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Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets

by

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Abstract. This work deals with various questions concerning Fourier multipliers on L^p, Schur multipliers on the Schatten class S^p as well as their completely bounded versions when L^p and S^p are viewed as operator spaces. For this purpose we use subsets of $\mathbb{Z}$ enjoying the non-commutative $\Lambda(p)$-property which is a new analytic property much stronger than the classical $\Lambda(p)$-property. We start by studying the notion of non-commutative $\Lambda(p)$-sets in the general case of an arbitrary discrete group before turning to the group $\mathbb{Z}$.

0. INTRODUCTION, BACKGROUND AND NOTATION

$M(L^p)$ stands for the algebra of all Fourier multipliers on the space L^p, and $M_{\mathrm{cb}}(L^p)$ for the algebra of those Fourier multipliers which are completely bounded on L^p when the latter is endowed with its natural operator space structure. $M(S^p)$ denotes the algebra of all Schur multipliers on the Schatten class S^p, and $M_{\mathrm{cb}}(S^p)$ the algebra of those Schur multipliers which are completely bounded on S^p equipped with its natural operator space structure.

Our first motivation was to show that the following contractive inclusion maps are all strict:

$$M_{\mathrm{cb}}(L^q) \subset M_{\mathrm{cb}}(L^p), \quad M(S^q) \subset M(S^p), \quad M_{\mathrm{cb}}(S^q) \subset M_{\mathrm{cb}}(S^p),$$
$$(M(L^\infty), M(L^2))_\theta \subset M_{\mathrm{cb}}(L^p), \quad (M(S^\infty), M(S^2))_\theta \subset M_{\mathrm{cb}}(S^p),$$

where in the first three inclusions p is an even integer and $2 < p < q \le \infty$, while in the last two $0 < \theta < 1$ is arbitrary and $p = 2/\theta$. The reader should note that the embeddings and isomorphisms we consider are natural in the sense that they send a given element simply to itself.

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For this purpose we introduce and study a non-commutative version of the usual $\Lambda(p)$-sets. The idea behind all the proofs is the existence for each even integer $2 < p < \infty$ of a non-commutative $\Lambda(p)$-set which is not a $\Lambda(q)$-set for any $q > p$.

In the rest of this section, we recall the facts and notations we need.

Section 1 is devoted to the study of the non-commutative $\Lambda(p)$-property in an arbitrary discrete group G. This is a new analytic property more restrictive in general than the classical $\Lambda(p)$-property. We start by recalling the definition of $\Lambda(p)$-sets and we point out their relationship to the set $M(L^p(\tau_0))$ of all Fourier multipliers on $L^p(\tau_0)$, the non-commutative L^p-space associated with a discrete group G equipped with its usual trace τ_0. Then we introduce the non-commutative $\Lambda(p)$-sets. We point out their relationship to the set $M_{\mathrm{cb}}(L^p(\tau_0))$ of all completely bounded Fourier multipliers on $L^p(\tau_0)$ when the latter is endowed with its natural operator space structure. This justifies the terminology "$\Lambda(p)_{\mathrm{cb}}$-sets" we use for "non-commutative $\Lambda(p)$-sets". The links between $\Lambda(p)$-sets and the algebra $M(L^p(\tau_0))$ on the one hand and between $\Lambda(p)_{\mathrm{cb}}$-sets and $M_{\mathrm{cb}}(L^p(\tau_0))$ on the other hand are proved by using the non-commutative version of the Khinchin inequalities proved in [26] (see also [27]). Then for all integers p we consider two combinatorial properties defined on subsets of G: the $B(p)$-property and the $Z(p)$-property. We show that the $B(p)$-property implies the $Z(p)$-property and that the $Z(p)$-property implies the $\Lambda(2p)_{\mathrm{cb}}$-property; the latter result is the crucial point of this work.

In Section 2, we consider the $\Lambda(p)_{\mathrm{cb}}$-property in the particular case of the group $\mathbb{Z}$. We prove that this property is very different from the usual $\Lambda(p)$-property. More precisely, we prove that there exists a set which is $\Lambda(p)$ for each $2 < p < \infty$ but not $\Lambda(p)_{\mathrm{cb}}$ for any $2 < p < \infty$. Then we show that for each even integer $p > 2$ there exists a $\Lambda(p)_{\mathrm{cb}}$-set which is not a $\Lambda(q)$-set for any $q > p$; this kind of sets will play a key rôle in the proofs.

In Section 3, we focus on Fourier multipliers. We prove that for $2 \le p < \infty$ an even integer, $M_{\mathrm{cb}}(L^p)$ cannot embed continuously into $M(L^q)$ for any $p < q \le \infty$. Recall that for $p = 2/\theta$ and $0 < \theta < 1$, the embedding of $(M(L^\infty), M(L^2))_\theta$ into $M(L^p)$ is strict (see [45], see also [40]). Then since as we recall the embedding of $M_{\mathrm{cb}}(L^p)$ into $M(L^p)$ is strict for any $2 < p < \infty$, it is natural to wonder whether the embedding of $(M(L^\infty), M(L^2))_\theta$ into $M_{\mathrm{cb}}(L^p)$ is again strict. We prove that this is indeed the case. More precisely, we show that $M_{\mathrm{cb}}(L^p)$ does not embed continuously into $(M(L^\infty), M(L^2))_\theta$ for any $0 < \theta < 1$.

In Section 4, we introduce and study the so-called $\sigma(p)$-sets and $\sigma(p)_{\mathrm{cb}}$-sets. These are subsets of $\mathbb{N} \times \mathbb{N}$ playing for $M(S^p)$ and $M_{\mathrm{cb}}(S^p)$ a rôle analogous to the one played by $\Lambda(p)$-sets and $\Lambda(p)_{\mathrm{cb}}$-sets for $M(L^p)$ and $M_{\mathrm{cb}}(L^p)$ respectively. We will see that from any given $\Lambda(p)_{\mathrm{cb}}$-set, we can obtain a $\sigma(p)_{\mathrm{cb}}$-set and thus we get for even integers p special $\sigma(p)_{\mathrm{cb}}$-sets. Indeed, we prove that for any even integer $p > 2$, there is a $\sigma(p)_{\mathrm{cb}}$-set $A \subset \mathbb{N} \times \mathbb{N}$ which is not a $\sigma(q)$-set for any $q > p$.

Section 5 is devoted to Schur multipliers. For each even integer $2 < p < \infty$, we prove the existence of an idempotent Schur multiplier which is completely bounded on S^p but not bounded on S^q for any $p < q \le \infty$. In fact, our idempotent Schur multiplier is not even bounded on the subspace of S^q formed by all Hankelian operators, denoted by $\mathfrak{S}^q$ in the sequel. This answers a question raised by J. Erdos. Therefore, the embeddings $M(S^q) \subset M(S^p)$ and $M_{\mathrm{cb}}(S^q) \subset M_{\mathrm{cb}}(S^p)$ are strict whenever $2 \le p < q \le \infty$ and p is an even integer. On the other hand, we show that for each $2 < p < \infty$, the set $M_{\mathrm{cb}}(S^p)$ does not embed continuously into $(M(S^\infty), M(S^2))_\theta$ for any $0 < \theta < 1$. This answers a question raised by V. Peller. We also establish links between Fourier and Schur multipliers as follows. Let $M(H^p)$ (resp. $M_{\mathrm{cb}}(H^p)$) be the algebra of Fourier multipliers (resp. completely bounded Fourier multipliers) on the Hardy space H^p, and let $M(\mathfrak{S}^p)$ and $M_{\mathrm{cb}}(\mathfrak{S}^p)$ be the corresponding algebras of Schur multipliers on $\mathfrak{S}^p$. The spaces H^p and $\mathfrak{S}^p$ are viewed as operator subspaces of L^p and S^p respectively. We show that $M(H^p)$ can be injected continuously into $M(\mathfrak{S}^p)$ in the same way as $M_{\mathrm{cb}}(H^p)$ is injected into $M_{\mathrm{cb}}(S^p)$. For this purpose, we are led to characterize the multipliers of $M(\mathfrak{S}^p)$ and $M_{\mathrm{cb}}(\mathfrak{S}^p)$ (our characterizations are easy consequences of [29], [30]).

Section 6 is included for the sake of completeness. Using probabilistic ideas, we exhibit a very "large" $Z(2)$-set, roughly the "largest" possible one which enjoys some additional properties. On the other hand, we introduce some simple combinatorial properties on the subsets of $\mathbb{N} \times \mathbb{N}$ ensuring the $\sigma(4)_{\mathrm{cb}}$ property; we call them property (C) and property (R). Then by using similar probabilistic ideas, we exhibit "large" sets satisfying one of these combinatorial properties.

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We now review the standard notation we use. Let E and F be Banach spaces. We denote by $E \otimes F$ the algebraic tensor product of E and F, and by $\mathcal{B}(E, F)$ the set of all bounded operators from E to F. $\mathcal{B}(E, E)$ is abbreviated to $\mathcal{B}(E)$. B_E stands for the open unit ball of E. id_E denotes the identity map on E. If E_i, F_i are Banach spaces and u_i is in $\mathcal{B}(E_i, F_i)$ for $i = 0, 1$ then $u_0 \otimes u_1$ denotes the operator which carries $x \otimes y$ in $E_0 \otimes E_1$ to $u_0(x) \otimes u_1(y)$ in $F_0 \otimes F_1$, extended linearly.

A contractive map $u : E \to F$ is said to be *μ-surjective* if $u(\mu B_E) \supset B_F$. 1-surjective maps are called *metric surjections.*

For $1 \le p \le \infty$, a measurable space (Ω, ν) and an arbitrary Banach space E, we let $L^p(\Omega, d\nu, E)$ be the set of all E-valued functions f on Ω which are Bochner measurable and such that

$$\|f\|_{L^p(\Omega,d\nu,E)} := \Big(\int_\Omega \|f(t)\|^p\, d\nu\Big)^{1/p} < \infty.$$

If Ω is the torus $\mathbb{T}$ and ν is the normalized Lebesgue measure, then $L^p(\mathbb{T}, d\nu, E)$ is simply denoted by $L^p(E)$. We let $H^p(E)$ be the E-valued Hardy space, consisting of all f in $L^p(E)$ such that the Fourier coefficients $\widehat{f}(n) = 0$ for all integers $n < 0$. $L^p(\mathbb{C})$ and $H^p(\mathbb{C})$ are simply denoted by L^p and H^p respectively.

More generally, let M be a von Neumann algebra endowed with a normal, faithful and semi-finite trace τ_M. For $1 \le p < \infty$, $L^p(\tau_M)$ denotes the *non-commutative L^p-space* associated with M equipped with τ_M. By definition, this is the Banach space obtained from the space of all x in M satisfying $\|x\|_{L^p(\tau_M)} := \tau_M((x^*x)^{p/2})^{1/p} < \infty$ after completion with respect to the norm $\|\cdot\|_{L^p(\tau_M)}$ (cf. [16], [28], [38]). By convention, $L^\infty(\tau_M)$ denotes M. The non-commutative L^p-space associated with $\mathcal{B}(H)$, where H denotes a separable Hilbert space, equipped with its usual trace, is nothing but the p-Schatten class on H. It will be denoted by $S^p(H)$ when $1 \le p < \infty$. $S^\infty(H)$ stands for the set of all compact operators on H. In the particular case $H = \ell_2$ (resp. ℓ_2^n, the n-dimensional Hilbert space), the usual trace on $\mathcal{B}(\ell_2)$ (resp. $M_n := \mathcal{B}(\ell_2^n)$) is denoted by tr (resp. tr_n) and the space $S^p(H)$ is simply denoted by S^p (resp. S_n^p) for each $1 \le p \le \infty$.

If τ_M and τ_N are normal, faithful and semi-finite traces on von Neumann algebras M and N respectively, then we let $\tau_M \otimes \tau_N$ denote the trace on the von Neumann algebra generated by $M \otimes N$ defined by: $\tau_M \otimes \tau_N(x \otimes y) := \tau_M(x)\tau_N(y)$ for $x \in M$ and $y \in N$. Then $\tau_M \otimes \tau_N$ is still normal, faithful and semi-finite, and thus we can consider unambiguously the space $L^p(\tau_M \otimes \tau_N)$.

Given a discrete group G, λ denotes the left regular representation of G into $\mathcal{B}(\ell_2(G))$, $L^p(\tau_0)$ denotes the non-commutative L^p-space associated with the von Neumann algebra generated by $\lambda(G)$ with respect to its usual trace denoted by τ_0, and $L^p(\tau)$ denotes the non-commutative L^p-space associated with the von Neumann algebra generated by $\lambda(G) \otimes \mathcal{B}(\ell_2)$ with respect to the trace $\tau = \tau_0 \otimes \mathrm{tr}$.

Given M and τ_M as above, the spaces $L^\infty(\tau_M \otimes \mathrm{tr})$ and $L^1(\tau_M \otimes \mathrm{tr})$ form a compatible couple for complex interpolation and we have isometrically (cf. [20])

$$\forall 1 < p < \infty, \qquad L^p(\tau_M \otimes \mathrm{tr}) = (L^\infty(\tau_M \otimes \mathrm{tr}), L^1(\tau_M \otimes \mathrm{tr}))_{1/p}.$$

This allows us to view the Banach spaces $L^p(\tau_M)$ as operator spaces in a natural way (cf. [33], [34]).

0.1. Complex interpolation. Let (E_0, E_1) be a compatible couple of Banach spaces, i.e. E_0 and E_1 are both continuously injected into the same topological space. Let

$$\Delta := \{z \in \mathbb{C} \mid 0 \le \mathrm{Re}(z) \le 1\}, \qquad \Delta_j := \{z \in \mathbb{C} \mid \mathrm{Re}(z) = j\}$$

for $j = 0, 1$. Then let $\mathcal{G}(E_0, E_1)$ be the set of all functions f of the form $f = \sum_{\text{finite}} f_k x_k$ where the x_k's are in $E_0 \cap E_1$, the functions $f_k : \Delta \to \mathbb{C}$ are continuous on Δ and analytic on its interior and vanishing at infinity. Denote by $\mathcal{F}(E_0, E_1)$ the completion of $\mathcal{G}(E_0, E_1)$ for the norm

$$\|f\| := \max\{\sup_{z\in\Delta_0} \|f(z)\|_{E_0}, \sup_{z\in\Delta_1} \|f(z)\|_{E_1}\}.$$

For $0 < \theta < 1$, consider the subset $\mathcal{N}_\theta(E_0, E_1)$ of $\mathcal{G}(E_0, E_1)$ of all functions which vanish at θ and let $\mathcal{S}_\theta(E_0, E_1)$ be its closure in $\mathcal{F}(E_0, E_1)$. By definition, the intermediate space E_θ obtained by complex interpolation between E_0 and E_1 corresponding to θ is the Banach space $\mathcal{F}(E_0, E_1)/\mathcal{S}_\theta(E_0, E_1)$ equipped with the quotient norm $\|\cdot\|_\theta$. We refer the reader to [39] for the proof that this definition of complex interpolation coincides with the one given in [2].

LEMMA 0.1. *Let (E_0, E_1) and (F_0, F_1) be two compatible couples such that $E_0 \cap E_1$ is dense in both E_0 and E_1. Then $(\mathcal{B}(E_0, F_0), \mathcal{B}(E_1, F_1))_\theta$ embeds contractively into $\mathcal{B}(E_\theta, F_\theta)$ for each $0 < \theta < 1$.*

Proof. Let E be the completion of $E_0 \cap E_1$ for the norm $\|x\|_E = \max\{\|x\|_{E_0}, \|x\|_{E_1}\}$ and F be any Banach space containing continuously F_0 and F_1. The assumption on (E_0, E_1) permits injecting continuously both $\mathcal{B}(E_0, F_0)$ and $\mathcal{B}(E_1, F_1)$ into $\mathcal{B}(E, F)$. Thus they form a compatible couple.

By density of $\mathcal{B}(E_0, F_0) \cap \mathcal{B}(E_1, F_1)$ in $(\mathcal{B}(E_0, F_0), \mathcal{B}(E_1, F_1))_\theta$ and of $E_0 \cap E_1$ in E_θ, we are reduced to showing that $\|Tx\|_\theta \le \|T\|_\theta \|x\|_\theta$ for each T in $\mathcal{B}(E_0, F_0) \cap \mathcal{B}(E_1, F_1)$ and x in $E_0 \cap E_1$. To check this, let φ be in $\mathcal{G}(\mathcal{B}(E_0, F_0), \mathcal{B}(E_1, F_1))$, f in $\mathcal{G}(E_0, E_1)$ such that $\varphi(\theta) = T$ and $f(\theta) = x$, and consider the function g which takes z in Δ to $\varphi(z)(f(z))$ in $F_0 \cap F_1$. Clearly g belongs to $\mathcal{G}(F_0, F_1)$ with $g(\theta) = Tx$. Moreover,

$$\|g\| = \max_{j=0,1} \sup_{z\in\Delta_j} \|\varphi(z)(f(z))\|_{F_j} \le \max_{j=0,1} \sup_{z\in\Delta_j} \|\varphi(z)\|_{\mathcal{B}(E_j,F_j)} \|f(z)\|_{E_j}$$
$$\le (\max_{j=0,1} \sup_{z\in\Delta_j} \|\varphi(z)\|_{\mathcal{B}(E_j,F_j)})(\max_{j=0,1} \sup_{z\in\Delta_j} \|f(z)\|_{E_j}) = \|\varphi\|\|f\|.$$

This gives the required inequality after taking the infimum over all such φ's and f's. ∎

0.2. Operator spaces. By an *operator space* E, we mean a closed subspace of $\mathcal{B}(H)$ for some Hilbert space H. Such an object has natural norms $\|\cdot\|_n$ on $M_n(E)$, the set of $n\times n$ matrices with entries in E. Indeed, $M_n(E)$ can be viewed as a subspace of $\mathcal{B}(\ell_2^n(H))$ via the natural identification between $M_n(\mathcal{B}(H))$ and $\mathcal{B}(\ell_2^n(H))$. This sequence of norms satisfies Ruan's axioms:

$$\forall a,b\in M_n, \forall x\in M_n(E)\qquad \|a\cdot x\cdot b\|_n\le \|a\|_{M_n}\|x\|_n\|b\|_{M_n},$$
$$\forall x\in M_n(E), \forall y\in M_m(E)\qquad \|x\oplus y\|_{n+m}=\max\{\|x\|_n,\|y\|_m\}.$$

Here the norm on M_n is the operator norm, the $\oplus$ denotes the direct sum of matrices and the dot denotes the usual matrix product.

In the operator setting, a map $u:E\to F$ is said to be c.b. (short for *completely bounded*) if the maps

$$u^n: M_n(E)\to M_n(F),\qquad (x_{ij})_{i,j}\mapsto (u(x_{ij}))_{i,j},$$

are uniformly bounded. We let $\mathcal{CB}(E,F)$ stand for the space of all c.b. maps endowed with the norm

$$\|u\|_{\mathrm{cb}}=\sup_n\|u^n\|.$$

$\mathcal{CB}(E)$ will stand for $\mathcal{CB}(E,E)$. An operator u is said to be a *complete contraction* (resp. *complete isometry*) if each u^n is contractive (resp. isometric).

Z. Ruan gave an abstract characterization of an operator space as a Banach space E with a sequence of norms on the $M_n(E)$'s which satisfy Ruan's axioms (see [36]). This abstract characterization allows defining for operator spaces the notions of duality, complex interpolation etc.

The *standard dual* of an operator space E is the usual Banach space E^* with the norms corresponding to the isometric identifications of $M_n(E^*)$ with $\mathcal{CB}(E,M_n)$ as in [4] and [17]. The complex interpolated space between two operator spaces E_0 and E_1 compatible as Banach spaces is the usual Banach space E_θ with the norms corresponding to the isometric identifications $M_n(E_\theta):=(M_n(E_0),M_n(E_1))_\theta$ (see [33]).

When E and F are two operator spaces, $\mathcal{CB}(E,F)$ is an operator space for the structure corresponding to the isometric identifications $M_n(\mathcal{CB}(E,F)) := \mathcal{CB}(E,M_n(F))$.

Note that the min. (short for *minimal*) tensor product is a very useful tool to describe completely the operator space structure of an operator space as well as the c.b. maps between operator spaces. Let $E\subset\mathcal{B}(H)$ be a concrete operator space. By $S^\infty\otimes_{\min}E$, we mean the completion of $S^\infty\otimes E$ for the norm induced by $\mathcal{B}(\ell_2(H))$. Then the operator space structure of the interpolated space E_θ and the one of the operator space dual E^* are completely described by the following isometric relations:

$$S^\infty\otimes_{\min}E^*\subset\mathcal{CB}(E,S^\infty)\quad\text{and}\quad S^\infty\otimes_{\min}E_\theta=(S^\infty\otimes_{\min}E_0,S^\infty\otimes_{\min}E_1)_\theta.$$

A map $u:E\to F$ is c.b. if and only if $\mathrm{id}_{S^\infty}\otimes u$ extends to a bounded operator from $S^\infty\otimes_{\min}E$ into $S^\infty\otimes_{\min}F$, and we have $\|u\|_{\mathrm{cb}}=\|\mathrm{id}_{S^\infty}\otimes u: S^\infty\otimes_{\min}E\to S^\infty\otimes_{\min}F\|$.

LEMMA 0.2. *Let (E_0,E_1) and (F_0,F_1) be two compatible couples. Assume that $E_0\cap E_1$ is dense in both E_0 and E_1. Then $(\mathcal{CB}(E_0,F_0),\mathcal{CB}(E_1,F_1))_\theta$ embeds completely contractively into $\mathcal{CB}(E_\theta,F_\theta)$ for each $0<\theta<1$.*

Proof. For arbitrary operator spaces E and F we may view $\mathcal{CB}(E,F)$ as a subspace of $\mathcal{B}(S^\infty\otimes_{\min}E,S^\infty\otimes_{\min}F)$ via the isometric embedding which carries $T\in\mathcal{CB}(E,F)$ to $\mathrm{id}_{S^\infty}\otimes T\in\mathcal{B}(S^\infty\otimes_{\min}E,S^\infty\otimes_{\min}F)$.

Now let E_0, E_1, F_0 and F_1 be as above. Lemma 0.1 applied to $(S^\infty\otimes_{\min}E_0,S^\infty\otimes_{\min}E_1)$ and $(S^\infty\otimes_{\min}F_0,S^\infty\otimes_{\min}F_1)$ implies that for each $0<\theta<1$, $(\mathcal{B}(S^\infty\otimes_{\min}E_0,S^\infty\otimes_{\min}F_0),\mathcal{B}(S^\infty\otimes_{\min}E_1,S^\infty\otimes_{\min}F_1))_\theta$ embeds contractively into $\mathcal{B}(S^\infty\otimes_{\min}E_\theta,S^\infty\otimes_{\min}F_\theta)$. This implies that $(\mathcal{CB}(E_0,F_0),\mathcal{CB}(E_1,F_1))_\theta$ embeds contractively into $\mathcal{CB}(E_\theta,F_\theta)$. Actually, the embedding is completely contractive. Indeed, for each integer $n\ge1$,

$$\begin{aligned}M_n(\mathcal{CB}(E_0,F_0),\mathcal{CB}(E_1,F_1))_\theta&=(M_n(\mathcal{CB}(E_0,F_0)),M_n(\mathcal{CB}(E_1,F_1)))_\theta\\&=(\mathcal{CB}(E_0,M_n(F_0)),\mathcal{CB}(E_1,M_n(F_1)))_\theta\\&\subset\mathcal{CB}(E_\theta,(M_n(F_0),M_n(F_1))_\theta)\\&=\mathcal{CB}(E_\theta,M_n(F_\theta)).\end{aligned}$$

Thus the embedding $M_n(\mathcal{CB}(E_0,F_0),\mathcal{CB}(E_1,F_1))_\theta\subset M_n(\mathcal{CB}(E_\theta,F_\theta))$ is contractive. ∎

G. Pisier proved in [34] that in fact the theory of operator spaces can be developed equivalently using other sequences of norms on the $M_n(E)$'s. Indeed, let $1\le p\le\infty$ be a *fixed* number, let E be an operator space and let E^* be its dual operator space. For an integer $n\ge1$, we let $S_n^p[E]$ denote the space $M_n(E)$ but equipped with the norm of

$$S_n^p[E]:=(S_n^\infty[E],S_n^1[E])_\theta$$

where $S_n^\infty[E]$ denotes $M_n(E)$ for convenience only, $S_n^1[E]$ is the space $M_n(E)$ viewed as a subspace of $(M_n(E^*))^*$ and $\theta=1/p$. Note that $S_n^p[E]$ embeds isometrically into $S_{n+1}^p[E]$; we let $S^p[E]$ be the completion of $\bigcup_{n\ge1}S_n^p[E]$.

PROPOSITION 0.3 ([34]). *For all x in $M_n(E)$, we have $\|x\|_{M_n(E)}=\sup\|a\cdot x\cdot b\|_{S_n^p[E]}$ where the supremum is over all a,b in the unit ball of S_n^{2p}. Therefore an operator $u:E\to F$ is c.b. if and only if the maps $u^n:S_n^p[E]\to S_n^p[F]$ are uniformly bounded, in which case we have $\|u\|_{\mathrm{cb}}=\sup_{n\ge1}\|u^n:S_n^p[E]\to S_n^p[F]\|$.*

Now let us go back to the case of non-commutative L^p-spaces. If M is a von Neumann algebra with a normal, faithful and semi-finite trace τ_M then since $L^\infty(\tau_M)$ is a C^*-algebra, it has a natural operator space structure given by any concrete realization as a C^*-subalgebra of some $\mathcal{B}(H)$. Since $L^1(\tau_M)$ coincides with the predual of $L^\infty(\tau_M)$, it also appears as an operator space in a natural way. Indeed, it is a subspace of the standard dual of $L^\infty(\tau_M)$. Hence the spaces $L^p(\tau_M)$ are also canonically endowed with an operator space structure, the one obtained by complex interpolation in the operator space category. Applying Proposition 0.3 we get a nice and simple characterization of the c.b. maps between these spaces since for each integer $n \geq 1$, we have the natural identifications

$$S_n^\infty[L^\infty(\tau_M)] = L^\infty(\tau_M \otimes \mathrm{tr}_n), \qquad S_n^1[L^1(\tau_M)] = L^1(\tau_M \otimes \mathrm{tr}_n).$$

These imply that we have isometrically

$$\begin{aligned} S_n^p[L^p(\tau_M)] &= (S_n^\infty[L^\infty(\tau_M)], S_n^1[L^1(\tau_M)])_\theta \\ &= (L^\infty(\tau_M \otimes \mathrm{tr}_n), L^1(\tau_M \otimes \mathrm{tr}_n))_\theta = L^p(\tau_M \otimes \mathrm{tr}_n). \end{aligned}$$

Therefore a density argument yields $S^p[L^p(\tau_M)] = L^p(\tau_M \otimes \mathrm{tr})$ isometrically. Thus Proposition 0.3 implies

PROPOSITION 0.4. *Let $1 \leq p < \infty$, $L^p(\tau_M)$ and $L^p(\tau_N)$ be two non-commutative L^p-spaces and $E \subset L^p(\tau_M)$, $F \subset L^p(\tau_N)$ arbitrary operator subspaces. Then an operator $u : E \to F$ is c.b. if and only if the operator $u \otimes \mathrm{id}_{S^p} : E \otimes S^p \to F \otimes S^p$ which takes $x \otimes y$ to $u(x) \otimes y$ where $x \in E$ and $y \in S^p$, extends to a bounded operator from $\overline{E \otimes S^p}^{L^p(\tau_M \otimes \mathrm{tr})}$ into $\overline{F \otimes S^p}^{L^p(\tau_N \otimes \mathrm{tr})}$. Moreover, $\|u\|_{\mathrm{cb}} = \|u \otimes \mathrm{id}_{S^p} : \overline{E \otimes S^p}^{L^p(\tau_M \otimes \mathrm{tr})} \to \overline{F \otimes S^p}^{L^p(\tau_N \otimes \mathrm{tr})}\|$.*

0.3. Non-commutative Khinchin inequalities. Let $\varepsilon_n : \{-1,1\}^{\mathbb{N}} \to \{-1,1\}$ be the nth coordinate projection, ν the uniform probability measure on $\{-1,1\}^{\mathbb{N}}$ and $1 \leq p < \infty$ an arbitrary real number. In the commutative case, the classical Khinchin inequalities say that there exists a constant $k_p > 0$ depending only on p such that for all integers $n \geq 1$ and all scalars $x_1, \ldots, x_n$ we have

$$(0.1) \qquad \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L^p(\{-1,1\}^{\mathbb{N}}, \nu)} \begin{cases} \geq k_p (\sum_{j=1}^n |x_j|^2)^{1/2} & \text{when } 1 \leq p \leq 2, \\ \leq k_p (\sum_{j=1}^n |x_j|^2)^{1/2} & \text{when } 2 \leq p < \infty. \end{cases}$$

See e.g. [23] for the proof. Later on these inequalities were generalized to the non-commutative case by F. Lust-Piquard for $1 < p < \infty$ (cf. [26]) and by F. Lust-Piquard and G. Pisier for $p = 1$ (cf. [27]) as follows. Let M be a von Neumann algebra with a normal, faithful and semi-finite trace τ_M. For each $1 \leq p < \infty$, there exists a positive constant $K_{L^p(\tau_M)}$ depending only on the pair (M, τ_M) and p such that for all $n \geq 1$ in $\mathbb{N}$ and all $x_1, \ldots, x_n$ in $L^p(\tau_M)$ we have, in the case of $1 \leq p \leq 2$,

$$(0.2) \qquad \begin{aligned} &\Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L^p(\{-1,1\}^{\mathbb{N}}, \nu, L^p(\tau_M))} \\ &\geq K_{L^p(\tau_M)} \inf \Big\{ \Big\| \Big(\sum_{j=1}^n y_j y_j^* \Big)^{1/2} \Big\|_{L^p(\tau_M)} + \Big\| \Big(\sum_{j=1}^n z_j^* z_j \Big)^{1/2} \Big\|_{L^p(\tau_M)} \Big\} \end{aligned}$$

where the infimum is over all decompositions of the x_j's in $L^p(\tau_M)$ as $y_j + z_j$, while

$$(0.3) \qquad \begin{aligned} &\Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L^p(\{-1,1\}^{\mathbb{N}}, \nu, L^p(\tau_M))} \\ &\leq K_{L^p(\tau_M)} \max \Big\{ \Big\| \Big(\sum_{j=1}^n x_j x_j^* \Big)^{1/2} \Big\|_{L^p(\tau_M)}, \Big\| \Big(\sum_{j=1}^n x_j^* x_j \Big)^{1/2} \Big\|_{L^p(\tau_M)} \Big\} \end{aligned}$$

in the case of $2 \leq p < \infty$. In the particular case of S^p the constant K_{S^p} will be denoted by K_p for simplicity.

0.4. An explicit description of $S^{p,\mathrm{unc}}$ and $S^{p,\mathrm{unc}}(S^p)$. An operator in $\mathcal{B}(\ell_2)$ will frequently be identified with its matrix relative to the canonical basis of ℓ_2. Let Ω_0 be the set $\{-1,1\}^{\mathbb{N} \times \mathbb{N}}$, ν be the uniform probability measure on Ω_0 and let $\varepsilon_{ij} : \Omega_0 \to \{-1,1\}$ be the (i,j)th coordinate projection. For $1 \leq p \leq \infty$, $S^{p,\mathrm{unc}}$ denotes the space of all operators $x = (x_{ij})_{i,j}$ in S^∞ such that the operators $(\varepsilon_{ij} x_{ij})_{i,j}$ belong to S^p for almost all choices of signs $(\varepsilon_{ij})_{i,j}$ on $\mathbb{N} \times \mathbb{N}$, equipped with the norm

$$\|x\|_{S^{p,\mathrm{unc}}} := \|(\varepsilon_{ij} x_{ij})_{i,j}\|_{L^p(\Omega_0, \nu, S^p)}.$$

We denote by $S^p(S^p)$ the set of all matrices $x = (x_{ij})_{i,j}$ with entries x_{ij} in S^p and which are—when viewed as operators on $\ell_2(\ell_2)$—in the p-Schatten class on the Hilbert space $\ell_2(\ell_2)$, equipped with the inherited norm (note that $S^p(S^p)$ is exactly the p-Schatten class on $\ell_2(\ell_2)$ via the identification mentioned above). Then, similarly, we let $S^{p,\mathrm{unc}}(S^p)$ be the set of all operators $x = (x_{ij})_{i,j}$ in $S^\infty(S^\infty)$ with entries in S^∞ such that the operators $(\varepsilon_{ij} x_{ij})_{i,j}$ are in $S^p(S^p)$ for almost all choices of signs $(\varepsilon_{ij})_{i,j}$ on $\mathbb{N} \times \mathbb{N}$, equipped with the norm

$$\|x\|_{S^{p,\mathrm{unc}}(S^p)} := \|(\varepsilon_{ij} x_{ij})_{i,j}\|_{L^p(\Omega_0, \nu, S^p(S^p))}.$$

The next result essentially goes back to F. Lust-Piquard [26].

LEMMA 0.5. *There is an explicit description of the space $S^{p,\mathrm{unc}}$ (resp. $S^{p,\mathrm{unc}}(S^p)$) for each $1 \leq p < \infty$ as follows. For all $x = (x_{ij})_{i,j}$ in $S^{p,\mathrm{unc}}$ (resp. $S^{p,\mathrm{unc}}(S^p)$) we have, for $2 \leq p < \infty$,*

$$\|x\|_{S^{p,\mathrm{unc}}} \cong \max \Big\{ \Big(\sum_i \Big(\sum_j |x_{ij}|^2 \Big)^{p/2} \Big)^{1/p}, \Big(\sum_j \Big(\sum_i |x_{ij}|^2 \Big)^{p/2} \Big)^{1/p} \Big\},$$

resp.

$$\|x\|_{S^{p,\mathrm{unc}}(S^p)} \cong \max\Big\{\Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}, \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\},$$

while for $1 \le p \le 2$,

$$\|x\|_{S^{p,\mathrm{unc}}} \cong \inf\Big\{\Big(\sum_i \Big(\sum_j |y_{ij}|^2\Big)^{p/2}\Big)^{1/p} + \Big(\sum_j \Big(\sum_i |z_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\},$$

resp.

$$\|x\|_{S^{p,\mathrm{unc}}(S^p)} \cong \inf\Big\{\Big(\sum_j \Big\|\Big(\sum_i y_{ij}^* y_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p} + \Big(\sum_i \Big\|\Big(\sum_j z_{ij} z_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\},$$

where the infimum is over all possible decompositions of x as a sum of $y = (y_{ij})_{i,j}$ and $z = (z_{ij})_{i,j}$, both in S^p (resp. $S^p(S^p)$).

Proof. We prove the lemma for $S^{p,\mathrm{unc}}(S^p)$ when $2 \le p < \infty$ only; the other cases are quite similar and left to the reader. We start by recalling that for each $2 \le p \le \infty$ and each $x = (x_{ij})_{i,j}$ in $S^p(S^p)$ we have

$$\|x\|_{S^p(S^p)} \ge \max\Big\{\Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}, \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\}. \tag{0.4}$$

Indeed, this holds for $p = 2$ and $p = \infty$. Therefore, by complex interpolation, it is also satisfied for all $2 < p < \infty$. Now fix $2 \le p < \infty$. If $x = (x_{ij})_{i,j} \in S^{p,\mathrm{unc}}(S^p)$, then the $\infty \times \infty$ matrices $(\varepsilon_{ij} x_{ij})_{i,j}$ satisfy (0.4) for almost all choices of signs $(\varepsilon_{ij})_{i,j}$ on Ω_0 since the matrices belong to $S^p(S^p)$ almost surely. Thus after integrating over all these choices of signs, for each $x = (x_{ij})_{i,j}$ in $S^{p,\mathrm{unc}}(S^p)$ we get

$$\|x\|_{S^{p,\mathrm{unc}}(S^p)} \ge \max\Big\{\Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}, \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\}.$$

The converse inequality is obtained with the help of (0.3). Indeed,

$$\begin{aligned}
\|x\|_{S^{p,\mathrm{unc}}(S^p)} &= \Big(\int_{\Omega_0} \Big\|\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \varepsilon_{ij} x_{ij} \otimes e_{ij}\Big\|_{S^p(S^p)}^p d\nu\Big)^{1/p}\\
&\le K_p \max\Big\{\Big\|\Big(\sum_{i,j} (x_{ij}\otimes e_{ij})^*(x_{ij}\otimes e_{ij})\Big)^{1/2}\Big\|_{S^p(S^p)}, \Big\|\Big(\sum_{i,j} (x_{ij}\otimes e_{ij})(x_{ij}\otimes e_{ij})^*\Big)^{1/2}\Big\|_{S^p(S^p)}\Big\}\\
&= K_p \max\Big\{\Big\|\Big(\sum_{i,j} x_{ij}^* x_{ij} \otimes e_{jj}\Big)^{1/2}\Big\|_{S^p(S^p)}, \Big\|\Big(\sum_{i,j} x_{ij} x_{ij}^* \otimes e_{ii}\Big)^{1/2}\Big\|_{S^p(S^p)}\Big\}\\
&= K_p \max\Big\{\Big\|\sum_j \Big(\sum_i x_{ij}^* x_{ij}\Big)^{1/2} \otimes e_{jj}\Big\|_{S^p(S^p)}, \Big\|\sum_i \Big(\sum_j x_{ij} x_{ij}^*\Big)^{1/2} \otimes e_{ii}\Big\|_{S^p(S^p)}\Big\}\\
&\le K_p \max\Big\{\Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}, \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\}. \ \blacksquare
\end{aligned}$$

0.5. Additional non-standard notations. In the sequel, we frequently use the following definitions and notations.

A matrix $x = (x_{kl})_{k,l}$ with entries x_{kl} in some fixed set is said to be *Hankelian* if $x_{kl} = x_{k'l'}$ whenever $k + l = k' + l'$.

For $1 \le p < \infty$, $\mathfrak{S}^p$ (resp. $\mathfrak{S}^p(S^p)$) stands for the subspace of S^p (resp. $S^p(S^p)$) formed by all Hankelian matrices $x = (x_{kl})_{k,l}$ in S^p (resp. $S^p(S^p)$).

For a set Λ, $\mathbf{1}_\Lambda$ stands for its indicator function, and $|\Lambda|$ is the cardinality of Λ.

If Λ is a subset of a discrete group G and if $\mathcal{F}$ denotes either $L^p(\tau_0)$ or $L^p(\tau)$ for some $1 \le p \le \infty$ then we let

$$\mathcal{F}_\Lambda := \{f \in \mathcal{F} \mid \widehat{f}(t) = 0,\ \forall t \in G \setminus \Lambda\}.$$

Recall that for f in $L^p(\tau_0)$ (resp. $L^p(\tau)$) the *Fourier coefficient* $\widehat{f}(t)$ is defined as follows:

$$\widehat{f}(t) = \tau_0[\lambda(t^{-1})f] \qquad (\text{resp. } \widehat{f}(t) = \tau_0 \otimes \mathrm{id}_{S^p}[(\lambda(t^{-1}) \otimes \mathrm{id}_{\ell_2})f]).$$

We denote $\mathcal{F}_G$ simply by $\mathcal{F}$, and when $G=\mathbb{Z}$ and $\Lambda=\mathbb{N}$, $L^p_{\mathbb{N}}$ will be still denoted by H^p. Similarly when $\mathcal{F}$ is a class of $\infty\times\infty$ matrices and A is a subset of $\mathbb{N}\times\mathbb{N}$ we let

$$\mathcal{F}_A := \{x=(x_{kl})_{k,l}\in\mathcal{F} \mid x_{kl}=0,\ \forall (k,l)\in\mathbb{N}\times\mathbb{N}\setminus A\}.$$

$\mathcal{F}_{\mathbb{N}\times\mathbb{N}}$ is denoted simply by $\mathcal{F}$. Moreover when $\mathcal{F}$ is a Banach or an operator space the sets $\mathcal{F}_\Lambda$ and $\mathcal{F}_A$ are automatically viewed as Banach or operator subspaces of $\mathcal{F}$.

For $\Lambda\subset\mathbb{N}$, we set $\widehat{\Lambda}:=\{(k,l)\in\mathbb{N}\times\mathbb{N}\mid k+l\in\Lambda\}$, and any subset of $\mathbb{N}\times\mathbb{N}$ which can be written as $\widehat{\Lambda}$ for some $\Lambda\subset\mathbb{N}$ is called a *Hankelian set.* Then given a map $\varphi:\Lambda\to\mathbb{C}$ we let

$$\widehat{\varphi}:\widehat{\Lambda}\to\mathbb{C},\qquad (k,l)\mapsto\varphi(k+l).$$

For an analytic function f on $\mathbb{T}$ and all z in $\mathbb{T}$ we let $f_{(0)}(z):=\widehat{f}(0)$, while for all integers $n\ge1$,

$$f_{(n)}(z):=\sum_{k=2^{n-1}}^{2^n-1}\widehat{f}(k)z^k.$$

Similarly given an $\infty\times\infty$ matrix x we let $x_{(0)}:=(x_{00})$ while for all integers $n\ge1$ we let

$$x_{(n)}:=x\mathbf{1}_{\{(k,l)\,|\,2^{n-1}\le k+l<2^n\}}\qquad\text{(Schur product)}.$$

0.6. Peller's theorem. The aim of Peller's theorem is to realize $\mathfrak{S}^p$ and more generally $\mathfrak{S}^p(S^p)$ as a space of functions on the torus $\mathbb{T}$ which we will describe here for $1<p<\infty$ only. Consider the Banach spaces (Besov spaces)

$$\mathcal{A}^p:=\{f:\mathbb{T}\to\mathbb{C}\text{ analytic}\mid \|f\|_{\mathcal{A}^p}<\infty\},$$
$$\mathcal{A}^p(S^p):=\{g:\mathbb{T}\to S^p\text{ analytic}\mid \|g\|_{\mathcal{A}^p(S^p)}<\infty\},$$

where

$$\|f\|_{\mathcal{A}^p}:=\Big(\sum_{n=0}^\infty 2^n\|f_{(n)}\|_{L^p}^p\Big)^{1/p},\qquad \|g\|_{\mathcal{A}^p(S^p)}:=\Big(\sum_{n=0}^\infty 2^n\|g_{(n)}\|_{L^p(S^p)}^p\Big)^{1/p}.$$

THEOREM 0.6. *The following maps are well defined, bounded and bijective*:

$$\mathcal{A}^p\to\mathfrak{S}^p,\qquad f\mapsto(\widehat{f}(k+l))_{k,l\ge0},$$
$$\mathcal{A}^p(S^p)\to\mathfrak{S}^p(S^p),\qquad g\mapsto(\widehat{g}(k+l))_{k,l\ge0}.$$

In other words, as Banach spaces, $\mathcal{A}^p$ is isomorphic to $\mathfrak{S}^p$ and $\mathcal{A}^p(S^p)$ is isomorphic to $\mathfrak{S}^p(S^p)$ in a canonical way.

In the case of $\mathfrak{S}^p$ we refer the reader to Section 2 of [29] (the norm of $\mathcal{A}^p$ as described above is given explicitly on page 450); while for the case of

$\mathfrak{S}^p(S^p)$, we refer to Section 3 of [30] (the norm of $\mathcal{A}^p(S^p)$ described above is implicit there). Therefore we have

$$\forall x\in\mathfrak{S}^p,\qquad \|x_{(n)}\|_{\mathfrak{S}^p}\cong 2^{n/p}\Big\|\sum_{k=2^{n-1}}^{2^n-1}x_{0k}z^k\Big\|_{L^p},$$
$$\|x\|_{\mathfrak{S}^p}\cong\Big(\sum_{n=0}^\infty 2^n\Big\|\sum_{k=2^{n-1}}^{2^n-1}x_{0k}z^k\Big\|_{L^p}^p\Big)^{1/p},$$

and similarly for all x in $\mathfrak{S}^p(S^p)$, we have

$$\|x_{(n)}\|_{\mathfrak{S}^p(S^p)}\cong 2^{n/p}\Big\|\sum_{k=2^{n-1}}^{2^n-1}x_{0k}z^k\Big\|_{L^p(S^p)},$$
$$\|x\|_{\mathfrak{S}^p(S^p)}\cong\Big(\sum_{n=0}^\infty 2^n\Big\|\sum_{k=2^{n-1}}^{2^n-1}x_{0k}z^k\Big\|_{L^p(S^p)}^p\Big)^{1/p}$$

($x_{0k}z^k:=0$ when $k=1/2$). These descriptions provide $\mathfrak{S}^p$ and $\mathfrak{S}^p(S^p)$ with very useful equivalent norms as follows.

COROLLARY 0.7. (i) *For each fixed $1<p<\infty$, the following are equivalent norms on $\mathfrak{S}^p$*:

$$\|x\|_{\mathfrak{S}^p}\cong\Big(\sum_{n=0}^\infty\|x_{(n)}\|_{\mathfrak{S}^p}^p\Big)^{1/p},\qquad\forall x\in\mathfrak{S}^p.$$

(ii) *For each fixed $1<p<\infty$, the following are equivalent norms on $\mathfrak{S}^p(S^p)$*:

$$\|x\|_{\mathfrak{S}^p(S^p)}\cong\Big(\sum_{n=0}^\infty\|x_{(n)}\|_{\mathfrak{S}^p(S^p)}^p\Big)^{1/p},\qquad\forall x\in\mathfrak{S}^p(S^p).$$

0.7. Some suitable operator norm inequalities

PROPOSITION 0.8. *Let $1\le q\le\infty$, let $\alpha,\beta>1$ be such that $1/\alpha+1/\beta=1$, y a positive operator in $S^{q\alpha}$ and $(x_n)_n$ a finite sequence of operators in $S^{2q\beta}$. Then*

$$\Big\|\sum_n x_n^*yx_n\Big\|_{S^q}\le\|y\|_{S^{q\alpha}}\max\Big\{\Big\|\sum_n x_n^*x_n\Big\|_{S^{q\beta}},\Big\|\sum_n x_nx_n^*\Big\|_{S^{q\beta}}\Big\}.$$

This proposition goes back to [26] when $(x_n)_n$ is a family of self-adjoint operators. The general case for which the proof uses basically the three line lemma can be found in [35]. The next corollary follows easily by a reiteration argument.

COROLLARY 0.9. *Let $1\le q\le\infty$, $r\ge1$ and for each $1\le j\le r$, let I_j be a finite set of indices, $\alpha_j>1$ with $\sum_{j=1}^r 1/\alpha_j=1$, and $(x^{(j)}_{n_j})_{n_j\in I_j}$ a family* *of operators in* $S^{2q\alpha_j}$. *Then*

$$\Big\| \sum_{\substack{n_j\in I_j\\ 1\le j\le r}} x_{n_r}^{(r)*}\ldots x_{n_1}^{(1)*}x_{n_1}^{(1)}\ldots x_{n_r}^{(r)}\Big\|_{S^q}$$
$$\le \prod_{j=1}^{r}\max\Big\{\Big\|\sum_{n_j\in I_j} x_{n_j}^{(j)*}x_{n_j}^{(j)}\Big\|_{S^{q\alpha_j}},\Big\|\sum_{n_j\in I_j} x_{n_j}^{(j)}x_{n_j}^{(j)^*}\Big\|_{S^{q\alpha_j}}\Big\}.$$

0.8. Fourier multipliers. A scalar-valued map φ on $\Lambda\subset G$ is said to be a *Fourier multiplier* on $L^p_\Lambda(\tau_0)$ if the associated operator

$$M_\varphi:\operatorname{span}\{\lambda(t)\mid t\in\Lambda\}\to\operatorname{span}\{\lambda(t)\mid t\in\Lambda\},\qquad \lambda(t)\mapsto\varphi(t)\lambda(t),$$

extends to a bounded operator on $L^p_\Lambda(\tau_0)$ (still denoted by M_φ); we let $M(L^p_\Lambda(\tau_0))$ stand for the set of all such maps. Then $M(L^p_\Lambda(\tau_0))$ is a unital Banach algebra for the pointwise product and the norm

$$\|\varphi\|_{M(L^p_\Lambda(\tau_0))}:=\|M_\varphi:L^p_\Lambda(\tau_0)\to L^p_\Lambda(\tau_0)\|.$$

Let $M_{\rm cb}(L^p_\Lambda(\tau_0))$ be the subalgebra of c.b. Fourier multipliers φ on $L^p_\Lambda(\tau_0)$ (i.e. the corresponding operators M_φ are c.b.), equipped with the norm

$$\|\varphi\|_{M_{\rm cb}(L^p_\Lambda(\tau_0))}:=\|M_\varphi:L^p_\Lambda(\tau_0)\to L^p_\Lambda(\tau_0)\|_{\rm cb}.$$

By Proposition 0.4, a multiplier φ belongs to $M_{\rm cb}(L^p_\Lambda(\tau_0))$ if and only if $M_\varphi\otimes \mathrm{id}_{S^p}$ is bounded on $L^p_\Lambda(\tau_0)\otimes S^p$ as a subspace of $L^p(\tau)$ with $\|\varphi\|_{M_{\rm cb}(L^p_\Lambda(\tau_0))}=\|M_\varphi\otimes\mathrm{id}_{S^p}\|$. By duality, it is very easy to see that for all $1\le p,q\le\infty$ where $1/p+1/q=1$ we have

$$M(L^p(\tau_0))=M(L^q(\tau_0)),\qquad M_{\rm cb}(L^p(\tau_0))=M_{\rm cb}(L^q(\tau_0))$$

isometrically. Note that the duality $\langle f,g\rangle=\tau_0(f\check g)$ for f in $L^p(\tau_0)$, g in $L^q(\tau_0)$ and $\check g:=\sum_{t\in G}\widehat g(t^{-1})\lambda(t)$ is the suitable choice to have the previous identifications via the identity map. Therefore we can restrict ourselves to the case where $2\le p\le\infty$. We see easily that

$$M(L^2(\tau_0))=M_{\rm cb}(L^2(\tau_0))=\ell_\infty(G)$$

isometrically. Since $M_{\rm cb}(L^\infty(\tau_0))\subset M(L^\infty(\tau_0))\subset M(L^2(\tau_0))=M_{\rm cb}(L^2(\tau_0))$ contractively, by complex interpolation we get

$$M(L^\infty(\tau_0))\subset M(L^p(\tau_0))\subset M(L^2(\tau_0)),$$
$$M_{\rm cb}(L^\infty(\tau_0))\subset M_{\rm cb}(L^p(\tau_0))\subset M(L^2(\tau_0))$$

contractively. By repeating the same argument, we see that for all $2\le q<p\le\infty$ we have

$$M(L^p(\tau_0))\subset M(L^q(\tau_0)),\qquad M_{\rm cb}(L^p(\tau_0))\subset M_{\rm cb}(L^q(\tau_0))$$

contractively. Thus $(M(L^p(\tau_0)))_{2\le p\le\infty}$ and $(M_{\rm cb}(L^p(\tau_0)))_{2\le p\le\infty}$ are two decreasing families of algebras.

Now assume moreover that G is Abelian and equip its dual group $\widehat G$ which is compact with its Haar measure. In this case, the von Neumann algebra generated by $\lambda(G)$ in $\mathcal B(\ell_2(G))$ coincides with $L^\infty(\widehat G)$, $L^p(\tau_0)$ coincides with $L^p(\widehat G)$ and $L^p(\tau)$ coincides with $L^p(\widehat G,S^p)$. This applies e.g. to the group $\mathbb Z$ which will be discussed later.

REMARK 0.10. It follows from well known results (cf. e.g. [7], [8]) that the canonical Hilbert transform defines a c.b. multiplier on L^p for $1<p<\infty$. Therefore the natural projections of L^p onto L^p_Λ which send f to $\sum_{k\in\Lambda}\widehat f(k)z^k$ are uniformly completely bounded when Λ runs over all intervals of $\mathbb Z$. In other words, the spaces L^p_Λ where $\Lambda\subset\mathbb Z$ is an arbitrary interval are uniformly complemented in L^p as operator spaces. Hence for $1<p<\infty$ the inclusion maps

$$M_{\rm cb}(L^p_\Lambda)\hookrightarrow M_{\rm cb}(L^p),\qquad \varphi\mapsto\widetilde\varphi,$$

where $\widetilde\varphi$ is the trivial extension of φ by zero outside Λ, are uniformly bounded when Λ runs over all intervals of $\mathbb Z$.

0.9. Schur multipliers. Let $\{e_{kl}\}_{k,l}$ be the canonical basis of S^p, $1\le p\le\infty$, and A be a subset of $\mathbb N\times\mathbb N$. A scalar map φ defined on A is said to be a *Schur multiplier* on S^p_A if the associated operator

$$T_\varphi:\operatorname{span}\{e_{kl}\mid(k,l)\in A\}\to\operatorname{span}\{e_{kl}\mid(k,l)\in A\},\qquad e_{kl}\mapsto\varphi(k,l)e_{kl},$$

extends to a bounded operator on S^p_A (still denoted by T_φ); we let $M(S^p_A)$ stand for the set of all Schur multipliers on S^p_A. Then $M(S^p_A)$ is a Banach algebra for the pointwise product and the norm

$$\|\varphi\|_{M(S^p_A)}:=\|T_\varphi:S^p_A\to S^p_A\|.$$

We denote by $M_{\rm cb}(S^p_A)$ the algebra of c.b. Schur multipliers φ on S^p_A, equipped with the norm

$$\|\varphi\|_{M_{\rm cb}(S^p_A)}=\|T_\varphi:S^p_A\to S^p_A\|_{\rm cb}.$$

We denote by $M^{\mathcal H}(S^p_A)$ and $M^{\mathcal H}_{\rm cb}(S^p_A)$ the subalgebras of $M(S^p_A)$ and $M_{\rm cb}(S^p_A)$ respectively formed by all Schur multipliers on S^p_A which have a Hankelian form (a multiplier φ is viewed as an $\infty\times\infty$ matrix).

When A has a Hankelian form (i.e. $A=\widehat\Lambda$ for some set $\Lambda\subset\mathbb N$), we let $M(\mathfrak S^p_A)$ (resp. $M_{\rm cb}(\mathfrak S^p_A)$) be the algebra of all scalar maps φ defined on A such that the corresponding operators map $\mathfrak S^p_A$ boundedly (resp. completely boundedly) into itself. Note that a multiplier on $\mathfrak S^p_A$ necessarily has a Hankelian form.

For an example of c.b. Hankelian Schur multipliers on S^p, we can quote the following. Fix z in $\mathbb{T}$ and consider the map $\varphi_z : (k,l) \mapsto z^{k+l}$. Then the corresponding operator is by definition

$$T_{\varphi_z} : S^p \to S^p, \quad (x_{kl})_{k,l} \mapsto (z^{k+l} x_{kl})_{k,l} = D_z x D_z,$$

where D_z is the unitary operator

$$D_z = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & z & 0 & \ldots \\ 0 & 0 & z^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly T_{φ_z} is an isometry and in fact T_{φ_z} is a complete isometry. Therefore $\varphi_z \in M_{\mathrm{cb}}^{\mathcal{H}}(S^p)$ for all $1 \le p \le \infty$.

For the study of the spaces $M(S^p)$ and $M_{\mathrm{cb}}(S^p)$, we can again restrict ourselves to the case where $2 \le p \le \infty$ since for $1 \le p, q \le \infty$ such that $1/p + 1/q = 1$ we have

$$M(S^p) = M(S^q), \quad M_{\mathrm{cb}}(S^p) = M_{\mathrm{cb}}(S^q)$$

isometrically. As noticed previously, these identifications can be done via the identity map if we wish, by a suitable choice of the duality between S^p and S^q. Namely, we set

$$\forall x \in S^p,\ \forall y \in S^q \quad \langle x, y\rangle := \mathrm{tr}({}^t xy).$$

There is a nice description of $M(S^p)$ for $p = 2$ and $p = \infty$:

$$M(S^2) = M_{\mathrm{cb}}(S^2) = \ell_\infty(\mathbb{N} \times \mathbb{N}), \quad M(S^\infty) = M_{\mathrm{cb}}(S^\infty) = \Gamma_2(\ell_1, \ell_\infty)$$

isometrically, where $\Gamma_2(\ell_1, \ell_\infty)$ is the space of all operators from ℓ_1 to ℓ_∞ which factor through a Hilbert space, equipped with the usual factorization norm. The case $p = 2$ is trivial while the case $p = \infty$ (for which [32] gives the precise statement repeated below and which goes back in essence to Grothendieck) is not (see [32] for more references).

THEOREM 0.11. *For $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$, the following are equivalent.*

(i) *φ is a Schur multiplier on S^∞ with norm less than* 1.

(ii) *There exist a Hilbert space H and sequences $(h_n)_n$ and $(k_m)_m$ of vectors in the unit ball of H such that $\varphi(n,m) = \langle h_n, k_m\rangle$ for all n, m in $\mathbb{N}$.*

(iii) *The operator $u_\varphi : \ell_1 \to \ell_\infty$ which takes e_n to $\sum_m \varphi(n,m) e_m$ belongs to $\Gamma_2(\ell_1, \ell_\infty)$ with norm less than* 1. *Here $\{e_n\}_n$ denotes the canonical basis of ℓ_1.*

(iv) *φ is a c.b. Schur multiplier on S^∞ with c.b. norm less than* 1.

It is useful to note that this description provides a class $\mathcal{C}$ of very simple multipliers in the unit ball of $M(S^\infty)$, those of the form $\varphi(n,m) = a_n b_m$ where $a = (a_n)_n$, $b = (b_m)_m$ are in B_{ℓ_∞}. The interest of this class $\mathcal{C}$ comes from the fact that there exists a universal constant $K > 0$ such that $\overline{\mathrm{conv}}(\mathcal{C}) \subset B_{M(S^\infty)} \subset K\overline{\mathrm{conv}}(\mathcal{C})$ where $\overline{\mathrm{conv}}(\mathcal{C})$ is the closure of the convex set generated by $\mathcal{C}$ in the simple convergence topology on $\mathbb{N} \times \mathbb{N}$ (cf. [32]).

For $2 < p < \infty$ we have contractive inclusions

$$M(S^\infty) \subset M_{\mathrm{cb}}(S^p) \subset M(S^p) \subset M(S^2).$$

Indeed, $M(S^\infty)$ embeds contractively into $M(S^2)$ and we use complex interpolation. More generally, we see that for $2 < p < q < \infty$ we have the following contractive embeddings:

$$M(S^\infty) \subset M(S^q) \subset M(S^p) \subset M(S^2),$$
$$M(S^\infty) \subset M_{\mathrm{cb}}(S^q) \subset M_{\mathrm{cb}}(S^p) \subset M(S^2).$$

Therefore $(M(S^p))_{2 \le p \le \infty}$ and $(M_{\mathrm{cb}}(S^p))_{2 \le p \le \infty}$ are two decreasing families of sets.

1. NON-COMMUTATIVE $\Lambda(p)$-SETS IN DISCRETE GROUPS

In this section G denotes an arbitrary discrete group with unit e, λ denotes the left regular representation of G into $\mathcal{B}(\ell_2(G))$, $L^p(\tau_0)$ denotes the non-commutative L^p-space associated with the von Neumann algebra generated by $\lambda(G)$ with respect to the usual trace τ_0, and $L^p(\tau)$ denotes the non-commutative L^p-space associated with the von Neumann algebra generated by $\lambda(G) \otimes \mathcal{B}(\ell_2)$ with respect to the trace $\tau = \tau_0 \otimes \mathrm{tr}$ where tr denotes the usual trace on $\mathcal{B}(\ell_2)$.

DEFINITION 1.1. Let $2 < p < \infty$ and $\Lambda \subset G$. We say that Λ is a *$\Lambda(p)$-set* if the spaces $L^p_\Lambda(\tau_0)$ and $L^2_\Lambda(\tau_0)$ are isomorphic, or equivalently, there exists a constant $\lambda > 0$ such that for all finitely supported families of scalars a_t we have

$$\Big\|\sum_{t\in\Lambda} a_t \lambda(t)\Big\|_{L^p(\tau_0)} \le \lambda\Big(\sum_{t\in\Lambda} |a_t|^2\Big)^{1/2}.$$

We let $\lambda_p(\Lambda)$ or sometimes simply λ_p stand for the smallest constant λ for which this happens.

The reader is referred to Subsection 0.8 for the definition of the algebra $M(L^p(\tau_0))$ of Fourier multipliers as well as its subalgebra $M_{\mathrm{cb}}(L^p(\tau_0))$.

DEFINITION 1.2. A set $\Lambda \subset G$ is said to be an *interpolation set for* $M(L^p(\tau_0))$ for some $1 \le p \le \infty$ if the restriction map

$$Q : M(L^p(\tau_0)) \to \ell_\infty(\Lambda), \quad \varphi \mapsto (\varphi(t))_{t\in\Lambda},$$

is μ-surjective for some constant μ. We let $\mu_p(\Lambda)$ or simply μ_p be the smallest constant μ for which this happens.

The following result shows that $\Lambda(p)$-sets can be viewed as classes of interpolation sets.

PROPOSITION 1.3. *Let $2<p<\infty$ and $\Lambda\subset G$. The assertions below are equivalent.*

(i) *Λ is a $\Lambda(p)$-set.*

(ii) *Λ is an interpolation set for $M(L^p(\tau_0))$.*

Moreover, $\mu_p(\Lambda)\le\lambda_p(\Lambda)\le K_{L^p(\tau_0)}\mu_p(\Lambda)$ where $K_{L^p(\tau_0)}$ is the constant defined in (0.3).

Proof. Assume that Λ is a $\Lambda(p)$-set. For $\varepsilon=(\varepsilon_t)_t$ in $\ell_\infty(\Lambda)$ we let $\widetilde{\varepsilon}$ be its trivial extension to $\ell_\infty(G)$ equal to zero outside Λ. Then for any f in $L^p(\tau_0)$,

$$\Big\|\sum_{t\in G}\widetilde{\varepsilon}_t\widehat{f}(t)\lambda(t)\Big\|_{L^p(\tau_0)}=\Big\|\sum_{t\in\Lambda}\varepsilon_t\widehat{f}(t)\lambda(t)\Big\|_{L^p(\tau_0)}\le\lambda_p\Big(\sum_{t\in\Lambda}|\varepsilon_t\widehat{f}(t)|^2\Big)^{1/2}$$
$$\le\lambda_p\|\varepsilon\|_{\ell_\infty(\Lambda)}\Big(\sum_{t\in G}|\widehat{f}(t)|^2\Big)^{1/2}$$
$$\le\lambda_p\|\varepsilon\|_{\ell_\infty(\Lambda)}\|f\|_{L^p(\tau_0)}.$$

Thus $\widetilde{\varepsilon}$ is in $M(L^p(\tau_0))$ and it satisfies $\|\widetilde{\varepsilon}\|_{M(L^p(\tau_0))}\le\lambda_p\|\varepsilon\|_{\ell_\infty(\Lambda)}$. This means $\mu_p\le\lambda_p$.

Conversely, assume that Λ is an interpolation set for $M(L^p(\tau_0))$. Then for any $\delta>0$, each choice of signs ε on Λ admits a lifting $\widetilde{\varepsilon}$ in $M(L^p(\tau_0))$ with $\|\widetilde{\varepsilon}\|_{M(L^p(\tau_0))}\le\mu_p+\delta$. This implies that for any f in $L^p_\Lambda(\tau_0)$, say with finitely supported Fourier transform $\widehat{f}$, we have

$$\|f\|_{L^p(\tau_0)}\le(\mu_p+\delta)\Big\|\sum_{t\in\Lambda}\varepsilon_t\widehat{f}(t)\lambda(t)\Big\|_{L^p(\tau_0)}$$

since $f=M_{\widetilde{\varepsilon}}(M_{\widetilde{\varepsilon}}f)$. Then we integrate the inequality over all the choices of signs on Λ:

$$\|f\|_{L^p(\tau_0)}\le(\mu_p+\delta)\Big(\int\limits_{\{-1,1\}^\Lambda}\Big\|\sum_{t\in\Lambda}\varepsilon_t\widehat{f}(t)\lambda(t)\Big\|^p_{L^p(\tau_0)}d\nu(\varepsilon)\Big)^{1/p}$$
$$=(\mu_p+\delta)\Big\|\sum_{t\in\Lambda}\varepsilon_t\widehat{f}(t)\lambda(t)\Big\|_{L^p(\{-1,1\}^\Lambda,\nu,L^p(\tau_0))}.$$

Now we apply the non-commutative version of the Khinchin inequalities (0.3) and let δ tend to 0 to obtain $\|f\|_{L^p(\tau_0)}\le K_{L^p(\tau_0)}\mu_p\|f\|_{L^2(\tau_0)}$. Hence Λ is a $\Lambda(p)$-set with $\lambda_p\le K_{L^p(\tau_0)}\mu_p$. ■

REMARKS 1.4. (i) Since the embeddings $L^\infty(\tau_0)\subset L^q(\tau_0)\subset L^p(\tau_0)\subset L^2(\tau_0)$ for all real numbers $2<p<q<\infty$ are bounded, we see that the $\Lambda(q)$-property implies the $\Lambda(p)$-property. Thus we have a decreasing family of sets $(\{\Lambda\subset G\mid\Lambda\text{ is a }\Lambda(p)\text{-set}\})_{2<p<\infty}$.

(ii) Although no significantly new examples are known, it is useful to consider also the case $1<p\le2$. A set Λ is called a $\Lambda(p)$-set in this case if $L^p_\Lambda(\tau_0)$ and $L^q_\Lambda(\tau_0)$ are equivalent Banach spaces for some and thus any $1\le q<p$. Similarly we denote by $\lambda_p(\Lambda)$ the smallest constant $\lambda>0$ such that for any f in $L^p_\Lambda(\tau_0)$ we have $\|f\|_{L^p(\tau_0)}\le\lambda\|f\|_{L^1(\tau_0)}$. With this terminology it is known by an extrapolation argument that if $q>2$ and Λ is a $\Lambda(q)$-set then Λ is a $\Lambda(2)$-set. Conversely, if Λ is a $\Lambda(2)$-set then Λ is a $\Lambda(q)$-set if and only if its indicator function $\mathbf{1}_\Lambda$ belongs to $M(L^q(\tau_0))$. Moreover for each set Λ we have

$$\lambda_2(\Lambda)\le\lambda_q(\Lambda)\le\lambda_2(\Lambda)\|\mathbf{1}_\Lambda\|_{M(L^q(\tau_0))}.\tag{1.5}$$

Now we extend the previous definitions and results to the non-commutative case. Namely we define subsets of G playing for the sets $M_{\rm cb}(L^p(\tau_0))$ a rôle similar to the one played by $\Lambda(p)$-sets for $M(L^p(\tau_0))$.

DEFINITION 1.5. Let $2<p<\infty$ and $\Lambda\subset G$. We say that Λ is a *$\Lambda(p)_{\rm cb}$-set* if there exists a constant $C>0$ such that for all finitely supported families of operators x_t in S^p we have

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}\le C\max\Big\{\Big\|\Big(\sum_{t\in\Lambda}x_t^*x_t\Big)^{1/2}\Big\|_{S^p},\Big\|\Big(\sum_{t\in\Lambda}x_tx_t^*\Big)^{1/2}\Big\|_{S^p}\Big\}.$$

Then we let $\lambda_p^{\rm cb}(\Lambda)$ stand for the smallest constant C for which the inequality above holds.

REMARKS 1.6. (i) Using Jensen's inequality it is very easy to see that when $p\ge2$, any f in $L^p(\tau)$ satisfies

$$\max\Big\{\Big\|\Big(\sum_{t\in G}\widehat{f}(t)^*\widehat{f}(t)\Big)^{1/2}\Big\|_{S^p},\Big\|\Big(\sum_{t\in G}\widehat{f}(t)\widehat{f}(t)^*\Big)^{1/2}\Big\|_{S^p}\Big\}\le\|f\|_{L^p(\tau)}.\tag{1.6}$$

Hence the $\Lambda(p)_{\rm cb}$-property means simply that the norms $\|\cdot\|_{L^p(\tau)}$ and $|||\cdot|||$ are equivalent on $L^p_\Lambda(\tau)$ where for any f in $L^p(\tau)$,

$$|||f|||:=\max\Big\{\Big\|\Big(\sum_{t\in G}\widehat{f}(t)^*\widehat{f}(t)\Big)^{1/2}\Big\|_{S^p},\Big\|\Big(\sum_{t\in G}\widehat{f}(t)\widehat{f}(t)^*\Big)^{1/2}\Big\|_{S^p}\Big\}.$$

Therefore if Λ is a $\Lambda(p)_{\rm cb}$-set then $\mathbf{1}_\Lambda$ is in $M_{\rm cb}(L^p(\tau_0))$, i.e. the natural projection of $L^p(\tau_0)$ onto $L^p_\Lambda(\tau_0)$ is c.b. with c.b. norm less than or equal to $\lambda_p^{\rm cb}(\Lambda)$.

(ii) Given two $\Lambda(p)_{\rm cb}$-subsets Λ_1 and Λ_2 of G, the set $\Lambda_1\cup\Lambda_2$ clearly has the $\Lambda(p)_{\rm cb}$-property with $\lambda_p^{\rm cb}(\Lambda_1\cup\Lambda_2)\le\lambda_p^{\rm cb}(\Lambda_1)+\lambda_p^{\rm cb}(\Lambda_2)$.

DEFINITION 1.7. Given $1\le p\le\infty$, a subset Λ of G is said to be an *interpolation set for $M_{\rm cb}(L^p(\tau_0))$* if the restriction map

$$Q : M_{\mathrm{cb}}(L^p(\tau_0)) \to \ell_\infty(\Lambda), \qquad \varphi \mapsto (\varphi(t))_{t\in\Lambda},$$

is surjective; then it is μ-surjective for some constant μ and we let $\mu_p^{\mathrm{cb}}(\Lambda)$ or simply μ_p^{cb} be the smallest constant μ for which this happens.

The following result shows that in this more general setting, $\Lambda(p)_{\mathrm{cb}}$-sets can also be viewed as classes of interpolation sets.

Proposition 1.8. *Let $2 < p < \infty$ and $\Lambda \subset G$. The assertions are equivalent.*

(i) *Λ is a $\Lambda(p)_{\mathrm{cb}}$-set.*

(ii) *Λ is an interpolation set for $M_{\mathrm{cb}}(L^p(\tau_0))$.*

Moreover, $\mu_p^{\mathrm{cb}}(\Lambda) \le \lambda_p^{\mathrm{cb}}(\Lambda) \le K_{L^p(\tau)}\mu_p^{\mathrm{cb}}(\Lambda)$ where $K_{L^p(\tau)}$ is defined in (0.3).

Proof. Assume that Λ has the $\Lambda(p)_{\mathrm{cb}}$-property. For $\varepsilon = (\varepsilon_t)_t$ in $\ell_\infty(\Lambda)$ we let $\widetilde{\varepsilon}$ be its trivial extension to $\ell_\infty(G)$ obtained by adding zeros. Then for any $f = \sum_{t\in G}\lambda(t)\otimes x_t$ in $L^p(\tau)$, say with finitely many non-zero operators x_t, we have

$$\begin{aligned}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)} &\le \lambda_p^{\mathrm{cb}}\max\Big\{\Big\|\Big(\sum_{t\in\Lambda}|\varepsilon_t|^2x_t^*x_t\Big)^{1/2}\Big\|_{S^p}, \Big\|\Big(\sum_{t\in\Lambda}|\varepsilon_t|^2x_tx_t^*\Big)^{1/2}\Big\|_{S^p}\Big\}\\ &\le \lambda_p^{\mathrm{cb}}\|\varepsilon\|_{\ell_\infty(\Lambda)}\max\Big\{\Big\|\Big(\sum_{t\in G}x_t^*x_t\Big)^{1/2}\Big\|_{S^p}, \Big\|\Big(\sum_{t\in G}x_tx_t^*\Big)^{1/2}\Big\|_{S^p}\Big\}.\end{aligned}$$

Hence using (1.6) we get

$$\Big\|\sum_{t\in G}\widetilde{\varepsilon}_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)} \le \lambda_p^{\mathrm{cb}}\|\varepsilon\|_{\ell_\infty(\Lambda)}\|f\|_{L^p(\tau)}.$$

Thus $\widetilde{\varepsilon}$ is in $M_{\mathrm{cb}}(L^p(\tau_0))$ with $\|\widetilde{\varepsilon}\|_{M_{\mathrm{cb}}(L^p(\tau_0))} \le \lambda_p^{\mathrm{cb}}\|\varepsilon\|_{\ell_\infty(\Lambda)}$, which means that $\mu_p^{\mathrm{cb}} \le \lambda_p^{\mathrm{cb}}$.

Conversely, let Λ be an interpolation set for $M_{\mathrm{cb}}(L^p(\tau_0))$ and $\delta > 0$. Any choice of signs ε on Λ admits a lifting $\widetilde{\varepsilon}$ in $M_{\mathrm{cb}}(L^p(\tau_0))$ with $\|\widetilde{\varepsilon}\|_{M_{\mathrm{cb}}(L^p(\tau_0))} \le \mu_p^{\mathrm{cb}} + \delta$. This implies that for any $f = \sum_{t\in\Lambda}\lambda(t)\otimes x_t$ in $L^p_\Lambda(\tau)$ we have

$$\|f\|_{L^p(\tau)} \le (\mu_p^{\mathrm{cb}} + \delta)\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}$$

(since $\varepsilon_t^2 = 1$ for all $t\in\Lambda$). Letting δ tend to 0 we see that each f in $L^p_\Lambda(\tau)$ satisfies, for each choice of signs ε,

$$\|f\|_{L^p(\tau)} \le \mu_p^{\mathrm{cb}}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}.$$

We integrate the right side above over all choices of signs on Λ to get

$$\begin{aligned}\|f\|_{L^p(\tau)} &\le \mu_p^{\mathrm{cb}}\Big(\int\limits_{\{-1,1\}^\Lambda}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|^p_{L^p(\tau)}d\nu(\varepsilon)\Big)^{1/p}\\ &= \mu_p^{\mathrm{cb}}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\{-1,1\}^\Lambda,\nu,L^p(\tau))}.\end{aligned}$$

Now we apply the non-commutative version of the Khinchin inequalities (0.3) to obtain

$$\|f\|_{L^p(\tau)} \le K_{L^p(\tau)}\mu_p^{\mathrm{cb}}\max\Big\{\Big\|\Big(\sum_{t\in\Lambda}x_tx_t^*\Big)^{1/2}\Big\|_{S^p}, \Big\|\Big(\sum_{t\in\Lambda}x_t^*x_t\Big)^{1/2}\Big\|_{S^p}\Big\}.$$

That is to say, Λ is a $\Lambda(p)_{\mathrm{cb}}$-set with $\lambda_p^{\mathrm{cb}} \le K_{L^p(\tau)}\mu_p^{\mathrm{cb}}$. ∎

Remark. As the embeddings $M_{\mathrm{cb}}(L^\infty(\tau_0)) \subset M_{\mathrm{cb}}(L^q(\tau_0)) \subset M_{\mathrm{cb}}(L^p(\tau_0)) \subset M(L^2(\tau_0))$ where $2 < p < q < \infty$ are bounded, we see that the $\Lambda(q)_{\mathrm{cb}}$-property implies the $\Lambda(p)_{\mathrm{cb}}$-property. Thus the family $(\{\Lambda \subset G \mid \Lambda$ is a $\Lambda(p)_{\mathrm{cb}}$-set$\})_{2<p<\infty}$ of sets is decreasing for each fixed discrete group G. On the other hand, the $\Lambda(p)_{\mathrm{cb}}$-property trivially implies the $\Lambda(p)$-property. Moreover for any $\Lambda \subset G$ and any $2 < p < q < \infty$ we have

$$\lambda_p^{\mathrm{cb}}(\Lambda) \le \lambda_q^{\mathrm{cb}}(\Lambda), \qquad \mu_p(\Lambda) \le \mu_p^{\mathrm{cb}}(\Lambda), \qquad \lambda_p(\Lambda) \le \lambda_p^{\mathrm{cb}}(\Lambda).$$

Comments 1.9. Clearly, we can naturally extend our definitions to the case of $1 < p \le 2$. We say that a set $\Lambda \subset G$ is a *$K(p)_{\mathrm{cb}}$-set* if there exists a constant $c > 0$ such that for any sequence $(x_t)_{t\in\Lambda}$ of operators in S^p, say a finitely supported one, we have

$$c^{-1}\inf\Big\{\Big\|\Big(\sum_{t\in\Lambda}y_ty_t^*\Big)^{1/2}\Big\|_{S^p} + \Big\|\Big(\sum_{t\in\Lambda}z_t^*z_t\Big)^{1/2}\Big\|_{S^p}\Big\} \le \Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}$$

where the infimum is over all decompositions of the x_t's in S^p as $x_t = y_t + z_t$. We let $K_p^{\mathrm{cb}}(\Lambda)$ stand for the smallest constant c for which this holds. Recall that the converse inequality (with constant 1 instead of c) is satisfied by any set Λ and note that the $K(2)_{\mathrm{cb}}$-property is trivial.

Let $1 < p < 2$ and $1/p + 1/p' = 1$. Then, for a given set $\Lambda \subset G$, the following are equivalent.

(i) Λ is a $K(p)_{\mathrm{cb}}$-set and each c.b. multiplier on $L^p_\Lambda(\tau_0)$ extends to a c.b. multiplier on $L^p(\tau_0)$.

(ii) Λ is a $\Lambda_{\mathrm{cb}}(p')$-set.

Indeed, assume (i). By Proposition 1.8, we need to prove that Λ is an interpolation set for $M_{\mathrm{cb}}(L^{p'}(\tau_0)) = M_{\mathrm{cb}}(L^p(\tau_0))$. Equivalently, we need to prove that the choices of signs on Λ extend uniformly completely boundedly to multipliers on $L^p(\tau_0)$. Let $\varepsilon=(\varepsilon_t)_{t\in\Lambda}$ be an arbitrary choice of signs on Λ. Then, for any finitely supported sequence $(x_t)_{t\in\Lambda}$ of operators in S^p, we have

$$\begin{aligned}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)} &\le \inf_{\substack{\varepsilon_t x_t=y_t+z_t\\ y_t,z_t\in S^p}}\Big\{\Big\|\Big(\sum_{t\in\Lambda}y_ty_t^*\Big)^{1/2}\Big\|_{S^p}+\Big\|\Big(\sum_{t\in\Lambda}z_t^*z_t\Big)^{1/2}\Big\|_{S^p}\Big\}\\ &= \inf_{\substack{x_t=y_t+z_t\\ y_t,z_t\in S^p}}\Big\{\Big\|\Big(\sum_{t\in\Lambda}y_ty_t^*\Big)^{1/2}\Big\|_{S^p}+\Big\|\Big(\sum_{t\in\Lambda}z_t^*z_t\Big)^{1/2}\Big\|_{S^p}\Big\}.\end{aligned}$$

Thus by our assumption on Λ we get

$$\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}\le K_p^{\mathrm{cb}}(\Lambda)\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}.$$

This means that ε defines a c.b. multiplier on $L^p_\Lambda(\tau_0)$ with c.b. norm less than or equal to $K_p^{\mathrm{cb}}(\Lambda)$. The second assumption on Λ says that the restriction map

$$M_{\mathrm{cb}}(L^p(\tau_0))\to M_{\mathrm{cb}}(L^p_\Lambda(\tau_0)),\qquad \varphi\mapsto\varphi|_\Lambda,$$

is surjective, hence μ-surjective for some constant $\mu>0$. Then, for all $\delta>0$, each choice of signs ε on Λ extends to a c.b. multiplier on $L^p(\tau_0)$ with norm less than or equal to $\mu K_p^{\mathrm{cb}}(\Lambda)+\delta$. Therefore we are done and $\mu_{p'}^{\mathrm{cb}}(\Lambda)=\mu_p^{\mathrm{cb}}(\Lambda)\le\mu K_p^{\mathrm{cb}}(\Lambda)$.

Conversely, assume (ii). Then by Proposition 1.8, Λ is an interpolation set for $M_{\mathrm{cb}}(L^{p'}(\tau_0))=M_{\mathrm{cb}}(L^p(\tau_0))$. A fortiori, each c.b. multiplier on $L^p_\Lambda(\tau_0)$ extends to a c.b. multiplier on $L^p(\tau_0)$ since every multiplier on $L^p_\Lambda(\tau_0)$ is in particular a bounded sequence on Λ. On the other hand, let $\delta>0$ be fixed. Then since every choice of signs ε on Λ admits a lifting $\widetilde\varepsilon$ with $\|\widetilde\varepsilon\|_{M_{\mathrm{cb}}(L^p(\tau_0))}\le\mu_p^{\mathrm{cb}}(\Lambda)+\delta$, for every $f=\sum_{t\in\Lambda}\lambda(t)\otimes x_t$ in $L^p_\Lambda(\tau)$ we get

$$\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}\le(\mu_p^{\mathrm{cb}}(\Lambda)+\delta)\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}.$$

Hence if we let δ tend to zero, integrate over all these choices of signs and apply the non-commutative version of Khinchin's inequalities (0.2), we obtain

$$\begin{aligned}K_{L^p(\tau)}\inf_{\substack{x_t=y_t+z_t\\ y_t,z_t\in S^p}}\Big\{\Big\|\Big(\sum_{t\in\Lambda}y_ty_t^*\Big)^{1/2}\Big\|_{S^p}+\Big\|\Big(\sum_{t\in\Lambda}z_t^*z_t\Big)^{1/2}\Big\|_{S^p}\Big\}\\ \le\mu_p^{\mathrm{cb}}(\Lambda)\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}\end{aligned}$$

where $K_{L^p(\tau)}$ is the constant of (0.2). This means that Λ is a $K(p)_{\mathrm{cb}}$-set. Moreover, $K_p^{\mathrm{cb}}(\Lambda)\le\mu_p^{\mathrm{cb}}(\Lambda)/K_{L^p(\tau)}=\mu_{p'}^{\mathrm{cb}}(\Lambda)/K_{L^p(\tau)}$. ■

REMARK. Let $1\le p_1<p<2<p'<\infty$. Then it is well known that any $\Lambda(p')$-set is a $\Lambda(p_1)$-set. However, we do not know whether a $\Lambda(p')_{\mathrm{cb}}$-set must necessarily be a $K(p_1)_{\mathrm{cb}}$-set.

In the sequel, we are interested in finding properties simpler and stronger than the $\Lambda(p)_{\mathrm{cb}}$-property. This aim is achieved for even integers by introducing two combinatorial properties called the $B(p)$- and $Z(p)$-properties. For the group $\mathbb{Z}$, the $B(p)$-property was first considered by W. Rudin in [37] while the $Z(2)$-property was introduced by A. Zygmund in his work [46], which justifies the name of the latter property. Thus our properties are nothing but an adaptation of Rudin's property to the case of arbitrary discrete groups, and a generalization of Zygmund's property to arbitrary positive integers and arbitrary discrete groups.

DEFINITION 1.10. Let $p\ge2$ be an integer. A subset $\Lambda\subset G$ has the *$B(p)$-property* if for all p-tuples $(t_1,\dots,t_p)$ and $(s_1,\dots,s_p)$ in Λ^p, the equality $t_1^{-1}s_1\dots t_p^{-1}s_p=e$ (e is the unit of G) holds if and only if $\{t_1,\dots,t_p\}=\{s_1,\dots,s_p\}$ [1].

EXAMPLE. When G is a free group, every free subset Λ of G has the $B(p)$-property. Indeed, let $(t_1,\dots,t_p),(s_1,\dots,s_p)$ in Λ^p be such that $t_1^{-1}s_1\dots t_p^{-1}s_p$ is the empty word and assume $\{t_1,\dots,t_p\}\ne\{s_1,\dots,s_p\}$. Denote by i_0 the first index such that $t_{i_0}\ne s_j$ for all $1\le j\le p$ and i_1 the last index for which $t_{i_0}=t_{i_1}$. Then we would have $t_{i_0}^{-1}s_{i_0}\dots s_{i_1-1}t_{i_1}^{-1}=s_{i_0-1}^{-1}\dots t_1s_p^{-1}\dots t_{i_1+1}s_{i_1}^{-1}$. The reduced word of $t_{i_0}^{-1}s_{i_0}\dots s_{i_1-1}t_{i_1}^{-1}$ is expressed with letters in Λ and contains necessarily the letter $t_{i_0}^{-1}$ while the reduced word of $s_{i_0-1}^{-1}\dots t_1s_p^{-1}\dots t_{i_1+1}s_{i_1}^{-1}$ which is also expressed with letters in Λ does not contain $t_{i_0}^{-1}$. This means that there exists a word which has two different reduced expressions, both with letters belonging to Λ, which contradicts the freeness of Λ. We will see later that this property which is adapted to free groups is well adapted to the case of the group $\mathbb{Z}$.

DEFINITION 1.11. Let $p\ge2$ be an integer. For each $1\le i\le p$, we set $\nu_i=1$ when i is even and $\nu_i=-1$ otherwise. Then we say that a set Λ has the *$Z(p)$-property* if $Z_p(\Lambda)<\infty$, where

$$Z_p(\Lambda):=\sup_{\gamma\in G}|\{(t_1,\dots,t_p)\in\Lambda^p\mid\forall i\ne j,\ t_i\ne t_j\ \&\ t_1^{\nu_1}\dots t_p^{\nu_p}=\gamma\}|.$$

PROPOSITION 1.12. *Let $p\ge2$ be an integer. Then the $B(p)$-property implies the $Z(p)$-property. Moreover, each $B(p)$-subset Λ of a discrete group G satisfies $Z_p(\Lambda)\le\left(\frac{p}{2}!\right)^2$ if p is even and $Z_p(\Lambda)\le\frac{p-1}{2}!\frac{p+1}{2}!$ if p is odd.*

[1] In each set, elements are repeated according to their multiplicity in the corresponding sequence.

Proof. Let $(t_1,\dots,t_p),(s_1,\dots,s_p)$ in Λ^p be such that $t_1^{\nu_1}\dots t_p^{\nu_p}=s_1^{\nu_1}\dots s_p^{\nu_p}$ with $(\nu_j)_{1\le j\le p}$ as in Definition 1.11 and $t_i\ne t_j$, $s_i\ne s_j$ for all $1\le i\ne j\le p$. Then $t_1^{\nu_1}\dots t_p^{\nu_p}s_p^{-\nu_p}\dots s_1^{-\nu_1}=e$. Since Λ has the $B(p)$-property, we have necessarily (with elements in each set repeated according to their multiplicity)

$$\begin{aligned}\{t_i\mid 1\le i\le p,\ i\text{ odd}\}&\cup\{s_i\mid 1\le i\le p,\ i\text{ even}\}\\ &=\{t_i\mid 1\le i\le p,\ i\text{ even}\}\cup\{s_i\mid 1\le i\le p,\ i\text{ odd}\}.\end{aligned}$$

But $t_i\ne t_j$ and $s_i\ne s_j$ for all $1\le i\ne j\le p$, therefore we have

$$\{t_i\mid 1\le i\le p,\ i\text{ even}\}=\{s_i\mid 1\le i\le p,\ i\text{ even}\},$$
$$\{t_i\mid 1\le i\le p,\ i\text{ odd}\}=\{s_i\mid 1\le i\le p,\ i\text{ odd}\},$$

and it is easy to deduce from this the announced control of the constant $Z_p(\Lambda)$. ∎

THEOREM 1.13. *Let $p\ge 2$ be an integer and let G be a discrete group. Then every subset Λ of G with the $Z(p)$-property is a $\Lambda(2p)_{\rm cb}$-set. Moreover, there exists a constant C_p depending only on p such that $\lambda_{2p}^{\rm cb}(\Lambda)\le 3\max\{Z_p(\Lambda)^{1/(2p)},C_p\}$ for each $\Lambda\subset G$.*

The proof is much easier to follow in the particular case $p=2$ for which it appears in the Appendix (Proposition 6.1), and we urge the reader to look at it first.

For the proof of Theorem 1.13, it will be convenient to make the following definitions and to use the inequality of Proposition 1.14 which was found by G. Pisier.

Given a partition $\mathbf P$ of $\{1,\dots,p\}$, we set $k\equiv l$ $(\mathbf P)$ for $1\le k,l\le p$ if k and l belong to the same element of $\mathbf P$. Now given two partitions $\mathbf P_1$ and $\mathbf P_2$ of $\{1,\dots,p\}$, we set $\mathbf P_1\le\mathbf P_2$ if for each $1\le k,l\le p$, $k\equiv l$ $(\mathbf P_1)$ whenever $k\equiv l$ $(\mathbf P_2)$, and we write $\mathbf P_1<\mathbf P_2$ if $\mathbf P_1\le\mathbf P_2$ and $|\mathbf P_1|<|\mathbf P_2|$. This provides the set of all partitions of $\{1,\dots,p\}$ with a partial order for which $\mathbf P_{\max}=\{\{1\},\dots,\{p\}\}$ is a (unique) maximal element and $\mathbf P_{\min}=\{\{1,\dots,p\}\}$ is a (unique) minimal one. Finally, if I is an arbitrary set and $\xi=(\xi_1,\dots,\xi_p)\in I^p$, then $\mathbf P_\xi$ is defined as the unique partition such that for all $1\le k,l\le p$, $k\equiv l$ $(\mathbf P_\xi)$ if and only if $\xi_k=\xi_l$.

PROPOSITION 1.14. *Let $(\Omega,\Sigma,\mathbb P)$ be a probability space and $(\varepsilon_i)_{i\in I}$ a family of independent random variables with $\mathbb P\{\varepsilon_i=1\}=1/2=\mathbb P\{\varepsilon_i=-1\}$ for each $i\in I$. Let $p\ge 2$ be an arbitrary integer and for $1\le j\le p$, let E_j be Banach spaces, $f_j:I\to E_j$ be finitely supported functions and $\varphi:E_1\times\dots\times E_p\to F$ be a p-linear map of norm not exceeding 1, where F is an arbitrary Banach space. Fix a partition $\mathbf P$ of $\{1,\dots,p\}$ and set $A_{\mathbf P}:=\{j\in\{1,\dots,p\}\mid\{j\}\in\mathbf P\}$. Then*

$$\Big\|\sum_{\substack{\xi\in I^p,\mathbf P_\xi\le\mathbf P\\ \xi=(\xi_1,\dots,\xi_p)}}\varphi(f_1(\xi_1),\dots,f_p(\xi_p))\Big\|_F\le\prod_{j\in A_{\mathbf P}}\Big\|\sum_{i\in I}f_j(i)\Big\|_{E_j}\prod_{\substack{j\notin A_{\mathbf P}\\ 1\le j\le p}}\Big(\int_\Omega\Big\|\sum_{i\in I}\varepsilon_if_j(i)\Big\|_{E_j}^p\,d\mathbb P\Big)^{1/p}.$$

Proof. We start by noticing the following. Given a finite set $\alpha=\{j_1,\dots,j_s\}$ of indices $(s\ge 2)$, we consider $s-1$ independent copies of the family $(\varepsilon_i)_{i\in I}$ on $(\Omega,\Sigma,\mathbb P)$ assumed large enough, denoted by $(Y_{j_1}(\alpha,i))_{i\in I},\dots,(Y_{j_{s-1}}(\alpha,i))_{i\in I}$. Then we set

$$\begin{aligned}Z_{j_1}(\alpha,i)&=Y_{j_1}(\alpha,i),\\ Z_{j_k}(\alpha,i)&=Y_{j_{k-1}}(\alpha,i)Y_{j_k}(\alpha,i),\qquad 2\le k\le s-1,\\ Z_{j_s}(\alpha,i)&=Y_{j_{s-1}}(\alpha,i).\end{aligned}$$

Clearly, each of the families $(Z_{j_1}(\alpha,i))_{i\in I},\dots,(Z_{j_s}(\alpha,i))_{i\in I}$ has the same distribution as $(\varepsilon_i)_{i\in I}$. Moreover, using successively the orthonormality of each of the families $(Y_{j_k}(\alpha,i))_{i\in I}$, we check easily that for any function $\eta:\alpha\to I$, the integral $\int_\Omega\prod_{k=1}^sZ_{j_k}(\alpha,\eta(j_k))\,d\mathbb P$ is 1 if η is constant on α, and 0 otherwise.

Now if we are given a partition $\mathbf P$ of $\{1,\dots,p\}$, say $\mathbf P=\{\alpha_1,\dots,\alpha_N\}$, then for each set α_k with $|\alpha_k|\ge 2$, we can define a family $(Z_j(\alpha_k,i))_{i\in I,\,j\in\alpha_k}$ as above. Moreover we can construct these families so that they are mutually independent. A simple verification shows that

$$\sum_{\xi\in I^p,\,\mathbf P_\xi\le\mathbf P}\varphi(f_1(\xi_1),\dots,f_p(\xi_p))=\int_\Omega\varphi(\phi_1(\omega),\dots,\phi_p(\omega))\,d\mathbb P(\omega)$$

where $\xi=(\xi_1,\dots,\xi_p)$ and for each integer $1\le j\le p$ we have set

$$\forall\omega\in\Omega,\qquad\phi_j(\omega):=\begin{cases}\sum_{i\in I}Z_j(\alpha_k,i)(\omega)f_j(i)&\text{if } j\in\alpha_k\text{ with }|\alpha_k|\ge 2,\\ \sum_{i\in I}f_j(i)&\text{if } j\in A_{\mathbf P}.\end{cases}$$

Hence by using Hölder's inequality we get

$$\begin{aligned}\Big\|\sum_{\xi\in I^p,\,\mathbf P_\xi\le\mathbf P}\varphi(f_1(\xi_1),\dots,f_p(\xi_p))\Big\|_F&\le\int_\Omega\|\phi_1(\omega)\|_{E_1}\dots\|\phi_p(\omega)\|_{E_p}\,d\mathbb P(\omega)\\ &\le\prod_{j\in A_{\mathbf P}}\|\phi_j\|_{E_j}\prod_{\substack{j\notin A_{\mathbf P}\\ 1\le j\le p}}\Big(\int_\Omega\|\phi_j\|_{E_j}^p\,d\mathbb P\Big)^{1/p}\end{aligned}$$

$$= \prod_{j\in A_{\mathbf{P}}} \Big\| \sum_{i\in I} f_j(i) \Big\|_{E_j} \prod_{\substack{j\notin A_{\mathbf{P}}\\ 1\le j\le p}} \Big(\int_\Omega \Big\| \sum_{i\in I} Z_j(\alpha_k, i) f_j(i) \Big\|_{E_j}^p d\mathbb{P}\Big)^{1/p}$$

$$= \prod_{j\in A_{\mathbf{P}}} \Big\| \sum_{i\in I} f_j(i) \Big\|_{E_j} \prod_{\substack{j\notin A_{\mathbf{P}}\\ 1\le j\le p}} \Big(\int_\Omega \Big\| \sum_{i\in I} \varepsilon_i f_j(i) \Big\|_{E_j}^p d\mathbb{P}\Big)^{1/p}. \ \blacksquare$$

Proof of Theorem 1.13. Let $f = \sum_{t\in\Lambda} \lambda(t)\otimes x_t$ with $t\mapsto x_t$ finitely supported. Then

$$\|f\|_{L^{2p}(\tau)}^{2p} = \tau((f^*f)^p) = \|f^{\mu_1}\ldots f^{\mu_p}\|_{L^2(\tau)}^2$$
$$= \Big\| \sum_{\gamma\in G} \lambda(\gamma) \otimes \sum_{\substack{\xi\in\Lambda^p\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{L^2(\tau)}^2$$

where, for each $1\le k\le p$, we have set

$$\begin{cases} \mu_k = 1, & \nu_k = 1 \quad \text{if } k \text{ is even},\\ \mu_k = *, & \nu_k = -1 \quad \text{if } k \text{ is odd}.\end{cases}$$

Then

$$\|f\|_{L^{2p}(\tau)}^{2p} = \sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2$$
$$= \sum_{\gamma\in G} \Big\| \sum_{\substack{\mathbf{P}\text{ partition}\\ \text{of }\{1,\ldots,p\}}} \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi=\mathbf{P}\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2$$
$$\le 2\sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi=\mathbf{P}_{\max}\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2$$
$$+ C_p \sum_{\mathbf{P}\ne\mathbf{P}_{\max}} \Big(\sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi=\mathbf{P}\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2\Big)$$

where C_p is a constant depending only on p and more precisely on the number of partitions of $\{1,\ldots,p\}$. Henceforth, all the constants which will appear during the proof and which depend on p only will be denoted by C_p for simplicity. On the other hand, let

$$\mathcal{S} := \max\Big\{ \Big\| \Big(\sum_{t\in\Lambda} x_t^* x_t\Big)^{1/2} \Big\|_{S^{2p}}, \Big\| \Big(\sum_{t\in\Lambda} x_t x_t^*\Big)^{1/2} \Big\|_{S^{2p}} \Big\}$$

and for each partition $\mathbf{P}$, let

$$\mathcal{S}(\mathbf{P}) := \sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi=\mathbf{P}\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2.$$

With these notations, the inequality above becomes

$$\|f\|_{L^{2p}(\tau)}^{2p} \le 2\mathcal{S}(\mathbf{P}_{\max}) + C_p \sum_{\mathbf{P}\ne\mathbf{P}_{\max}} \mathcal{S}(\mathbf{P}). \tag{1.7}$$

Our aim is to prove that $\mathcal{S}(\mathbf{P}_{\max}) \le Z_p(\Lambda)\mathcal{S}^{2p}$ and that there exists C_p such that for each partition $\mathbf{P}\ne\mathbf{P}_{\max}$, we have

$$\mathcal{S}(\mathbf{P}) \le C_p \mathcal{S}^2 \|f\|_{L^{2p}(\tau)}^{2p-2}. \tag{1.8}$$

Step 1. The assumption that Λ has the $Z(p)$-property implies $\mathcal{S}(\mathbf{P}_{\max}) \le Z_p(\Lambda)\mathcal{S}^{2p}$. Indeed,

$$\mathcal{S}(\mathbf{P}_{\max}) = \sum_{\gamma\in G} \Big\| \sum_{\substack{\xi_1,\ldots,\xi_p\in\Lambda \text{ distinct}\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2$$
$$\le Z_p(\Lambda) \sum_{\gamma\in G} \sum_{\substack{\xi_1,\ldots,\xi_p\in\Lambda \text{ distinct}\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} \|x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\|_{S^2}^2$$
$$\le Z_p(\Lambda) \sum_{\xi_1,\ldots,\xi_p\in\Lambda} \|(x_{\xi_p}^{\mu_p})^*\ldots(x_{\xi_1}^{\mu_1})^* x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\|_{S^1}$$
$$= Z_p(\Lambda) \Big\| \sum_{\xi_1,\ldots,\xi_p\in\Lambda} (x_{\xi_p}^{\mu_p})^*\ldots(x_{\xi_1}^{\mu_1})^* x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^1}$$
$$\le Z_p(\Lambda) \prod_{i=1}^p \max\Big\{ \Big\| \sum_{\xi_i\in\Lambda} x_{\xi_i}^* x_{\xi_i} \Big\|_{S^p}, \Big\| \sum_{\xi_i\in\Lambda} x_{\xi_i} x_{\xi_i}^* \Big\|_{S^p} \Big\}$$

where for the last inequality we applied Corollary 0.9. Therefore $\mathcal{S}(\mathbf{P}_{\max}) \le Z_p(\Lambda)\mathcal{S}^{2p}$.

Step 2. Given an integer $1\le k\le p-2$, we show that if (1.8) is satisfied for all the partitions $\mathbf{P}$ with $|\mathbf{P}|\le k$, then it is also satisfied for all $\mathbf{P}$ with $|\mathbf{P}|\le k+1$. Indeed, let $\mathbf{P}_0$ be a fixed partition with $|\mathbf{P}_0| = k+1$; then

$$\mathcal{S}(\mathbf{P}_0) = \sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} - \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi<\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2$$
$$\le 2\sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2 + 2\sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi<\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p} \Big\|_{S^2}^2$$

$$\le 2\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2 + 2C_p\sum_{\mathbf{P}<\mathbf{P}_0}\mathcal{S}(\mathbf{P}).$$

According to the induction hypothesis, each $\mathbf{P}<\mathbf{P}_0$ satisfies (1.8) since $|\mathbf{P}|<|\mathbf{P}_0|=k+1$, hence we are reduced to proving the inequality

$$\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2 \le C_p\mathcal{S}^2\|f\|_{L^{2p}(\tau)}^{2p-2}.$$

Now

$$\begin{aligned}
\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2
&=\Big\|\sum_{\gamma\in G}\lambda(\gamma)\otimes\Big(\sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\Big)\Big\|_{L^2(\tau)}^2\\
&=\Big\|\sum_{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0}(\lambda(\xi_1)\otimes x_{\xi_1})^{\mu_1}(\lambda(\xi_2)\otimes x_{\xi_2})^{\mu_2}\ldots(\lambda(\xi_p)\otimes x_{\xi_p})^{\mu_p}\Big\|_{L^2(\tau)}^2\\
&=\Big\|\sum_{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0} f_1(\xi_1)\ldots f_p(\xi_p)\Big\|_{L^2(\tau)}^2
\end{aligned}$$

where for each $1\le j\le p$, f_j is defined on Λ by setting $f_j(t)=(\lambda(t)\otimes x_t)^{\mu_j}$ for $t\in\Lambda$. The f_j's belong to $L^{2p}(\tau)$. At this level, we apply Proposition 1.14 to $\{-1,1\}^{\mathbb N}$ equipped with the counting probability ν, to the n_tth coordinate projection on $\{-1,1\}^{\mathbb N}$ denoted by ε_t where $\{n_t \mid t\in\Lambda\}$ is an enumeration of the set Λ, to the functions $f_j:\Lambda\to L^{2p}(\tau)$ above and to the p-linear contractive map which is the product from $L^{2p}(\tau)\times\ldots\times L^{2p}(\tau)$ (p times) into $L^2(\tau)$. Hence, letting $A_{\mathbf{P}_0}:=\{j\in\{1,\ldots,p\}\mid\{j\}\in\mathbf{P}_0\}$, we get

$$\begin{aligned}
\Big\|\sum_{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0} f_1(\xi_1)\ldots f_p(\xi_p)\Big\|_{L^2(\tau)}
&\le\prod_{j\in A_{\mathbf{P}_0}}\Big\|\sum_{t\in\Lambda}f_j(t)\Big\|_{L^{2p}(\tau)}\prod_{\substack{1\le j\le p\\ j\notin A_{\mathbf{P}_0}}}\Big(\int_{\{-1,1\}^{\mathbb N}}\Big\|\sum_{t\in\Lambda}\varepsilon_t f_j(t)\Big\|_{L^{2p}(\tau)}^p\,d\nu\Big)^{1/p}\\
&=\|f\|_{L^{2p}(\tau)}^{|A_{\mathbf{P}_0}|}\Big(\int_{\{-1,1\}^{\mathbb N}}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^p\,d\nu\Big)^{(p-|A_{\mathbf{P}_0}|)/p}
\end{aligned}$$

since for each $1\le j\le p$, we have

$$\sum_{t\in\Lambda}f_j(t)=\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^{\mu_j}=f^{\mu_j},\qquad \sum_{t\in\Lambda}\varepsilon_t f_j(t)=\Big(\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big)^{\mu_j}.$$

On the other hand, applying Jensen's inequality followed by the non-commutative version of the Khinchin inequalities proved in [26] and [27], for each integer $1\le j\le p$ we get

$$\begin{aligned}
&\Big(\int_{\{-1,1\}^{\mathbb N}}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^p\,d\nu\Big)^{1/p}\\
&\le\Big(\int_{\{-1,1\}^{\mathbb N}}\Big\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^{2p}\,d\nu\Big)^{1/(2p)}\\
&\le K_{L^{2p}(\tau)}\max\Big\{\Big\|\Big(\sum_{t\in\Lambda}(\lambda(t)\otimes x_t)^*(\lambda(t)\otimes x_t)\Big)^{1/2}\Big\|_{L^{2p}(\tau)},\\
&\qquad\qquad\Big\|\Big(\sum_{t\in\Lambda}(\lambda(t)\otimes x_t)(\lambda(t)\otimes x_t)^*\Big)^{1/2}\Big\|_{L^{2p}(\tau)}\Big\}\\
&=K_{L^{2p}(\tau)}\max\Big\{\Big\|\Big(\lambda(e)\otimes\sum_{t\in\Lambda}x_t^*x_t\Big)^{1/2}\Big\|_{L^{2p}(\tau)},\Big\|\Big(\lambda(e)\otimes\sum_{t\in\Lambda}x_tx_t^*\Big)^{1/2}\Big\|_{L^{2p}(\tau)}\Big\}\\
&=K_{L^{2p}(\tau)}\max\Big\{\Big\|\Big(\sum_{t\in\Lambda}x_t^*x_t\Big)^{1/2}\Big\|_{S^{2p}},\Big\|\Big(\sum_{t\in\Lambda}x_tx_t^*\Big)^{1/2}\Big\|_{S^{2p}}\Big\}=K_{L^{2p}(\tau)}\mathcal{S}.
\end{aligned}$$

This means that

$$\Big\|\sum_{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0} f_1(\xi_1)\ldots f_p(\xi_p)\Big\|_{L^2(\tau)}\le(K_{L^{2p}(\tau)}\mathcal{S})^{p-|A_{\mathbf{P}_0}|}\|f\|_{L^{2p}(\tau)}^{|A_{\mathbf{P}_0}|}.$$

Therefore we have

$$\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2\le(K_{L^{2p}(\tau)}\mathcal{S})^{2(p-|A_{\mathbf{P}_0}|)}\|f\|_{L^{2p}(\tau)}^{2|A_{\mathbf{P}_0}|}.$$

Now since the inequality $\mathcal{S}\le\|f\|_{L^{2p}(\tau)}$ always holds and since $\mathbf{P}_0\ne\mathbf{P}_{\max}$ so that $|A_{\mathbf{P}_0}|<p$, from the above inequality we get

$$\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\,\mathbf{P}_\xi\le\mathbf{P}_0\\ \xi_1^{\nu_1}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}\ldots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2\le K_{L^{2p}(\tau)}^{2p}\mathcal{S}^2\|f\|_{L^{2p}(\tau)}^{2p-2}.$$

STEP 3. The minimal partition satisfies the inequality (1.8). Indeed, Proposition 1.14 applied similarly to $\mathbf{P}_{\min}$ instead of $\mathbf{P}_0$ gives the much more precise inequality $\mathcal{S}(\mathbf{P}_{\min})\le K_{L^{2p}(\tau)}^{2p}\mathcal{S}^{2p}$.

CONCLUSION. By means of Steps 2 and 3, the inequality (1.8) is satisfied for all partitions $\mathbf{P}\ne\mathbf{P}_{\max}$. Therefore, upon taking into account Step 1, the inequality (1.7) gives

$$\|f\|_{L^{2p}(\tau)}^{2p} \le 2Z_p(\Lambda)\mathcal{S}^{2p} + C_p\mathcal{S}^2\|f\|_{L^{2p}(\tau)}^{2p-2}.$$

This means that letting $x = \mathcal{S}^{-1}\|f\|_{L^{2p}(\tau)}$ we have $x^{2p} - C_p x^{2(p-1)} - 2Z_p(\Lambda) \le 0$, which easily leads to $x \le 3\max\{Z_p(\Lambda)^{1/(2p)}, C_p\}$. Hence we are done. ■

As an illustration of Theorem 1.13, we derive the following result already obtained in [19].

COROLLARY 1.15. *Let $\{g_n \mid n \in \mathbb{N}\}$ denote an arbitrary free subset of the free group $\mathbb{F}_\infty$. Then for each $2 < p < \infty$, there exists a constant $C_p > 0$ depending only on p such that for each finitely supported sequence $(x_n)_{n\ge 0}$ of operators in S^p, we have*

$$\Big\|\sum_{n\in\mathbb{N}} \lambda(g_n)\otimes x_n\Big\|_{L^p(\tau)} \le C_p \max\Big\{\Big\|\Big(\sum_{n\in\mathbb{N}} x_n^* x_n\Big)^{1/2}\Big\|_{S^p}, \Big\|\Big(\sum_{n\in\mathbb{N}} x_n x_n^*\Big)^{1/2}\Big\|_{S^p}\Big\}. \tag{1.9}$$

Proof. Since $\{g_n \mid n \in \mathbb{N}\}$ is a free set in $\mathbb{F}_\infty$, it has the $B(p)$-property for all integers $2 \le p < \infty$. Then, by Theorem 1.13, it has the $\Lambda(2p)_{\rm cb}$-property for all integers $2 \le p < \infty$. Therefore the inequality (1.9) is satisfied for all real $2 < p < \infty$. ■

COMMENTS 1.16. Taking inverses in the definition of property $Z(p)$ is compulsory. Indeed, let us say that a subset Λ of G has *property* $Z^+(p)$ if the constant

$$Z_p^+(\Lambda) = \sup_{\gamma\in G} |\{(t_1,\dots,t_p)\in\Lambda^p \mid t_1\dots t_p = \gamma\}|$$

is finite. When G is Abelian, such a set is certainly a $\Lambda(2p)$-set as shown in [37] but it is not a $\Lambda(2p)_{\rm cb}$-set in general. As an example, take $\Lambda = \{2^i + 2^j \mid i,j \ge 0\}$ in $G = \mathbb{Z}$. Then Λ has the $Z^+(p)$-property for all p but does not have the $\Lambda(p)_{\rm cb}$-property for any $2 < p < \infty$ as we will point out later in Corollary 2.9. However, a $Z^+(p)$-subset of an arbitrary discrete group enjoys a weaker and actually a strictly weaker analytical property, namely: for all x_t in S^{2p}, we have

$$\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|_{L^2(\tau)} \le \sqrt{Z_p^+(\Lambda)}\max\Big\{\Big\|\Big(\sum_{t\in\Lambda} x_t x_t^*\Big)^{p/2}\Big\|_{S^2}, \Big\|\Big(\sum_{t\in\Lambda} x_t^* x_t\Big)^{p/2}\Big\|_{S^2}\Big\}. \tag{1.10}$$

Indeed,

$$\begin{aligned}\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|_{L^2(\tau)}^2 &= \Big\|\sum_{\gamma\in G}\lambda(\gamma)\otimes\Big(\sum_{\substack{t_1,\dots,t_p\in\Lambda\\ t_1\dots t_p=\gamma}} x_{t_1}\dots x_{t_p}\Big)\Big\|_{L^2(\tau)}^2\\ &= \sum_{\gamma\in G}\Big\|\sum_{\substack{t_1,\dots,t_p\in\Lambda\\ t_1\dots t_p=\gamma}} x_{t_1}\dots x_{t_p}\Big\|_{S^2}^2\\ &\le Z_p^+(\Lambda)\sum_{\gamma\in G}\sum_{\substack{t_1,\dots,t_p\in\Lambda\\ t_1\dots t_p=\gamma}}\|x_{t_1}\dots x_{t_p}\|_{S^2}^2\\ &\le Z_p^+(\Lambda)\Big\|\sum_{t_1,\dots,t_p\in\Lambda} x_{t_p}^*\dots x_{t_1}^* x_{t_1}\dots x_{t_p}\Big\|_{S^1}.\end{aligned}$$

Using Corollary 0.9, we get

$$\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|_{L^2(\tau)}^2 \le Z_p^+(\Lambda)\prod_{j=1}^p \max\Big\{\Big\|\sum_{t_j\in\Lambda} x_{t_j}^* x_{t_j}\Big\|_{S^p}, \Big\|\sum_{t_j\in\Lambda} x_{t_j} x_{t_j}^*\Big\|_{S^p}\Big\},$$

$$\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|_{L^2(\tau)}^2 \le Z_p^+(\Lambda)\max\Big\{\Big\|\sum_{t\in\Lambda} x_t^* x_t\Big\|_{S^p}^p, \Big\|\sum_{t\in\Lambda} x_t x_t^*\Big\|_{S^p}^p\Big\}. ■$$

In the Abelian case, if a subset Λ has the $Z^+(p)$-property then $L_\Lambda^{2p}(\tau)$ satisfies an inequality analogous to "type 2", i.e. for every finitely supported sequence $(x_t)_{t\in\Lambda}$ in S^{2p}, we have

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)} \le (Z_p^+(\Lambda))^{1/(2p)}\Big(\sum_{t\in\Lambda}\|x_t\|_{S^{2p}}^2\Big)^{1/2}. \tag{1.11}$$

The proof of (1.11) sketched below is similar to the one given in [43]:

$$\begin{aligned}\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^{2p} &= \Big\|\sum_{\gamma\in G}\lambda(\gamma)\otimes\Big(\sum_{\substack{(t_1,\dots,t_p)\in\Lambda^p\\ (\xi_1,\dots,\xi_p)\in\Lambda^p\\ t_1\xi_1^{-1}\dots t_p\xi_p^{-1}=\gamma}} x_{t_1}x_{\xi_1}^*\dots x_{t_p}x_{\xi_p}^*\Big)\Big\|_{L^1(\tau)}\\ &= \Big\|\sum_{\substack{(t_1,\dots,t_p)\in\Lambda^p\\ (\xi_1,\dots,\xi_p)\in\Lambda^p\\ t_1\xi_1^{-1}\dots t_p\xi_p^{-1}=e}} x_{t_1}x_{\xi_1}^*\dots x_{t_p}x_{\xi_p}^*\Big\|_{S^1}\\ &\le \sum_{\substack{(t_1,\dots,t_p)\in\Lambda^p\\ (\xi_1,\dots,\xi_p)\in\Lambda^p\\ t_1\dots t_p=\xi_1\dots\xi_p}}\|x_{t_1}x_{\xi_1}^*\dots x_{t_p}x_{\xi_p}^*\|_{S^1}.\end{aligned}$$

For a compact operator y, $(s_j(y))_{j\geq 1}$ stands for the decreasing sequence of the eigenvalues of the operator $(y^*y)^{1/2}$ (repeated according to their multiplicities). With this notation, we have

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^{2p}\leq \sum_{\substack{(t_1,\ldots,t_p)\in\Lambda^p\\(\xi_1,\ldots,\xi_p)\in\Lambda^p\\t_1\ldots t_p=\xi_1\ldots\xi_p}}\sum_{j\geq 1}s_j(x_{t_1}x_{\xi_1}^*\ldots x_{t_p}x_{\xi_p}^*).$$

Then, using the general Horn inequality proved in [43], we obtain

$$\begin{aligned}\Big\|\sum_{t\in\Lambda}&\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^{2p}\\
&\leq \sum_{\substack{(t_1,\ldots,t_p)\in\Lambda^p\\(\xi_1,\ldots,\xi_p)\in\Lambda^p\\t_1\ldots t_p=\xi_1\ldots\xi_p}}\sum_{j\geq 1}s_j(x_{t_1})s_j(x_{\xi_1}^*)\ldots s_j(x_{t_p})s_j(x_{\xi_p}^*)\\
&=\sum_{\gamma\in G}\sum_{\substack{(t_1,\ldots,t_p)\in\Lambda^p\\(\xi_1,\ldots,\xi_p)\in\Lambda^p\\t_1\ldots t_p=\gamma=\xi_1\ldots\xi_p}}\sum_{j\geq 1}s_j(x_{t_1})s_j(x_{\xi_1})\ldots s_j(x_{t_p})s_j(x_{\xi_p})\\
&=\sum_{j\geq 1}\sum_{\gamma\in G}\Big(\sum_{\substack{t_1,\ldots,t_p\in\Lambda\\t_1\ldots t_p=\gamma}}s_j(x_{t_1})\ldots s_j(x_{t_p})\Big)^2.\end{aligned}$$

Using the assumption on Λ, we get

$$\begin{aligned}\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}^{2p}&\leq Z_p^+(\Lambda)\sum_{j\geq 1}\sum_{\gamma\in G}\sum_{\substack{t_1,\ldots,t_p\in\Lambda\\t_1\ldots t_p=\gamma}}s_j^2(x_{t_1})\ldots s_j^2(x_{t_p})\\
&=Z_p^+(\Lambda)\sum_{j\geq 1}\sum_{t_1,\ldots,t_p\in\Lambda}s_j^2(x_{t_1})\ldots s_j^2(x_{t_p})\\
&=Z_p^+(\Lambda)\sum_{j\geq 1}\Big(\sum_{t\in\Lambda}s_j^2(x_t)\Big)^p.\end{aligned}$$

This implies that

$$\begin{aligned}\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)}&\leq (Z_p^+(\Lambda))^{1/(2p)}\Big(\sum_{j\geq 1}\Big(\sum_{t\in\Lambda}s_j^2(x_t)\Big)^p\Big)^{1/(2p)}\\
&\leq (Z_p^+(\Lambda))^{1/(2p)}\Big(\sum_{t\in\Lambda}\Big(\sum_{j\geq 1}s_j^{2p}(x_t)\Big)^{1/p}\Big)^{1/2}\\
&\leq (Z_p^+(\Lambda))^{1/(2p)}\Big(\sum_{t\in\Lambda}\|x_t\|_{S^{2p}}^2\Big)^{1/2}. \ \blacksquare\end{aligned}$$

2. NON-COMMUTATIVE $\Lambda(p)$-SETS IN $\mathbb{Z}$

We start by stating the definitions and the results proved in Section 1 for the group $\mathbb{Z}$, in which case $L^p(\tau_0)$ coincides with L^p and $L^p(\tau)$ with $L^p(S^p)$ via the identification for each integer n between the operator $\lambda(n)$ and the function z^n defined on the torus $\mathbb{T}$.

Definition 2.1. (i) Let $2<p<\infty$. A subset $\Lambda\subset\mathbb{Z}$ is called a *$\Lambda(p)$-set* (resp. *$\Lambda(p)_{\rm cb}$-set*) if there exists a constant $c>0$ such that for all f in L^p_Λ (resp. $L^p_\Lambda(S^p)$), say with $\widehat{f}$ finitely supported, we have

$$\|f\|_{L^p}\leq c\|f\|_{L^2},$$

resp.

$$\|f\|_{L^p(S^p)}\leq c\max\Big\{\Big\|\Big(\sum_{n\in\Lambda}\widehat{f}(n)^*\widehat{f}(n)\Big)^{1/2}\Big\|_{S^p},\Big\|\Big(\sum_{n\in\Lambda}\widehat{f}(n)\widehat{f}(n)^*\Big)^{1/2}\Big\|_{S^p}\Big\}.$$

We denote by $\lambda_p(\Lambda)$ (resp. $\lambda_p^{\rm cb}(\Lambda)$) or simply λ_p (resp. $\lambda_p^{\rm cb}$) the smallest constant c for which the inequality above holds.

(ii) Let $1\leq p\leq\infty$. A set $\Lambda\subset\mathbb{Z}$ is said to be an *interpolation set for* $M(L^p)$ (resp. $M_{\rm cb}(L^p)$) if the restriction map $\mathcal{Q}$ defined on $M(L^p)$ (resp. $M_{\rm cb}(L^p)$) by sending a Fourier multiplier φ on L^p to the sequence $(\varphi(n))_{n\in\Lambda}$ in $\ell_\infty(\Lambda)$, is surjective and thus μ-surjective for some constant μ. We let $\mu_p(\Lambda)$ (resp. $\mu_p^{\rm cb}(\Lambda)$) or simply μ_p (resp. $\mu_p^{\rm cb}$) be the smallest constant μ for which this happens.

Proposition 2.2. *Let $2<p<\infty$. For $\Lambda\subset\mathbb{Z}$, the following properties are equivalent.*

(i) *Λ is a $\Lambda(p)$-set (resp. $\Lambda(p)_{\rm cb}$-set).*
(ii) *Λ is an interpolation set for $M(L^p)$ (resp. $M_{\rm cb}(L^p)$).*

Moreover, for each set $\Lambda\subset\mathbb{Z}$, we have

$$\mu_p(\Lambda)\leq\lambda_p(\Lambda)\leq k_p\mu_p(\Lambda)\qquad(\text{resp. }\mu_p^{\rm cb}(\Lambda)\leq\lambda_p^{\rm cb}(\Lambda)\leq K_p\mu_p^{\rm cb}(\Lambda))$$

where k_p (resp. K_p) is the constant defined in the Khinchin inequality (0.1) (*resp.* (0.3)).

The following known facts show that there is a restriction on the size of $\Lambda(p)$-sets (thus a fortiori on $\Lambda(p)_{\rm cb}$-sets).

Facts 2.3. (i) ([37]) *There exists a constant $c_1>0$ such that for any $\Lambda(p)$-subset Λ of $\mathbb{Z}$ $(2<p<\infty)$ and any integers a,b,N with $N\geq 1$, we have*

$$|\Lambda\cap[a,a+Nb]|\leq c_1(\lambda_p(\Lambda))^2N^{2/p}.$$

(ii) ([9], see also [42]) *For any fixed $2 < p < \infty$, there exists a constant c_2 such that for any integer $N \geq 1$, there exists $\Lambda_N \subset [0, N]$ satisfying*

$$c_2 N^{2/p} \leq |\Lambda_N| \ \&\ \sup_{N\geq 1} \lambda_p(\Lambda_N) < \infty.$$

Thus the decreasing family $(\{\Lambda \subset \mathbb{Z} \mid \Lambda \text{ is } \Lambda(p)\})_{2<p<\infty}$ of sets is in fact strictly decreasing.

In the sequel, we are interested in the size of $\Lambda(p)_{\mathrm{cb}}$-sets where p is an even integer. More precisely, our goal is to construct large and actually the largest $\Lambda(p)_{\mathrm{cb}}$-sets possible. For this purpose, we use the combinatorial properties introduced in Section 1, namely the $B(p)$- and $Z(p)$-properties which we recall below.

DEFINITION 2.4. Let $p \geq 2$ be an integer. We say that a subset Λ of $\mathbb{Z}$ has the *$B(p)$-property* if for all p-tuples $(n_1, \ldots, n_p)$ and $(m_1, \ldots, m_p)$ in Λ^p, $\sum_{k=1}^p n_k = \sum_{k=1}^p m_k$ implies $\{n_k \mid 1 \leq k \leq p\} = \{m_k \mid 1 \leq k \leq p\}$, where in each set, the integers are repeated according to their multiplicity in the corresponding sequence. We say that Λ has the *$Z(p)$-property* if $Z_p(\Lambda) < \infty$, where

$$Z_p(\Lambda) := \sup_{\gamma\in\mathbb{Z}} \left|\left\{(n_1, \ldots, n_p) \in \Lambda^p \,\middle|\, \forall i \neq j,\ n_i \neq n_j \ \&\ \sum_{k=1}^p (-1)^k n_k = \gamma\right\}\right|.$$

Theorem 2.5 below was proved previously (cf. Section 1) in the more general case of subsets of discrete groups.

THEOREM 2.5. (i) *If $\Lambda \subset \mathbb{Z}$ has the $B(p)$-property then it has the $Z(p)$-property with $Z_p(\Lambda) \leq (\frac{p}{2}!)^2$ if p is even and $Z_p(\Lambda) \leq (\frac{p+1}{2}!)^2$ if p is odd.*

(ii) *Every set $\Lambda \subset \mathbb{Z}$ with the $Z(p)$-property has the $\Lambda(2p)_{\mathrm{cb}}$-property. Moreover, there exists a constant C_p depending on p only such that $\lambda_{2p}^{\mathrm{cb}}(\Lambda) \leq 3\max\{Z_p(\Lambda)^{1/2p}, C_p\}$.*

COROLLARY 2.6. *For each even integer $p > 2$, there exists a $\Lambda(p)_{\mathrm{cb}}$-set which is not a $\Lambda(q)$-set for any $q > p$.*

Proof. Let $p > 2$ be a fixed even integer. By a construction in [37], there exists a set $\Lambda \subset \mathbb{N}$ which has the $B(p/2)$-property and satisfies

$$\overline{\lim_{N\to\infty}} \sup_{a,b\in\mathbb{N}} \frac{|\Lambda \cap [a, a+Nb]|}{N^{2/p}} > 0.$$

So there exist sequences $(a_k)_k$, $(b_k)_k$, $(N_k)_k$ of integers with $\lim_{k\to\infty} N_k = \infty$ and a positive constant c such that for each k, we have

$$cN_k^{2/p} \leq |\Lambda \cap [a_k, a_k + N_k b_k]|.$$

By Fact 2.3(i), if Λ has the $\Lambda(q)$-property for some $q > p$ then for each integer k, it also satisfies (c_1 is the constant appearing in this fact)

$$(\lambda_q(\Lambda))^2 c_1 N_k^{2/q} \geq |\Lambda \cap [a_k, a_k + N_k b_k]|.$$

Since $q > p$ and N_k can be arbitrarily large, we see that this cannot hold. Thus Λ is not a $\Lambda(q)$-set and we are done. ■

From the Rudin set appearing in the proof above, we can construct a sequence of sets as in the following corollary. The corollary will be used for the proof of Theorem 4.9 while a reformulation of it will be used to prove Theorem 4.8.

COROLLARY 2.7. *For each even integer $p > 2$, there exists a sequence of sets $\Lambda_n \subset [2^n, 2^{n+1}[$ such that*

$$\inf_{n\geq 0} 2^{-2n/p}|\Lambda_n| > 0, \qquad \sup_{n\geq 0} \lambda_p^{\mathrm{cb}}(\Lambda_n) < \infty.$$

The next result shows that the $\Lambda(p)_{\mathrm{cb}}$-property is much more restrictive than the usual $\Lambda(p)$-property.

PROPOSITION 2.8. *There exists a numerical constant $\delta > 0$ such that for each $2 < p < \infty$ and each $\Lambda(p)_{\mathrm{cb}}$-set Λ, if Λ contains the sum $A + A$ of an arbitrary finite set A then $|A| < \delta(2\lambda_p^{\mathrm{cb}}(\Lambda))^{8p/(p-2)}$.*

Proof. STEP 1. If a $\Lambda(p)_{\mathrm{cb}}$-set Λ contains the sum $B + B$ of some finite set B with property $(\star)$ below, then $|B| \leq 2(2\lambda_p^{\mathrm{cb}}(\Lambda))^{2p/(p-2)}$.

We say that a set B of integers has *property* $(\star)$ if given an enumeration $B = \{b_1, b_2, \ldots\}$ ($b_k \neq b_l$ whenever $k \neq l$), the following holds: if $\beta_k = -1, 0$ or 1 for each $k \geq 1$, $\sum_{k\geq 1} |\beta_k| \leq 4$, and $\sum_{k\geq 1} \beta_k b_k = 0$, then $\beta_k b_k = 0$ for all $k \geq 1$.

Let Λ be a $\Lambda(p)_{\mathrm{cb}}$-set and assume that there exists a set $B = \{b_1, \ldots, b_n\}$ with cardinality $n \geq 1$ satisfying $(\star)$ and such that $\Lambda \supset B + B = \{b_k + b_l \mid 1 \leq k, l \leq n\}$. Let $\varepsilon = (\varepsilon_{kl})_{1\leq k,l\leq n}$ be such that for all $1 \leq k, l \leq n$,

$$\varepsilon_{kl} = \varepsilon_{lk} = \pm 1, \qquad \forall k \neq l,$$
$$\varepsilon_{kk} = \begin{cases} 1 & \text{if } \forall i \neq j,\ 2b_k \neq b_i + b_j, \\ \varepsilon_{ij} & \text{if } \exists i \neq j,\ 2b_k = b_i + b_j. \end{cases}$$

Note that ε is well defined since B satisfies $(\star)$. We are interested in controlling the norm of the operator T_ε defined on S_n^p by sending $x = (x_{kl})_{1\leq k,l\leq n}$ to $(\varepsilon_{kl}x_{kl})_{1\leq k,l\leq n}$. Let $x = (x_{kl})_{1\leq k,l\leq n}$ be a fixed operator in S_n^p and consider the function f_x which takes z in $\mathbb{T}$ to $(z^{b_k+b_l}x_{kl})_{1\leq k,l\leq n}$ in S_n^p. This function is clearly well defined and belongs to $L^p(S_n^p)$. Moreover, $f_x(z) = D_z x D_z$ for all z in $\mathbb{T}$, where by definition D_z is the unitary operator

$$D_z = \begin{pmatrix} z^{b_1} & 0 & \dots & 0 \\ 0 & z^{b_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^{b_n} \end{pmatrix}.$$

Therefore $\|f_x\|_{L^p(S_n^p)} = \|x\|_{S_n^p}$. We define the map ν on $\mathbb{Z}$ by setting $\nu(b_k + b_l) := \varepsilon_{kl}$ for all $1 \le k, l \le n$ and $\nu(s) := 0$ for $s \in \mathbb{Z} \setminus B + B$. Note that ν is well defined since B has property $(\star)$, and ν is the trivial extension to $\mathbb{Z}$ of some choice of signs on $B + B$. Hence $\|\nu\|_{M_{\mathrm{cb}}(L^p)} \le \lambda_p^{\mathrm{cb}}(B+B) \le \lambda_p^{\mathrm{cb}}(\Lambda)$, that is, the operator M_ν associated with the multiplier ν is such that $M_\nu \otimes \mathrm{id}_{S^p}$ is bounded on $L^p(S^p)$ with $\|M_\nu \otimes \mathrm{id}_{S^p}\| \le \lambda_p^{\mathrm{cb}}(\Lambda)$. We check easily that $M_\nu \otimes \mathrm{id}_{S_n^p}(f_x) = f_{T_\varepsilon(x)}$. Thus

$$\|T_\varepsilon(x)\|_{S_n^p} = \|f_{T_\varepsilon(x)}\|_{L^p(S_n^p)} \le \lambda_p^{\mathrm{cb}}(\Lambda)\|f_x\|_{L^p(S_n^p)} = \lambda_p^{\mathrm{cb}}(\Lambda)\|x\|_{S_n^p}.$$

Hence T_ε has norm at most $\lambda_p^{\mathrm{cb}}(\Lambda)$. This implies that if we denote by α_m the unconditionality constant of the canonical basis of S_m^p where $m = n/2$ if n is even and $m = (n-1)/2$ if n is odd, then $\alpha_m \le \lambda_p^{\mathrm{cb}}(\Lambda)$. Indeed, given a choice $\varepsilon = (\varepsilon_{kl})_{1\le k,l\le m}$ of signs we can consider the map ε' defined on $\{1,\dots,n\} \times \{1,\dots,n\}$ by setting for each $1 \le k \le n$,

$$\varepsilon'(k,k) = \begin{cases} 1 & \text{if } \forall i \ne j,\ 2b_k \ne b_i + b_j, \\ \varepsilon'(i,j) & \text{if } \exists i \ne j,\ 2b_k = b_i + b_j, \end{cases}$$

and for all $1 \le l, k \le m$,

$$\varepsilon'(k+m,l) = \varepsilon_{kl}, \qquad \varepsilon'(k,l+m) = \varepsilon_{lk},$$
$$\varepsilon'(k,l) = \varepsilon'(k+m,l+m) = 1 \quad \text{if } k \ne l$$

when n is even, and

$$\varepsilon'(k+m+1,l) = \varepsilon_{kl}, \qquad \varepsilon'(k,l+m+1) = \varepsilon_{lk},$$
$$\varepsilon'(k,l) = \varepsilon'(k+m+1,l+m+1) = 1 \quad \text{if } k \ne l,$$
$$\varepsilon'(m+1,l) = \varepsilon'(m+1,l+m+1) = 1$$

when n is odd. By the above, $\|T_\varepsilon\|_{B(S_m^p)} \le \|T_{\varepsilon'}\|_{B(S_n^p)} \le \lambda_p^{\mathrm{cb}}(\Lambda)$. By the results of [22] we have $m^{1/2} \le 2\alpha_m m^{1/p}$. Thus $m^{1/2} \le 2\lambda_p^{\mathrm{cb}}(\Lambda) m^{1/p}$, equivalently $|B| \le 2(2\lambda_p^{\mathrm{cb}}(\Lambda))^{2p/(p-2)}$.

STEP 2. There exists a numerical constant $\delta > 0$ such that each finite subset A of $\mathbb{Z}$ contains a subset B with $|B| > \delta^{-1}|A|^{1/4}$ satisfying $(\star)$.

We start by picking up an arbitrary b_1 in A and assume that for some integer $k \ge 1$, we have k elements $b_1, \dots, b_k$ in A such that $\{b_1, \dots, b_k\}$ enjoys $(\star)$. Then consider the set

$$[b_1,\dots,b_k] := \Big\{ \sum_{l=1}^k \beta_l b_l \,\Big|\, \forall 1 \le l \le k,\ \beta_l = 0, 1, -1 \ \&\ \sum_{l=1}^k |\beta_l| \le 4 \Big\}.$$

Then

$$|[b_1,\dots,b_k]| \le 3^4 \frac{k!}{4!(k-4)!} < \frac{27}{8}k^4.$$

Assume that $|A| > \frac{27}{8}k^4$. Then there exists at least one b_{k+1} in $A \setminus [b_1,\dots,b_k]$. We check easily that $\{b_1,\dots,b_k,b_{k+1}\}$ still has property $(\star)$. Indeed, suppose that $\sum_{l=1}^4 \beta_l b_{i_l} = 0$ for some $\beta_l = -1, 0$ or -1 and $1 \le i_l \le k+1$. Since $\{b_1,\dots,b_k\}$ has property $(\star)$, the only case to check is $\sum_{l=1}^3 \beta_l b_{i_l} - b_{k+1} = 0$. But this does not hold since it would contradict the assumption that $b_{k+1} \notin [b_1,\dots,b_k]$. This means that we can find by induction a set $B = \{b_1,\dots,b_n,b_{n+1}\}$ with property $(\star)$ where $\frac{27}{8}n^4 < |A| < \frac{27}{8}(n+1)^4$ and thus $|B| > (\frac{8}{27}|A|)^{1/4}$.

CONCLUSION. If Λ contains $A + A$ for some finite set A, then by Step 2 it contains $B + B$ for some B with property $(\star)$ and $|B| > (\frac{8}{27}|A|)^{1/4}$. Applying Step 1 we get $|B| \le 2(2\lambda_p^{\mathrm{cb}}(\Lambda))^{2p/(p-2)}$. Therefore

$$|A| < 54(2\lambda_p^{\mathrm{cb}}(\Lambda))^{8p/(p-2)} =: \delta(2\lambda_p^{\mathrm{cb}}(\Lambda))^{8p/(p-2)}. \ \blacksquare$$

COROLLARY 2.9. *There exists a set which is $\Lambda(p)$ for each $2 < p < \infty$ but not $\Lambda(p)_{\mathrm{cb}}$ for any $2 < p < \infty$.*

Proof. Consider $\Lambda = \{2^k + 2^l \mid k, l \ge 0\}$. It is well known that Λ is a $\Lambda(p)$-set for all $2 < p < \infty$ (cf. [24], see also Comments 1.16). But by Proposition 2.8, Λ cannot be a $\Lambda(p)_{\mathrm{cb}}$-set for any $2 < p < \infty$. $\blacksquare$

3. APPLICATIONS TO FOURIER MULTIPLIERS

PROPOSITION 3.1. *For each $2 < p < \infty$, the inclusion $M_{\mathrm{cb}}(L^p) \subset M(L^p)$ is strict.*

Proof. Let $2 < p < \infty$. By Corollary 2.9, there exists a set Λ which is $\Lambda(p)$ but not $\Lambda(p)_{\mathrm{cb}}$. Hence, by Proposition 2.2, it is an interpolation set for $M(L^p)$ but not for $M_{\mathrm{cb}}(L^p)$. Thus, a fortiori, the embedding of $M_{\mathrm{cb}}(L^p)$ into $M(L^p)$ is strict. $\blacksquare$

COMMENT. It was shown in [21] that a Banach space X is a subspace of a quotient of L^p if and only if there exists a constant c such that, for any bounded operator T on L^p, the operator $T \otimes \mathrm{id}_X$ extends to a bounded operator on $L^p(X)$ with $\|T \otimes \mathrm{id}_X\| \le c\|T\|$. Proposition 3.1 implies that there exists an operator T on L^p such that $T\otimes \mathrm{id}_{S^p}$ is not bounded, which means that S^p is not a subspace of a quotient of L^p (cf. [31]).

PROPOSITION 3.2. *For each $2<p<q\le\infty$ where p is an even integer, the inclusion map $M_{\mathrm{cb}}(L^q)\subset M_{\mathrm{cb}}(L^p)$ is strict. Moreover, $M_{\mathrm{cb}}(L^p)$ does not embed continuously into $M(L^q)$.*

Proof. Let $2<p<\infty$ be an even integer. By Corollary 2.6, there exists a $\Lambda(p)_{\mathrm{cb}}$-set which is not a $\Lambda(q)$-set for any $q>p$. By Proposition 2.2, it is an interpolation set for $M_{\mathrm{cb}}(L^p)$ but not for $M(L^q)$. Thus, $M_{\mathrm{cb}}(L^p)$ does not embed continuously into $M(L^q)$. ∎

As a direct application of Lemma 0.2, the interpolated space $(M(L^\infty), M(L^2))_{2/p}$ embeds contractively into $M_{\mathrm{cb}}(L^p)$ for each $2<p<\infty$. Thus, it is natural to wonder whether we do have equality. The following lemmas will be used for the study of this as well as for the proof of Theorem 5.2.

LEMMA 3.3. *Let X, Y be two Banach spaces and $u: X\to Y$ be a bounded operator. Assume there exist $c>0$ and $0<r<1$ such that for all y in B_Y, there exists x in cB_X with $\|ux-y\|<r$. Then u is surjective. More precisely, u is μ-surjective for some constant $\mu\le c/(1-r)$.*

Proof. The proof is elementary. Indeed, for y in B_Y, we can produce a sequence of vectors y_k in B_Y with $y_1=y$, and a sequence of vectors x_k in X such that $\|x_k\|<c$, $y_{k+1}=\frac{1}{r}(y_k-ux_k)$ and $\|ux_k-y_k\|<r$. Then let $x=\sum_{k\ge0} r^k x_{k+1}$. We clearly have $y=ux$ and $\|x\|\le c/(1-r)$. This means that u is μ-surjective for some $\mu\le c/(1-r)$. ∎

LEMMA 3.4. *Let $0<\theta<1$ and x in $(X_0,X_1)_\theta$. Then for all $s,\delta>0$ there exist x_0 in X_0 and x_1 in X_1 such that $x=x_0+x_1$ and $\|x_0\|_{X_0}+s\|x_1\|_{X_1}\le s^\theta\|x\|_\theta+\delta$.*

Proof. See page 103 of [2]. ∎

LEMMA 3.5. *Let (X_0,X_1) be a compatible couple of Banach spaces and Y be an arbitrary Banach space. Consider two operators $u_0: X_0\to Y$, $u_1: X_1\to Y$ which agree on $X_0\cap X_1$. Assume that the operator $u_\theta: X_\theta\to Y$ obtained by complex interpolation is μ-surjective. Let $\alpha(\theta)=\frac{1}{1-\theta}\theta^{\theta/(\theta-1)}$. Then u_0 (resp. u_1) is necessarily surjective. Moreover, it is $\alpha_0(\theta,\mu)$-surjective (resp. $\alpha_1(\theta,\mu)$-surjective) for some constant satisfying*

$$\alpha_0(\theta,\mu)\le\alpha(\theta)\mu^{1/(1-\theta)}\|u_1\|^{\theta/(1-\theta)},$$

resp.

$$\alpha_1(\theta,\mu)\le\alpha(1-\theta)\mu^{1/\theta}\|u_0\|^{(1-\theta)/\theta}.$$

Proof. It suffices to prove the part concerning u_0 since for each $0<\theta<1$, $(X_0,X_1)_\theta=(X_1,X_0)_{1-\theta}$. By replacing u_0, u_1 and μ by $u_0\|u_1\|^{-1}$, $u_1\|u_1\|^{-1}$ and $\mu\|u_1\|$ respectively, we can assume that u_1 has norm one ($u_1\ne0$). Let y be in B_Y. Since u_θ is μ-surjective, there exists x in X_θ with $\|x\|_\theta<\mu$ and $u_\theta(x)=y$. For fixed $s>0$, by Lemma 3.4, there exists a decomposition $x=x_0+x_1$ with $\|x_0\|_{X_0}<s^\theta\mu$ and $\|x_1\|_{X_1}<s^{\theta-1}\mu$. Let us start from s such that $s^{\theta-1}\mu<1$, equivalently $s>\mu^{1/(1-\theta)}$. Then $u_0: X_0\to Y$ satisfies the conditions of Lemma 3.3 with $r=r(s)=s^{\theta-1}\mu$ and $c=c(s)=s^\theta\mu$. Hence, Lemma 3.3 implies that u_0 is $\mu(s)$-surjective for some $\mu(s)\le c(s)/(1-r(s))$. The infimum of $s\mapsto c(s)/(1-r(s))$ when s runs over $]\mu^{1/(1-\theta)},\infty[$ is attained at $s_{\min}=(\theta/\mu)^{1/(\theta-1)}$, and its value is $\alpha(\theta)\mu^{1/(1-\theta)}$ with $\alpha(\theta)$ as in the statement. ∎

PROPOSITION 3.6. *For each $2<p<\infty$, the interpolated space $(M(L^\infty), M(L^2))_{2/p}$ embeds strictly into $M_{\mathrm{cb}}(L^p)$. Moreover, $M_{\mathrm{cb}}(L^p)$ does not embed into $(M(L^\infty), M(L^2))_\theta$ for any $0<\theta<1$.*

Proof. Since $M_{\mathrm{cb}}(L^p)$ embeds into $M_{\mathrm{cb}}(L^s)$ for each $2<s<p<\infty$, we can restrict ourselves to even integers p. Let $\Lambda\subset\mathbb{Z}$ be any interpolation set for $M_{\mathrm{cb}}(L^p)$ which is not an interpolation set for any $M(L^q)$ with $q>p$. Corollary 2.6 implies that such a set Λ exists. This means that the restriction map $\mathcal{Q}$ which carries a multiplier φ in $M_{\mathrm{cb}}(L^p)$ to the sequence $(\varphi(n))_{n\in\Lambda}$ in $\ell_\infty(\Lambda)$ is surjective and a fortiori if we suppose that $M_{\mathrm{cb}}(L^p)$ embeds into $(M(L^\infty), M(L^2))_\theta$ for some $0<\theta<1$, then $\mathcal{Q}: (M(L^\infty), M(L^2))_\theta\to\ell_\infty(\Lambda)$ is also surjective. Lemma 3.5 implies then that $\mathcal{Q}: M(L^\infty)\to\ell_\infty(\Lambda)$ is surjective. Hence, Λ is an interpolation set for $M(L^\infty)$ and thus for all $M(L^q)$. This contradiction completes the proof. ∎

4. $\sigma(p)$-SETS AND $\sigma(p)_{\mathrm{cb}}$-SETS

DEFINITION 4.1. Let $2<p<\infty$. A subset A of $\mathbb{N}\times\mathbb{N}$ is called a *$\sigma(p)$-set* if there exists a constant $C>0$ such that for all $x=(x_{ij})_{i,j}$ in S_A^p, we have

$$\|x\|_{S^p}\le C\max\Big\{\Big(\sum_{i=1}^\infty\Big(\sum_{j=1}^\infty|x_{ij}|^2\Big)^{p/2}\Big)^{1/p},\Big(\sum_{j=1}^\infty\Big(\sum_{i=1}^\infty|x_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\}.$$

We denote by $\sigma_p(A)$ the smallest constant $C>0$ in this inequality.

Recall that if $2\le p\le\infty$ then for all $x=(x_{ij})_{i,j}$ in S^p,

$$\|x\|_{S^p}\ge\max\Big\{\Big(\sum_{i=1}^\infty\Big(\sum_{j=1}^\infty|x_{ij}|^2\Big)^{p/2}\Big)^{1/p},\Big(\sum_{j=1}^\infty\Big(\sum_{i=1}^\infty|x_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\},$$

so that A is a $\sigma(p)$-set if and only if the following are equivalent norms on S_A^p:

$$\|x\|_{S^p}\cong\max\Big\{\Big(\sum_{i=1}^\infty\Big(\sum_{j=1}^\infty|x_{ij}|^2\Big)^{p/2}\Big)^{1/p},\Big(\sum_{j=1}^\infty\Big(\sum_{i=1}^\infty|x_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\},\quad\forall x=(x_{ij})_{i,j}\in S_A^p.$$

In other words, A is a $\sigma(p)$-set if and only if the spaces S^p_A and $S^{p,\mathrm{unc}}_A$ are isomorphic.

REMARKS 4.2. As first properties of $\sigma(p)$-sets, we mention the following.

(i) Every subset A_1 of a $\sigma(p)$-set A_2 is a $\sigma(p)$-set with $\sigma_p(A_1) \le \sigma_p(A_2)$.

(ii) If A_1 and A_2 are $\sigma(p)$-sets, then so is $A_1 \cup A_2$, and $\sigma_p(A_1 \cup A_2) \le \sigma_p(A_1) + \sigma_p(A_2)$.

(iii) A is a $\sigma(p)$-set if and only if ${}^tA := \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid (j,i) \in A\}$ is a $\sigma(p)$-set. Moreover, $\sigma_p({}^tA) = \sigma_p(A)$.

(iv) Let $(n_k)_{k\ge1}$ be a strictly increasing sequence of integers and let A_k be a subset of $[n_{k-1}, n_k[\times [n_{k-1}, n_k[$ for each $k \ge 1$. Then $A = \bigcup_{k\ge1} A_k$ is a $\sigma(p)$-set if and only if $\sup_{k\ge1} \sigma_p(A_k)$ is finite. Moreover, $\sup_{k\ge1} \sigma_p(A_k) \le \sigma_p(A) \le 2\sup_{k\ge1}\sigma_p(A_k)$.

The reader is referred to Subsection 0.9 for the definition of Schur multipliers on S^p.

PROPOSITION 4.3. *Let $2 < p < \infty$ and $A \subset \mathbb{N}\times\mathbb{N}$. Then the following assertions are equivalent.*

(i) *A is a $\sigma(p)$-set.*

(ii) *The canonical basis $\{e_{ij} \mid (i,j) \in A\}$ is an unconditional basis of S^p_A.*

(iii) *The restriction map below is surjective*:

$$Q : M(S^p) \to \ell_\infty(A), \qquad \varphi \mapsto \varphi|_A.$$

Letting $\alpha_p(A)$ denote the unconditionality constant of the canonical basis of S^p_A and $\mu_p(A)$ denote the smallest constant μ for which Q is μ-surjective, we see that for each set $A \subset \mathbb{N}\times\mathbb{N}$ (with $K_p = K_{S^p}$ defined in (0.3)),

$$\alpha_p(A) \le \sigma_p(A) \le K_p\alpha_p(A), \qquad \mu_p(A) \le \sigma_p(A) \le K_p\mu_p(A).$$

Proof. The proof, analogous to that of Proposition 1.8, is left to the reader. ■

DEFINITION 4.4. Let $2 < p < \infty$. A subset A of $\mathbb{N}\times\mathbb{N}$ is called a *$\sigma(p)_{\mathrm{cb}}$-set* if there exists a constant $C > 0$ such that for all $x = (x_{ij})_{i,j}$ in $S^p_A(S^p)$, we have

$$\|x\|_{S^p(S^p)} \le C\max\Big\{\Big(\sum_i \Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{1/2}\Big\|^p_{S^p}\Big)^{1/p}, \Big(\sum_j \Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{1/2}\Big\|^p_{S^p}\Big)^{1/p}\Big\}.$$

We denote by $\sigma_p^{\mathrm{cb}}(A)$ the infimum of the constants C for which this inequality holds.

Since (0.4) always holds, A is a $\sigma(p)_{\mathrm{cb}}$-set if and only if for all $x = (x_{ij})_{i,j}$ in $S^p_A(S^p)$,

$$\|x\|_{S^p(S^p)} \cong \max\Big\{\Big(\sum_i \Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{1/2}\Big\|^p_{S^p}\Big)^{1/p}, \Big(\sum_j \Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{1/2}\Big\|^p_{S^p}\Big)^{1/p}\Big\}.$$

Equivalently, A is a $\sigma(p)_{\mathrm{cb}}$-set if and only if $S^p_A(S^p)$ and $S^{p,\mathrm{unc}}_A(S^p)$ are isomorphic. All the properties in Remarks 4.2 have an analogous version for $\sigma(p)_{\mathrm{cb}}$-sets. Proposition 4.3 also has a c.b. version as follows. We will skip all the proofs.

PROPOSITION 4.5. *Let $2 < p < \infty$ and $A \subset \mathbb{N}\times\mathbb{N}$. Then the following are equivalent.*

(i) *A is a $\sigma(p)_{\mathrm{cb}}$-set.*

(ii) *The restriction map Q which takes $\varphi \in M_{\mathrm{cb}}(S^p)$ to $\varphi|_A \in \ell_\infty(A)$ is surjective.*

(iii) *The operators T_ε where $\varepsilon = (\varepsilon_{ij})_{i,j}$ with $\varepsilon_{ij} = 1$ or -1 if $(i,j) \in A$ and $\varepsilon_{ij} = 0$ if not, defined on S^p by sending an operator $x = (x_{ij})_{i,j}$ to $T_\varepsilon(x) = (\varepsilon_{ij}x_{ij})_{i,j}$, are uniformly c.b.*

Letting $\alpha_p^{\mathrm{cb}}(A)$ denote the unconditionality constant of the canonical basis of S^p_A viewed as an operator space, and $\mu_p^{\mathrm{cb}}(A)$ the smallest constant μ for which Q is μ-surjective, we see that for each $A \subset \mathbb{N}\times\mathbb{N}$ (with $K_p = K_{S^p}$ defined in (0.3)),

$$\alpha_p^{\mathrm{cb}}(A) \le \sigma_p^{\mathrm{cb}}(A) \le K_p\alpha_p^{\mathrm{cb}}(A), \qquad \mu_p^{\mathrm{cb}}(A) \le \sigma_p^{\mathrm{cb}}(A) \le K_p\mu_p^{\mathrm{cb}}(A).$$

REMARKS 4.6. (i) Since $M(S^q) \subset M(S^p)$, $M_{\mathrm{cb}}(S^q) \subset M_{\mathrm{cb}}(S^p)$ for all $2 < p < q < \infty$ and the embeddings are both contractive, the $\sigma(q)$-property implies the $\sigma(p)$-property, the $\sigma(q)_{\mathrm{cb}}$-property implies the $\sigma(p)_{\mathrm{cb}}$-property, and $\sigma_p(A) \le \sigma_q(A)$, $\sigma_p^{\mathrm{cb}}(A) \le \sigma_q^{\mathrm{cb}}(A)$ for each set A. On the other hand, the $\sigma(p)_{\mathrm{cb}}$-property implies trivially the $\sigma(p)$-property, and $\sigma_p(A) \le \sigma_p^{\mathrm{cb}}(A)$ for each A.

(ii) Each bounded map $\varphi = (\varphi_{ij})_{i,j}$ supported by A defines a Schur multiplier on S^p (resp. a c.b. Schur multiplier on S^p) whenever A is a $\sigma(p)$-set (resp. $\sigma(p)_{\mathrm{cb}}$-set), and

$$\|\varphi\|_{M(S^p)} \le \sigma_p(A)\|\varphi\|_{\ell_\infty(\mathbb{N}\times\mathbb{N})} \qquad (\text{resp. } \|\varphi\|_{M_{\mathrm{cb}}(S^p)} \le \sigma_p^{\mathrm{cb}}(A)\|\varphi\|_{\ell_\infty(\mathbb{N}\times\mathbb{N})}).$$

This applies in particular to the indicator function $\mathbf{1}_A$ of a $\sigma(p)$-set (resp. a $\sigma(p)_{\mathrm{cb}}$-set).

(iii) The preceding results can be extended to the case $p = \infty$, but then the resulting notion is entirely elucidated by the work of N. Varopoulos (cf.

[44]) who characterized the sets $A \subset \mathbb{N}\times\mathbb{N}$ for which the restriction map

$$\mathcal{Q} : M(S^\infty) \to \ell_\infty(A), \qquad \varphi \mapsto \varphi|_A,$$

is surjective. His work shows that this holds if and only if A can be written as a finite union of 1-*sections* and 2-*sections* in the following sense. We say that a subset $A \subset \mathbb{N}\times\mathbb{N}$ is a 1-*section* (resp. 2-*section*) if the first (resp. second) coordinate projection is injective when restricted to A.

(iv) The definition of $\sigma(p)$-sets and $\sigma(p)_{\mathrm{cb}}$-sets can be extended to the case where $1 \le p \le 2$ as follows. Roughly speaking, a subset A of $\mathbb{N}\times\mathbb{N}$ is called a $\sigma(p)$-set if S_A^p is isomorphic to $S_A^{p,\mathrm{unc}}$, and it is called a $\sigma(p)_{\mathrm{cb}}$-set if $S_A^p(S^p)$ is isomorphic to $S_A^{p,\mathrm{unc}}(S^p)$. Moreover, we let $\sigma_p(A)$ be the smallest constant $C>0$ such that for all $x=(x_{ij})_{i,j}$ in S_A^p, we have

$$\|x\|_{S^p} \ge C^{-1}\inf\Big\{\Big(\sum_{i=1}^\infty\Big(\sum_{j=1}^\infty |y_{ij}|^2\Big)^{p/2}\Big)^{1/p} + \Big(\sum_{j=1}^\infty\Big(\sum_{i=1}^\infty |z_{ij}|^2\Big)^{p/2}\Big)^{1/p}\Big\}$$

where the infimum is over all decompositions $x = y+z$ with $y=(y_{ij})_{i,j}$ and $z=(z_{ij})_{i,j}$ both in S^p. We let $\sigma_p^{\mathrm{cb}}(A)$ be the smallest constant $C>0$ such that for all $x=(x_{ij})_{i,j}$ in $S_A^p(S^p)$, we have

$$\|x\|_{S^p(S^p)} \ge C^{-1}\inf\Big\{\Big(\sum_{i=1}^\infty\Big\|\Big(\sum_{j=1}^\infty y_{ij}y_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p} + \Big(\sum_{j=1}^\infty\Big\|\Big(\sum_{i=1}^\infty z_{ij}^*z_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\}$$

where the infimum is over all decompositions $x=y+z$ with $y=(y_{ij})_{i,j}$ and $z=(z_{ij})_{i,j}$ both in $S^p(S^p)$. Then, in analogy with Comments 1.9, it is easy to check that if $1\le p<2$, a subset $A\subset\mathbb{N}\times\mathbb{N}$ is a $\sigma(p')$-set (resp. $\sigma(p')_{\mathrm{cb}}$-set) where $1/p+1/p'=1$ if and only if it is a $\sigma(p)$-set (resp. $\sigma(p)_{\mathrm{cb}}$-set) in the above sense and its indicator function $\mathbf{1}_A$ defines a bounded (resp. c.b.) Schur multiplier on S^p.

PROPOSITION 4.7. *Let Λ be a a subset of $\mathbb{N}$ and let*

$$\widehat{\Lambda} := \{(i,j)\in\mathbb{N}\times\mathbb{N} \mid i+j\in\Lambda\}.$$

(i) *For $1\le p\le 2$, $\widehat{\Lambda}$ is a $\sigma(p)_{\mathrm{cb}}$-set whenever Λ is a $K(p)_{\mathrm{cb}}$-set with $\sigma_p^{\mathrm{cb}}(\widehat{\Lambda}) \le K_p^{\mathrm{cb}}(\Lambda)$.*

(ii) *For $2<p<\infty$, $\widehat{\Lambda}$ is a $\sigma(p)_{\mathrm{cb}}$-set whenever Λ is a $\Lambda(p)_{\mathrm{cb}}$-set with $\sigma_p^{\mathrm{cb}}(\widehat{\Lambda}) \le \lambda_p^{\mathrm{cb}}(\Lambda)$.*

Proof. We sketch only (ii). Let $x=(x_{ij})_{i,j}$ be in $S_{\widehat{\Lambda}}^p(S^p)$. Using the assumption on Λ, we get

$$\begin{aligned}\|(x_{ij})_{i,j}\|_{S^p(S^p)} &= \|(z^{i+j}x_{ij})_{i,j}\|_{L^p(S^p(S^p))}\\ &= \Big\|\sum_{n\in\Lambda} z^n\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda}\\ i+j=n}} x_{ij}\otimes e_{ij}\Big)\Big\|_{L^p(S^p(S^p))}\\ &\le \lambda_p^{\mathrm{cb}}(\Lambda)\max\Big\{\Big\|\sum_j\Big(\sum_i x_{ij}^*x_{ij}\Big)^{1/2}\otimes e_{jj}\Big\|_{S^p(S^p)},\\ &\qquad\qquad \Big\|\sum_i\Big(\sum_j x_{ij}x_{ij}^*\Big)^{1/2}\otimes e_{ii}\Big\|_{S^p(S^p)}\Big\}\end{aligned}$$

since

$$\Big(\sum_{n\in\Lambda}\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda}\\ i+j=n}} x_{ij}\otimes e_{ij}\Big)^*\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda}\\ i+j=n}} x_{ij}\otimes e_{ij}\Big)\Big)^{1/2} = \sum_j\Big(\sum_i x_{ij}^*x_{ij}\Big)^{1/2}\otimes e_{jj},$$

$$\Big(\sum_{n\in\Lambda}\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda}\\ i+j=n}} x_{ij}\otimes e_{ij}\Big)\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda}\\ i+j=n}} x_{ij}\otimes e_{ij}\Big)^*\Big)^{1/2} = \sum_i\Big(\sum_j x_{ij}x_{ij}^*\Big)^{1/2}\otimes e_{ii}.$$

Hence we get

$$\|(x_{ij})_{i,j}\|_{S^p(S^p)} \le \lambda_p^{\mathrm{cb}}(\Lambda)\max\Big\{\Big(\sum_j\Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}, \Big(\sum_i\Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{1/2}\Big\|_{S^p}^p\Big)^{1/p}\Big\}.$$

This implies that $\widehat{\Lambda}$ is a $\sigma(p)_{\mathrm{cb}}$-set with $\sigma_p^{\mathrm{cb}}(\widehat{\Lambda}) \le \lambda_p^{\mathrm{cb}}(\Lambda)$. ∎

THEOREM 4.8. *Let $p>2$ be an even integer. Then, for each $n\ge 1$, we can find a Hankelian subset A_n of $[1,n]\times[1,n]$ satisfying*

$$\sup_{n\ge1}\sigma_p^{\mathrm{cb}}(A_n) < \infty, \qquad \inf_{n\ge1} n^{-(1+2/p)}|A_n| > 0.$$

Proof. A reformulation of Corollary 2.7 implies that for each integer $n\ge1$, there exists $\Lambda_n\subset[0,n]$ such that $\sup_{n\ge1}\lambda_p^{\mathrm{cb}}(\Lambda_n)<\infty$ and $\inf_{n\ge1} n^{-2/p}|\Lambda_n|>0$. Then for each n, we let $A_n := \widehat{\Lambda}_n$; thus $A_n\subset[0,n]\times[0,n]$. To ensure $|A_n|\ge n|\Lambda_n|$, we can clearly assume that $\Lambda_n\subset[n/2,n]$. Therefore, using Proposition 4.7, we get a sequence $(A_n)_{n\ge1}$ of sets satisfying $\sup_{n\ge1}\sigma_p^{\mathrm{cb}}(A_n)<\infty$ along with $\inf_{n\ge1} n^{-(1+2/p)}|A_n|>0$. ∎

THEOREM 4.9. *For any even integer $p>2$, there is a $\sigma(p)_{\mathrm{cb}}$-set $A\subset\mathbb{N}\times\mathbb{N}$ which is not a $\sigma(q)$-set for any $q>p$. More precisely, the indicator function of A is not in $M(S^q)$ for any $q>p$.*

Proof. Let $p>2$ be an even integer. Then, by Corollary 2.7, there exist a constant $c>0$ and sets $\Lambda_n\subset[2^{n-1},2^n[$ for each integer $n\ge1$ satisfying

$c2^{2(n-1)/p} \le |\Lambda_n|$ and $\sup_{n\ge1}\lambda_p^{\mathrm{cb}}(\Lambda_n) < \infty$. Such sets necessarily satisfy $\sup_{n\ge1}\lambda_q(\Lambda_n)=\infty$ for all $p<q<\infty$. Indeed, by Fact 2.3(i), there exists a constant $c_1>0$ such that

$$c2^{2(n-1)/p} \le |\Lambda_n| \le c_1 2^{2(n-1)/q}\lambda_q^2(\Lambda_n)$$

for all integers $n\ge1$, that is to say,

$$c2^{2(n-1)(1/p-1/q)} \le c_1\lambda_q^2(\Lambda_n) \le c_1 \sup_{k\ge1}\lambda_q^2(\Lambda_k).$$

Since $1/p-1/q>0$ and n can be arbitrarily large, $\sup_{k\ge1}\lambda_q(\Lambda_k)=\infty$. On the other hand, we let

$$A_n := (2^n,2^n)+\widehat{\Lambda}_n = \{(2^n+i,2^n+j)\mid i+j\in\Lambda_n\},\qquad \forall n\ge1.$$

Note that for all $n\ge1$, we have

$$A_n \subset [2^n,2^{n+1}[\times[2^n,2^{n+1}[\cap\{(i,j)\mid i+j\in 2^{n+1}+\Lambda_n\}.$$

Then we consider the set $A:=\bigcup_{n\ge1}A_n$. We apply successively the c.b. version of Remark 4.2(iv), Proposition 4.7 and the fact that the $\Lambda(p)_{\mathrm{cb}}$-property is stable under translations, to see that A is a $\sigma(p)_{\mathrm{cb}}$-set, as follows:

$$\sigma_p^{\mathrm{cb}}(A) \le 2\sup_{n\ge1}\sigma_p^{\mathrm{cb}}(A_n) \le 2\sup_{n\ge1}\lambda_p^{\mathrm{cb}}(2^{n+1}+\Lambda_n) = 2\sup_{n\ge1}\lambda_p^{\mathrm{cb}}(\Lambda_n)<\infty.$$

Now we check that $\mathbf{1}_A\notin M(S^q)$ for all $q>p$. Indeed, taking the supremum after applying (1.5) to each Λ_n, we get

$$\begin{aligned}\sup_{n\ge1}\lambda_q(\Lambda_n) &\le \sup_{n\ge1}\lambda_2(\Lambda_n)\sup_{n\ge1}\|\mathbf{1}_{\Lambda_n}\|_{M(L^q)} \le \sup_{n\ge1}\lambda_p(\Lambda_n)\sup_{n\ge1}\|\mathbf{1}_{\Lambda_n}\|_{M(L^q)}\\ &\le \sup_{n\ge1}\lambda_p^{\mathrm{cb}}(\Lambda_n)\sup_{n\ge1}\|\mathbf{1}_{\Lambda_n}\|_{M(L^q)}.\end{aligned}$$

Since by our choice of $\{\Lambda_n\}_{n\ge1}$, $\sup_{n\ge1}\lambda_p^{\mathrm{cb}}(\Lambda_n)<\infty$ and $\sup_{n\ge1}\lambda_q(\Lambda_n)=\infty$, we have necessarily $\sup_{n\ge1}\|\mathbf{1}_{\Lambda_n}\|_{M(L^q)}=\infty$. We can easily see that

$$\|\mathbf{1}_{A_n}\|_{M(S^q)} = \|\mathbf{1}_{\widehat{\Lambda}_n}\|_{M(S^q)} \ge \|\mathbf{1}_{\widehat{\Lambda}_n}\|_{M(\mathfrak{S}^q)},\qquad \forall n\ge1.$$

Then, using Peller's results (see Subsection 0.6), we get

$$\sup_{n\ge1}\|\mathbf{1}_{A_n}\|_{M(S^q)} \ge \sup_{n\ge1}\|\mathbf{1}_{\widehat{\Lambda}_n}\|_{M(\mathfrak{S}^q)} \cong \sup_{n\ge1}\|\mathbf{1}_{\Lambda_n}\|_{M(L^q)}=\infty.$$

Thus $\mathbf{1}_A\notin M(S^q)$ and so A is not a $\sigma(q)$-set by Remark 4.6(ii). ■

5. APPLICATIONS TO SCHUR MULTIPLIERS

The last assertion of the following proposition answers a question raised by J. Erdos as Remark 2 in [18] (I am grateful to Professor E. Katsoulis for indicating this reference; see [1] for related work).

THEOREM 5.1. *For all $2<p<q\le\infty$ where p is an even integer, the inclusion maps*

$$M_{\mathrm{cb}}(S^q)\subset M_{\mathrm{cb}}(S^p),\qquad M(S^q)\subset M(S^p)$$

are strict. Moreover, there is an idempotent Schur multiplier which is c.b. on S^p but not bounded on S^q for any $q>p$.

Proof. Let $p>2$ be an even integer. Then by Theorem 4.9, there exists a $\sigma(p)_{\mathrm{cb}}$-set $A\subset\mathbb{N}\times\mathbb{N}$ which is not a $\sigma(q)$-set for any $q>p$. Moreover, $\mathbf{1}_A\notin M(S^q)$. Using Remark 4.6(ii), we see that $\mathbf{1}_A\in M_{\mathrm{cb}}(S^p)$ and we are done. ■

THEOREM 5.2. *For $2<p<\infty$, the following canonical inclusion map is contractive:*

$$(M(S^\infty),M(S^2))_{2/p}\subset M_{\mathrm{cb}}(S^p).$$

Moreover, $M_{\mathrm{cb}}(S^p)$ does not embed into any interpolated space $(M(S^\infty), M(S^2))_\theta$ when θ runs over $]0,1[$. Therefore, the inclusion above is strict.

Proof. The first part follows immediately from Lemma 0.2. For the last part, we can clearly restrict ourselves to even integers p. Thus, fix an even integer $p>2$ and let $A\subset\mathbb{N}\times\mathbb{N}$ be a $\sigma(p)_{\mathrm{cb}}$-set which is not $\sigma(q)$ for any $q>p$. Such a set exists by Theorem 4.9. Thus, A is an interpolation set for $M_{\mathrm{cb}}(S^p)$ but not for $M(S^q)$ according to Proposition 4.5, i.e. the restriction map $\mathcal{Q}:M_{\mathrm{cb}}(S^p)\to\ell_\infty(A)$ is surjective but $\mathcal{Q}:M(S^q)\to\ell_\infty(A)$ is not. If we assume that $M_{\mathrm{cb}}(S^p)$ embeds into $(M(S^\infty),M(S^2))_\theta$ for some $0<\theta<1$, we see that $\mathcal{Q}:(M(S^\infty),M(S^2))_\theta\to\ell_\infty(A)$ is again surjective. Lemma 3.5 implies then that $\mathcal{Q}:M(S^\infty)\to\ell_\infty(A)$ is also surjective. This contradicts the assumption that A is not an interpolation set for $M(S^q)$ for any $q>p$. ■

Now we will be interested in $M(\mathfrak{S}^p)$, $M_{\mathrm{cb}}(\mathfrak{S}^p)$ and in establishing some links between Fourier and Schur multipliers. We start by recalling a few notations. For a set $\Lambda\subset\mathbb{N}$, we define $\widehat{\Lambda}:=\{(k,l)\in\mathbb{N}\times\mathbb{N}\mid k+l\in\Lambda\}$ and we say that a map $\varphi:\mathbb{N}\times\mathbb{N}\to\mathbb{C}$ is Hankelian if $\varphi(k,l)=\varphi(k',l')$ whenever $k+l=k'+l'$ for all $(k,l),(k',l')$ in $\mathbb{N}\times\mathbb{N}$. Recall also that for all integers $n\ge1$ and all $\varphi:\mathbb{N}\times\mathbb{N}\to\mathbb{C}$, we let $\varphi_{(n)}:=\mathbf{1}_{\widehat{I}_n}\varphi$ (Schur product) where $I_0:=\{0\}$ and $I_n:=\{k\in\mathbb{N}\mid 2^{n-1}\le k<2^n\}$ for $n\ge1$.

PROPOSITION 5.3. *For a Hankelian map $\varphi:\mathbb{N}\times\mathbb{N}\to\mathbb{C}$, the following are equivalent.*

(i) $\varphi\in M(\mathfrak{S}^p)$.
(ii) *The multipliers $\varphi_{(n)}$ are uniformly bounded in $M(\mathfrak{S}^p_{\widehat{I}_n})$.*
(iii) *The multipliers $\varphi_{(n)}$ are uniformly bounded in $M(\mathfrak{S}^p)$.*

Moreover, the two norms defined below are equivalent on $M(\mathfrak{S}^p)$:

$$\|\varphi\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0} \|\varphi_{(n)}\|_{M(\mathfrak{S}^p_{\widehat{I}_n})} \cong \sup_{n\geq 0} \|\varphi_{(n)}\|_{M(\mathfrak{S}^p)}.$$

Proof. The equivalence between (ii) and (iii) is easy since the spaces $\mathfrak{S}^p_{\widehat{I}_n}$ are uniformly complemented in $\mathfrak{S}^p$. To prove the equivalence between (i) and (iii), consider x in $\mathfrak{S}^p$. We have $T_\varphi(x) = \sum_{n\geq 0} T_{\varphi_{(n)}}(x_{(n)})$. Thus by Corollary 0.7(i),

$$\begin{aligned}\|T_\varphi(x)\|_{\mathfrak{S}^p} &\cong \Big(\sum_{n\geq 0} \|T_{\varphi_{(n)}}(x_{(n)})\|^p_{\mathfrak{S}^p}\Big)^{1/p} \leq \Big(\sum_{n\geq 0} \|T_{\varphi_{(n)}}\|^p_{\mathcal{B}(\mathfrak{S}^p)} \|x_{(n)}\|^p_{\mathfrak{S}^p}\Big)^{1/p} \\ &\leq \sup_{n\geq 0} \|T_{\varphi_{(n)}}\|_{\mathcal{B}(\mathfrak{S}^p)} \Big(\sum_{n\geq 0} \|x_{(n)}\|^p_{\mathfrak{S}^p}\Big)^{1/p} \\ &\cong \sup_{n\geq 0} \|\varphi_{(n)}\|_{M(\mathfrak{S}^p)} \|x\|_{\mathfrak{S}^p} \leq \|\varphi\|_{M(\mathfrak{S}^p)} \|x\|_{\mathfrak{S}^p}. \ \blacksquare\end{aligned}$$

PROPOSITION 5.4. *For a Hankelian map* $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{C}$, *the following are equivalent.*

(i) $\varphi \in M_{\mathrm{cb}}(\mathfrak{S}^p)$.

(ii) *The multipliers* $\varphi_{(n)}$ *are uniformly bounded in* $M_{\mathrm{cb}}(\mathfrak{S}^p_{\widehat{I}_n})$.

(iii) *The multipliers* $\varphi_{(n)}$ *are uniformly bounded in* $M_{\mathrm{cb}}(\mathfrak{S}^p)$.

Moreover, the two norms defined below are equivalent on $M_{\mathrm{cb}}(\mathfrak{S}^p)$:

$$\|\varphi\|_{M_{\mathrm{cb}}(\mathfrak{S}^p)} \cong \sup_{n\geq 0} \|\varphi_{(n)}\|_{M_{\mathrm{cb}}(\mathfrak{S}^p_{\widehat{I}_n})} \cong \sup_{n\geq 0} \|\varphi_{(n)}\|_{M_{\mathrm{cb}}(\mathfrak{S}^p)}.$$

Proof. By the characterization of c.b. maps given in Proposition 0.4, we can prove Proposition 5.4 exactly in the same way as we did for Proposition 5.3 by applying (ii) of Corollary 0.7 instead of (i). ■

PROPOSITION 5.5. *For all* $1 < p < \infty$, $M(H^p)$ *can be continuously injected into* $M(\mathfrak{S}^p)$ *via the map which takes* $\varphi \in M(H^p)$ *to* $\widehat{\varphi} \in M(\mathfrak{S}^p)$, *where* $\widehat{\varphi}$ *sends* (k,l) *in* $\mathbb{N}\times\mathbb{N}$ *to* $\varphi(k+l)$.

Proof. We assume first that φ has support in I_n. Then by Corollary 0.7(i) we have

$$\begin{aligned}\|\widehat{\varphi}\|_{M(\mathfrak{S}^p)} \cong \|\widehat{\varphi}\|_{M(\mathfrak{S}^p_{\widehat{I}_n})} &= \sup\{\|T_{\widehat{\varphi}}(x)\|_{\mathfrak{S}^p_{\widehat{I}_n}} \mid \|x\|_{\mathfrak{S}^p_{\widehat{I}_n}} \leq 1\} \\ &\cong \sup\{\|M_\varphi(f)\|_{\mathcal{A}^p_{I_n}} \mid \|f\|_{\mathcal{A}^p_{I_n}} \leq 1\} \\ &= \sup\{\|M_\varphi(f)\|_{H^p_{I_n}} \mid \|f\|_{H^p_{I_n}} \leq 1\}\end{aligned}$$

(see Subsection 0.6 for the definition of $\mathcal{A}^p$). Applying Remark 0.10, we get

$$\|\widehat{\varphi}\|_{M(\mathfrak{S}^p)} \cong \|\varphi\|_{M(H^p_{I_n})} \cong \|\varphi\|_{M(H^p)}.$$

Using Proposition 5.3, we see that for all φ in $M(H^p)$,

$$\|\widehat{\varphi}\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0} \|\widehat{\varphi}_{(n)}\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0} \|\varphi_{(n)}\|_{M(H^p)} \leq \|\varphi\|_{M(H^p)}. \ \blacksquare$$

PROPOSITION 5.6. $M_{\mathrm{cb}}(H^p)$ *can be contractively injected into* $M^{\mathcal{H}}_{\mathrm{cb}}(S^p)$ *via the map which takes* $\varphi \in M_{\mathrm{cb}}(H^p)$ *to* $\widehat{\varphi} \in M^{\mathcal{H}}_{\mathrm{cb}}(S^p)$.

Proof. Let φ be fixed in $M_{\mathrm{cb}}(H^p)$ and $M_\varphi : H^p \to H^p$ be the associated operator. Since φ is c.b., $M_\varphi \otimes \mathrm{id}_{S^p}$ extends to a bounded operator on $H^p(S^p)$. On the other hand, for an operator x in S^p, we consider one more time the function $f_x(z) = D_z x D_z$ defined on $\mathbb{T}$ where D_z denotes the unitary $\infty\times\infty$ matrix

$$D_z = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & z & 0 & \ldots \\ 0 & 0 & z^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

f_x lies in $H^p(S^p)$ and satisfies $\|f_x\|_{H^p(S^p)} = \|x\|_{S^p}$. Now write $T_{\widehat{\varphi}}(x)$ for the $\infty\times\infty$ matrix $(\varphi(k+l)x_{kl})_{k,l}$. We see that $M_\varphi \otimes \mathrm{id}_{S^p}(f_x)(z) = D_z T_{\widehat{\varphi}}(x) D_z$. Hence, the operator

$$T_{\widehat{\varphi}} : S^p \to S^p, \qquad (x_{kl})_{k,l} \mapsto (\varphi(k+l)x_{kl})_{k,l},$$

is well defined, has a Hankelian form and satisfies

$$\begin{aligned}\|T_{\widehat{\varphi}}x\|_{S^p} &= \|M_\varphi \otimes \mathrm{id}_{S^p}(f_x)\|_{L^p(S^p)} \\ &\leq \|M_\varphi\|_{C\mathcal{B}(H^p)} \|f_x\|_{H^p(S^p)} = \|\varphi\|_{M_{\mathrm{cb}}(H^p)} \|x\|_{S^p}.\end{aligned}$$

This means that $\widehat{\varphi}$ is in $M^{\mathcal{H}}(S^p)$ with $\|\widehat{\varphi}\|_{M^{\mathcal{H}}(S^p)} \leq \|\varphi\|_{M_{\mathrm{cb}}(H^p)}$. In a similar way, using Proposition 0.4, we prove that in fact $\widehat{\varphi} \in M^{\mathcal{H}}_{\mathrm{cb}}(S^p)$ with $\|\widehat{\varphi}\|_{M^{\mathcal{H}}_{\mathrm{cb}}(S^p)} \leq \|\varphi\|_{M_{\mathrm{cb}}(H^p)}$. ■

REMARK. The case $p = 1$ is quite interesting since $M_{\mathrm{cb}}(H^1)$ and $M^{\mathcal{H}}_{\mathrm{cb}}(S^1)$ coincide isometrically. See [32] for the proof. The question seems to be open for other non-trivial values of p.

6. APPENDIX

For the sake of completeness, we include a different way to show the existence of "large" $\Lambda(4)_{\mathrm{cb}}$-sets by using probabilistic ideas to exhibit "large" sets having the combinatorial property $Z(2)$ and satisfying moreover some additional assumptions. We check first that the $Z(2)$-property implies the $\Lambda(4)_{\mathrm{cb}}$-property directly (without using Theorem 1.13). Recall that a set

$\Lambda \subset \mathbb{Z}$ is said to have the $Z(2)$-property whenever

$$Z_2(\Lambda) := \sup_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |\{(n_1,n_2)\in\Lambda\times\Lambda \mid n_1-n_2=k\}| < \infty.$$

PROPOSITION 6.1. *If $\Lambda\subset\mathbb{Z}$ has the $Z(2)$-property then it has the $\Lambda(4)_{\mathrm{cb}}$-property and $\lambda_4^{\mathrm{cb}}(\Lambda)\le(1+Z_2(\Lambda))^{1/4}$.*

Proof. Let $f=\sum_{n\in\Lambda}x_n e^{int}$ be in $L^4(S^4)$, say with finitely many $x_n \neq 0$. We have

$$\|f\|^4_{L^4(S^4)} = \|f^*f\|^2_{L^2(S^2)} = \sum_{k\in\mathbb{Z}} \|\widehat{f^*f}(k)\|^2_{S^2}.$$

For all integers k, we have

$$\widehat{f^*f}(k) = \sum_{n_1,n_2\in\Lambda,\,-n_1+n_2=k} x^*_{n_1}x_{n_2}.$$

Thus, for each $k\neq 0$,

$$\|\widehat{f^*f}(k)\|^2_{S^2} \le \Big(\sum_{\substack{n_1,n_2\in\Lambda\\ -n_1+n_2=k}} \|x^*_{n_1}x_{n_2}\|_{S^2}\Big)^2 \le Z_2(\Lambda)\sum_{\substack{n_1,n_2\in\Lambda\\ -n_1+n_2=k}}\|x^*_{n_1}x_{n_2}\|^2_{S^2}.$$

Hence, using the trace property, we get

$$\begin{aligned}
\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \|\widehat{f^*f}(k)\|^2_{S^2} &\le Z_2(\Lambda)\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{\substack{n_1,n_2\in\Lambda\\ -n_1+n_2=k}} \mathrm{tr}(x^*_{n_2}x_{n_1}x^*_{n_1}x_{n_2})\\
&= Z_2(\Lambda)\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{\substack{n_1,n_2\in\Lambda\\ -n_1+n_2=k}} \mathrm{tr}(x_{n_1}x^*_{n_1}x_{n_2}x^*_{n_2})\\
&\le Z_2(\Lambda)\sum_{-n_1,n_2\in\Lambda}\mathrm{tr}(x_{n_1}x^*_{n_1}x_{n_2}x^*_{n_2})\\
&\le Z_2(\Lambda)\,\mathrm{tr}\Big(\sum_{n_1\in\Lambda}x_{n_1}x^*_{n_1}\sum_{n_2\in\Lambda}x_{n_2}x^*_{n_2}\Big)\\
&= Z_2(\Lambda)\Big\|\sum_{n\in\Lambda}x_nx^*_n\Big\|^2_{S^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|f\|^4_{L^4(S^4)} &\le \Big\|\sum_{n\in\Lambda}x^*_nx_n\Big\|^2_{S^2} + Z_2(\Lambda)\Big\|\sum_{n\in\Lambda}x_nx^*_n\Big\|^2_{S^2}\\
&\le (1+Z_2(\Lambda))\max\Big\{\Big\|\sum_{n\in\Lambda}x^*_nx_n\Big\|^2_{S^2}, \Big\|\sum_{n\in\Lambda}x_nx^*_n\Big\|^2_{S^2}\Big\}.
\end{aligned}$$

Finally, we get

$$\|f\|_{L^4(S^4)} \le (1+Z_2(\Lambda))^{1/4}\max\Big\{\Big\|\Big(\sum_{n\in\Lambda}x^*_nx_n\Big)^{1/2}\Big\|_{S^4}, \Big\|\Big(\sum_{n\in\Lambda}x_nx^*_n\Big)^{1/2}\Big\|_{S^4}\Big\}. \ ■$$

The following is essentially a result known in the folklore of harmonic analysis but it does not seem to appear in print anywhere.

PROPOSITION 6.2. *For all small $\delta>0$ and all $C>2/\delta$, there exist constants C_1,C_2 depending only on δ such that for all sequences $(u_n)_{n\ge1}$ of non-negative integers with $\sum_{n\ge1}u_n^{-1/2+\delta}<\infty$, there exists a $Z(2)$-set Λ satisfying:*

- *$Z_2(\Lambda)\le C$.*

•• *For all $n\ge n_0$, where n_0 is an integer depending only on the convergence speed of $\sum_{n\ge1}u_n^{-1/2+\delta}$ (e.g. $\sum_{n\ge n_0}u_n^{-1/2+\delta}\le C_1/4$), we have*

$$C_1u_n^{1/2-\delta}\le|\Lambda\cap[u_n,2u_n[|\le C_2u_n^{1/2-\delta}.$$

Proof. Let $\{\xi_k\}_{k\ge1}$ be a sequence of independent random variables on a standard probability space (for example the torus $\mathbb{T}$ equipped with the normalized Lebesgue measure $d\mathbb{P}=dt/(2\pi)$) such that for each k, ξ_k takes its values in $\{0,1\}$ and has expectation $\mathbb{E}\xi_k=\beta/k^\alpha$ where $\alpha=1/2+\delta$ for simplicity, with $\alpha<1$, and β is a constant depending only on δ, to be fixed later. Thus $\mathbb{P}(\xi_k=1)=\beta/k^\alpha$ and $\mathbb{P}(\xi_k=0)=1-\beta/k^\alpha$.

With each ω in $\mathbb{T}$, we associate the subset $\Lambda_\omega:=\{k\in\mathbb{N}^*\mid\xi_k(\omega)=1\}$. We will show that by a convenient choice of β, most of these random sets Λ_ω have the required properties. Indeed, for $\gamma\in\mathbb{Z}^*$ we set

$$Z_2(\gamma,\Lambda_\omega):=|\{(k,l)\in\Lambda_\omega\times\Lambda_\omega\mid k-l=\gamma\}|=\sum_{\substack{k,l\ge1\\ k-l=\gamma}}\xi_k(\omega)\xi_l(\omega).$$

Then

$$Z_2(\gamma,\Lambda_\omega)=\sum_{k\ge1}\xi_k(\omega)\xi_{k+|\gamma|}(\omega)=Z_2(|\gamma|,\Lambda_\omega). \tag{6.12}$$

We split $\mathbb{N}^*$ into J_1 and J_2 where

$$J_1=\bigcup_{s\ge0}[1+2s|\gamma|,(1+2s)|\gamma|],\qquad J_2=\bigcup_{s\ge0}](1+2s)|\gamma|,(2+2s)|\gamma|]$$

in order to have the variables $\{\xi_k\xi_{k+|\gamma|}\}_{k\in J_j}$ independent for each j. Then we let

$$Z_{2,j}(\gamma,\Lambda_\omega)=\sum_{k\in J_j}\xi_k(\omega)\xi_{k+|\gamma|}(\omega).$$

STEP 1. We start by selecting among the sets Λ_ω those with the $Z(2)$-property. Using (6.12), we see that for every integer m,

$$\mathbb{P}\{\sup_{\gamma\neq0}Z_2(\gamma,\Lambda_\omega)\ge2m\}=\mathbb{P}\{\sup_{\gamma\ge1}Z_2(\gamma,\Lambda_\omega)\ge2m\}.$$

Hence,

$$\mathbb{P}\{\sup_{\gamma\neq 0} Z_2(\gamma,\Lambda_\omega)\ge 2m\}\le \mathbb{P}\{\sup_{\gamma\ge 1} Z_{2,1}(\gamma,\Lambda_\omega)\ge m\} + \mathbb{P}\{\sup_{\gamma\ge 1} Z_{2,2}(\gamma,\Lambda_\omega)\ge m\}.$$

For each $\gamma\ge 1$, we have

$$\begin{aligned}\mathbb{P}\{Z_{2,1}(\gamma,\Lambda_\omega)\ge m\} &\le \sum_{\substack{k_1,\dots,k_m\in J_1\\ \text{distinct}}} \mathbb{P}\{\xi_{k_j}(\omega)\xi_{k_j+\gamma}(\omega)=1\mid j=1,\dots,m\}\\ &= \sum_{\substack{k_1,\dots,k_m\in J_1\\ \text{distinct}}} \prod_{j=1}^m \mathbb{P}\{\xi_{k_j}(\omega)\xi_{k_j+\gamma}(\omega)=1\}\\ &\le \sum_{k_1,\dots,k_m\in J_1} \prod_{j=1}^m \mathbb{P}\{\xi_{k_j}(\omega)\xi_{k_j+\gamma}(\omega)=1\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}\{Z_{2,1}(\gamma,\Lambda_\omega)\ge m\} &\le \sum_{k_1,\dots,k_m\in J_1}\prod_{j=1}^m \frac{\beta}{k_j^\alpha}\cdot\frac{\beta}{(k_j+\gamma)^\alpha}\\ &= \beta^{2m}\Big(\sum_{k\in J_1}\frac{1}{k^\alpha(k+\gamma)^\alpha}\Big)^m.\end{aligned}$$

Now we use the fact that for all $K\ge 1$,

$$\sum_{k=1}^K \frac{1}{k^\alpha}\le 1+\int_1^K \frac{1}{t^\alpha}\,dt = 1+\frac{1}{1-\alpha}(K^{1-\alpha}-1).$$

This gives us (recall that $2\alpha>1$)

$$\begin{aligned}\sum_{k\in J_1}\frac{1}{k^\alpha(k+\gamma)^\alpha} &= \sum_{k=1}^{\gamma}\frac{1}{k^\alpha(k+\gamma)^\alpha}+\sum_{s=1}^\infty\Big(\sum_{k=1+2s\gamma}^{(1+2s)\gamma}\frac{1}{k^\alpha(k+\gamma)^\alpha}\Big)\\ &\le \frac{1}{\gamma^\alpha}\sum_{k=1}^{\gamma}\frac{1}{k^\alpha}+\sum_{s=1}^\infty\Big(\sum_{k=1+2s\gamma}^{(1+2s)\gamma}\frac{1}{k^{2\alpha}}\Big)\\ &\le \frac{1}{(1-\alpha)\gamma^{2\alpha-1}}+\sum_{s=1}^\infty\frac{\gamma}{(2s\gamma)^{2\alpha}}\\ &\le \frac{1}{(1-\alpha)\gamma^{2\alpha-1}}+\frac{1}{4^\alpha\gamma^{2\alpha-1}}\sum_{s=1}^\infty\frac{1}{s^{2\alpha}}\\ &\le \frac{1}{(1-\alpha)\gamma^{2\alpha-1}}+\frac{2\alpha}{4^\alpha\gamma^{2\alpha-1}(2\alpha-1)}\\ &\le \Big(\frac{1}{1-\alpha}+\frac{2\alpha}{2\alpha-1}\Big)\frac{1}{\gamma^{2\alpha-1}}.\end{aligned}$$

Thus, we obtain

$$\sum_{k\in J_1}\frac{1}{k^\alpha(k+\gamma)^\alpha}\le\frac{4\alpha}{(2\alpha-1)\gamma^{2\alpha-1}}$$

by assuming for simplicity δ small enough, say $\delta\le 1/\sqrt{2}-1/2$ so that $1/(1-\alpha)\le 2\alpha/(2\alpha-1)$. By the same calculation, we get

$$\sum_{k\in J_2}\frac{1}{k^\alpha(k+\gamma)^\alpha}\le\frac{2\alpha}{(2\alpha-1)\gamma^{2\alpha-1}}.$$

Hence

$$\mathbb{P}\{Z_{2,1}(\gamma,\Lambda_\omega)\ge m\}\le\frac{c^m}{\gamma^{(2\alpha-1)m}},\qquad \mathbb{P}\{Z_{2,2}(\gamma,\Lambda_\omega)\ge m\}\le\frac{c^m}{2^m\gamma^{(2\alpha-1)m}},$$

where $c:=4\alpha\beta^2/(2\alpha-1)$. This implies the estimate $\mathbb{P}\{Z_2(\gamma,\Lambda_\omega)\ge 2m\}\le 2c^m/\gamma^{(2\alpha-1)m}$ and therefore

$$\mathbb{P}\{\sup_{\gamma\neq 0}Z_2(\gamma,\Lambda_\omega)\ge 2m\}\le 2c^m\sum_{\gamma\ge 1}1/\gamma^{2\delta m}.$$

By taking $m>1/(2\delta)$ and β such that $c=(1+2\delta)\beta^2/\delta<1$, say $\beta=(\delta/(8(1+2\delta)))^{1/2}$, we obtain

$$\mathbb{P}\{\sup_{\gamma\neq 0}Z_2(\gamma,\Lambda_\omega)\ge 2m\}\le 2c^m\Big(1-\frac{1}{1-2\delta m}\Big)=8^{-m}\frac{4\delta m}{2\delta m-1}=:P_1(m).$$

Therefore, we conclude that

$$\mathbb{P}\{Z_2(\Lambda_\omega)<2m\}\ge 1-P_1(m)>0.$$

Note that if we take $C=2m>2/\delta$ and δ small enough (say $\delta<\delta_0$ for some $\delta_0>0$) then we can ensure that $P_1(m)<1/2$.

Step 2. Among the random sets Λ_ω satisfying $Z_2(\Lambda_\omega)<2m$, we will select the "largest" one in the sense of our condition ••.

For each integer $n\ge 1$, I_n stands for the interval $[u_n,2u_n[$. We have

$$\mathbb{E}|\Lambda_\omega\cap I_n|=\mathbb{E}\Big(\sum_{k=u_n}^{2u_n-1}\xi_k(\omega)\Big)=\sum_{k=u_n}^{2u_n-1}\frac{\beta}{k^\alpha}.$$

Since

$$C_1(u_n):=\beta\int_{u_n}^{2u_n}\frac{1}{t^\alpha}\,dt\le\beta\sum_{u_n}^{2u_n-1}\frac{1}{k^\alpha}\le C_2(u_n):=\beta\int_{u_n-1}^{2u_n-1}\frac{1}{t^\alpha}\,dt$$

we get

$$C_1(u_n)\le\mathbb{E}|\Lambda_\omega\cap I_n|\le C_2(u_n).$$

We define the constants C_1 and C_2 depending on δ only as follows:

$$C_1(u_n)=\beta\frac{1}{1-\alpha}((2u_n)^{1-\alpha}-u_n^{1-\alpha})=\beta\frac{2^{1-\alpha}-1}{1-\alpha}u_n^{1-\alpha}=:2C_1u_n^{1/2-\delta},$$

$$C_2(u_n) = \frac{\beta}{1-\alpha}((2u_n-1)^{1-\alpha} - (u_n-1)^{1-\alpha}) \le \frac{\beta}{1-\alpha}(2u_n)^{1-\alpha}$$
$$= \beta\,\frac{2^{1-\alpha}}{1-\alpha}u_n^{1-\alpha} =: \frac{2}{3}C_2 u_n^{1/2-\delta}.$$

Since $(\xi_k - \mathbb{E}\xi_k)_k$ is a sequence of centered and independent random variables, the variance of $|\Lambda_\omega \cap I_n|$ is

$$\|\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,\|_{L^2}^2 = \Big\|\sum_{k=u_n}^{2u_n-1}(\xi_k - \mathbb{E}\xi_k)\Big\|_{L^2}^2 = \sum_{k=u_n}^{2u_n-1}\|\xi_k - \mathbb{E}\xi_k\|_{L^2}^2.$$

Hence,

$$\|\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,\|_{L^2}^2 \le \sum_{k=u_n}^{2u_n-1}\|\xi_k\|_{L^2}^2 = \sum_{k=u_n}^{2u_n-1}\mathbb{E}(\xi_k^2) = \sum_{k=u_n}^{2u_n-1}\frac{\beta}{k^\alpha} = \mathbb{E}|\Lambda_\omega \cap I_n|.$$

Using Chebyshev's inequality, we obtain

$$\mathbb{P}\{\,|\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,| \ge \tfrac{1}{2}\mathbb{E}|\Lambda_\omega \cap I_n|\} \le \frac{4}{\mathbb{E}|\Lambda_\omega \cap I_n|} \le \frac{4}{C_1(u_n)} = \frac{2}{C_1}u_n^{-1/2+\delta}.$$

For each fixed integer N, we have

$$\{\exists n \ge N,\ |\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,| \ge \tfrac{1}{2}\mathbb{E}|\Lambda_\omega \cap I_n|\} = \bigcup_{n=N}^{\infty}\{\,|\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,| \ge \tfrac{1}{2}\mathbb{E}|\Lambda_\omega \cap I_n|\}.$$

Therefore

$$\mathbb{P}\{\exists n \ge N,\ |\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,| \ge \tfrac{1}{2}\mathbb{E}|\Lambda_\omega \cap I_n|\} \le \frac{2}{C_1}\sum_{n=N}^{\infty}u_n^{-1/2+\delta} =: P_2(N).$$

Then since

$$\{\,|\,|\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\,| < \tfrac{1}{2}\mathbb{E}|\Lambda_\omega \cap I_n|,\ \forall n \ge N\} \subset \{C_1 u_n^{1/2-\delta} \le |\Lambda_\omega \cap I_n| \le C_2 u_n^{1/2-\delta},\ \forall n \ge N\}$$

we clearly get, for each fixed integer N,

$$\mathbb{P}\{C_1 u_n^{1/2-\delta} \le |\Lambda_\omega \cap I_n| \le C_2 u_n^{1/2-\delta},\ \forall n \ge N\} \ge 1 - P_2(N).$$

Thus we finally obtain

$$\mathbb{P}\{Z_2(\Lambda_\omega) < m\ \&\ C_1 u_n^{1/2-\delta} \le |\Lambda_\omega \cap I_n| \le C_2 u_n^{1/2-\delta},\ \forall n \ge N\} \ge 1 - P_1(m) - P_2(N).$$

Since by our assumption on $(u_n)_{n\ge1}$, $P_2(N)$ tends to zero at infinity, there exists an integer n_0 such that $1 - P_1(m) - P_2(n_0) > 0$. Note that $\lim_{m,N\to\infty}(1 - P_1(m) - P_2(N)) = 1$.

CONCLUSION. There exists at least one set with the properties required in Proposition 6.2. ■

CONSEQUENCE. For each $4 < p \le \infty$, there exists an idempotent *Hankelian* Schur multiplier which is c.b. on S^4 but not bounded on $\mathfrak{S}^p$.

Indeed, let $p > 4$ and choose $0 < \delta < (p-4)/(2p)$ small enough. Consider the sequence $(u_n)_n$ defined by $u_n = 2^{n-1}$. According to Proposition 6.1 and 6.2, there exists a set $\Lambda \subset \mathbb{N}$ which has the $\Lambda(4)_{\mathrm{cb}}$-property and satisfies, for all $n \ge n_0$,

$$C_1 2^{(n-1)(1/2-\delta)} \le |\Lambda_n| \le C_2 2^{(n-1)(1/2-\delta)}$$

where $\Lambda_n := \Lambda \cap I_n$ with $I_n := [2^{n-1}, 2^n[$, $I_0 := \{0\}$ and where the constants C_1, C_2 and the integer n_0 are defined as in Proposition 6.2. Fact 2.3(i) implies that there exists a constant $c_1 > 0$ such that for all integers $n \ge n_0$ we have

$$C_1 2^{(n-1)(1/2-\delta)} \le |\Lambda_n| \le c_1 2^{2(n-1)/p}\lambda_p^2(\Lambda_n) \le c_1 2^{2(n-1)/p}\sup_{k\ge n_0}\lambda_p^2(\Lambda_k),$$

that is to say,

$$C_1 2^{(n-1)(1/2-2/p-\delta)} \le c_1 \sup_{k\ge0}\lambda_p^2(\Lambda_k).$$

Since $1/2 - 2/p - \delta > 0$ and n can be arbitrarily large, $\sup_{k\ge0}\lambda_p(\Lambda_k) = \infty$. Using Proposition 2.2, we see that $\sup_{k\ge0}\mu_p(\Lambda_k) = \infty$. On each interval Λ_n, we may find a choice of signs ε_n such that its extension to $\mathbb{Z}$ by adding 0's on $\mathbb{Z}\setminus I_n$ and 1's on $I_n \setminus \Lambda_n$, denoted by ξ_n, satisfies $\|\xi_n\|_{M(L^p)} \ge \frac{1}{3}\mu_p(\Lambda_n)$. This is clearly possible by using the definition of the constant $\mu_p(\Lambda_n)$ and an extreme point argument. Then we consider $\varepsilon := \sum_{n\ge0}\xi_n|_{\mathbb{N}}$. Note that $\varepsilon(k) = \pm1$ for each integer $k \ge 0$. Using Proposition 5.3 and Peller's results (see Subsection 0.6) together with Remark 0.10, we get

$$\|\widehat{\varepsilon}\|_{M(\mathfrak{S}^p)} \cong \sup_{n\ge0}\|\widehat{\varepsilon}_{(n)}\|_{M(\mathfrak{S}^p_{\widehat{I}_n})} \cong \sup_{n\ge0}\|\xi_n|_{I_n}\|_{M(L^p_{I_n})}$$
$$\cong \sup_{n\ge0}\|\xi_n\|_{M(L^p)} \ge \tfrac{1}{3}\sup\nolimits_{n\ge0}\mu_p(\Lambda_n) = \infty.$$

Hence $\widehat{\varepsilon}$ does not belong to $M(\mathfrak{S}^p)$. Now we consider $\eta := \frac{1}{2}(\varepsilon + \mathbf{1}_{\mathbb{N}})$. Recall that $\mathbf{1}_{\mathbb{N}\setminus\Lambda}$ is a c.b. multiplier on H^4 since Λ is a $\Lambda(4)_{\mathrm{cb}}$-set, and that the constant function $\mathbf{1}_{\mathbb{N}}$ is trivially a c.b. multiplier on H^r for all r. Therefore, $\eta \in M_{\mathrm{cb}}(H^4)$. The idempotent Hankelian multiplier $\widehat{\eta}$ is in $M_{\mathrm{cb}}(S^4)$ by Proposition 5.6 but is not in $M(\mathfrak{S}^p)$ by the above and the fact that the constant function 1 is trivially a c.b. multiplier on S^r for all r, so we are done.

Now we show the existence of "large" $\sigma(4)_{\rm cb}$-sets by using probabilistic ideas to exhibit "large" sets having the combinatorial properties (C) or (R) defined below after checking of course that they imply the $\sigma(4)_{\rm cb}$-property.

Definition 6.3. We say that a subset A of $\mathbb{N}\times\mathbb{N}$ has *property* (C) if $C(A)<\infty$ and that it has *property* (R) if $R(A)<\infty$ where

$$C(A):=\sup_{i\neq i'}|\{j\in\mathbb{N}\mid (i,j)\in A\ \&\ (i',j)\in A\}|,$$
$$R(A):=\sup_{j\neq j'}|\{i\in\mathbb{N}\mid (i,j)\in A\ \&\ (i,j')\in A\}|.$$

Remarks 6.4. (i) If $\Lambda\subset\mathbb{N}$ has the $Z(2)$-property then the Hankelian set $\widehat{\Lambda}$ associated with Λ has (C) and (R) with both $C(\widehat{\Lambda})$ and $R(\widehat{\Lambda})$ less than $Z_2(\Lambda)$.

(ii) (C) and (R) are different combinatorial properties. As an example, the set $A:=\mathbb{N}\times\{1\}\cup\mathbb{N}\times\{2\}$ has property (C) but not (R) (note that neither (C) nor (R) is stable under finite unions). However, A has property (C) if and only if the set ${}^tA:=\{(i,j)\mid (j,i)\in A\}$ has property (R) and we have $C(A)=R({}^tA)$.

(iii) 1-sections (resp. 2-sections) are not 2-sections (resp. 1-sections) and are not finite unions of 2-sections (resp. 1-sections) in general but they have both (C) and (R). Assume that $A_1,\ldots,A_n$ are 1-sections (resp. 2-sections). Then $A:=\bigcup_{i=1}^n A_i$ necessarily has property (C) (resp. (R)) with $C(A)\le n$ (resp. $R(A)\le n$) but it does not have property (R) (resp. (C)) in general. However, a set with (C) (resp. (R)) is not a finite union of 1-sections (resp. 2-sections) in general. As an example, consider an increasing sequence $(k_i)_i$ of integers tending to infinity with $k_{i+1}\ge k_i^2$ for each i and let

$$A:=\bigcup_{i=1}^{\infty}\{(k_i,l),(k,k_i),(k,k_i^2)\mid k_i\le l\le k_i^2,\ k_i<k<k_{i+1}\}.$$

Then A satisfies $C(A)=2$ but cannot be written as a finite union of 1-sections; moreover $R(A)=\infty$.

Proposition 6.5. *Let $A\subset\mathbb{N}\times\mathbb{N}$. Then A is a $\sigma(4)_{\rm cb}$-set whenever A has property (C) (resp. property (R)). Moreover,*

$$\sigma_4^{\rm cb}(A)\le(1+C(A))^{1/4}\qquad (\text{resp. } \sigma_4^{\rm cb}(A)\le(1+R(A))^{1/4}).$$

Thus, a finite union of sets having properties either (C) or (R) is necessarily a $\sigma(4)_{\rm cb}$-set.

Proof. According to the c.b. version of Remark 4.2(iii) and Remark 6.4(ii), we can restrict ourselves to the case where A has property (R). Let $x=(x_{ij})_{i,j}$ be in $S_A^4(S^4)$, say with only finitely many non-zero entries x_{ij}.

We have

$$\|x\|_{S^4(S^4)}^4=\operatorname{tr}\Big(\Big(\sum_{i,j}x_{ij}\otimes e_{ij}\Big)^*\Big(\sum_{i,j}x_{ij}\otimes e_{ij}\Big)\Big)^2$$
$$=\operatorname{tr}\Big(\sum_{i,j,k}x_{ij}^*x_{ik}\otimes e_{jk}\Big)^2=\sum_{i,j,k,r}\operatorname{tr}(x_{ij}^*x_{ik}x_{rk}^*x_{rj})$$
$$=\sum_{j,k}\operatorname{tr}\Big(\sum_i x_{ij}^*x_{ik}\Big)\Big(\sum_r x_{rk}^*x_{rj}\Big)=\sum_{j,k}\Big\|\sum_i x_{ij}^*x_{ik}\Big\|_{S^2}^2$$
$$=\sum_j\Big\|\sum_i x_{ij}^*x_{ij}\Big\|_{S^2}^2+\sum_{j\neq k}\Big\|\sum_i x_{ij}^*x_{ik}\Big\|_{S^2}^2.$$

Using the assumption on A as well as the trace property, we get

$$\|x\|_{S^4(S^4)}^4\le\sum_j\Big\|\sum_i x_{ij}^*x_{ij}\Big\|_{S^2}^2+R(A)\sum_{j\neq k}\sum_i\|x_{ij}^*x_{ik}\|_{S^2}^2$$
$$\le\sum_j\Big\|\sum_i x_{ij}^*x_{ij}\Big\|_{S^2}^2+R(A)\sum_i\sum_{j,k}\operatorname{tr}(x_{ik}^*x_{ij}x_{ij}^*x_{ik})$$
$$\le\sum_j\Big\|\sum_i x_{ij}^*x_{ij}\Big\|_{S^2}^2+R(A)\sum_i\operatorname{tr}\Big(\Big(\sum_j x_{ij}x_{ij}^*\Big)^*\Big(\sum_j x_{ij}x_{ij}^*\Big)\Big)$$
$$\le\sum_j\Big\|\sum_i x_{ij}^*x_{ij}\Big\|_{S^2}^2+R(A)\sum_i\Big\|\sum_j x_{ij}x_{ij}^*\Big\|_{S^2}^2.$$

This implies that for each $x=(x_{ij})_{i,j}$ in $S_A^4(S^4)$, we have

$$\|x\|_{S^4(S^4)}\le(1+R(A))^{1/4}\max\Big\{\Big(\sum_j\Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{1/2}\Big\|_{S^4}^4\Big)^{1/4},\ \Big(\sum_i\Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{1/2}\Big\|_{S^4}^4\Big)^{1/4}\Big\}.\ \blacksquare$$

Remark. Incidentally, the converse of Proposition 6.5 might be true: perhaps every $\sigma(4)_{\rm cb}$-set is a finite union of sets satisfying either (C) or (R).

Proposition 6.6. *For all small $\delta>0$ and all $c\ge1/\delta$, there exist constants $n_0,C_1,C_2>0$ depending on δ and c only such that for each integer $n\ge n_0$, there exists a subset A_n of $[1,n]\times[1,n]$ satisfying $C(A_n)\le c$ (resp. $R(A_n)\le c$) and $C_1n^{3/2-\delta}\le|A_n|\le C_2n^{3/2-\delta}$.*

Proof. Let $\{\xi_{ij}\}_{1\le i,j\le n}$ be a sequence of independent random variables, say on the torus $\mathbb{T}$ equipped with the normalized Lebesgue measure $d\mathbb{P}=dt/(2\pi)$, such that for each $1\le i,j\le n$, ξ_{ij} takes values in $\{0,1\}$ and has expectation $\mathbb{E}\xi_{ij}=\beta/n^{1/2+\delta}$ where β is a non-negative constant to be fixed later. For each ω in $\mathbb{T}$, we let

$$A_\omega = \{(i,j) \in [1,n] \times [1,n] \mid \xi_{ij}(\omega) = 1\}.$$

Clearly $|A_\omega| = \sum_{1\le i,j\le n} \xi_{ij}(\omega)$ and thus $\mathbb{E}|A_\omega| = \beta n^{3/2-\delta}$. For each $1 \le i \ne i' \le n$, we let

$$C(i,i',A_\omega) := |\{j \in [1,n] | (i,j),(i',j) \in A_\omega\}| = \sum_{1\le j\le n} \xi_{ij}(\omega)\xi_{i'j}(\omega).$$

Then, given an integer $m \ge 1$, by the independence of the ξ_{ij}'s we get

$$\begin{aligned}\mathbb{P}\{\omega \mid C(i,i',A_\omega) \ge m\} &= \mathbb{P}\Big\{\omega \Big| \sum_{1\le j\le n} \xi_{ij}(\omega)\xi_{i'j}(\omega) \ge m\Big\}\\ &= \mathbb{P}\{\omega \mid \exists 1 \le j_1 \ne j_2 \ne \ldots \ne j_m \le n \text{ such that}\\ &\qquad\qquad \xi_{ij}(\omega)\xi_{i'j}(\omega) = 1,\ \forall j = j_1,\ldots,j_m\}\\ &= \sum_{1\le j_1\ne\ldots\ne j_m\le n} \Big(\prod_{k=1}^m \mathbb{P}\{\omega \mid \xi_{ij_k}(\omega)\xi_{i'j_k}(\omega) = 1\}\Big)\\ &= \sum_{1\le j_1\ne\ldots\ne j_m\le n} \Big(\prod_{k=1}^m \frac{\beta^2}{n^{1+2\delta}}\Big) \le \frac{\beta^{2m}}{n^{2m\delta}}.\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{P}\{\omega \mid C(A_\omega) \ge m\} = \mathbb{P}\{\omega \mid \sup_{1\le i\ne i'\le n} C(i,i',A_\omega) \ge m\} &\le \sum_{1\le i\ne i'\le n} \frac{\beta^{2m}}{n^{2m\delta}}\\ &\le n^{2(1-m\delta)}\beta^{2m}.\end{aligned}$$

By choosing β such that $\beta^{2c} < 1/2$, we get $\mathbb{P}\{\omega \mid C(A_\omega) \ge c\} < 1/2$ since $1 - c\delta \le 0$. On the other hand, $(\xi_{ij} - \mathbb{E}\xi_{ij})_{i,j}$ is a sequence of centered and independent random variables and hence the variance of $|A_\omega| - \mathbb{E}|A_\omega|$ is

$$\| |A_\omega| - \mathbb{E}|A_\omega| \|_{L^2}^2 \le \beta n^{3/2-\delta} = \mathbb{E}|a_\omega|.$$

Thus, using Chebyshev's inequality, we get

$$\mathbb{P}\big\{\omega \big| | |A_\omega| - \mathbb{E}|A_\omega| | \ge \tfrac12 \mathbb{E}|A_\omega|\big\} \le 4/(\mathbb{E}|A_\omega|) = \tfrac{4}{\beta} n^{-3/2+\delta} =: P(n),$$

$$\mathbb{P}\big\{\omega \big| \tfrac12 \beta n^{3/2-\delta} \le |A_\omega| \le \tfrac32 \beta n^{3/2-\delta}\big\} \ge 1 - P(n).$$

This completes the proof since $P(n)$ tends to zero at infinity. ∎

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Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process

by

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Abstract. Stochastic partial differential equations on $\mathbb{R}^d$ are considered. The noise is supposed to be a spatially homogeneous Wiener process. Using the theory of stochastic integration in Banach spaces we show the existence of a Markovian solution in a certain weighted L^q-space. Then we obtain the existence of a space continuous solution by means of the Da Prato, Kwapień and Zabczyk factorization identity for stochastic convolutions.

0. Introduction. The paper is concerned with the following stochastic partial differential equation:

$$
\begin{aligned}
&\frac{\partial X}{\partial t}(t,x) = AX(t,x) + f(t,x,X(t,x)) + b(t,x,X(t,x))\dot{\mathcal{W}}(t),\\
&X(0,x) = \zeta(x).
\end{aligned}
\tag{0.1}
$$

Throughout the paper we assume that $A = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ is a uniformly elliptic differential operator on $\mathbb{R}^d$, and $\mathcal{W}$ is a spatially homogeneous Wiener process taking values in the space of tempered distributions on $\mathbb{R}^d$.

By a solution to (0.1) we understand the so-called *mild solution*, that is, a solution of the integral equation

$$X(t) = S(t)\zeta + \int_0^t S(t-s)F(s,X(s))\,ds + \int_0^t S(t-s)B(s,X(s))\,d\mathcal{W}(s), \tag{0.2}$$

where S is the semigroup generated by A, and F, B are Nemytskiĭ operators corresponding to f and b, that is, for $t \ge 0$, $x \in \mathbb{R}^d$, and functions $u, \psi : \mathbb{R}^d \to \mathbb{R}$,

$$F(t,u)(x) = f(t,x,u(x)) \quad \text{and} \quad (B(t,u)\psi)(x) = b(t,x,u(x))\psi(x). \tag{0.3}$$

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