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Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process

by

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Abstract. Stochastic partial differential equations on $\mathbb{R}^d$ are considered. The noise is supposed to be a spatially homogeneous Wiener process. Using the theory of stochastic integration in Banach spaces we show the existence of a Markovian solution in a certain weighted L^q-space. Then we obtain the existence of a space continuous solution by means of the Da Prato, Kwapień and Zabczyk factorization identity for stochastic convolutions.

0. Introduction. The paper is concerned with the following stochastic partial differential equation:

$$\begin{aligned} &\frac{\partial X}{\partial t}(t,x) = AX(t,x) + f(t,x,X(t,x)) + b(t,x,X(t,x))\dot{\mathcal{W}}(t), \\ &X(0,x) = \zeta(x). \end{aligned} \tag{0.1}$$

Throughout the paper we assume that $A = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ is a uniformly elliptic differential operator on $\mathbb{R}^d$, and $\mathcal{W}$ is a spatially homogeneous Wiener process taking values in the space of tempered distributions on $\mathbb{R}^d$.

By a solution to (0.1) we understand the so-called *mild solution*, that is, a solution of the integral equation

$$X(t) = S(t)\zeta + \int_0^t S(t-s)F(s,X(s))\,ds + \int_0^t S(t-s)B(s,X(s))\,d\mathcal{W}(s), \tag{0.2}$$

where S is the semigroup generated by A, and F, B are Nemytskiĭ operators corresponding to f and b, that is, for $t \ge 0$, $x \in \mathbb{R}^d$, and functions $u, \psi : \mathbb{R}^d \to \mathbb{R}$,

$$F(t,u)(x) = f(t,x,u(x)) \quad \text{and} \quad (B(t,u)\psi)(x) = b(t,x,u(x))\psi(x). \tag{0.3}$$

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For $\varrho \in \mathbb{R}$ let $L^q_\varrho = L^q(\mathbb{R}^d, e^{-\varrho|x|}\,dx)$, $q \in [2,\infty)$, and let $\mathcal{C}_\varrho$ be the space of all continuous functions u such that $|u(x)|e^{-\varrho|x|} \to 0$ as $|x| \to \infty$. Note that $L^q_\varrho \subseteq L^q_\kappa$ for $\varrho \le \kappa$, and that $C_b(\mathbb{R}^d) \subset L^q_\varrho$ for $\varrho > 0$. In the paper we are interested in the existence of a Markovian L^q_ϱ or $L^q_\varrho \cap \mathcal{C}_{\varrho/q}$ valued solution to (0.1).

The reason for introducing weighted spaces is that we would like to cover the case of initial values being constant functions. Observe that if the coefficients f and b satisfy the linear growth condition then, as the constant function $x \mapsto 1$ is not integrable on $\mathbb{R}^d$, the nonlinear operators F and B defined by (0.3) need not take values in any $L^q(\mathbb{R}^d, dx)$-space. This is another reason for working in weighted spaces.

It is very important to show the space-time continuity of a solution to (0.1). Unfortunately it is not possible to establish a theory of stochastic integration in the space of continuous functions. To overcome this difficulty we first develop a theory of stochastic integration in L^q_ϱ-spaces. Then space-time continuity follows from a generalization of the Da Prato, Kwapień and Zabczyk factorization (see [10] and [11]) and some regularity properties of the semigroup generated by A.

One can show the space-time continuity of a solution to (0.1) by means of Kolmogorov's test (see e.g. [11], [31], [32], and [41]). The method based on the theory of stochastic integration in L^q-spaces seems to be easier, more natural and more general.

In this paper we prove the existence of a solution to (0.1) under various conditions on the coefficients A, f and b. First (see Theorem 1.1), we study the case of f and b being Lipschitz continuous with respect to the third variable. In order to ensure that the stochastic integral takes values in the appropriate function space we have to impose a certain condition involving the dimension d, the order of A, and some analytic characteristics of the spectral measure of the noise. Under this condition we show that (0.1) defines a unique Markov family in the state spaces L^q_ϱ and $L^q_\varrho \cap \mathcal{C}_{\varrho/q}$. Next we consider the case of f having polynomial growth, and b being Lipschitz and bounded. To ensure the global existence of a solution we impose on f a certain semi-dissipativity condition. We also need to assume that the elliptic operator A satisfies the weak dissipativity condition

$$\int_{\mathbb{R}^d} Au(x)|u(x)|^{q-2}u(x)e^{-\varrho|x|}\,dx \le C|u|^q_{L^q_\varrho} \quad \text{for } u \in \operatorname{Dom} A$$

on any L^q_ϱ-space. Since only second order operators exhibit the above property, we have to restrict our considerations to the case of order $A = 2$. In the third theorem we also deal with a semi-dissipative drift. However, we assume that $\mathcal{W}$ is a random field. This allows us to dispose of the assumption on the boundedness of the diffusion term.

The techniques used in the paper can be applied to the more general class of stochastic parabolic equations. We believe that for some more specific problems our assumptions on the drift and diffusion can be relaxed. In particular, in the framework of Theorems 1.2 and 1.3, the semi-dissipativity condition might be replaced by the weaker "one-sided linear growth" assumption from [1]. Furthermore, in Theorem 1.2, one should be able to show the existence of a solution without assuming the boundedness of the diffusion term b, possibly using either the method from [1], or some comparison techniques (see e.g. [16], [19], [23], or [26]).

In the present paper we concentrate only on the existence of a strong solution to (0.1) in the probability sense. The existence of martingale solutions to Navier–Stokes and Euler equations driven by a spatially homogeneous Wiener random field is studied in [7] and [9]. Nonlinear wave equations on $\mathbb{R}^d$ with a spatially homogeneous Wiener process have recently been considered in [35]. For the existence of invariant measures for the process given by (0.1) we refer the reader to [14] and [38].

Linear equations of the type (0.1) with $A = \Delta$, a multiplicative noise, and $\zeta \equiv 1$ were introduced by Dawson and Salehi [14] as models of the growth of a population in a random environment. The same class of equations has been considered by Nobel [29]. Equations with an additive spatially homogeneous Wiener process were studied by Da Prato and Zabczyk in [12]. Nonlinear stochastic partial differential equations with finite-dimensional noise on $\mathbb{R}^d$ were investigated by Funaki [18]. The present paper is to a large extent based on the work by Peszat and Zabczyk [34] on the L^2-theory of SPDE's driven by a spatially homogeneous Wiener process, and on Brzeźniak's work [4] and [5] on stochastic evolution equations in Banach spaces. From [34] we use the representation of the noise as a cylindrical Wiener process on a properly chosen Hilbert space. However, the theory of stochastic integrals in Banach spaces, in our case L^q_ϱ-spaces, enables us to obtain better regularity of a solution. In particular, we can easily show the space-time continuity of solutions. Moreover, we have relaxed the assumption from [34] on the noise and the coefficients.

The results of the present paper have already been used by Tessitore and Zabczyk [39] to investigate the positivity of solutions to stochastic heat equations on $\mathbb{R}^d$, and by Brzeźniak and Peszat [7] to study stochastic Euler equations.

The paper is organized as follows. In Section 1 we introduce the notation and we formulate the main results on the existence and uniqueness of a solution to (0.1).

In Section 2 we present basic facts concerning stochastic integration in L^q-spaces. We also evaluate the so-called γ-radonifying norm of an integral operator acting from an abstract Hilbert space to an L^q-space (see

Proposition 2.1). Finally, we recall from [34] the fact that every spatially homogeneous Wiener process can be regarded as a cylindrical Wiener process on a Hilbert space $\mathcal{H}_\mu$ called the *reproducing kernel Hilbert space* or the *Cameron–Martin space* of $\mathcal{W}$. The properties of stochastic integrals in L^q-spaces follow from the general theory of stochastic integration in Banach spaces. In order to make our presentation self-contained we summarize this theory in Section 7.

In Section 3 we establish some analytical properties of the semigroup generated by A and then we evaluate the γ-radonifying norm of convolution operators acting from $\mathcal{H}_\mu$ into L^q_ϱ. These estimates are crucial for our proofs of the main results. Sections 4, 5, and 6 are devoted to the proofs of the theorems formulated in Section 1. Some auxiliary results are gathered in the two appendices. Namely, Appendix A is concerned with Itô's formula for processes taking values in Banach spaces, whereas Appendix B deals with the equation obtained from (0.1) by replacing A by its Yosida approximation. The approximation result from Appendix B allows us to apply Itô's formula to the process given by (0.1).

1. Notation and formulation of the main results. Let us denote by $\mathcal{S}$ the space of all infinitely differentiable real-valued functions ψ on $\mathbb{R}^d$ for which the seminorms $p_{\alpha,\beta}(\psi)=\sup_{x\in\mathbb{R}^d}|x^\alpha D^\beta\psi(x)|$ are finite. The dual $\mathcal{S}'$ of $\mathcal{S}$ is the space of tempered distributions. We denote by $\langle\xi,\psi\rangle$ the value of $\xi\in\mathcal{S}'$ on $\psi\in\mathcal{S}$. For $x\in\mathbb{R}^d$ we define the translation operator $\tau_x:\mathcal{S}\to\mathcal{S}$ by $\tau_x\psi(\cdot)=\psi(\cdot+x)$, $\psi\in\mathcal{S}$. Then $\tau'_x:\mathcal{S}'\to\mathcal{S}'$ is defined to be the adjoint operator, that is, $\langle\tau'_x\xi,\psi\rangle=\langle\xi,\tau_x\psi\rangle$ for all $\xi\in\mathcal{S}'$ and $\psi\in\mathcal{S}$.

DEFINITION 1.1. A Gaussian $\mathcal{S}'$-valued process $\mathcal{W}$ defined on a filtered probability space $\mathfrak{U}=(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ is called a *spatially homogeneous Wiener process* iff it fulfils the following conditions:

(i) For each $\psi\in\mathcal{S}$, $\{\langle\mathcal{W}(t),\psi\rangle\}_{t\in[0,\infty)}$ is a real-valued Wiener process.

(ii) For each $t\in[0,\infty)$ the law $\mathcal{L}(\mathcal{W}(t))$ of $\mathcal{W}(t)$ is invariant with respect to all translation operators τ'_x, $x\in\mathbb{R}^d$, that is, $\mathcal{L}(\mathcal{W}(t))\circ(\tau'_x)^{-1}=\mathcal{L}(\mathcal{W}(t))$ for every $x\in\mathbb{R}^d$.

For a real-valued function ψ on $\mathbb{R}^d$, we set $\psi_{(s)}(x)=\psi(-x)$, $x\in\mathbb{R}^d$. Later, when ψ can be a complex-valued function, we shall write $\psi_{(s)}(x)=\overline{\psi(-x)}$, $x\in\mathbb{R}^d$. We have the following result (see [34], Section 1).

PROPOSITION 1.1. *Let $\mathcal{W}$ be an $\mathcal{S}'$-valued process satisfying condition* (i) *from Definition* 1.1. *Then $\mathcal{W}$ also satisfies* (ii) *iff there is a positive symmetric tempered measure μ on $\mathbb{R}^d$ such that*

$$\mathbb{E}\langle\mathcal{W}(1),\psi\rangle\langle\mathcal{W}(1),\varphi\rangle=\langle\widehat{\mu},\psi*\varphi_{(s)}\rangle\quad\textit{for all } \psi,\varphi\in\mathcal{S}. \tag{1.4}$$

We call μ appearing in (1.4) the *spectral measure* of $\mathcal{W}$, and $\Gamma=\widehat{\mu}$ its *correlation function*. If μ is finite then (see [34]) there is a stationary Wiener random field W such that

$$\langle\mathcal{W}(t),\psi\rangle=\int_{\mathbb{R}^d}W(t,x)\psi(x)\,dx\quad\text{for } t\ge0,\ \psi\in\mathcal{S}.$$

Moreover, $\mathbb{E}W(t,x)W(s,y)=t\wedge s\,\Gamma(x-y)$ for $t,s\ge0$, $x,y\in\mathbb{R}^d$. If μ is Lebesgue measure then (see [34] or [11]) $\mathcal{W}$ is a *cylindrical Wiener process* on L^2. Its time derivative $\dot{\mathcal{W}}$ is then called a *white noise* on $L^2([0,\infty)\times\mathbb{R}^d)$, or *space-time white noise* (see [41]).

Let $\vartheta\in C^\infty(\mathbb{R}^d)$ be a fixed strictly positive even function equal to $e^{-|x|}$ for $|x|\ge1$. Let $q\in[1,\infty]$ and $\varrho\in\mathbb{R}$. For brevity we denote by L^q_ϱ the weighted space $L^q(\mathbb{R}^d,\vartheta_\varrho(x)\,dx)$, where $\vartheta_\varrho(x)=(\vartheta(x))^\varrho$. However, we write L^q instead of L^q_0. The scalar product and norm on L^2_ϱ are denoted by $\langle\cdot,\cdot\rangle_\varrho$ and $|\cdot|_\varrho$. The products on L^2 and on $\mathbb{R}^d$ are denoted by the same symbol $\langle\cdot,\cdot\rangle$.

Note that $L^q_\varrho=L^q(\mathbb{R}^d,e^{-\varrho|x|}\,dx)$. For technical reasons (see the proof of Lemma 3.1), it is convenient to replace the non-differentiable weight $e^{-\varrho|\cdot|}$ by the equivalent and smooth ϑ_ϱ. In the present paper we have chosen a system of exponential weights. However, all our results remain true if ϑ_ϱ is replaced by $\widetilde{\vartheta}_\varrho(x)=(1+|x|^2)^{-\varrho}$. What we really require of the weights is the strong continuity on L^q_ϱ of the semigroup generated by A (see Lemma 3.1), and this is guaranteed by estimate (3.9). Note that the weighted space $\mathcal{C}_\varrho$ of continuous functions equipped with the norm $|u|_{\mathcal{C}_\varrho}=\sup_{x\in\mathbb{R}^d}|u(x)|\vartheta_\varrho(x)$ is a Banach space.

In this paper we shall show the existence and uniqueness of a solution X to (0.1) from the classes $\mathcal{K}(L^q_\varrho)$ and $\mathcal{K}(L^q_\varrho\cap\mathcal{C}_{\varrho/q})$ defined below. Let $T>0$, $\varrho\in\mathbb{R}$, $q\in[2,\infty)$ and $p>1$. We denote by $\mathcal{K}^p_T(L^q_\varrho)$ the space of all measurable $(\mathcal{F}_t)$-adapted processes Z having continuous trajectories in L^q_ϱ such that

$$|||Z|||_{\mathcal{K}^p_T(L^q_\varrho)}:=(\mathbb{E}\sup_{t\in[0,T]}|Z(t)|^p_{L^q_\varrho})^{1/p}<\infty.$$

Furthermore, we set $\mathcal{K}(L^q_\varrho)=\bigcap\mathcal{K}^p_T(L^q_\varrho)$, where the intersection is taken over all $T>0$ and $p>1$. In the same manner we define the classes $\mathcal{K}^p_T(L^q_\varrho\cap\mathcal{C}_\kappa)$ and $\mathcal{K}(L^q_\varrho\cap\mathcal{C}_\kappa)$.

In the main theorems we need the following additional condition on $\mathcal{W}$.

(H.1) The spectral measure μ of $\mathcal{W}$ is of the form $\mu=\sum_{i=0}^n\mu_i$, where $n\in\mathbb{N}$, μ_0 is a finite measure, and for each $i=1,\ldots,n$ there is a subspace L_i of $\mathbb{R}^d$, possibly $L_i=\mathbb{R}^d$, such that μ_i is absolutely continuous with respect to Lebesgue measure m_{L_i} on L_i, and the density $\eta_i=d\mu_i/d\mathrm{m}_{L_i}$ belongs to $L^{p_i}(L_i,\mathrm{m}_{L_i})$ for a $p_i\in[1,\infty)$.

Assume that (H.1) holds, and write

$$a_{\mathcal{W}} = \max_{i=1,\dots,n}\left(1-\frac{1}{\widetilde{p}_i}\right)\dim L_i, \tag{1.5}$$

where

$$\widetilde{p}_i = \inf\{p\in[1,\infty] : d\mu_i/d\mathrm{m}_{L_i}\in L^p(L_i,\mathrm{m}_{L_i})\}.$$

Our main existence and regularity results are contained in the following three theorems. The first one deals with the case of Lipschitz coefficients. Before its formulation it is convenient to introduce some notation.

DEFINITION 1.2. Let $\varrho\in\mathbb{R}$ and $q\in[2,\infty)$. We say that the pair (f,b) of coefficients belongs to the class $\mathrm{Lip}(\varrho,q)$ iff f and b are measurable, and for every $T>0$ there exist a constant L and a function $l_0\in L^q_\varrho$ such that for all $t\in[0,T]$, $x\in\mathbb{R}^d$ and $y,z\in\mathbb{R}$,

$$|f(t,x,y)-f(t,x,z)|+|b(t,x,y)-b(t,x,z)|\le L|y-z|,$$
$$|f(t,x,y)|+|b(t,x,y)|\le L(l_0(x)+|y|).$$

THEOREM 1.1. *Let $\varrho\in\mathbb{R}$. Assume that* (H.1) *is fulfilled, and $a_{\mathcal{W}}$ given by* (1.5) *satisfies $a_{\mathcal{W}}<2m$.*

(i) *Let $q\in[2,\infty)$, and let $(f,b)\in\mathrm{Lip}(\varrho,q)$. Then for every $\zeta\in L^q_\varrho$ there exists a unique solution X_ζ to* (0.1) *from the class $\mathcal{K}(L^q_\varrho)$, and $\{X_\zeta\}_{\zeta\in L^q_\varrho}$ is a Markov family with the Feller property.*

(ii) *Let $q>2d(2m-a_{\mathcal{W}})^{-1}$, and let $(f,b)\in\mathrm{Lip}(\varrho,q)$. Then for every $\zeta\in L^q_\varrho\cap\mathcal{C}_{\varrho/q}$ there exists a unique solution X_ζ to* (0.1) *from the class $\mathcal{K}(L^q_\varrho\cap\mathcal{C}_{\varrho/q})$, and $\{X_\zeta\}_{\zeta\in L^q_\varrho\cap\mathcal{C}_{\varrho/q}}$ is a Markov family with the Feller property.*

Our next theorems deal with the case of a second order operator A, and the coefficient f satisfying a certain semi-dissipativity condition. In particular, f can have polynomial growth. This causes some technical difficulties. They are concerned with the fact that generally the semigroup S does not transform L^{rq}_ϱ into L^q_ϱ. This explains why we consider (0.1) on the state spaces L_ϱ, $\mathcal{C}^+$, and $\mathcal{C}^-$ defined as follows:

$$L_\varrho=\bigcap_{q\in[2,\infty)}L^q_\varrho,\quad \mathcal{C}^+=\bigcap_{\varrho>0}\mathcal{C}_\varrho,\quad \mathcal{C}^-=\bigcap_{\varrho<0}\mathcal{C}_\varrho.$$

Note that $C_{\mathrm{b}}(\mathbb{R}^d)\subset\mathcal{C}^+\subset L_\varrho$ for every $\varrho>0$, and $C_0(\mathbb{R}^d)\subset\mathcal{C}^-\subset L_\varrho$ for every $\varrho<0$. Next we set

$$\mathcal{K}(L_\varrho)=\bigcap_{q\in[2,\infty)}\mathcal{K}(L^q_\varrho),\quad \mathcal{K}(\mathcal{C}^+)=\bigcap_{\varrho>0}\mathcal{K}(\mathcal{C}_\varrho),\quad \mathcal{K}(\mathcal{C}^-)=\bigcap_{\varrho<0}\mathcal{K}(\mathcal{C}_\varrho).$$

DEFINITION 1.3. Let $r\in[1,\infty)$. We say that (f,b) belongs to the class $\mathrm{Dis}(\varrho,r)$ iff f and b are measurable, and for every $T\in(0,\infty)$ there are a constant L and a function $l_0\in L_\varrho$ such that for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and $y,z\in\mathbb{R}$,

$$|f(t,x,y)|\le L(l_0(x)+|y|+|y|^r),$$
$$|f(t,x,y)-f(t,x,z)|\le L|y-z|(1+|y|^{r-1}+|z|^{r-1}), \tag{1.6}$$
$$f(t,x,y+z)y\le L(l_0(x)+|z|+|z|^r+|y|)|y|$$

and

$$|b(t,x,y)|\le L(l_0(x)+|y|),$$
$$|b(t,x,y)-b(t,x,z)|\le L|y-z|.$$

If, additionally, b satisfies the boundedness condition $|b(t,x,y)|\le l_0(x)$ with $l_0\in L_\varrho$, then we say that the pair (f,b) belongs to the class $\mathrm{BDis}(\varrho,r)$.

Before proceeding further we present an example of a function satisfying (1.6).

EXAMPLE 1.1. Let $f(t,x,y)=\sum_{j=0}^{2n-1}a_j(t,x)y^j$, $t\ge0$, $x\in\mathbb{R}^d$, and $y\in\mathbb{R}$, where $n\in\mathbb{N}$, and $a_j\in C_{\mathrm{b}}([0,T]\times\mathbb{R}^d)$ for $j=0,\dots,2n-1$ and $T>0$. Let $\varrho\in\mathbb{R}$ and let $a_0\in L_\varrho$. Assume that for any $T>0$ there is a $C<0$ such that $a_{2n-1}(t,x)<C$ for all $t\in[0,T]$, $x\in\mathbb{R}^d$. Then f satisfies (1.6).

THEOREM 1.2. *Let $r\in[1,\infty)$. Assume that A is a second order uniformly elliptic operator with coefficients from C^∞_{b}. Assume that* (H.1) *is fulfilled, and $a_{\mathcal{W}}<2$.*

(i) *Let $\varrho\in\mathbb{R}$, and let $(f,b)\in\mathrm{BDis}(\varrho,r)$. Then for every $\zeta\in L_\varrho$ there exists a unique solution X_ζ to* (0.1) *from the class $\mathcal{K}(L_\varrho)$, and $\{X_\zeta\}_{\zeta\in L_\varrho}$ is a Markov family.*

(ii) *Let $\varrho>0$, and let $(f,b)\in\mathrm{BDis}(\varrho,r)$. Then for every $\zeta\in\mathcal{C}^+$ there exists a unique solution X_ζ to* (0.1) *from the class $\mathcal{K}(\mathcal{C}^+)$, and $\{X_\zeta\}_{\zeta\in\mathcal{C}^+}$ is a Markov family.*

(iii) *Let $(f,b)\in\mathrm{BDis}(\varrho,r)$ for every $\varrho<0$. Then for every $\zeta\in\mathcal{C}^-$ there is a unique solution X_ζ to* (0.1) *from the class $\mathcal{K}(\mathcal{C}^-)$, and $\{X_\zeta\}_{\zeta\in\mathcal{C}^-}$ is a Markov family.*

As an illustration of Theorem 1.2 (see also Example 1.1), we have:

EXAMPLE 1.2 (Reaction-diffusion equation on $\mathbb{R}$). Consider the following particular case of equation (0.1):

$$\frac{\partial X}{\partial t}(t,x)=\frac{\partial^2X}{\partial x^2}(t,x)+\sum_{j=0}^{3}a_jX^j(t,x)+b(t,x,X(t,x))\dot{\mathcal{W}}(t), \tag{1.7}$$
$$X(0,x)=\zeta(x),$$

where a_j, $j = 0, \ldots, 3$, are constants, $\dot{W}$ is the space-time white noise on $\mathbb{R}$, and b is a measurable function. Assume that $a_3 < 0$, and that for every $T > 0$ there are a constant K and a function l such that for $t \in [0,T]$, $x \in \mathbb{R}^d$, and $y, z \in \mathbb{R}$,

$$|b(t,x,y) - b(t,x,z)| \le K|y-z| \quad \text{and} \quad |b(t,x,y) \le l(x).$$

Then as a consequence of Theorem 1.2 we have:

(i) If $l \in \mathcal{C}^+$ then (1.7) defines uniquely a Markov family on $\mathcal{C}^+$.

(ii) If $l \in \mathcal{C}^-$ and $a_0 = 0$, then (1.7) defines uniquely a Markov family on $\mathcal{C}^-$.

Our last theorem is concerned with the case of $\mathcal{W}$ being a random field, and f of polynomial growth. Since we can apply the Itô formula in this case, we can drop the assumption on the boundedness of the diffusion term.

THEOREM 1.3. *Let $r \in [1,\infty)$. Assume that A is a uniformly elliptic second order operator, and the spectral measure μ of $\mathcal{W}$ is finite.*

(i) *Let $\varrho \in \mathbb{R}$, and let $(f,b) \in \mathrm{Dis}(\varrho, r)$. Then for every $\zeta \in L_\varrho$ there exists a unique solution X_ζ to (0.1) from the class $\mathcal{K}(L_\varrho)$, and $\{X_\zeta\}_{\zeta\in L_\varrho}$ is a Markov family.*

(ii) *Let $\varrho > 0$, and let $(f,b) \in \mathrm{Dis}(\varrho, r)$. Then for every $\zeta \in \mathcal{C}^+$ there exists a unique solution X_ζ to (0.1) from the class $\mathcal{K}(\mathcal{C}^+)$, and $\{X_\zeta\}_{\zeta\in \mathcal{C}^+}$ is a Markov family.*

(iii) *Let $(f,b) \in \mathrm{Dis}(\varrho, r)$ for every $\varrho < 0$. Then for every $\zeta \in \mathcal{C}^-$ there is a unique solution X_ζ to (0.1) from the class $\mathcal{K}(\mathcal{C}^-)$, and $\{X_\zeta\}_{\zeta\in \mathcal{C}^-}$ is a Markov family.*

As illustrations of Theorem 1.3 we have the following two examples.

EXAMPLE 1.3. Consider the following particular case of equation (0.1):

$$\begin{aligned} &\frac{\partial X}{\partial t}(t,x) = \frac{\partial^2 X}{\partial x^2}(t,x) + \sum_{j=0}^{2n-1} a_j X^j(t,x) + \sum_{j=0}^{1} b_j X^j(t,x)\frac{\partial \mathcal{W}}{\partial t}(t,x), \\ &X(0,x) = \zeta(x), \end{aligned} \tag{1.8}$$

where a_j, b_j are constants, and $\mathcal{W}$ is a spatially homogeneous Wiener random field on $\mathbb{R}^d$. Assume that $a_{2n-1} < 0$. Then it follows from Theorem 1.3 that (1.8) defines uniquely a Markov family on $\mathcal{C}^+$. If additionally $a_0 = b_0 = 0$, then (1.8) defines uniquely a Markov family on $\mathcal{C}^-$.

EXAMPLE 1.4. Consider (0.1) with A being a second order elliptic operator on $\mathbb{R}$, $b(t,x,y) = y$, and $f(t,x,y) = -y^2 + ay$ for $y \ge 0$ and 0 otherwise. Then as a consequence of Theorem 1.3, (0.1) defines a unique Markov family in $\mathcal{C}^+$ and in $\mathcal{C}^-$. Using a comparison theorem (see e.g. [16], [19], [23], or [26]), one can show that if the initial value is a non-negative function, then the solution takes values in the set of non-negative functions.

REMARK 1.1. Assume that the hypotheses of one of Theorems 1.1 to 1.3 are fulfilled. Then we have the pathwise uniqueness of a solution. But it is easy to deduce from the proofs that law uniqueness holds as well.

Let $\mathcal{E}$ be any of the state spaces listed in Theorems 1.1 to 1.3. Assume that ζ is an $\mathcal{F}_0$-measurable random variable in $\mathcal{E}$ such that $\mathbb{E}|\zeta|_{\mathcal{E}}^p < \infty$ for every $p \in [1,\infty)$. Then it also follows from the proofs that there is a unique solution in $\mathcal{K}(\mathcal{E})$ to (0.1) starting from ζ.

Assume that ζ is stationary, or equivalently that its law is spatially homogeneous. Then for every $t > 0$ the law of $X_\zeta(t)$ is spatially homogeneous provided that the coefficients f and b do not depend on the x-variable. Indeed, to show that $\tau_x(X_\zeta(t))$ and $X_\zeta(t)$ have the same law it suffices to observe that $\tau_x(X_\zeta)$ is a solution to (0.1) driven by $\tau_x\mathcal{W}$, and starting from $\tau_x(\zeta)$, and then use law uniqueness.

2. Stochastic integration in L^q-spaces. Let $(\mathcal{O}, \mathcal{B}, \theta)$ be a measurable space, and let $L^q(\mathcal{O}) = L^q(\mathcal{O}, \mathcal{B}, \theta)$. Let H be a real separable Hilbert space, and let $\{e_k\}$ be an orthonormal basis of H. In this section $\{\beta_k\}_{k=1}^\infty$ is a sequence of independent standard (i.e. $\mathbb{E}\beta_k = 0$ and $D^2\beta_k = 1$) normal real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A bounded linear operator $K : H \to L^q(\mathcal{O})$ is *γ-radonifying* iff the series $\sum_{k=1}^\infty \beta_k K e_k$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; L^q(\mathcal{O}))$ (see also Definition 7.1 and Proposition 7.1). We use $R(H, L^q(\mathcal{O}))$ to denote the class of all γ-radonifying operators from H into $L^q(\mathcal{O})$. Note that $R(H, L^q(\mathcal{O}))$ equipped with the norm

$$\|K\|_{R(H,L^q(\mathcal{O}))} = \Big(\mathbb{E}\Big|\sum_{k=1}^\infty \beta_k K e_k\Big|^2_{L^q(\mathcal{O})}\Big)^{1/2}$$

is a Banach space.

The proposition below provides a useful estimate for the γ-radonifying norm of an operator given by a kernel. We supply a simple proof and remark that the proposition is a special case of a more general result of Kwapień (see Proposition 2.1, Chapter II and Theorem 5.4, Chapter VI of [40]).

PROPOSITION 2.1. *Let $q \in [2,\infty)$. Assume that $K \in L(H, L^q(\mathcal{O}))$ is given by $(K\psi)(x) = \langle \mathcal{K}(x), \psi\rangle_H$, $x \in \mathcal{O}$, $\psi \in H$, where $\mathcal{K} \in L^q(\mathcal{O}, \mathcal{F}, \theta; H)$. Then K belongs to $R(H, L^q(\mathcal{O}))$ and*

$$\|K\|_{R(H,L^q(\mathcal{O}))} \le C\Big(\int_{\mathcal{O}} |\mathcal{K}(x)|_H^q \, d\theta(x)\Big)^{1/q}.$$

Proof. Since the real-valued random variable $\sum_{k=n}^{n+l} \beta_k \langle \mathcal{K}(x), e_k\rangle_H$ is Gaussian for each x, there exists a constant C_1 depending only on q such that

$$\Big(\mathbb{E}\Big|\sum_{k=n}^{n+l}\beta_k K e_k\Big|^2_{L^q(\mathcal{O})}\Big)^{q/2} = \Big(\mathbb{E}\Big(\int_{\mathcal{O}}\Big|\sum_{k=n}^{n+l}\beta_k\langle\mathcal{K}(x),e_k\rangle_H\Big|^q\,d\theta(x)\Big)^{2/q}\Big)^{q/2}$$
$$\le \mathbb{E}\int_{\mathcal{O}}\Big|\sum_{k=n}^{n+l}\beta_k\langle\mathcal{K}(x),e_k\rangle_H\Big|^q\,d\theta(x)$$
$$\le C_1\int_{\mathcal{O}}\Big(\mathbb{E}\Big|\sum_{k=n}^{n+l}\beta_k\langle\mathcal{K}(x),e_k\rangle_H\Big|^2\Big)^{q/2}\,d\theta(x)$$
$$\le C_1\int_{\mathcal{O}}\Big|\sum_{k=n}^{n+l}\langle\mathcal{K}(x),e_k\rangle_H^2\Big|^q\,d\theta(x).$$

Thus the series $\sum\beta_k K e_k$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P};L^q(\mathcal{O}))$ and

$$\|K\|^2_{R(H,L^q(\mathcal{O}))} = \mathbb{E}\Big|\sum_{k=1}^{\infty}\beta_k K e_k\Big|^2_{L^q(\mathcal{O})} \le C_1^{2/q}\Big(\int_{\mathcal{O}}|\mathcal{K}(x)|^q_H\,d\theta(x)\Big)^{2/q},$$

which gives the desired conclusion. ∎

Let W be a cylindrical Wiener process in H (see e.g. [11]). The stochastic integral with respect to W can be defined first for simple processes and then extended to the space $\mathcal{L}^2(0,\infty;R(H,L^q(\mathcal{O})))$ of all measurable, adapted $R(H,L^q(\mathcal{O}))$-valued processes σ such that $\mathbb{E}\int_0^T\|\sigma(t)\|^2_{R(H,L^q(\mathcal{O}))}\,dt<\infty$ for $T<\infty$. In fact we have the following consequence of general theorems on stochastic integration in Banach spaces and some special properties of L^q-spaces (see Theorems 7.2 and 7.3, and Proposition 7.2 of the present paper).

THEOREM 2.1. *Let $q\in[2,\infty)$. Then for any $\sigma\in\mathcal{L}^2(0,\infty;R(H,L^q(\mathcal{O})))$ the stochastic integral $\int_0^t\sigma(s)\,dW(s)$, $t\ge 0$, is an $L^q(\mathcal{O})$-valued square integrable martingale with continuous modification and zero mean. Moreover, for every $p\in(1,\infty)$ there is a constant C, independent of T and σ, such that*

$$\mathbb{E}\sup_{0\le t\le T}\Big|\int_0^t\sigma(s)\,dW(s)\Big|^p_{L^q(\mathcal{O})} \le C\mathbb{E}\Big(\int_0^T\|\sigma(s)\|^2_{R(H,L^q(\mathcal{O}))}\,ds\Big)^{p/2}.$$

In the present paper $W=\mathcal{W}$ is a spatially homogeneous Wiener process with a spectral measure μ. Recall (see Section 1) that $\psi_{(\mathrm{s})}(x)=\overline{\psi(-x)}$. Denote by $L^2_{(\mathrm{s})}(\mathbb{R}^d,\mu)$ the subspace of $L^2(\mathbb{R}^d,\mu;\mathbb{C})$ consisting of all ψ such that $\psi=\psi_{(\mathrm{s})}$. Set

$$\mathcal{H}_\mu=\{\widehat{\psi\mu}:\psi\in L^2_{(\mathrm{s})}(\mathbb{R}^d,\mu)\},$$
$$\langle\widehat{\psi\mu},\widehat{\varphi\mu}\rangle_{\mathcal{H}_\mu}=\int_{\mathbb{R}^d}\psi(x)\overline{\varphi(x)}\,d\mu(x),\qquad \psi,\varphi\in L^2_{(\mathrm{s})}(\mathbb{R}^d,\mu).$$

Obviously, $\mathcal{H}_\mu\subset\mathcal{S}'$, and $(\mathcal{H}_\mu,\langle\cdot,\cdot\rangle_{\mathcal{H}_\mu})$ is a real separable Hilbert space. For further reference we present without proof the following version of Proposition 1.1 of [34].

THEOREM 2.2. *$(\mathcal{H}_\mu,\langle\cdot,\cdot\rangle_{\mathcal{H}_\mu})$ is the Cameron–Martin space of $\mathcal{W}$. In particular $\mathcal{W}$ is a cylindrical Wiener process on $(\mathcal{H}_\mu,\langle\cdot,\cdot\rangle_{\mathcal{H}_\mu})$, and $\mathcal{W}$ takes values in any Banach space V such that the embedding $\mathcal{H}_\mu\hookrightarrow V$ is γ-radonifying.*

We present without proof the following slight generalization of Proposition 1.3 of [34].

LEMMA 2.1. *Assume that* (H.1) *is fulfilled. Denote by Π_{L_i} the orthogonal projection of $\mathbb{R}^d$ onto L_i, and identify $\psi\in L^2(L_i,\mathrm{m}_{L_i})$ with $\widetilde{\psi}:\mathbb{R}^d\to\mathbb{R}$ such that $\widetilde{\psi}(x)=\psi(\Pi_{L_i}x)$, $x\in\mathbb{R}^d$. Then*

$$\mathcal{H}_\mu\subset C_{\mathrm{b}}(\mathbb{R}^d)+L^2(L_1,\mathrm{m}_{L_i})+\ldots+L^2(L_n,\mathrm{m}_{L_n}).$$

3. Analytic preliminaries. Recall that A is a uniformly elliptic operator of order $2m$. Set

$$g(t,x)=Kt^{-d/(2m)}\exp\{-(|x|^{2m}/t)^{1/(2m-1)}\},\qquad t>0,\ x\in\mathbb{R}^d,$$

the constant K being such that $\int_{\mathbb{R}^d}g(t,x)\,dx=1$ for every $t>0$. In the sequel we shall make use of the following property of the weights ϑ_ϱ:

$$\forall\varrho\ \forall T>0\ \exists K_1\ \forall x\in\mathbb{R}^d\ \forall t\in[0,T],\qquad g(t,\cdot)*\vartheta_\varrho(x)\le K_1\vartheta_\varrho(x). \tag{3.9}$$

Recall (see [17]) that the fundamental solution G exists for the operator $\partial/\partial t-A$. Moreover, $G\in C^\infty(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d)$ and for each T there exists a constant $K_2>0$ such that

$$|G(t,x,y)|\le K_2^{-1}g(K_2t,x-y)\qquad\text{for all } t\in(0,T],\ x,y\in\mathbb{R}^d. \tag{3.10}$$

Let $n\in\mathbb{N}$ and $q\in[2,\infty)$. Denote by $H^{n,q}$ the Sobolev space of functions on $\mathbb{R}^d$ with q-integrable derivatives of order $\le n$. Then for any q, A with domain $H^{2m,q}$ is the generator of a holomorphic semigroup S on L^q and

$$S(t)u(x)=\int_{\mathbb{R}^d}G(t,x,y)u(y)\,dy\qquad\text{for } u\in L^q,\ t>0,\ x\in\mathbb{R}^d. \tag{3.11}$$

Moreover, since $G(t,x,\cdot)\in\mathcal{S}$ for all $t\ge0$ and $x\in\mathbb{R}^d$, the semigroup S has an extension to a semigroup on $\mathcal{S}'$.

LEMMA 3.1. *Let $q\in[2,\infty)$ and $\varrho\in\mathbb{R}$.*

(i) *If $\varrho\ge0$ then the semigroup S defined on L^q by (3.11) has a unique extension to a holomorphic semigroup on L^q_ϱ.*
(ii) *If $\varrho<0$ then the restriction of S to L^q_ϱ is a holomorphic semigroup.*
(iii) *For every ϱ, S is a C_0-semigroup on C_ϱ.*

(iv) *For arbitrary* $\varrho \in \mathbb{R}^d$, $q \in [2,\infty)$ *and* $t > 0$, $S(t)$ *is a bounded linear operator acting from* L^q_ϱ *into* $C_{\varrho/q}$. *Moreover, for each* T *there exists a constant* C *such that*

$$\|S(t)\|_{L(L^q_\varrho, C_{\varrho/q})} \le Ct^{-d/(2mq)} \qquad \text{for } t \in (0,T].$$

Proof. (i) & (ii). Let $\varrho \in \mathbb{R}$, $T > 0$, $t \in (0,T]$ and $u \in \mathcal{S}$. Combining (3.11) with (3.10), (3.9) and Young's inequality we get

$$\begin{aligned}|S(t)u|^q_{L^q_\varrho} &= \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} G(t,x,y)u(y)\,dy\Big|^q \vartheta_\varrho(x)\,dx \\ &\le K_2^{-q} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} g(K_2 t, x-y)|u(y)|\,dy\Big)^q \vartheta_\varrho(x)\,dx \\ &\le K_2^{-q} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(K_2 t, x-y)|u(y)|^q\,dy\,\vartheta_\varrho(x)\,dx \\ &\le C \int_{\mathbb{R}^d} |u(y)|^q \vartheta_\varrho(y)\,dy,\end{aligned}$$

with the constant C being independent of u and $t \in (0,T]$. This implies that the extension, or restriction, of S to L^q_ϱ is a C_0-semigroup. In this paper we shall use the same letter to designate the semigroup on L^q and its extension or restriction to L^q_ϱ. To prove that S is holomorphic on L^q_ϱ we take the isometry $j_\varrho : L^q_\varrho \to L^q$ given by $j_\varrho(u) = u\vartheta_{\varrho/q}$. Set $\widetilde{S}(t) = j_\varrho S(t) j_\varrho^{-1}$. Clearly, $\widetilde{S} = \{\widetilde{S}(t)\}$ is a C_0-semigroup on L^q, and its generator $\widetilde{A}$ is equal to $j_\varrho A j_\varrho^{-1}$. It is easy to see that $\widetilde{A}$ is a uniformly elliptic operator with coefficients from $C_{\mathrm{b}}^\infty(\mathbb{R}^d)$. Thus $\widetilde{S}$ is holomorphic on L^q, and consequently S is holomorphic on L^q_ϱ for it is unitarily equivalent to $\widetilde{S}$.

(iii) & (iv). Fix $T \in (0,\infty)$, $q \in [2,\infty)$ and $\varrho \in \mathbb{R}$. Part (iii) is proven in [18] and [34]. Since $S(t)$, $t > 0$, are given by continuous kernels, it is easy to see that $S(t)u$, $t > 0$, are continuous functions for any $u \in L^q_\varrho$. Thus it is enough to show that there is a constant C such that

$$\sup_{x\in\mathbb{R}^d} |S(t)u(x)|\vartheta_{\varrho/q}(x) \le Ct^{-d/(2mq)}|u|_{L^q_\varrho} \qquad \text{for } u \in L^q_\varrho.$$

To do this note that using (3.10), and then Hölder's inequality and (3.9) we get

$$\begin{aligned}|S(t)u(x)|^q &\le \Big(K_2^{-1} \int_{\mathbb{R}^d} g(K_2 t, x-y)|u(y)|\,dy\Big)^q \\ &\le K_2^{-q}\Big(\int_{\mathbb{R}^d} g(K_2 t, x-y)|u(y)|^{q/2}\,dy\Big)^2 \\ &\le K_2^{-q}\Big(\int_{\mathbb{R}^d} g(K_2 t, x-y)\vartheta_{-\varrho/2}|u(y)|^{q/2}\vartheta_{\varrho/2}\,dy\Big)^2 \\ &\le K_2^{-q}|u|^q_{L^q_\varrho} \int_{\mathbb{R}^d} g^2(K_2 t, x-y)\vartheta_{-\varrho}(y)\,dy \\ &\le C_1|u|^q_{L^q_\varrho} t^{-d/(2m)}\vartheta_{-\varrho}(x),\end{aligned}$$

which gives the desired conclusion. ∎

Recall that $\mathcal{H}_\mu$ is the Cameron–Martin space of a spatially $\mathcal{W}$ homogeneous Wiener process with a spectral measure μ. Note that if $u \in L^q_\varrho$ and $\xi \in \mathcal{H}_\mu$, then as a consequence of Lemma 2.1 we have $u\xi \in \mathcal{S}'$. Since the semigroup S generated by A is also a semigroup on $\mathcal{S}'$,

$$K(t,u)\xi = S(t)(u\xi) \quad \text{for } t \ge 0,\ u \in L^q_\varrho,\ \xi \in \mathcal{H}_\mu \tag{3.12}$$

is a well defined element of $\mathcal{S}'$. We shall show that in fact $K(t,u)\xi$ belongs to L^q_ϱ, and for $\xi = \widehat{\psi\mu}$ we have

$$K(t,u)(\xi)(x) = \int_{\mathbb{R}^d} [G(t,x,\cdot)u]^\wedge(y)\psi(y)\,d\mu(y). \tag{3.13}$$

THEOREM 3.1. *Let* (H.1) *be fulfilled, and let* K *be given by* (3.12). *Then for all* $u \in L^q_\varrho$ *and* $t > 0$, $K(t,u) \in R(\mathcal{H}_\mu, L^q_\varrho)$. *Moreover, for any* $T \in (0,\infty)$ *there is a constant* C *such that for all* $u \in L^q_\varrho$ *and* $t \in (0,T]$,

$$\|K(t,u)\|^q_{R(\mathcal{H}_\mu, L^q_\varrho)} \le C|u|_{L^q_\varrho}\Big(1 + \sum_{i=1}^n t^{-q\frac{d_i}{4m}(1-1/p_i)}\Big).$$

COROLLARY 3.1. *Let the assumptions of Theorem* 3.1 *be fulfilled, let* K *be given by* (3.12), *and let* $a_{\mathcal{W}}$ *be given by* (1.5). *Then for every* $a > a_{\mathcal{W}}/(2m)$ *and* $T \in (0,\infty)$ *there is a constant* C *such that for all* $u \in L^q_\varrho$ *and* $t \in (0,T]$,

$$\|K(t,u)\|_{R(\mathcal{H}_\mu, L^q_\varrho)} \le C|u|_{L^q_\varrho} t^{-a/2}.$$

We have divided the proof of the theorem into two lemmas.

LEMMA 3.2. *Assume that* (H.1) *holds. Then for any* $T \in (0,\infty)$ *there is a constant* C *such that for all* $u \in L^q_\varrho$ *and* $t \in (0,T]$,

$$\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |[G(t,x,\cdot)u]^\wedge(y)|^2\,d\mu(y)\Big)^{q/2}\vartheta_\varrho(x)\,dx \le C|u|_{L^q_\varrho}\Big(1+\sum_{i=1}^n t^{-q\frac{d_i}{4m}(1-1/p_i)}\Big).$$

Proof. For $i = 0,1,\ldots,n$ we write

$$I_i = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |[G(t,x,\cdot)u]^\wedge(y)|^2\,d\mu_i(y)\Big)^{q/2}\vartheta_\varrho(x)\,dx.$$

Using (3.10), Jensen's inequality and then (3.9), we get

$$
\begin{aligned}
I_0 &\le (\mu_0(\mathbb{R}^d))^{q/2} \int_{\mathbb{R}^d} \sup_{y\in\mathbb{R}^d} |[G(t,x,\cdot)u]^\wedge(y)|^q \vartheta_\varrho(x)\,dx \\
&\le (\mu_0(\mathbb{R}^d))^{q/2} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} K_2^{-1} g(K_2 t, x-z)|u(z)|\,dz\Big)^q \vartheta_\varrho(x)\,dx \\
&\le (\mu_0(\mathbb{R}^d))^{q/2} K_2^{-q} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(K_2 t, x-z)|u(z)|^q\,dz\,\vartheta_\varrho(x)\,dx \\
&\le C_1 \int_{\mathbb{R}^d} |u(z)|^q \vartheta_\varrho(z)\,dz = C_1 |u|^q_{L^q_\varrho},
\end{aligned}
$$

where C_1 is independent of u and $t\in[0,T]$.

Let $i\in\{1,\ldots,n\}$. Using Hölder's inequality we get

$$
\begin{aligned}
I_i &= \int_{\mathbb{R}^d} \Big(\int_{L_i} |[G(t,x,\cdot)u]^\wedge(y)|^2 \frac{d\mu_i}{d\mathrm{m}_{L_i}}(y)\,d\mathrm{m}_{L_i}(y)\Big)^{q/2} \vartheta_\varrho(x)\,dx \\
&\le \Big|\frac{d\mu_i}{d\mathrm{m}_{L_i}}\Big|^{q/2}_{L^{p_i}(L_i,\mathrm{m}_{L_i})} \\
&\quad\times \int_{\mathbb{R}^d} \Big(\int_{L_i} |[G(t,x,\cdot)u]^\wedge(y)|^{2p_i^*}\,d\mathrm{m}_{L_i}(y)\Big)^{q/(2p_i^*)} \vartheta_\varrho(x)\,dx.
\end{aligned}
$$

Using first the Hausdorff–Young inequality and then (3.10), we get

$$
\begin{aligned}
&\Big(\int_{L_i} |[G(t,x,\cdot)u]^\wedge(y)|^{2p_i^*}\,d\mathrm{m}_{L_i}(y)\Big)^{1/(2p_i^*)} \\
&\le C_2 \Big(\int_{L_i}\Big(\int_{L_i^\perp} |G(t,x,z+v)u(z+v)|\,d\mathrm{m}_{L_i^\perp}(v)\Big)^{(2p_i^*)^*} d\mathrm{m}_{L_i}(z)\Big)^{1/(2p_i^*)^*} \\
&\le C_2 \Big(\int_{L_i}\Big(\int_{L_i^\perp} K_2^{-1} g(K_2 t, x-z-v) \\
&\qquad\times |u(z+v)|\,d\mathrm{m}_{L_i^\perp}(v)\Big)^{2p_i/(p_i+1)} d\mathrm{m}_{L_i}(z)\Big)^{(p_i+1)/(2p_i)} \\
&\le C_2 \Big\{\int_{L_i}\Big(\int_{L_i^\perp} K_2^{-1} g(K_2 t, x-z-v)|u(z+v)|^{2p_i/(p_i+1)}\,d\mathrm{m}_{L_i^\perp}(v) \\
&\qquad\times \Big(\int_{L_i^\perp} K_2^{-1} g(K_2 t, x-z-v)\,d\mathrm{m}_{L_i^\perp}(v)\Big)^{2p_i/(p_i+1)-1}\Big)\,d\mathrm{m}_{L_i}(z)\Big\}^{(p_i+1)/(2p_i)}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{L_i^\perp} K_2^{-1} g(K_2 t, x-y-v)\,d\mathrm{m}_{L_i^\perp}(v) \\
&\le C_3 \int_{\mathbb{R}^{d-d_i}} K_2^{-1}(K_2 t)^{-d_i/(2m)} \exp\{-|v|^{2m/(2m-1)}\}\,dv \le C_4 t^{-d_i/(2m)},
\end{aligned}
$$

we have

$$
\begin{aligned}
I_i &\le C_5 t^{-\frac{d_i}{2m}\frac{q(p_i-1)}{2p_i}} \\
&\quad\times \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} K_2^{-1} g(K_2 t, x-z)|u(z)|^{2p_i/(p_i+1)}\,dz\Big)^{q(p_i+1)/(2p_i)} \vartheta_\varrho(x)\,dx \\
&\le C_6 t^{-\frac{d_i}{2m}\frac{q(p_i-1)}{2p_i}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(K_2 t, x-z)|u(z)|^q\,dz\vartheta_\varrho(x)\,dx \\
&\le C_7 t^{-\frac{d_i}{2m}\frac{q(p_i-1)}{2p_i}} |u|^q_{L^q_\varrho} = C_7 t^{-q\frac{d_i}{4m}(1-1/p_i)} |u|^q_{L^q_\varrho},
\end{aligned}
$$

which gives the desired conclusion. ■

LEMMA 3.3. *Let* (H.1) *hold and let* K *be given by* (3.12). *Then for all* $u\in L^q_\varrho$, $t>0$ *and* $\xi=\widehat{\psi\mu}\in\mathcal{H}_\mu$, *the identity* (3.13) *holds and* $K(t,u)\in R(\mathcal{H}_\mu, L^q_\varrho)$. *Moreover, for any* $T\in(0,\infty)$ *there is a constant* C *independent of* $u\in L^q_\varrho$ *and* $t\in(0,T]$ *such that*

$$
\|K(t,u)\|^q_{R(\mathcal{H}_\mu,L^q_\varrho)} \le C\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |[G(t,x,\cdot)u]^\wedge(y)|^2\,d\mu(y)\Big)^{q/2}\vartheta_\varrho(x)\,dx.
$$

Proof. To show (3.13) we take $\varphi\in\mathcal{S}$. Then

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} [G(t,x,\cdot)u]^\wedge(y)\,\psi(y)\,d\mu(y)\,\varphi(x)\,dx \\
&\qquad= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (2\pi)^{-d/2} e^{-i\langle y,z\rangle} G(t,x,z)u(z)\,\psi(y)\,d\mu(y)\,\varphi(x)\,dx \\
&\qquad= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t,x,z)u(z)\widehat{\psi\mu}(z)\,\varphi(x)\,dx\,dz \\
&\qquad= \int_{\mathbb{R}^d} K(t,u)(\widehat{\psi\mu})(x)\varphi(x)\,dx,
\end{aligned}
$$

which proves (3.13).

Let $j\psi=\widehat{\psi\mu}$. Then j is an isometry between $L_{(s)}=L_{(s)}(\mathbb{R}^d,\mu)$ and $\mathcal{H}_\mu$. Clearly $K(t,u)\in R(\mathcal{H}_\mu,L^q_\varrho)$ iff $K(t,u)j\in R(L_{(s)},L^q_\varrho)$. Moreover,

$$
\|K(t,u)\|_{R(\mathcal{H}_\mu,L^q_\varrho)} = \|K(t,u)j\|_{R(L_{(s)},L^q_\varrho)}.
$$

Observe now that by (3.13) we have

$$(K(t,u)j)\psi(x) = \langle [G(t,x,\cdot)u]^\wedge, \psi\rangle_{L_{(s)}}.$$

Thus by Proposition 2.1 there is a constant C such that

$$\|K(t,u)j\|^q_{R(L_{(s)},L^q_\varrho)} \le C \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |[G(t,x,\cdot)u]^\wedge(y)|^2\, d\mu(y)\Big)^{q/2} \vartheta_\varrho(x)\, dx,$$

and the proof is complete. ∎

4. Proof of Theorem 1.1. In the proof of the first part we use the Banach contraction principle. As in [31] and [5] we use the factorization identity from [10] to obtain some maximal estimates for stochastic convolution in the supremum norm in $C([0,T];L^q_\varrho)$. In the second part we use smoothness properties of the semigroup S in much the same way as in [34]. However, as we integrate in L^q_ϱ-spaces we are able to show the space-time continuity of a solution for an elliptic operator A of an arbitrary order.

Proof of (i). Let F and B be given by (0.3), and let $\widetilde{B}(t,u)(x) = b(t,x,u(x))$ for $x \in \mathbb{R}^d$ and a function $u : \mathbb{R}^d \to \mathbb{R}$. First observe that $F(s,\cdot)$ and $\widetilde{B}(t,\cdot)$, $t \ge 0$, map L^q_ϱ into L^q_ϱ. Moreover, for any $T \in [0,\infty)$, there is a constant C_1 such that for all $t \in [0,T]$ and $u, v \in L^q_\varrho$,

$$\begin{aligned} |F(t,u) - F(t,v)|_{L^q_\varrho} + |\widetilde{B}(t,u) - \widetilde{B}(t,v)|_{L^q_\varrho} &\le C_1 |u-v|_{L^q_\varrho}, \\ |F(t,u)|_{L^q_\varrho} + |\widetilde{B}(t,u)|_{L^q_\varrho} &\le C_1(1+|u|_{L^q_\varrho}). \end{aligned} \tag{4.14}$$

From the assumptions of the theorem there is a constant a such that

$$\frac{a_{\mathcal{W}}}{2m} < a < 1. \tag{4.15}$$

Thus there is an $\alpha \in (0,1/2)$ such that $2\alpha + a < 1$. Let $p \in [2,\infty)$ be such that $p > \alpha^{-1}$. Recall that the spaces $\mathcal{K}^p_T(L^q_\varrho)$, $T > 0$, were introduced in Section 1. Let $T \in (0,\infty)$. Observe that if $Z \in \mathcal{K}^p_T(L^q_\varrho)$, then for any $t \in [0,T]$ the processes

$$\Psi^Z_t(s) := \begin{cases} 0 & \text{if } s \notin [0,t], \\ S(t-s)B(s,Z(s)) & \text{if } s \in [0,t], \end{cases}$$

and

$$\Psi^Z_{t,\alpha}(s) := \begin{cases} 0 & \text{if } s \notin [0,t], \\ (t-s)^{-\alpha} S(t-s)B(s,Z(s)) & \text{if } s \in [0,t], \end{cases}$$

belong to $\mathcal{L}^2(0,\infty;R(\mathcal{H}_\mu,L^q_\varrho))$. Indeed, they are predictable, and for $s \in [0,t]$,

$$\Psi^Z_t(s) = K(t-s,\widetilde{B}(s,Z(s))), \quad \Psi^Z_{t,\alpha}(s) = (t-s)^{-\alpha} K(t-s,\widetilde{B}(s,Z(s))),$$

where K is given by (3.12). Thus, by Theorem 3.1, $\Psi^Z_t(s), \Psi^Z_{t,\alpha}(s) \in R(\mathcal{H}_\mu, L^q_\varrho)$ and for $s \in [T_1,t]$,

$$\begin{aligned} \|\Psi^Z_t(s)\|_{R(\mathcal{H}_\mu,L^q_\varrho)} &\le C_2 |\widetilde{B}(s,Z(s))|_{L^q_\varrho} \Big(1 + \sum_{i=1}^n (t-s)^{-\frac{\dim L_i}{4m}(1-1/p_i)}\Big) \\ &\le C_3(1+|Z(s)|_{L^q_\varrho})(t-s)^{-a/2}, \end{aligned}$$

and consequently

$$\|\Psi^Z_{t,\alpha}(s)\|_{R(\mathcal{H}_\mu,L^q_\varrho)} \le C_4(1+|Z(s)|_{L^q_\varrho})(t-s)^{-\alpha-a/2}.$$

Thus, for any $\widetilde{T} > 0$,

$$\mathbb{E}\int_0^{\widetilde{T}} (\|\Psi^Z_t(s)\|^2_{R(\mathcal{H}_\mu,L^q_\varrho)} + \|\Psi^Z_{t,\alpha}(s)\|^2_{R(\mathcal{H}_\mu,L^q_\varrho)})\, ds < \infty.$$

For $Z \in \mathcal{K}^p_T(L^q_\varrho)$ and $t \in [0,T]$ we write

$$\begin{aligned} J(Z)(t) &= \int_0^t S(t-s)B(s,Z(s))\, d\mathcal{W}(s), \\ J_\alpha(Z)(t) &= \int_0^t (t-s)^{-\alpha} S(t-s)B(s,Z(s))\, d\mathcal{W}(s), \\ I(Z)(t) &= \int_0^t S(t-s)F(s,Z(s))\, ds. \end{aligned} \tag{4.16}$$

Finally, for $Y \in L^p(0,T;L^q_\varrho)$ and $t \in [0,T]$ we set

$$J^{\alpha-1}Y(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} S(t-s)Y(s)\, ds. \tag{4.17}$$

Using Burkholder's inequality from Theorem 2.1 one can easily show that

$$J_\alpha(Z) \in L^p(\Omega\times[0,T];L^q_\varrho) \quad \text{for } Z \in \mathcal{K}_T(L^q_\varrho).$$

Since S is a C_0-semigroup on L^q_ϱ and $\alpha > p^{-1}$ one can easily show that

$$J^{\alpha-1} \in L(L^p(0,T;L^q_\varrho), C([0,T];L^q_\varrho)). \tag{4.18}$$

Now employing the factorization identity $J \equiv J^{\alpha-1}J_\alpha$ from [10] (see also [5], [11], or [31]), we see that J maps $\mathcal{K}^p_T(L^q_\varrho)$ into $\mathcal{K}^p_T(L^q_\varrho)$ for any T. Moreover, there is a constant $C(T)$ such that $C(T) \downarrow 0$ as $T \downarrow 0$ and for all $Z, Y \in \mathcal{K}_{T_1,T_2}$,

$$|||J(Z) - J(Y)|||_{\mathcal{K}^p_T(L^q_\varrho)} \le C(T)|||Z-Y|||_{\mathcal{K}^p_T(L^q_\varrho)}. \tag{4.19}$$

It is a simple matter to show that the estimate (4.19) holds true if we replace J by I.

Set $\mathcal{L}^p = L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^q_\varrho)$. For $z \in \mathcal{L}^p$ and $Z \in \mathcal{K}^p_T(L^q_\varrho)$ we set

$$\begin{aligned}\mathcal{J}(z,Z)(t) &= S(t)z + \int_0^t S(t-s)F(s,Z(s))\,ds + \int_0^t S(t-s)B(s,Z(s))\,d\mathcal{W}(s)\\ &= S(t)z + I(Z)(t) + J(Z)(t).\end{aligned}$$

Then $\mathcal{J}$ is a continuous mapping from $\mathcal{L}^p \times \mathcal{K}^p_T(L^q_\varrho)$ into $\mathcal{K}^p_T(L^q_\varrho)$ for any T. Moreover, for any sufficiently small $T > 0$ there is a constant $C < 1$ such that for all $z \in \mathcal{L}^p$ and $Z, Y \in \mathcal{K}^p_T(L^q_\varrho)$,

$$|||\mathcal{J}(z,Z) - \mathcal{J}(z,Y)|||_{\mathcal{K}^p_T(L^q_\varrho)} \le C|||Z - Y|||_{\mathcal{K}^p_T(L^q_\varrho)}.$$

Therefore, the Banach fixed point theorem yields the existence and uniqueness of a solution to (0.1) from the class $\mathcal{K}^p_T(L^q_\varrho)$ for T small enough. The continuous dependence of the solution on the initial data follows from the well known fact that the fixed point depends continuously on the parameter (see e.g. [5], [11], [20], or [33]). It is a simple matter to show that this solution can be extended to a solution on any interval. Since this holds for every $p > \alpha^{-1}$, the solution belongs to $\mathcal{K}(L^q_\varrho)$. The Feller property of $\{X_\zeta\}$, $\zeta \in L^q_\varrho$, follows from the continuous dependence of X_ζ on ζ. ∎

Proof of (ii). Recall that $q \in [2,\infty)$ satisfies $q > 2d(2m - a_{\mathcal{W}})^{-1}$. Note that there are a satisfying (4.15) and $\alpha \in (0,1)$ such that

$$\alpha - 1 - \frac{d}{2mq} > -1 \quad\text{and}\quad 2\alpha + a < 1, \tag{4.20}$$

Let $\zeta \in L^q_\varrho \cap \mathcal{C}_{\varrho/q}$. Then, by Theorem 1.1(i), there exists a unique solution X_ζ to (0.1) from the class $\mathcal{K}(L^q_\varrho)$. What is left to show is that X_ζ belongs to $\mathcal{K}(L^q_\varrho \cap \mathcal{C}_{\varrho/q})$. By the definition of a solution we have

$$X_\zeta(t) = S(t)\zeta + I(X_\zeta)(t) + J(X_\zeta)(t),$$

where I and J are defined by (4.16). Since $\zeta \in L^q_\varrho \cap \mathcal{C}_{\varrho/q}$, and since S is a C_0-semigroup on L^q_ϱ and $\mathcal{C}_{\varrho/q}$, we have $S(\cdot)\zeta \in C([0,\infty); L^q_\varrho \cap \mathcal{C}_{\varrho/q})$. Taking Lemma 3.1(iv) and (4.14) into account, one can show that $I(X_\zeta) \in \mathcal{K}(L^q_\varrho \cap \mathcal{C}_{\varrho/q})$. To show that $J(X_\zeta) \in \mathcal{K}(L^q_\varrho \cap \mathcal{C}_{\varrho/q})$ we use again the factorization $J \equiv J^{\alpha-1}J_\alpha$. We have proved (see the proof of Theorem 1.1(i)) that $J_\alpha(X_\zeta) \in L^p(\Omega \times [0,T]; L^q_\varrho)$ for all $T > 0$, $p > 2$. Thus, it is enough to show that

$$J^{\alpha-1} \in L(L^p(0,T;L^q_\varrho), C([0,T];\mathcal{C}_{\varrho/q})) \quad \text{for } p > \left(\alpha - \frac{d}{2mq}\right)^{-1}. \tag{4.21}$$

To do this observe that, by Lemma 3.1(iv) and Hölder's inequality, we have

$$\begin{aligned}|J^{\alpha-1}Z(t)|_{\mathcal{C}_{\varrho/q}} &\le \frac{\sin\alpha\pi}{\pi}\int_0^t (t-s)^{\alpha-1}\|S(t-s)\|_{L(L^q_\varrho,\mathcal{C}_{\varrho/q})}|Z(s)|_{L^q_\varrho}\,ds\\ &\le C_1\int_0^t (t-s)^{\alpha-1-d/(2mq)}|Z(s)|_{L^q_\varrho}\,ds\\ &\le C_1\Big(\int_0^t (t-s)^{(\alpha-1-d/(2mq))p^*}\,ds\Big)^{1/p^*}\Big(\int_0^T |Z(s)|^p_{L^q_\varrho}\,ds\Big)^{1/p}.\end{aligned}$$

Hence $J^{\alpha-1}$ is a bounded operator from $L^p(0,T;L^q_\varrho)$ into $L^\infty(0,T;\mathcal{C}_{\varrho/q})$ as from (4.20) we have $(\alpha - 1 - d/(2mq))p^* > -1$. The L^∞-space above can be replaced by the space of continuous functions as the map $t \mapsto J^{\alpha-1}Z(t)$ is continuous for every Z from the dense subset $C^\infty_0([0,T];L^q_\varrho)$ of $L^p(0,T;L^q_\varrho)$ (see [10] for more details). Thus we have shown (4.21), which completes the proof. ∎

5. Proof of Theorem 1.2. The proof makes use of the concept of the subdifferential of a norm. This technique was used by Da Prato and Zabczyk [11] for equations with a constant diffusion term, and then extended by Peszat [31] to equations with a bounded diffusion term. In the proof of the first part of the theorem we follow [31]. Roughly speaking, assuming that the coefficients are locally Lipschitz we obtain the existence of a local solution. To prove the global existence we take the norm in L^q_ϱ as a Lyapunov function. In the proofs of the second and third parts we employ the regularity properties of the semigroup generated by A.

DEFINITION 5.1. Let $(E, |\cdot|_E)$ be a Banach space, let E^* be the dual space, and let $(\cdot,\cdot)_{E,E^*}$ be the canonical bilinear form on $E \times E^*$. Let $e \in E$. The *subdifferential* $\partial|e|_E$ of $|e|_E$ is the subset of E^* defined as follows:

$$\partial|e|_E = \{e^* \in E^* : (e,e^*)_{E,E^*} = |e|_E \text{ and } |e^*|_{E^*} = 1\}.$$

Let $q \ge 2$, and $\varrho \in \mathbb{R}$. Then, identifying $(L^q_\varrho)^*$ with $L^{q^*}_\varrho$, we obtain $\partial|u|_{L^q_\varrho} = \{|u|^{1-q}_{L^q_\varrho}|u|^{q-2}u\}$ for $u \ne 0$ (see [11] for more details). The motivation for introducing the notion of subdifferential is the following well known lemma. For its proof we refer the reader to [11], Proposition D.4, or [37], Lemma 3.1.

LEMMA 5.1. *Let $y : [0,T] \to E$ be differentiable at some $t_0 \in (0,T)$. Then the function $t \mapsto |y(t)|_E$ is differentiable on the right and on the left at t_0 and*

$$\frac{d^+}{dt}|y(t_0)|_E = \sup\{(y'(t_0), y^*_{t_0})_{E,E^*} : y^*_{t_0} \in \partial|y(t_0)|_E\},$$
$$\frac{d^-}{dt}|y(t_0)|_E = \inf\{(y'(t_0), y^*_{t_0})_{E,E^*} : y^*_{t_0} \in \partial|y(t_0)|_E\}.$$

In the proof of the theorem we need the following two deterministic lemmas.

LEMMA 5.2. *Let $\varrho \in \mathbb{R}$ and $q \geq 2$. Let A be a second order uniformly elliptic operator with coefficients from $C_b^\infty(\mathbb{R}^d)$. If we treat A as the generator of a semigroup on L_ϱ^q, then there is a constant C such that*

$$(5.22)\qquad (Au, u^*)_{L_\varrho^q, L_\varrho^{q^*}} \leq C|u|_{L_\varrho^q} \quad \text{for } u \in \operatorname{Dom} A,\ u^* \in \partial|u|_{L_\varrho^q}.$$

Proof. Using the isomorphism $j_\varrho : L_\varrho^q \to L^q$ from the proof of Lemma 3.1 we can assume that $\varrho = 0$. Since $\mathcal{S}$ is dense in L_ϱ^q and invariant with respect to S, it is a core for A (see [13], Theorem 1.9). Thus it is enough to find a constant C such that (5.22) holds for every $u \in \mathcal{S}$, with $\varrho = 0$.

Assume first that A is in divergence form, i.e.

$$A = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big).$$

Then (see e.g. [30], Theorem 3.6 in Chapter 7) A generates a contraction semigroup on L^q, and so by the Lumer–Phillips theorem (see e.g. [30], Chapter 1) it satisfies (5.22) with $C = 0$. Although Theorem 3.6 from [30] deals only with the case of a bounded domain we note that the same proof works for $\mathbb{R}^d$ as well.

Note that in the general case A can be written as a sum of an elliptic operator in divergence form and of a first order operator. Thus it is enough to show that (5.22) holds for an arbitrary first order operator $A_1 = a\cdot\nabla + c$ with $a \in C_b^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $c \in C_b^\infty$. Obviously, we can assume that $c = 0$. Since the function $x \mapsto |x|^{q-2}x$ is C^1 on $\mathbb{R}$, the function u^* is also C^1 for any $u \in \mathcal{S}$. Integrating by parts, we obtain

$$\begin{aligned}(A_1u, u^*) &= ((a\cdot\nabla)u, u^*)\\ &= -\int_{\mathbb{R}^d}\{(\operatorname{div} a(x))u(x)u^*(x)\\ &\quad - (q-1)|u|_{L^q}^{1-q}|u(x)|^{q-2}u(x)(a(x)\nabla)u(x)\}\,dx\\ &= -\int_{\mathbb{R}^d}(\operatorname{div} a(x))u(x)u^*(x)\,dx - (q-1)(A_0u, u^*).\end{aligned}$$

Hence

$$(A_0u, u^*) = -\frac{1}{q}|u|_{L^q}^{1-q}\int_{\mathbb{R}^d}\operatorname{div} a(x)|u(x)|^q\,dx,$$

which, as $\operatorname{div} a$ is a bounded function, gives the required inequality. ∎

LEMMA 5.3. *Let ϱ, q and A be as in Lemma 5.2, and let V be a Banach space continuously imbedded into L_ϱ^q. Let $T \in (0,\infty)$, and let $F : [0,T]\times L_\varrho^q \to L_\varrho^q$ be a measurable mapping. Assume that there are constants $C > 0$ and $r \geq 1$ such that*

$$(F(t,u+v), u^*)_{L_\varrho^q, L_\varrho^{q^*}} \leq C(1+|v|_V^r+|u|_{L_\varrho^q}) \quad \text{for } t\in[0,T],\ u\in\operatorname{Dom} A,\ v\in V,$$

and

$$|F(t,u)|_E \leq C(1+|u|_V^r) \quad \text{for } t\in[0,T],\ u\in V.$$

Let $y, \psi \in C([0,T];V)$ be such that

$$y(t) = S(t)y(0) + \int_0^t S(t-s)F(s,y(s)+\psi(s))\,ds \quad \text{for } t\in[0,T].$$

Then

$$|y(t)|_{L_\varrho^q} \leq e^{Ct}\Big(|y(0)|_{L_\varrho^q} + C\int_0^t(1+|\psi(s)|_V^r)\,ds\Big) \quad \text{for } t\in[0,T].$$

Proof. Note that $F(\cdot, y+\psi) \in L^\infty(0,T;L_\varrho^q)$ and $y = y_1 + y_2$, where $y_1(t) = S(t)y(0)$ and

$$y_2(t) = \int_0^t S(t-s)F(s,y(s)+\psi(s))\,ds.$$

By [25], Chapter 4, Section 9, $y_2 \in W^{1,q}(0,T;L_\varrho^q)\cap L^q(0,T;\operatorname{Dom} A)$. Hence, as S is holomorphic, we have

$$\frac{d}{dt}y(t) = Au(t) + F(t,y(t)+\psi(t)) \quad \text{for almost all } t\in(0,T].$$

Therefore, by Lemma 5.1, for almost all $t \in (0,T]$ and for every $y_t^* \in \partial|y(t)|_{L_\varrho^q}$ we have

$$\frac{d^-}{dt}|y(t)|_{L_\varrho^q} \leq (Ay(t)+F(t,y(t)+\psi(t)), y_t^*)_{L_\varrho^q, L_\varrho^{q^*}}$$

and, consequently, by Lemma 5.2 we have

$$\frac{d^-}{dt}|y(t)|_{L_\varrho^q} \leq C(1+|\psi(t)|_V^r + |y(t)|_{L_\varrho^q}).$$

Hence, by Gronwall's inequality,

$$|y(t)|_{L_\varrho^q} \leq e^{Ct}\Big(|y(0)|_{L_\varrho^q} + C\int_0^t(1+|\psi(s)|_V^r)\,ds\Big),$$

which is the desired estimate. ∎

Proof of Theorem 1.2(i). Fix a $\varrho \in \mathbb{R}$, and let $\phi \in C_0^\infty(\mathbb{R})$ be a function such that $0 \leq \phi \leq 1$, $\phi(y) = 0$ for $|y| \geq 2$ and $\phi(y) = 1$ for $|y| \leq 1$. For $n \in \mathbb{N}$ we set

$$f_n(t,x,y) = f(t,x,y)\phi(n^{-1}y), \quad t\geq 0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}.$$

Note that for all $n \in \mathbb{N}$ and $q \in [2,\infty)$ we have $(f_n, b) \in \operatorname{Lip}(\varrho, q)$. Let $n \in \mathbb{N}$, $t \geq 0$, and let $u, \psi : \mathbb{R}^d \to \mathbb{R}$. Write

$$F^{(n)}(t,u)(x) = f_n(t,x,u(x)) = f(t,x,u(x))\phi(n^{-1}u(x)).$$

Let $\zeta \in L_\varrho$. By Theorem 1.1(i), for every n there is a unique solution $X^{(n)} \in \mathcal{K}(L_\varrho)$ to the stochastic equation

$$X^{(n)}(t) = S(t)\zeta + \int_0^t S(t-s)F^{(n)}(s, X^{(n)}(s))\,ds + \int_0^t S(t-s)B(s, X^{(n)}(s))\,d\mathcal{W}(s). \tag{5.23}$$

Let

$$\Psi^{(n)}(t) = \int_0^t S(t-s)B(s, X^{(n)}(s))\,d\mathcal{W}(s),$$

$$Y^{(n)}(t) = S(t)\zeta + \int_0^t S(t-s)F^{(n)}(s, X^{(n)}(s))\,ds.$$

Note that $Y^{(n)} = X^{(n)} - \Psi^{(n)}$. Our first goal is to show that

$$\sup_n |||\Psi^{(n)}|||_{\mathcal{K}_T^p(L_\varrho^{rq})} < \infty \quad \text{for all } T > 0 \text{ and } p, q \in [2,\infty). \tag{5.24}$$

We can do this using the arguments from the proof of Theorem 1.1. Namely, we fix a $q \in [2,\infty)$, and for a satisfying (4.15) we take an $\alpha \in (0,1)$ such that $2\alpha + a < 1$. Let $p \in [2,\infty)$ be such that $p > \alpha^{-1}$. Let

$$\Psi^{(n,\alpha)}(t) = \int_0^t (t-s)^{-\alpha}B(s, X^{(n)}(s))\,d\mathcal{W}(s) \tag{5.25}$$

and let $J^{\alpha-1}$ be given by (4.17). Then $\Psi^{(n)} = J^{\alpha-1}\Psi^{(n,\alpha)}$, and taking into account (4.18), we only have to show that

$$\sup_n \mathbb{E}\int_0^T |\Psi^{(n,\alpha)}(t)|_{L_\varrho^q}^p\,dt < \infty. \tag{5.26}$$

By Corollary 3.1 we have

$$\|(t-s)^{-\alpha}S(t-s)B(s, X^{(n)}(s))\|_{R(\mathcal{H}_\mu, L_\varrho^q)} \le c_1(t-s)^{-\alpha-a/2}|b(s,\cdot,X^{(n)}(s))|_{L_\varrho^q}$$

and hence, as b is bounded by an element of L_ϱ^q, there is a constant c_2 such that

$$\|(t-s)^{-\alpha}S(t-s)B(s, X^{(n)}(s))\|_{R(\mathcal{H}_\mu, L_\varrho^q)} \le c_2(t-s)^{-\alpha-a/2} \quad \text{for all } n \in \mathbb{N}.$$

Using now the Burkholder inequality from Theorem 2.1 we obtain (5.26). Now as $X^{(n)}, \Psi^{(n)} \in \mathcal{K}(L_\varrho)$, we have $Y^{(n)} \in \mathcal{K}(L_\varrho)$.

Our next step is to prove that

$$\sup_n |||Y^{(n)}|||_{\mathcal{K}_T^p(L_\varrho^q)} < \infty \quad \text{for all } T > 0 \text{ and } p, q \in [2,\infty), \tag{5.27}$$

and consequently

$$\sup_n |||X^{(n)}|||_{\mathcal{K}_T^p(L_\varrho^q)} < \infty \quad \text{for all } T > 0 \text{ and } p, q \in [2,\infty). \tag{5.28}$$

To show (5.27) note that $Y^{(n)}$ is the mild solution to the problem

$$\frac{\partial Y^{(n)}}{\partial t} = AY^{(n)} + F^{(n)}(t, Y^{(n)} + \Psi^{(n)}), \qquad Y^{(n)}(0) = \zeta.$$

Now fix $T \in (0,\infty)$ and $q \in [2,\infty)$. Then there is a constant C such that for all $n \in \mathbb{N}$, $t \in [0,T]$, $u, v \in L_\varrho^{rq} \cap L_\varrho^q$, $u^* \in \partial|u|_{L_\varrho^q}$ one has

$$\langle F^{(n)}(t, u+v), u^*\rangle_\varrho \le C(1 + |v|_{L_\varrho^q} + |v|_{L_\varrho^{rq}}^r + |u|_{L_\varrho^q}).$$

Indeed, take $u \in L_\varrho^{qr}$ and $v \in L_\varrho^{rq}$. Clearly we can assume that $u \ne 0$. Then, as $\partial|u|_{L_\varrho^q} = \{u^* = |u|_{L_\varrho^q}^{1-q}|u|^{q-2}u\}$, and since $0 \le \phi \le 1$, we have

$$\begin{aligned}
&(F^{(n)}(t, u+v), u^*)_{L_\varrho^q, L_\varrho^{q^*}} \\
&\quad = |u|_{L_\varrho^q}^{1-q}\int_{\mathbb{R}^d} f(t,x,u(x)+v(x)) \\
&\qquad \times \phi(n^{-1}u(x) + n^{-1}v(x))|u(x)|^{q-2}u(x)\vartheta_\varrho(x)\,dx \\
&\quad \le L|u|_{L_\varrho^q}^{1-q}\int_{\mathbb{R}^d} (l_0(x) + |v(x)| + |v(x)|^r + |u(x)|)|u(x)|^{q-1}\vartheta_\varrho(x)\,dx.
\end{aligned}$$

Using Hölder's inequality, we get the desired estimate. Combining this with Lemma 5.3, we get

$$\begin{aligned}
&|Y^{(n)}(t)|_{L_\varrho^q} \\
&\quad \le C\Big(1 + |\zeta|_{L_\varrho^q} + \int_0^t (|\Psi^{(n)}(s)|_{L_\varrho^q}^r + |\Psi^{(n)}(s)|_{L_\varrho^{rq}}^r)\,ds\Big), \qquad t \in [0,T],
\end{aligned} \tag{5.29}$$

where C is a constant depending on T, q and ϱ, and independent of n. Clearly, (5.29) and (5.24) give (5.27), and (5.27) and (5.24) give (5.28).

Now we are going to show that for all $T \in (0,\infty)$ and $p, q \ge 2$, $X^{(n)}$ is a Cauchy sequence in $\mathcal{K}_T^p(L_\varrho^q)$ and its limit is the unique solution to (0.1). To do this we fix T, q, and p. Let $l \le n$, and let $u, v \in L_\varrho$. Then

$$|F^{(n)}(t,u) - F^{(l)}(t,v)|_{L_\varrho^q} \le I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \Big(\int_{\mathbb{R}^d} |f(t,x,u(x)) - f(t,x,v(x))|^q\phi^q(n^{-1}u(x))\vartheta_\varrho(x)\,dx\Big)^{1/q} \\
&\le \Big(\int_{\mathbb{R}^d} L^q(1 + |u(x)|^{r-1} + |v(x)|^{r-1})^q|u(x)-v(x)|^q\vartheta_\varrho(x)\,dx\Big)^{1/q}
\end{aligned}$$

$$\le \Big(\int_{\mathbb{R}^d} L^{2q}(1+|u(x)|^{r-1}+|v(x)|^{r-1})^{2q}|u(x)-v(x)|^q\vartheta_\varrho(x)\,dx\Big)^{1/(2q)}$$
$$\times |u-v|^{1/2}_{L^q_\varrho}$$
$$\le C_1\{|u-v|^{1/2}_{L^q_\varrho}+(1+|u|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|v|^{4q(r-1)}_{L^{4q(r-1)}_\varrho})^{1/(4q)}|u-v|^{1/2}_{L^{2q}_\varrho}\}$$
$$\times |u-v|^{1/2}_{L^q_\varrho}$$
$$\le C_2(1+|u|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|v|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|u|_{L^{2q}_\varrho}+|v|_{L^{2q}_\varrho}|u|_{L^q_\varrho}+|v|_{L^q_\varrho})$$
$$\times |u-v|^{1/2}_{L^q_\varrho},$$

and, since ϕ is Lipschitz continuous,

$$I_2=\Big(\int_{\mathbb{R}^d}|f(t,x,v(x))|^q|\phi(n^{-1}u(x))-\phi(l^{-1}v(x))|^q\vartheta_\varrho(x)\,dx\Big)^{1/q}$$
$$\le C_3\Big(\int_{\mathbb{R}^d}L^q(l_0(x)+|v(x)|^r)^q|n^{-1}u(x)-l^{-1}v(x)|^q\vartheta_\varrho(x)\,dx\Big)^{1/q}$$
$$\le C_4l^{-1}(|l_0|^{2q}_{L^{2q}_\varrho}+|v|^{2qr}_{L^{2qr}_\varrho})^{1/(2q)}(|u|^{2q}_{L^{2q}_\varrho}+|v|^{2q}_{L^{2q}_\varrho})^{1/(2q)}$$
$$\le C_5l^{-1}(1+|v|^{2qr}_{L^{2qr}_\varrho}+|u|^{2q}_{L^{2q}_\varrho}+|v|^{2q}_{L^{2q}_\varrho}),$$

the constants C_i being independent of u, v, l, and n. Thus, there is a constant C_6 such that for all $t\in[0,T]$, $u,v\in L_\varrho$, and $l\le n$,

$$|F^{(n)}(t,u)-F^{(l)}(t,v)|_{L^q_\varrho}$$
$$\le C_6(1+|u|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|v|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|v|^{2qr}_{L^{2qr}_\varrho}+|u|^{2q}_{L^{2q}_\varrho}+|v|^{2q}_{L^{2q}_\varrho}+|u|^q_{L^q_\varrho}+|v|^q_{L^q_\varrho})$$
$$\times(l^{-1}+|u-v|^{1/2}_{L^q_\varrho}).$$

Consequently, since S is a C_0-semigroup on L^q_ϱ, we have

$$|Y^{(n)}(t)-Y^{(l)}(t)|_{L^q_\varrho}\le C_7\int_0^t|F^{(n)}(s,X^{(n)}(s))-F^{(l)}(s,X^{(l)}(s))|_{L^q_\varrho}\,ds$$
$$\le C_8\int_0^t I(n,l)(s)(l^{-1}+|X^{(n)}(s)-X^{(l)}(s)|^{1/2}_{L^q_\varrho})\,ds,$$

where

$$I(n,l)(s)=1+|X^{(n)}(s)|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|X^{(l)}(s)|^{4q(r-1)}_{L^{4q(r-1)}_\varrho}+|X^{(l)}(s)|^{2qr}_{L^{2qr}_\varrho}$$
$$+|X^{(n)}(s)|^{2q}_{L^{2q}_\varrho}+|X^{(l)}(s)|^{2q}_{L^{2q}_\varrho}+|X^{(n)}(s)|^q_{L^q_\varrho}+|X^{(l)}(s)|^q_{L^q_\varrho}.$$

Hence

$$|||Y^{(n)}-Y^{(l)}|||^p_{\mathcal{K}^p_t(L^q_\varrho)}=\mathbb{E}\sup_{s\in[0,t]}|Y^{(n)}(s)-Y^{(l)}(s)|^p_{L^q_\varrho}$$
$$\le C_9\mathbb{E}(\sup_{s\in[0,T]}I(n,l)(s))^p\Big(l^{-1}+\int_0^t|X^{(n)}(s)-X^{(l)}(s)|^{1/2}_{L^q_\varrho}\,ds\Big)^p$$
$$\le C_9\mathbb{E}(\sup_{s\in[0,T]}I(n,l)(s))^{2p}\mathbb{E}\Big(l^{-1}+\int_0^t|X^{(n)}(s)-X^{(l)}(s)|^{1/2}_{L^q_\varrho}\,ds\Big)^{2p}.$$

Note that from (5.28) we have

$$\sup_{n,l}\mathbb{E}(\sup_{s\in[0,T]}I(n,l)(s))^{2p}<\infty.$$

Thus

$$|||Y^{(n)}-Y^{(l)}|||^p_{\mathcal{K}^p_t(L^q_\varrho)}\le C_{10}\mathbb{E}\Big(l^{-1}+\int_0^t|X^{(n)}(s)-X^{(l)}(s)|^{1/2}_{L^q_\varrho}\,ds\Big)^{2p}$$
$$\le C_{11}\mathbb{E}\Big(l^{-2}+\int_0^t|X^{(n)}(s)-X^{(l)}(s)|_{L^q_\varrho}\,ds\Big)^p,$$

and consequently there is a constant C_{12} such that for all $t\in[0,T]$ and $l\le n$,

$$|||Y^{(n)}-Y^{(l)}|||_{\mathcal{K}^p_t(L^q_\varrho)}\le C_{12}\Big(l^{-2}+\int_0^t|||X^{(n)}-X^{(l)}|||_{\mathcal{K}^p_s(L^q_\varrho)}\,ds\Big). \tag{5.30}$$

Let a satisfy (4.15), and let $\alpha\in(0,1)$ be such that $2\alpha+a<1$. Let $q\in[2,\infty)$, and let $\Psi^{(n,\alpha)}$ be given by (5.25). Then for any $p\in[2,\infty)$ such that $(\alpha-1)p^*>-1$ we have

$$|||\Psi^{(n)}-\Psi^{(l)}|||_{\mathcal{K}^p_t(L^q_\varrho)}=|||J^{\alpha-1}(\Psi^{(n,\alpha)}-\Psi^{(l,\alpha)})|||_{\mathcal{K}^p_t(L^q_\varrho)}$$
$$=\frac{\sin\alpha\pi}{\pi}\Big[\mathbb{E}\sup_{s\in[0,t]}\Big|\int_0^s(s-\tau)^{\alpha-1}S(s-\tau)(\Psi^{(n,\alpha)}(\tau)-\Psi^{(l,\alpha)}(\tau))\,d\tau\Big|^p_{L^q_\varrho}\Big]^{1/p}$$
$$\le C_{13}\Big[\mathbb{E}\sup_{s\in[0,t]}\Big(\int_0^s(s-\tau)^{\alpha-1}|\Psi^{(n,\alpha)}(\tau)-\Psi^{(l,\alpha)}(\tau)|_{L^q_\varrho}\,d\tau\Big)^p\Big]^{1/p}$$
$$\le C_{13}\Big(\int_0^T t^{(\alpha-1)p^*}\,dt\Big)^{1/p^*}\Big(\mathbb{E}\int_0^t|\Psi^{(n,\alpha)}(s)-\Psi^{(l,\alpha)}(s)|^p_{L^q_\varrho}\,ds\Big)^{1/p}.$$

Since $(\alpha-1)p^*>-1$, we have

$$|||\Psi^{(n)}-\Psi^{(l)}|||_{\mathcal{K}^p_t(L^q_\varrho)}\le C_{14}\Big(\mathbb{E}\int_0^t|\Psi^{(n,\alpha)}(s)-\Psi^{(l,\alpha)}(s)|^p_{L^q_\varrho}\,ds\Big)^{1/p}.$$

Using Burkholder's inequality and then Theorem 3.1 and the Lipschitz continuity of the diffusion term B, we obtain

$$\begin{aligned}\mathbb{E}|\Psi^{(n,\alpha)}(s)-\Psi^{(l,\alpha)}(s)|^p_{L^q_\varrho} &= \mathbb{E}\Big|\int_0^s (s-\tau)^{-\alpha}S(s-\tau)(B(\tau,X^{(n)}(\tau))-B(\tau,X^{(l)}(\tau)))\,d\mathcal{W}(\tau)\Big|^p_{L^q_\varrho}\\ &\le C_{15}\mathbb{E}\Big(\int_0^s (s-\tau)^{-2\alpha-a}|X^{(n)}(\tau)-X^{(l)}(\tau)|^2_{L^q_\varrho}\,d\tau\Big)^{p/2}.\end{aligned}$$

Thus

$$\begin{aligned}|||\Psi^{(n)}-\Psi^{(l)}|||_{\mathcal{K}^p_t(L^q_\varrho)} &\le C_{14}\Big(\mathbb{E}\int_0^t |\Psi^{(n,\alpha)}(s)-\Psi^{(l,\alpha)}(s)|^p_{L^q_\varrho}\,ds\Big)^{1/p}\\ &\le C_{16}\Big(\int_0^t \mathbb{E}\Big(\int_0^s (s-\tau)^{-2\alpha-a}|X^{(n)}(\tau)-X^{(l)}(\tau)|^2_{L^q_\varrho}\,d\tau\Big)^{p/2}ds\Big)^{1/p}\\ &\le C_{17}\Big(\int_0^t |||X^{(n)}-X^{(l)}|||^p_{\mathcal{K}^p_s(L^q_\varrho)}\Big(\int_0^s (s-\tau)^{-2\alpha-a}\,d\tau\Big)^{p/2}ds\Big)^{1/p},\end{aligned}$$

and consequently

$$|||\Psi^{(n)}-\Psi^{(l)}|||_{\mathcal{K}^p_t(L^q_\varrho)} \le C_{18}\int_0^t |||X^{(n)}-X^{(l)}|||_{\mathcal{K}^p_s(L^q_\varrho)}\,ds. \tag{5.31}$$

Combining (5.31) with (5.30), we obtain

$$|||X^{(n)}-X^{(l)}|||_{\mathcal{K}^p_t(L^q_\varrho)} \le C_{19}\Big(l^{-2}+\int_0^t |||X^{(n)}-X^{(l)}|||_{\mathcal{K}^p_s(L^q_\varrho)}\,ds\Big).$$

Applying Gronwall's inequality we obtain

$$|||X^{(n)}-X^{(l)}|||_{\mathcal{K}^p_T(L^q_\varrho)} \le C_{19}l^{-2}e^{C_{19}T}.$$

Therefore, $\{X^{(n)}\}$ is a Cauchy sequence in $\mathcal{K}^p_T(L^q_\varrho)$, for any p, q, T and ϱ. Hence there is an $X\in\mathcal{K}(L_\varrho)$ such that

$$\lim_{n\to\infty}|||X^{(n)}-X|||_{\mathcal{K}^p_T(L^q_\varrho)}=0 \quad \text{for } p,q\in[2,\infty) \text{ and } T\in[0,\infty). \tag{5.32}$$

Now we can show that X is a solution to (0.1). To do this it is enough to show that for all $p,q\in[2,\infty)$ and $T\in(0,\infty)$,

$$\lim_{n\to\infty}|||Y^{(n)}-Y|||_{\mathcal{K}^p_T(L^q_\varrho)}=0 \tag{5.33}$$

and

$$\lim_{n\to\infty}|||\Psi^{(n)}-\Psi|||_{\mathcal{K}^p_T(L^q_\varrho)}=0, \tag{5.34}$$

where

$$Y^{(n)}(t)=\int_0^t S(t-s)F^{(n)}(s,X^{(n)}(s))\,ds,\qquad Y(t)=\int_0^t S(t-s)F(s,X(s))\,ds,$$

and

$$\Psi^{(n)}(t)=\int_0^t S(t-s)B(s,X^{(n)}(s))\,d\mathcal{W}(s),$$

$$\Psi(t)=\int_0^t S(t-s)B(s,X(s))\,d\mathcal{W}(s).$$

Fix $p,q\in[2,\infty)$ and T. Then using the calculation from the proof of (5.30) we obtain

$$|||Y^{(n)}-Y|||_{\mathcal{K}^p_T(L^q_\varrho)} \le C_{12}\Big(n^{-2}+\int_0^T |||X^{(n)}-X|||_{\mathcal{K}^p_t(L^q_\varrho)}\,dt\Big),$$

and taking into account (5.32) we get (5.33). Now let $\alpha\in(0,1)$ and let $p\in[2,\infty)$ be such that $2\alpha+a<1$ and $(\alpha-1)p^*>-1$. Then, using the calculation from the proof of (5.31), we obtain

$$|||\Psi^{(n)}-\Psi|||^p_{\mathcal{K}^p_T(L^q_\varrho)} \le C_{18}\int_0^T |||X^{(n)}-X|||^p_{\mathcal{K}^p_t(L^q_\varrho)}\,dt,$$

and consequently (5.34).

To show uniqueness, it is enough to observe that for any solution X to (0.1) from the class $\mathcal{K}(L_\varrho)$, formulae (5.33) and (5.34) hold true. Finally, the Markov property of $\{X_\zeta\}$ follows from the fact that $X_\zeta(t+s)=X^{(t)}_{X_\zeta(t)}(s)$, where $X^{(t)}$ is the solution to the equation with the coefficients $f^{(t)}(s,x,u)=f(t+s,x,u)$ and $b^{(t)}(s,x,u)=b(t+s,x,u)$. ∎

Proof of Theorem 1.2(ii). Let $\zeta\in\mathcal{C}^+$. Since $\mathcal{C}^+\subset L_\varrho$, the first part of the theorem yields the existence and uniqueness of a solution X_ζ from the class $\mathcal{K}(L_\varrho)$. Let

$$I_1(t)=S(t)\zeta,\qquad I_2(t)=\int_0^t S(t-s)F(s,X_\zeta(s))\,ds,$$

$$I_3(t)=\int_0^t S(t-s)B(s,X_\zeta(s))\,d\mathcal{W}(s).$$

The proof of (ii) will be completed as soon as we show that $I_i\in\mathcal{K}(\mathcal{C}^+)$, $i=1,2,3$. Since S is a C_0-semigroup on each $\mathcal{C}_\varrho$-space (see Lemma 3.1(iii)), we have $I_1\in\mathcal{K}(\mathcal{C}^+)$. To show that $I_2\in\mathcal{K}(\mathcal{C}^+)$ note that

$$F(\cdot,X_\zeta(\cdot))\in L^p(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;L^q_\varrho)) \quad \text{for all } T>0,\ p,q\in[2,\infty).$$

Now using Lemma 3.1(iv), we easily get $I_2 \in \mathcal{K}(\mathcal{C}_{\varrho/q})$. Since this holds for every $q \in [2,\infty)$ and since $\mathcal{C}_\varrho \subset \mathcal{C}_\kappa$ for $\varrho \le \kappa$, we get the desired conclusion. To show that $I_3 \in \mathcal{K}(\mathcal{C}^+)$ we use the argument from the proof of Theorem 1.1(ii). Namely, first we take a satisfying (4.15) and then $\alpha \in (0,1)$ such that $2\alpha + a < 1$. Then, since b is bounded by an element of L_ϱ, the process $J_\alpha(X_\zeta)$ given by (4.16) satisfies

$$J_\alpha(X_\zeta) \in L^p(\Omega,\mathcal{F},\mathbb{P}; L^h(0,T;L^q_\varrho)) \quad \text{for all } p,q,h \in [2,\infty),\ T>0.$$

Finally, the factorization $I_3 = J^{\alpha-1}J_\alpha(X_\zeta)$, where $J^{\alpha-1}$ is given by (4.17), and Lemma 2.1(iv) yield $I_3 \in \mathcal{K}(\mathcal{C}_{\varrho/q})$ for q sufficiently large, and consequently $I_3 \in \mathcal{K}(\mathcal{C}^+)$, which completes the proof. ∎

Proof of Theorem 1.2(iii). Since $(f,b) \in \mathrm{BDis}(\varrho,r)$ for every $\varrho < 0$, Theorem 1.2(i) gives the existence of a solution $X_\zeta \in \mathcal{K}(L_\varrho)$ for every $\varrho < 0$. Using the method from the proof of Theorem 1.2(ii) we get the desired regularity of X_ζ. It is worth pointing out that in this part we need the fact that $(f,b) \in \mathrm{BDis}(\varrho,r)$ for every $\varrho < 0$. Indeed, we have to show that $X_\zeta \in \mathcal{K}(\mathcal{C}_\varrho)$ for all $\varrho < 0$, and for this we employ the boundedness of $S(t)$ acting from $L^q_{\varrho q}$ into $\mathcal{C}_\varrho$. To do this we need to have the solution in any $L^q_{\varrho q}$-space. ∎

6. Proof of Theorem 1.3.

Let $\varrho \in \mathbb{R}$ and let $(f,b) \in \mathrm{Dis}(\varrho,r)$. As in the proof of Theorem 1.2 we take a function $\phi \in C_0^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi(y) = 0$ for $|y| \ge 2$ and $\phi(y) = 1$ for $|y| \le 1$. For $n \in \mathbb{N}$ we set

$$f_n(t,x,y) = f(t,x,y)\phi(n^{-1}y) \quad \text{for } t \ge 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}.$$

Clearly, for all $n \in \mathbb{N}$ and $q \in [2,\infty)$ we have $(f_n,b) \in \mathrm{Lip}(\varrho,q)$. Thus by Theorem 1.1(i), for every $\zeta \in L_\varrho$ and $n \in \mathbb{N}$ there is a unique solution $X^{(n)} \in \mathcal{K}(L_\varrho)$ to the stochastic equation

$$X^{(n)}(t) = S(t)\zeta + \int_0^t S(t-s)F^{(n)}(s,X^{(n)}(s))\,ds + \int_0^t S(t-s)B(s,X^{(n)}(s))\,d\mathcal{W}(s),$$

where

$$F^{(n)}(t,u)(x) = f_n(t,x,u(x)). \tag{6.35}$$

To finish the proof we only have to show that

$$\sup_n |||X^{(n)}|||_{\mathcal{K}^p_T(L^q_\varrho)} < \infty \quad \text{for all } T>0 \text{ and } p,q \in [2,\infty). \tag{6.36}$$

Indeed, having proved (6.36), one can use the techniques and calculations from the proof of Theorem 1.2, and show that $\{X^{(n)}\}$ is a Cauchy sequence

in every $\mathcal{K}^p_T(L^q_\varrho)$-space, and that its limit has the desired regularity and satisfies (0.1).

To show (6.36) we fix q and ϱ, and we treat A as the generator of a C_0-semigroup on L^q_ϱ. By Lemma 5.2 there is a constant C such that

$$((A-C\mathrm{I})u,u^*)_{L^q_\varrho,L^{q^*}_\varrho} \le 0 \quad \text{for } u \in \mathrm{Dom}\,A,\ u^* \in \partial|u|_{L^q_\varrho}.$$

Replacing f in (0.1) by $f + C\mathrm{I}$ we may assume that $(Au,u^*)_{L^q_\varrho,L^{q^*}_\varrho} \le 0$ for all $u \in \mathrm{Dom}\,A$, $u^* \in \partial|u|_{L^q_\varrho}$. Thus, by the Lumer–Phillips theorem A is the generator of a contraction semigroup on L^q_ϱ. Let $\lambda > 0$. Then λ is from the resolvent set of A, and its Yosida approximation $A_\lambda = \lambda A(\lambda\mathrm{I} - A)^{-1}$ is a bounded operator. Let S_λ be the semigroup generated by A_λ. Then, since S_λ is a contraction semigroup (see [30], Lemma 3.4, p. 10), the Lumer–Phillips theorem gives $(A_\lambda u,u^*)_{L^q_\varrho,L^{q^*}_\varrho} \le 0$ for all $\lambda > 0$, $u \in L^q_\varrho$, $u^* \in \partial|u|_{L^q_\varrho}$. Hence, as $u^* = \{|u|^{1-q}_{L^q_\varrho}|u|^{q-2}u\}$ for $u \ne 0$, we have

$$\langle |u|^{q-2}u, A_\lambda u\rangle_\varrho \le 0 \quad \text{for all } \lambda > 0 \text{ and } u \in L^q_\varrho, \tag{6.37}$$

where $\langle\cdot,\cdot\rangle_\varrho$ stands for the bilinear mapping on $L^{q^*}_\varrho \times L^q_\varrho$. Consider the equation

$$Y^{(n)}_\lambda(t) = S_\lambda(t)\zeta + \int_0^t S_\lambda(t-s)F^{(n)}(s,Y^{(n)}_\lambda(s))\,ds + \int_0^t S_\lambda(t-s)B(s,Y^{(n)}_\lambda(s))\,d\mathcal{W}(s), \tag{6.38}$$

where $F^{(n)}$ is given by (6.35). Note that the coefficients of (6.38) are Lipschitz continuous. Moreover (see Theorem 3.1), $B(s,Y^{(n)}_\lambda(s))$ is a γ-radonifying operator from $\mathcal{H}_\mu$ into L^q_ϱ and for every $T > 0$ there is a constant c_1 such that

$$\|B(s,u)\|_{R(\mathcal{H}_\mu,L^q_\varrho)} \le c_1(1 + |u|_{L^q_\varrho}), \quad s \in [0,T],\ u \in L^q_\varrho. \tag{6.39}$$

Thus there is a unique solution $Y^{(n)}_\lambda$ to (6.38) from the class $\mathcal{K}(L_\varrho)$. Moreover, $Y^{(n)}$ is a strong solution to (6.38), that is,

$$Y^{(n)}_\lambda(t) = \zeta + \int_0^t (A_\lambda Y^{(n)}_\lambda(s) + F^{(n)}(s,Y^{(n)}_\lambda(s)))\,ds + \int_0^t B(s,Y^{(n)}_\lambda(s))\,d\mathcal{W}(s).$$

Let $p \ge q$. Applying Itô's formula to the function $\Psi(u) = |u|^p_{L^q_\varrho}$ (see Theorem A.2 and Remark A.1 in Appendix A), we obtain

$$|Y^{(n)}_\lambda(t)|^p_{L^q_\varrho} \le |\zeta|^p_{L^q_\varrho} + p\int_0^t |Y^{(n)}_\lambda(s)|^{p-q}_{L^q_\varrho}\langle |Y^{(n)}_\lambda(s)|^{q-2}Y^{(n)}_\lambda(s), A_\lambda Y^{(n)}_\lambda(s)\rangle_\varrho\,ds$$

$$+p\int_0^t |Y_\lambda^{(n)}(s)|_{L_\varrho^q}^{p-q}\langle |Y_\lambda^{(n)}(s)|^{q-2}Y_\lambda^{(n)}(s), F^{(n)}(s,Y_\lambda^{(n)}(s))\rangle_\varrho\,ds$$
$$+\frac{p(p-1)}{2}\int_0^t |Y_\lambda^{(n)}(s)|_{L_\varrho^q}^{p-2}\|B(s,Y_\lambda^{(n)}(s))\|^2_{R(\mathcal{H}_\mu,L_\varrho^q)}\,ds$$
$$+p\int_0^t |Y_\lambda^{(n)}(s)|_{L_\varrho^q}^{p-q}\langle |Y_\lambda^{(n)}(s)|^{q-2}Y_\lambda^{(n)}(s), B(s,Y_\lambda^{(n)}(s))\,dW(s)\rangle_\varrho.$$

Note that from (6.37) we have

$$p\int_0^t |Y_\lambda^{(n)}(s)|_{L_\varrho^q}^{p-q}\langle |Y_\lambda^{(n)}(s)|^{q-2}Y_\lambda^{(n)}(s), A_\lambda Y_\lambda^{(n)}(s)\rangle_\varrho\,ds\le 0.$$

Fix $T>0$. Then using (1.6) for $z=0$ we obtain

$$\langle |u|^{q-2}u, F^{(n)}(t,u)\rangle_\varrho \le \int_{\mathbb{R}^d} L(l_0(x)+|u(x)|)|u(x)|^{q-1}\vartheta_\varrho(x)\,dx$$
$$\le L(|l_0|_{L_\varrho^q}|u|_{L_\varrho^q}^{q-1}+|u|_{L_\varrho^q}^q).$$

Combining this with (6.39), we get

$$p|u|_{L_\varrho^q}^{p-q}\langle |u|^{q-2}u, F^{(n)}(s,u)\rangle_\varrho+\frac{p(p-1)}{2}|u|_{L_\varrho^q}^{p-2}\|B(s,u)\|^2_{R(\mathcal{H}_\mu,L_\varrho^q)}$$
$$\le p|u|_{L_\varrho^q}^{p-q}L(|l_0|_{L_\varrho^q}|u|_{L_\varrho^q}^{q-1}+|u|_{L_\varrho^q}^q)+\frac{p(p-1)}{2}c_1^2(1+|u|^2_{L_\varrho^q})^2$$
$$\le c_2(1+|u|^p_{L_\varrho^q}),$$

with c_2 being independent of n, λ and u. Hence

$$|Y_\lambda^{(n)}|^p_{L_\varrho^q}\le |\zeta|^p_{L_\varrho^q}+\int_0^t c_2(1+|Y_\lambda^{(n)}(s)|^p_{L_\varrho^q})\,ds$$
$$+p\int_0^t |Y_\lambda^{(n)}(s)|_{L_\varrho^q}^{p-q}\langle |Y_\lambda^{(n)}(s)|^{q-2}Y_\lambda^{(n)}(s), B(s,Y_\lambda^{(n)}(s))\,dW(s)\rangle_\varrho.$$

Applying Burkholder's inequality for real martingales, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|Y_\lambda^{(n)}|^{2p}_{L_\varrho^q}$$
$$\le 3|\zeta|^{2p}_{L_\varrho^q}+3\mathbb{E}\Big(\int_0^t c_2(1+|Y_\lambda^{(n)}(s)|^p_{L_\varrho^q})\,ds\Big)^2$$
$$+c_3\mathbb{E}\int_0^t |Y_\lambda^{(n)}(s)|^{2(p-q)}_{L_\varrho^q}|Y_\lambda^{(n)}(s)|^{2(q-2)}|Y_\lambda^{(n)}(s)|^2_{L_\varrho^q}(1+|Y_\lambda^{(n)}(s)|_{L_\varrho^q})^2\,ds.$$

Consequently, there is a constant c_4 independent of n and λ such that for $t\in[0,T]$ we have

$$\mathbb{E}\sup_{s\in[0,t]}|Y_\lambda^{(n)}(s)|^{2p}_{L_\varrho^q}\le c_4(|\zeta|^{2p}_{L_\varrho^q}+1)+c_4\mathbb{E}\int_0^t|Y_\lambda^{(n)}(s)|^{2p}_{L_\varrho^q}\,ds.$$

Applying Gronwall's inequality we obtain

$$\mathbb{E}\sup_{s\in[0,T]}|Y_\lambda^{(n)}(s)|^{2p}_{L_\varrho^q}\le c_4(|\zeta|^{2p}_{L_\varrho^q}+1)e^{c_4T}.$$

Since, by Proposition B.1 from Appendix B,

$$\mathbb{E}\sup_{s\in[0,T]}|Y^{(n)}(s)|^{2p}_{L_\varrho^q}\le\limsup_{\lambda\to\infty}\mathbb{E}\sup_{s\in[0,T]}|Y_\lambda^{(n)}(s)|^{2p}_{L_\varrho^q},$$

we have (6.36), which is the desired conclusion. ∎

7. Stochastic integration in Banach spaces. In this section $(H,\langle\cdot,\cdot\rangle_H)$ and $(E,|\cdot|_E)$ are real separable Hilbert and Banach spaces, respectively, and $\{e_k\}$ is a fixed orthonormal basis of H. By Π_l and $\Pi_{n,n+l}$ we denote the orthogonal projections onto the spaces spanned by $\{e_1,\dots,e_l\}$ and $\{e_n,\dots,e_{n+l}\}$. Let $\mathcal{B}_0(H)$ denote the class of all subsets U of H having the form

$$U=\{v\in H:(\langle v,h_1\rangle,\dots,\langle v,h_n\rangle)\in U_0\}\tag{7.40}$$

for a certain n, an orthonormal system $h_1,\dots,h_n$ in H, and $U_0\in\mathcal{B}(\mathbb{R}^n)$. Denote by γ the standard Gaussian cylindrical distribution on H, that is, $\gamma(U)=(2\pi)^{-n/2}\int_{U_0}\exp\{-|x|^2/2\}\,dx$ for U given by (7.40). If $\dim H=\infty$, then γ is only finitely additive.

DEFINITION 7.1. A bounded linear operator $K:H\to E$ is called *γ-radonifying* iff the image $\gamma\circ K^{-1}$ of γ under K has an extension to a σ-additive measure γ_K on E.

The set of all γ-radonifying operators from H into E is denoted by $R(H,E)$. Note that if $K\in R(H,E)$, then γ_K is a centered Gaussian Borel measure on E. Thus by the Fernique–Landau–Shepp theorem $\int_E|e|_E^2\,\gamma_K(de)<\infty$. We endow $R(H,E)$ with the norm

$$\|K\|_{R(H,E)}:=\Big(\int_E|e|_E^2\,d\gamma_K(e)\Big)^{1/2}.$$

The following result was proven partly by Neidhardt [28] and partly by Baxendale [2].

THEOREM 7.1. (i) *$R(H,E)$ with the norm $\|\cdot\|_{R(H,E)}$ is a separable Banach space.*

(ii) *For any* $K \in R(H,E)$, *we have* $\|K\Pi_n\|_{R(H,E)} \le \|K\|_{R(H,E)}$ *and* $\|K\Pi_n - K\|_{R(H,E)} \to 0$ *as* $n \to \infty$.

We need the following result which is well known to the specialists on Gaussian measures. It can be seen as a special case of a result due to Linde–Pietsch (see Lemma 5.6, Chapter VI of the monograph [40]).

PROPOSITION 7.1. *Let* $\{\beta_k\}_{k=1}^\infty$ *be a system of independent standard normal real-valued random variables, and let* $K \in L(H,E)$. *Then the following two conditions are equivalent.*

(i) $K \in R(H,E)$.

(ii) *The series* $\sum_{k=1}^\infty \beta_k K e_k$ *converges in* $L^2(\Omega,\mathcal{F},\mathbb{P};E)$.

Moreover, if $K \in R(H,E)$ *then* $\|K\|_{R(H,E)} = (\mathbb{E}|\sum_{k=1}^\infty \beta_k K e_k|_E^2)^{1/2}$.

Let W be a cylindrical Wiener process on a Hilbert space H, defined on a filtered probability space $\mathfrak{U} = (\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$. Let V be a Banach space, and denote by $\mathcal{L}^p(0,\infty;V)$ the space of all measurable, adapted V-valued processes σ such that $\mathbb{E}\int_0^T |\sigma(t)|_V^p\,dt < \infty$ for $T < \infty$. Clearly, $\mathcal{L}^p(0,\infty;V)$ is a Fréchet space with a natural system of seminorms. Denote by $\mathcal{L}_0$ the class of all $\sigma \in \mathcal{L}^2(0,\infty;R(H,E))$ such that

$$\sigma(\omega,t) = \sum_{j=1}^n \sigma_j(\omega)\Pi_i \chi_{(t_j,t_{j+1}]}(t)$$

for some $n, i \in \mathbb{N}$, $0 \le t_1 < \ldots < t_{n+1} < \infty$ and $\sigma_j \in L^2(\Omega,\mathcal{F}_{t_j},\mathbb{P};R(H,E))$. For $\sigma \in \mathcal{L}_0$ and $t \in [0,\infty)$ we put

$$\mathcal{I}_t^W(\sigma) = \sum_{j=1}^n \sum_{k=1}^i (W(t_{j+1}\wedge t) - W(t_j \wedge t))e_k \sigma_j e_k.$$

So far we have defined the stochastic integral $\mathcal{I}^W(\sigma)$ for processes $\sigma \in \mathcal{L}_0$. In general $\mathcal{I}^W$ cannot be extended continuously to the whole $\mathcal{L}^2(0,\infty;R(H,E))$. This requires some additional assumptions on the Banach space E.

DEFINITION 7.3. A Banach space E is of *M-type* 2, or 2-*smoothable*, iff there is a constant $C > 0$ such that for any finite E-valued martingale $\{M_k\}$ one has

$$\sup_k \mathbb{E}|M_k|_E^2 \le C \sum_k \mathbb{E}|M_k - M_{k-1}|_E^2.$$

For the proof of the theorem below we refer the reader to [28], or [5], Theorem 2.3.

THEOREM 7.2. *Assume that* E *is an M-type* 2 *Banach space. Then* $\mathcal{L}_0$ *is a dense subspace of* $\mathcal{L}^2(0,\infty;R(H,E))$, *and for every* $t \in [0,\infty)$ *there exists a unique extension of* $\mathcal{I}_t^W$ *to a linear bounded operator acting from* $\mathcal{L}^2(0,\infty;R(H,E))$ *into* $L^2(\Omega,\mathcal{F}_t,\mathbb{P};E)$. *Moreover, there exists* $C > 0$ *such that for any* $T > 0$ *and* $\sigma \in \mathcal{L}^2(0,\infty;R(H,E))$ *one has*

$$\mathbb{E}|\mathcal{I}_T^W(\sigma)|_E^2 \le C\mathbb{E}\int_0^T \|\sigma(t)\|_{R(H,E)}^2\,dt.$$

We denote the value of the extension of $\mathcal{I}_t^W$ at $\sigma \in \mathcal{L}^2(0,\infty;R(H,E))$ by $\int_0^t \sigma(s)\,dW(s)$. The above theorem constitutes the first step in the theory of Banach space valued stochastic Itô integrals. The next one is the Burkholder type inequality. Pisier [36] stated that the Burkholder inequality is valid in any 2-smoothable Banach space. This was later proven by Dettweiler in [15], Theorems 2.4 and 3.3. An independent proof of a weaker result, i.e. that the Burkholder inequality is valid for any UMD type 2 Banach space, is given in [5], Theorem 2.4.

THEOREM 7.3. *Assume that* E *is an M-type* 2 *Banach space. Then for every* $\sigma \in \mathcal{L}^2(0,\infty;R(H,E))$, $\int_0^t \sigma(s)\,dW(s)$, $t \ge 0$, *is an E-valued square integrable martingale with continuous modification and zero mean. Moreover, for every* $p \in [2,\infty)$ *there is a constant* C *independent of* T *and* σ *such that*

$$\mathbb{E}\sup_{0\le t\le T}\Big|\int_0^t \sigma(s)\,dW(s)\Big|_E^p \le C\mathbb{E}\Big(\int_0^T \|\sigma(s)\|_{R(H,E)}^2\,ds\Big)^{p/2}.$$

In the present paper we integrate $R(\mathcal{H}_\mu,E)$-valued processes, where $\mathcal{H}_\mu$ is the reproducing kernel Hilbert space of a homogeneous Wiener process, and E is an L_ϱ^q-space. To do this we need the following fact (see [36]).

PROPOSITION 7.2. *Let* $(\mathcal{O},\mathcal{B},\theta)$ *be a measurable space, and let* $q \in [2,\infty)$. *Then* $L^q(\mathcal{O},\mathcal{B},\theta)$ *is an M-type* 2 *Banach space.*

Appendix A. Our aim here is to state Itô's lemma. We do this first for an Itô process with values in an abstract M-type 2 Banach space E and for a Fréchet differentiable mapping $\Psi : [0,T]\times E \to \mathbb{R}$. For the convenience of the reader we repeat the relevant material from [28] without proof, thus making our exposition self-contained. Then we derive Itô's lemma for an L_ϱ^q-valued process, and for $\Psi = |\cdot|_{L_\varrho^q}^p$, where $p \ge q$.

We need to introduce some notation. Let $(H,\langle\cdot,\cdot\rangle_H)$ and $(E,|\cdot|_E)$ be real separable Hilbert and Banach spaces, and let γ be a standard Gaussian distribution on H (see Section 7). Let $K \in R(H,E)$ and let $\gamma_K = \gamma \circ K^{-1}$. For any Banach space V and any bounded bilinear map $L : E\times E \to V$ we define

$$\mathrm{tr}_K L = \int_E L(x,x)\,d\gamma_K(x).$$

Denote by $L_2(E,V)$ the space of all bounded bilinear operators acting from E into V. Then by the Fernique–Landau–Shepp theorem tr_K is a bounded linear operator from $L_2(E,V)$ into V. Moreover,

$$|\mathrm{tr}_K L|_V \le \|L\|_{L_2(E,V)}\|K\|^2_{R(H,E)} \quad \text{for all } K\in R(H,E),\ L\in L_2(E,V). \tag{A.1}$$

Let W be a cylindrical Wiener process on H. Let us denote by $C^2_{\mathrm{b}}(E)$ the class of all twice Fréchet differentiable functions $\Psi : E\to\mathbb{R}$ with bounded derivatives. The following theorem is proven in [28].

THEOREM A.1. *Assume that E is an M-type 2 Banach space. Let $\Psi\in C^2_{\mathrm{b}}(E)$. Let $a\in\mathcal{L}^1(0,\infty;E)$ and $\sigma\in\mathcal{L}^2(0,\infty;R(H,E))$. Let*

$$\xi(t)=\xi(0)+\int_0^t a(s)\,ds+\int_0^t\sigma(s)\,dW(s),\quad t\ge 0.$$

Then for all $t\ge 0$,

$$\Psi(\xi(t))=\Psi(\xi(0))+\int_0^t\Psi'(\xi(s))a(s)\,ds+\int_0^t\Psi'(\xi(s))\sigma(s)\,dW(s)$$
$$+\frac12\int_0^t\mathrm{tr}_{\sigma(s)}\Psi''(\xi(s))\,ds.$$

Let $\varrho\in\mathbb{R}$ and $q\in L^q_\varrho$. Let $u\in L^q_\varrho$. Note that $|u|^{q-2}u\in L^{q^*}_\varrho=(L^q_\varrho)^*$. Below, $\langle\cdot,\cdot\rangle_\varrho$ denotes the duality form on $(L^q_\varrho)^*\times L^q_\varrho$.

THEOREM A.2. *Let $\varrho\in\mathbb{R}$, $q\in[2,\infty)$, and $p\ge q$. Assume that*

$$\xi(t)=\xi(0)+\int_0^t a(s)\,ds+\int_0^t\sigma(s)\,dW(s),\quad t\in[0,T],$$

with $a\in\mathcal{L}^p(0,\infty;L^q_\varrho)$ and $\sigma\in\mathcal{L}^p(0,\infty;R(H,L^q_\varrho))$. Then for all $t\ge0$,

$$|\xi(t)|^p_{L^q_\varrho}=|\xi(0)|^p_{L^q_\varrho}+p\int_0^t|\xi(s)|^{p-q}_{L^q_\varrho}\langle|\xi(s)|^{q-2}\xi(s),a(s)\rangle_\varrho\,ds$$
$$+p\int_0^t|\xi(s)|^{p-q}_{L^q_\varrho}\langle|\xi(s)|^{q-2}\xi(s),\sigma(s)\,dW(s)\rangle_\varrho$$
$$+\frac12\int_0^t\mathrm{tr}_{\sigma(s)}\Psi''(\xi(s))\,ds.$$

Proof. Note that the function $\Psi:L^q_\varrho\ni f\mapsto|f|^p_{L^q_\varrho}\in\mathbb{R}$ is of class $C^2(E)$ and for $v,v_1,v_2\in L^q_\varrho$ we have $\Psi'(u)v=p|u|^{p-q}_{L^q_\varrho}\langle|u|^{q-2}u,v\rangle_\varrho$ and

$$\begin{aligned}\Psi''(u)(v_1,v_2)&=p(q-1)|u|^{p-q}_{L^q_\varrho}\int_{\mathbb{R}^d}|u(x)|^{q-2}v_1(x)v_2(x)\vartheta_\varrho(x)\,dx\\&\quad+p(p-q)|u|^{p-2q}_{L^q_\varrho}\int_{\mathbb{R}^d}|u(x)|^{q-2}u(x)v_1(x)\vartheta_\varrho(x)\,dx\\&\quad\times\int_{\mathbb{R}^d}|u(x)|^{q-2}u(x)v_2(x)\vartheta_\varrho(x)\,dx.\end{aligned}\tag{A.2}$$

Let $\eta\in C^\infty_0(\mathbb{R})$ with $\eta(x)=1$ for $x\in[0,1]$. Set $\Psi_n(u)=\eta(n^{-1}\Psi(u))\Psi(u)$ for $n\in\mathbb{N}$, $u\in L^q_\varrho$. Applying to Ψ_n Itô's formula from Theorem A.1 and then letting $n\to\infty$, we obtain the desired conclusion. ∎

REMARK A.1. Let $\Psi(u)=|u|^p_{L^q_\varrho}$. Then, combining (A.1) with (A.2), we obtain

$$\mathrm{tr}_\sigma\Psi''(u)\le p(p-1)|u|^{p-2}_{L^q_\varrho}\|\sigma\|^2_{R(H,L^q_\varrho)}.\tag{A.3}$$

Appendix B. Assume that H is a separable Hilbert space, and E is of M-type 2. Suppose that A is the generator of a C_0 contraction semigroup $\{S(t)\}_{t\ge0}$ in E. Let

$$A_\lambda:=\lambda A(\lambda\mathrm{I}-A)^{-1}=\lambda^2(\lambda\mathrm{I}-A)^{-1}-\lambda\mathrm{I},\quad\lambda>0,$$

be the Yosida approximation of A. Recall (see e.g. [30]) that A_λ, $\lambda>0$, are bounded generators of C_0 contraction semigroups $\{S_\lambda(t)\}_{t\ge0}$ in E. Furthermore, $S_\lambda(t)x\to S(t)x$ as $\lambda\to\infty$ for all $x\in E$, uniformly in t on each bounded interval. In what follows we adopt the convention that $A_\infty=A$ and $S_\infty=S$.

Let $\mathfrak{A}=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$ be a filtered probability space, and let W be an $(\mathcal{F}_t)$-adapted cylindrical Wiener process on H. Let V be a Banach space. Denote by $Z_T(V)$ the space of all predictable processes u such that

$$\|u\|_{T,V}:=\Big(\mathbb{E}\int_0^T|u(t)|^p_V\,dt\Big)^{1/p}<\infty.$$

Let $F:[0,\infty)\times E\to E$ and $B:[0,\infty)\times E\to R(H,E)$. Assume that for each $T>0$ there is a constant L such that for all $u,v\in E$ and $t\in[0,T]$,

$$|F(t,u)|_E+\|B(t,u)\|_{R(H,E)}\le L(1+|u|_E),$$
$$|F(t,u)-F(t,v)|_E+\|B(t,u)-B(t,v)\|_{R(H,E)}\le L|u-v|_E.$$

The existence and uniqueness part of the result below is quite standard. However, its proof is essential for the second part of the theorem.

THEOREM B.1. *Let $p\in(2,\infty)$. Then for all $\zeta\in L^p(\Omega,\mathcal{F}_0,\mathbb{P};E)$ and $\lambda\in(0,\infty]$, there is a unique mild solution $Y_\lambda\in\mathcal{L}^p(0,\infty;E)$ to the problem*

$$dY_\lambda=(A_\lambda Y_\lambda+F(t,Y_\lambda))\,dt+B(t,Y_\lambda)\,dW,\quad Y_\lambda(0)=\zeta.\tag{B.1}$$

Moreover,

$$\lim_{\lambda\to\infty} \|Y_\lambda - Y_\infty\|_{T,E} = 0 \quad \textit{for every } T > 0. \tag{B.2}$$

Proof. It follows from Young's inequality that for $u \in Z_T(E)$ the process $I_\lambda u$ defined by

$$I_\lambda u(t) = \int_0^t S_\lambda(t-s)u(s)\,ds, \quad t \in [0,T],$$

belongs to $Z_T(E)$ and

$$\|I_\lambda u\|_{T,E} \le T\|u\|_{T,E}. \tag{B.3}$$

Furthermore, employing the Young and Burkholder inequalities (for the latter see Theorem 6.3), we infer that for $u \in Z_T(R(H,E))$ the process $J_\lambda u$ defined by

$$J_\lambda u(t) = \int_0^t S_\lambda(t-s)u(s)\,dW(s), \quad t \in [0,T],$$

belongs to $Z_T(E)$, and there is a constant c_1 independent of u and λ such that

$$\|J_\lambda u\|_{T,E} \le c_1 T^{1/2}\|u\|_{T,R(H,E)}. \tag{B.4}$$

Note that

$$F(\cdot,u) \in Z_T(E) \quad \text{and} \quad B(\cdot,u) \in Z_T(E) \quad \text{for } u \in Z_T. \tag{B.5}$$

For $u \in Z_T(E)$ we define $v = \mathcal{J}_{T,\lambda}(u)$ by

$$\begin{aligned} v(t) = S_\lambda(t)\zeta &+ \int_0^t S_\lambda(t-s)F(s,u(s))\,ds \\ &+ \int_0^t S_\lambda(t-s)B(s,u(s))\,dW(s). \end{aligned} \tag{B.6}$$

From (B.3) to (B.5), $\mathcal{J}_{T,\lambda}$ is a Lipschitz mapping from $Z_T(E)$ into $Z_T(E)$ with Lipschitz constant $\kappa(T)$ independent of λ. Furthermore, it is easy to see that for T small enough, $\kappa(T) < 1$. This proves the existence and uniqueness of a mild solution to (B.1) on a small time interval. Since the length of this interval does not depend on the initial value, the solution can be prolonged uniquely to the whole half line $[0,\infty)$.

Observe that for all $T > 0$ and $u \in Z_T$,

$$\lim_{\lambda\to\infty} \|I_\lambda u - I_\infty u\|_{T,E} = 0 \tag{B.7}$$

and

$$\lim_{\lambda\to\infty} \|J_\lambda u - J_\infty u\|_{T,E} = 0. \tag{B.8}$$

This is because by Young's and Fubini's theorems

$$\|I_\lambda u - I_\infty u\|_{T,E}^p \le T^{p-1}\int_0^T\int_0^t \mathbb{E}|((S_\lambda(t-s) - S(t-s))u(s)|_E^p\,ds\,dt.$$

Since

$$|(S_\lambda(t-s) - S(t-s))u(s)|_E^p \to 0 \quad \text{a.s. on } \{0 \le s \le t \le T\},$$

a simple application of Lebesgue's dominated convergence theorem yields (B.7). By Burkholder's inequality and Fubini's theorem there is a constant c_2 such that

$$\|J_\lambda u - J_\infty u\|_{T,E}^p \le c_2 T^{p/2-1}\int_0^T\int_0^t \mathbb{E}|(S_\lambda(t-s) - S(t-s))u(s)|_E^p\,ds\,dt.$$

Thus (B.8) follows, and (B.3) is a consequence of the continuous dependence of fixed points on the parameter λ (see [20] for more details). ∎

PROPOSITION B.1. *In the framework of Theorem* B.1 *we have*

$$\mathbb{E}|Y_\infty|_{L^\infty(0,T;E)}^p \le \sup_\lambda \mathbb{E}|Y_\lambda|_{L^\infty(0,T;E)}^p. \tag{B.9}$$

Proof. Take $q > 1$. Applying Theorem B.1 for $\widetilde{p} = pq$, we obtain

$$\mathbb{E}|Y_\infty|_{L^{pq}(0,T;E)}^p \le \sup_\lambda \mathbb{E}|Y_\lambda|_{L^{pq}(0,T;E)}^p \le T^{1/q}\sup_\lambda \mathbb{E}|Y_\lambda|_{L^\infty(0,T;E)}^p. \tag{B.10}$$

Hence, since the L^∞-norm is the limit of L^q-norms, Fatou's lemma and (B.10) give

$$\begin{aligned} \mathbb{E}|Y_\infty|_{L^\infty(0,T;E)}^p &\le \liminf_{q\to\infty} \mathbb{E}|Y_0|_{L^{pq}(0,T;E)}^p \le \liminf_{q\to\infty} T^{1/q}\sup_\lambda \mathbb{E}|Y_\lambda|_{L^{pq}(0,T;E)}^p \\ &\le \sup_\lambda \mathbb{E}|Y_\lambda|_{L^\infty(0,T;E)}^p, \end{aligned}$$

which is the desired conclusion. ∎

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