} \varrho_2(\omega_0)$. Then $u(\varrho_1, \phi_0) = u(\varrho_2, \phi_0)$ iff $\operatorname{supp} \varrho_1(\omega_0) = \operatorname{supp} \varrho_2(\omega_0)$ and $\{\phi_0/\omega_0\}\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\} = \{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}\{\phi_0/\omega_0\}$.*

Proof. Let $u(\varrho_1, \phi_0) = u(\varrho_2, \phi_0)$; then $\operatorname{supp} \varrho_1(\phi_0) = \operatorname{supp} \varrho_2(\phi_0)$ since the former is the initial projection of $u(\varrho_1, \phi_0)$ and the latter the initial projection of $u(\varrho_2, \phi_0)$. So $\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}$ is well defined, as are $\{\phi_0/\omega_0\} \circ \{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}$ and $\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}\{\phi_0/\omega_0\}$, whose domains coincide with $\{a'\xi(\varrho(\omega_0)) : a' \in M'\}$. By Lemma 5.1 there is a selfdual positive cone U for M in H to which both the vectors

$$\xi_0(\varrho_1(\phi_0)) = \{\phi_0/\omega_0\}\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}\xi(\varrho_2(\omega_0))$$

and

$$\xi_0(\varrho_2(\phi_0)) = \{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0)),$$

and therefore $\{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}\{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0))$, belong. The state $\varrho_1(\phi_0)$ on M is implemented by the vector $\{\phi_0/\omega_0\}\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}\xi(\varrho_2(\omega_0))$ and the vector $\{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}\{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0))$, which are in the same selfdual positive cone for M in H; so they must coincide. We can now conclude by noticing that both the operators $\{\phi_0/\omega_0\}\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}$ and $\{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}\{\phi_0/\omega_0\}$ commute with M'.

Conversely, $\operatorname{supp} \varrho_1(\phi_0) = \operatorname{supp} \varrho_2(\phi_0)$ guarantees that the vector $\{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0)) = \xi_0(\varrho_2(\phi_0))$ belongs to any selfdual positive cone for M in H to which $\{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}\{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0))$ belongs. This last vector coincides by our hypothesis with $\{\phi_0/\omega_0\}\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}\xi(\varrho_2(\omega_0)) = \xi_0(\varrho_1(\phi_0))$. Now Lemma 5.1 and $\operatorname{supp} \varrho_1(\phi_0) = \operatorname{supp} \varrho_2(\phi_0)$ imply our claim.

5.4. DEFINITION. Let ϱ_1 and ϱ_2 be in $R(M, M_0)$, and S_0 be a subset of $((M_0)_*)^+$. We denote by $\{\varrho_1(S_0)/\varrho_2(S_0)\}$, when it exists, the linear (not necessarily bounded or even closable) operator which coincides, for all ϕ_0 in S_0, with $(D\varrho_1(\phi_0) : D\varrho_2(\phi_0))_{i/2}$ on $D(H, \varrho_2(\phi_0))$ and is zero on the linear subspace of H orthogonal to the linear span of $\bigcup_{\phi_0 \in S_0} \operatorname{supp} \varrho_2(\phi_0)$.

If $\{\varrho_1(S_0)/\varrho_2(S_0)\}$ is bounded we denote its closure by $[\varrho_1(S_0)/\varrho_2(S_0)]$; if $S_0 = ((M_0)_*)^+$ we shorten $\{\varrho_1(S_0)/\varrho_2(S_0)\}$ and $[\varrho_1(S_0)/\varrho_2(S_0)]$ to $\{\varrho_1/\varrho_2\}$ and $[\varrho_1/\varrho_2]$.

Our next proposition shows that the existence of $[\varrho_1/\varrho_2]$ is a rather restrictive condition.

5.5. PROPOSITION. *Let ϱ_1 and ϱ_2 be in $R(M, M_0)$, S_0 be a subset of $S(M_0)$ and assume $[\varrho_1(S_0)/\varrho_2(S_0)]$ exists. Then for all ϕ_0 in S_0 we have*

$$[\varepsilon(\varrho_2(\phi_0))](|[\varrho_1(S_0)/\varrho_2(S_0)]|^2) = \operatorname{supp} \phi_0.$$

Proof. For all a_0 in M_0 we have

$$\begin{aligned}&\langle a_0\xi(\varrho_2(\phi_0)), J_0[\varepsilon(\varrho_2(\phi_0))](|[\varrho_1(S_0)/\varrho_2(S_0)]|^2)\xi(\varrho_2(\phi_0))\rangle \\ &\quad = \langle a_0\xi(\varrho_2(\phi_0)), J(|[\varrho_1(S_0)/\varrho_2(S_0)]|^2)\xi(\varrho_2(\phi_0))\rangle \\ &\quad = \langle a_0[\varrho_1(\phi_0)/\varrho_2(\phi_0)]\xi(\varrho_2(\phi_0)), J[\varrho_1(\phi_0)/\varrho_2(\phi_0)]\xi(\varrho_2(\phi_0))\rangle \\ &\quad = \langle a_0\xi(\varrho_1(\phi_0)), \xi(\varrho_1(\phi_0))\rangle = \phi_0(a_0) = \langle a_0\xi(\varrho_2(\phi_0)), \xi(\varrho_2(\phi_0))\rangle,\end{aligned}$$

so $[\varepsilon(\varrho_2(\phi_0))](|[\varrho_1(S_0)/\varrho_2(S_0)]|^2)\xi(\varrho_2(\phi_0)) = \xi(\varrho_2(\phi_0))$, which implies our claim.

5.6. THEOREM. *Let* ϱ_1 *and* ϱ_2 *be in* $R(M, M_0)$ *and assume* $\{\varrho_1/\varrho_2\}$ *exists. Then* $u(\varrho_1, \phi_0) = u(\varrho_2, \phi_0)$ *for all* ϕ_0 *in* $((M_0)_*)^+$ *iff for all* ϕ_0 *in* $m(\phi_0)$ *the operators* $[\phi_0/\omega_0]$ *and* $\{\varrho_1/\varrho_2\}$ *commute and* $\operatorname{supp} \varrho_1(\phi_0) = \operatorname{supp} \varrho_2(\phi_0)$.

Proof. The "if" part is proved by noticing first that our hypothesis implies by Proposition 5.3 that $u(\varrho_1, \phi_0) = u(\varrho_2, \phi_0)$ for all ϕ_0 in $m(\phi_0)$. As $m(\phi_0)$ is norm dense in $((M_0)_*)^+$, Theorem 4.2 yields our claim.

The converse implication follows immediately from Proposition 5.3.

5.7. THEOREM. *Let* ϱ_1 *and* ϱ_2 *in* $R(M, M_0)$ *be such that* $\operatorname{supp} \varrho_1(\omega_0) = \operatorname{supp} \varrho_2(\omega_0)$, *and assume there is a linear extension* T *of* $\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}$ *whose domain is the linear span of the union of all* $D(H, \varrho_2(\phi_0))$ *with* ϕ_0 *in* $((M_0)_*)^+$ *and which commutes with* M' *and with* $\{\phi_0/\omega_0\}$ *for all* ϕ_0 *in* $((M_0)_*)^+$. *If* $u(\varrho_1, \phi_0) = u(\varrho_2, \phi_0)$ *for all* ϕ_0 *in* $((M_0)_*)^+$ *then* $\{\varrho_1/\varrho_2\}$ *exists and coincides with* T.

Proof. It is enough to prove that for all a' in M' and all ϕ_0 in $((M_0)_*)^+$ we have
$$Ta'\xi(\varrho_2(\phi_0)) = \{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}a'\xi(\varrho_2(\phi_0)).$$
By hypothesis and Proposition 5.3 we get
$$\begin{aligned} Ta'\xi(\varrho_2(\phi_0)) &= Ta'\{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0)) = a'\{\phi_0/\omega_0\}T\xi(\varrho_2(\omega_0)) \\ &= a'\{\phi_0/\omega_0\}\{\varrho_1(\omega_0)/\varrho_2(\omega_0)\}\xi(\varrho_2(\omega_0)) \\ &= a'\{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}\{\phi_0/\omega_0\}\xi(\varrho_2(\omega_0)) \\ &= \{\varrho_1(\phi_0)/\varrho_2(\phi_0)\}a'\xi(\varrho_2(\phi_0)). \end{aligned}$$

6. Dominated canonical functional extensions

6.1. DEFINITION. Let ϱ_1 and ϱ be in $R(M, M_0)$. We say that ϱ_1 is *dominated* by ϱ if $\varrho_1(\phi_0)$ is dominated by $\varrho(\phi_0)$ for all ϕ_0 in $((M_0)_*)^+$.

We recall that from [5] it follows that if $\varrho_1(\omega_0)$ in $S_f(M)$ is dominated by $\varrho(\omega_0)$ then ϱ_1 is dominated by ϱ.

6.2. THEOREM. *Let* ϱ_1 *and* ϱ *be in* $R(M, M_0)$, $\varrho_1(\omega_0)$ *be in* $(M_*)^+$ *and faithful, and* ϱ_1 *be dominated by* ϱ. *Then for all* ϕ_0 *in* $((M_0)_*)^+$,
$$u(\varrho_1, \phi_0)[\varrho_1(\phi_0)/\varrho(\phi_0)] = [\varrho_1(\omega_0)/\varrho(\omega_0)]u(\varrho, \phi_0).$$

Proof. We have
$$\begin{aligned} &u(\varrho_1, \phi_0)[\varrho_1(\phi_0)/\varrho(\phi_0)]\xi(\varrho(\phi_0)) \\ &\quad = u(\varrho_1, \phi_0)\{\varrho_1(\phi_0)/\varrho_1(\omega_0)\}[\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) \\ &\quad = J\{\phi_0/\omega_0\}[\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) = [\varrho_1(\omega_0)/\varrho(\omega_0)]J\{\phi_0/\omega_0\}J\xi(\varrho(\omega_0)) \\ &\quad = [\varrho_1(\omega_0)/\varrho(\omega_0)]u(\varrho, \phi_0)\{\varrho(\phi_0)/\varrho(\omega_0)\}\xi(\varrho(\omega_0)) \\ &\quad = [\varrho_1(\omega_0)/\varrho(\omega_0)]u(\varrho, \phi_0)\xi(\varrho(\phi_0)) \end{aligned}$$
and our claim follows.

6.3. COROLLARY. *Under the above hypothesis,*
$$|[\varrho_1(\phi_0)/\varrho(\phi_0)]|^2 = u(\varrho, \phi_0)^+|[\varrho_1(\omega_0)/\varrho(\omega_0)]|^2u(\varrho, \phi_0)$$
and, for all a *in* M^+,
$$[\varrho_1(\phi_0)](a) \le \|[\varrho_1(\omega_0)/\varrho(\omega_0)]\|^2[\varrho(\phi_0)](a).$$

Proof. The first statement is an immediate consequence of Theorem 6.2; we also have, for a in M^+,
$$\begin{aligned} [\varrho_1(\phi_0)](a) &= \|a^{1/2}[\varrho_1(\phi_0)/\varrho(\phi_0)]\xi(\varrho(\phi_0))\|^2 \\ &= \|J[\varrho_1(\phi_0)/\varrho(\phi_0)]Ja^{1/2}\xi(\varrho(\phi_0))\|^2 \\ &= \|u(\varrho_1, \phi_0)^+[\varrho_1(\omega_0)/\varrho(\omega_0)]u(\varrho, \phi_0)Ja^{1/2}\xi(\varrho(\phi_0))\|^2 \\ &\le \|u(\varrho_1, \phi_0)^+[\varrho_1(\omega_0)/\varrho(\omega_0)]u(\varrho, \phi_0)\|^2\|a^{1/2}\xi(\varrho(\phi_0))\|^2 \\ &\le \|[\varrho_1(\omega_0)/\varrho(\omega_0)]\|^2[\varrho(\phi_0)](a). \end{aligned}$$

By combining Theorems 5.6, 5.7, and 6.2 we get as an immediate consequence:

6.4. COROLLARY. *Under the above hypothesis consider the following statements:*

(a) $u(\varrho_1, \phi_0) = u(\varrho, \phi_0)$ *for all* ϕ_0 *in* $((M_0)_*)^+$.

(b) $[\varrho_1/\varrho]$ *exists.*

(c) $[\varrho_1(\omega_0)/\varrho(\omega_0)]$ *commutes with* $u(\varrho, \phi_0)$ *for all* ϕ_0 *in* $((M_0)_*)^+$.

(d) *For all* ϕ_0 *in* $m(\phi_0)$ *the operators* $[\varrho_1(\omega_0)/\varrho(\omega_0)]$ *and* $[\phi_0/\omega_0]$ *commute, and* $\operatorname{supp} \varrho_1(\phi_0) = \operatorname{supp} \varrho(\phi_0)$.

The following implications hold: (a)&(b) $\Rightarrow$ (c)&(d); (a)&(d) – (b)&(c); (b)&(d) $\Rightarrow$ (a)&(c).

6.5. *The particular case of norm one projections* (cf. also [6]). Let ϱ_1 and ϱ be in $R(M, M_0)$, with ϱ_1 dominated by ϱ, $\operatorname{supp} \varrho_1(\omega_0) = \operatorname{supp} \varrho(\omega_0) = I$, and ε a norm one projection from M to M_0 such that $\varrho(\omega_0) = \omega_0 \circ \varepsilon$ for all ω_0 in $((M_0)_*)^+$. Obviously in this situation (a) of Corollary 6.4 is equivalent to the existence of a norm one projection ε_1 from M to M_0 such that $\varrho_1(\omega_0) = \omega_0 \circ \varepsilon_1$ for all ω_0 in $((M_0)_*)^+$. In this case we also have $\operatorname{supp} \varrho_1(\phi_0) = \operatorname{supp} \varrho(\phi_0)$ for all ϕ_0 in $((M_0)_*)^+$.

(a) In our situation (a) of Corollary 6.4 implies $[\varrho_1(\omega_0)/\varrho(\omega_0)]$ and $\{\phi_0/\omega_0\}$ commute for all ϕ_0 in $((M_0)_*)^+$ and therefore (d); indeed, since $\{\varrho_1(\phi_0)/\varrho_1(\omega_0)\} = \{\phi_0/\omega_0\}$ for all ϕ_0 in $((M_0)_*)^+$, we get

$$\{\phi_0/\omega_0\}[\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0))$$
$$= \{\varrho_1(\phi_0)/\varrho_1(\omega_0)\}[\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) = \{\varrho_1(\phi_0)/\varrho(\omega_0)\}\xi(\varrho(\omega_0))$$
$$= J\{\varrho_1(\phi_0)/\varrho(\omega_0)\}\xi(\varrho(\omega_0)) = J\{\varrho_1(\phi_0)/\varrho_1(\omega_0)\}J[\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0))$$
$$= [\varrho_1(\omega_0)/\varrho(\omega_0)]J\{\varrho_1(\phi_0)/\varrho_1(\omega_0)\}\xi(\varrho(\omega_0))$$
$$= [\varrho_1(\omega_0)/\varrho(\omega_0)]\{\varrho_1(\phi_0)/\varrho_1(\omega_0)\}\xi(\varrho(\omega_0))$$
$$= [\varrho_1(\omega_0)/\varrho(\omega_0)]\{\phi_0/\omega_0\}\xi(\varrho(\omega_0)).$$

By Corollary 6.4, $[\varrho_1/\varrho]$ exists and coincides with $[\varrho_1(\omega_0)/\varrho(\omega_0)]$. Let Δ (resp. Δ_0) be the modular operator associated with M and $\xi(\varrho(\omega_0))$ (resp. M_0 and $\xi(\varrho(\omega_0))$). For all positive a_0 in M_0 there is some ϕ_0 in $m(\omega_0)$ such that $\Delta^{1/4}a_0\Delta^{-1/4}$ $(= \Delta_0^{1/4}a_0\Delta_0^{-1/4})$ is the closure of $[\phi_0/\omega_0]$ (cf. [2]); let also a be the positive operator in M which is the closure of $\Delta^{-1/4}[\varrho_1/\varrho]\Delta^{1/4}$. We then have

$$\begin{aligned} a_0a\xi(\varrho(\omega_0)) &= \Delta^{-1/4}[\phi_0/\omega_0][\varrho_1/\varrho]\xi(\varrho(\omega_0)) \\ &= \Delta^{-1/4}[\varrho_1/\varrho][\phi_0/\omega_0]\xi(\varrho(\omega_0)) = aa_0\xi(\varrho(\omega_0)), \end{aligned}$$

which implies that a is in $(M_0)'$ and therefore in the relative commutant of M_0 in M. Since there is a norm one projection from M to the relative commutant of M_0 in M, also $[\varrho_1/\varrho]$ belongs to it.

(b) Assume $[\varrho_1/\varrho]$ exists and let ϕ_0 be in $m(\omega_0)$. Then

$$\begin{aligned} [\phi_0/\omega_0][\varrho_1/\varrho]\xi(\varrho(\omega_0)) &= [\phi_0/\omega_0][\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) \\ &= [\phi_0/\omega_0]J[\varrho_1(\omega_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) \\ &= J[\varrho_1(\omega_0)/\varrho(\omega_0)]J[\phi_0/\omega_0]\xi(\varrho(\omega_0)) \\ &= J[\varrho_1/\varrho]J[\varrho(\phi_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) \\ &= J[\varrho_1(\phi_0)/\varrho(\phi_0)][\varrho(\phi_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) \\ &= [\varrho_1(\phi_0)/\varrho(\omega_0)]\xi(\varrho(\omega_0)) \\ &= [\varrho_1/\varrho][\phi_0/\omega_0]\xi(\varrho(\omega_0)), \end{aligned}$$

which implies $[\phi_0/\omega_0]$ and $[\varrho_1/\varrho]$ commute. Then by Corollary 6.4, $u(\varrho_1, \phi_0) = u(\varrho, \phi_0) = \operatorname{supp}\varrho(\phi_0)$ for all ϕ_0 in $((M_0)_*)^+$, and there is a norm one projection ε_1 from M to M_0 such that $\varrho_1(\omega_0) = \omega_0\circ\varepsilon_1$ for all ω_0 in $((M_0)_*)^+$. We can now prove as in (a) that $[\varrho_1/\varrho]$ is in the relative commutant of M_0 in M.

We can prove the analogue of Theorem 6.2 by assuming ϕ_0 to be dominated by ω_0 rather than ϱ_1 by ϱ as follows:

6.6. Proposition. *Let ϱ_1 and ϱ be in $R(M, M_0)$, $\varrho_1(\omega_0)$ and $\varrho(\omega_0)$ be faithful in $(M_*)^+$ and ϕ_0 in $m(\omega_0)$. Then $\{\varrho_1(\omega_0)/\varrho(\omega_0)\}u(\varrho, \phi_0)$ is an extension of $u(\varrho_1, \phi_0)\{\varrho_1(\phi_0)/\varrho(\phi_0)\}$.*

Proof. The equality

$$u(\varrho_1, \phi_0)\{\varrho_1(\phi_0)/\varrho(\phi_0)\}\xi(\varrho(\phi_0)) = \{\varrho_1(\omega_0)/\varrho(\omega_0)\}u(\varrho, \phi_0)\xi(\varrho(\phi_0))$$

can be proved as in Theorem 6.2, since

$$\begin{aligned} J[\phi_0/\omega_0]\{\varrho_1(\omega_0)/\varrho(\omega_0)\}\xi(\varrho(\omega_0)) &= J[\phi_0/\omega_0]J\{\varrho_1(\omega_0)/\varrho(\omega_0)\}\xi(\varrho(\omega_0)) \\ &= \{\varrho_1(\omega_0)/\varrho(\omega_0)\}J[\phi_0/\omega_0]J\xi(\varrho(\omega_0)), \end{aligned}$$

as $\{\varrho_1(\omega_0)/\varrho(\omega_0)\}$ commutes with M' to which $J[\phi_0/\omega_0]J$ belongs. If a' is in M' we get

$$u(\varrho_1, \phi_0)\{\varrho_1(\phi_0)/\varrho(\phi_0)\}a'\xi(\varrho(\phi_0)) = \{\varrho_1(\omega_0)/\varrho(\omega_0)\}u(\varrho, \phi_0)a'\xi(\varrho(\phi_0)),$$

which is our claim.

For a normal faithful state ω on M, set $L(\omega) = \{a \in M : [\varepsilon(\omega)](|a|^2) = |[\varepsilon(\omega)](a)|^2\}$, $R(\omega) = \{a \in M : [\varepsilon(\omega)](|a^+|^2) = |[\varepsilon(\omega)](a^+)|^2\}$, and $LR(\omega) = L(\omega) \cap R(\omega)$.

6.7. Lemma. *Let ϕ be a normal state in $m(\omega)$ such that $[\sigma(\omega)]^t([\phi/\omega])$ is in $L(\omega)$ for all real t. Then $[\sigma(\omega)]^t([\phi/\omega])$ is in $L(\omega)$ for all real t.*

Proof. Since $[\sigma(\omega)]^t([\phi/\omega]) = [[\sigma(\omega)]^t(\phi)/\omega]$ with $[[\sigma(\omega)]^t(\phi)](a) = \phi([\sigma(\omega)]^t(a))$, it is enough to show our claim for $t = 0$. If Δ is the modular operator associated with M and $\xi(\omega)$, our hypothesis is equivalent to

$$\begin{aligned} J_0E\Delta^{-it}[\phi/\omega]\xi(\omega) &= J_0EJ\Delta^{it}[\phi/\omega]\xi(\omega) = [\varepsilon(\omega)]([\sigma(\omega)]^t([\phi/\omega]))\xi(\omega) \\ &= J_0J\Delta^{it}[\phi/\omega]\xi(\omega) = J_0\Delta^{-it}[\phi/\omega]\xi(\omega) \end{aligned}$$

for all real t. So $\Delta^{it}[\phi/\omega]\xi(\omega)$ is in H_0 for all real t. This implies that $\Delta^{1/2}[\phi/\omega]\xi(\omega)$ is in H_0, $[\phi/\omega]^+\xi(\omega)$ is in $L(\omega)$, and our claim follows.

6.8. Proposition. *Let ϱ_1 and ϱ be in $R(M, M_0)$, ϱ_1 be dominated by ϱ and $\varrho(\omega_0)$ be faithful. Then $\varrho_1 = \varrho$ iff $[\sigma(\varrho(\omega_0))]^t([\varrho_1(\omega_0)/\varrho(\omega_0)])$ is in $L(\varrho(\omega_0))$ for all real t.*

Proof. The "if" part can be proved by noticing that by Lemma 6.7, $[\varrho_1(\omega_0)/\varrho(\omega_0)]$ is in $LR(\varrho(\omega_0))$ and therefore so is $|[\varrho_1(\omega_0)/\varrho(\omega_0)]|^2$. As in the proof of Proposition 5.5, $[\varepsilon(\varrho(\omega_0))](|[\varrho_1(\omega_0)/\varrho(\omega_0)]|^2) = I$; so

$$J_0J|[\varrho_1(\omega_0)/\varrho(\omega_0)]|^2\xi(\varrho(\omega_0)) = \xi(\varrho(\omega_0)),$$

which implies that $|[\varrho_1(\omega_0)/\varrho(\omega_0)]|^2$ is the identity, $\varrho_1(\omega_0) = \varrho(\omega_0)$ and finally $\varrho_1 = \varrho$.

The converse implication is trivial.

6.9. Theorem. *Let $\xi(\varrho(\omega_0))$ be cyclic and separating for both M and M_0. Then there is no canonical ϱ_1 in $R(M, M_0)$ dominated by ϱ except ϱ itself.*

Proof. In this case E is the identity, $LR(\varrho(\omega_0)) = M$ and therefore any ϱ_1 in $R(M, M_0)$ dominated by ϱ satisfies the hypothesis in Proposition 6.8.

6.10. REMARK. In general a necessary and sufficient condition for $\xi(\omega)$ to be cyclic and separating for both M and M_0 is that $LR(\omega) = M$.

Theorem 6.9 shows how far the subset of $R(M, M_0)$ of its elements dominated by ϱ can be from being dense in $R(M, M_0)$, while this is the case for $m(\omega)$ in $(M_*)^+$.

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Interpolation of the measure of non-compactness by the real method

by

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Dedicated to Professor David E. Edmunds on the occasion of his 65th birthday

Abstract. We investigate the behaviour of the measure of non-compactness of an operator under real interpolation. Our results refer to general Banach couples. An application to the essential spectral radius of interpolated operators is also given.

Introduction. In 1960 Krasnosel'skiĭ [11] proved that compactness of an operator can be interpolated between L_p-spaces. A motivation for this result might have been a remark by S. G. Kreĭn on the interpolation character that certain compactness results for integral operators between L_p-spaces established by Kantorovich in 1956 seemed to have (see [12], p. 118).

At the beginning of the sixties, with the foundation of abstract interpolation theory, Krasnosel'skiĭ's result led to the investigation of interpolation properties of compact operators between abstract Banach spaces. The main contributions during that period are due to J. L. Lions, J. Peetre, E. Gagliardo, A. Calderón, A. Persson, S. G. Kreĭn, Yu. I. Petunin and K. Hayakawa (see [2] and [16] for precise references).

More recently, the paper [4] by Cobos, Edmunds and Potter opened a new research period in this area, and in 1992, culminating the efforts of several authors (see the paper [6] by Cobos and Peetre for references) M. Cwikel [9] proved that compactness of an operator can be interpolated between any Banach couples by the real method.

In the present paper we investigate the behaviour under real interpolation of the measure of non-compactness, a concept that means more than only continuity but not so much as compactness. Previous results on this

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