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GRADIENT METHOD FOR NON-INJECTIVE OPERATORS IN HILBERT SPACE WITH APPLICATION TO NEUMANN PROBLEMS

Abstract. The gradient method is developed for non-injective non-linear operators in Hilbert space that satisfy a translation invariance condition. The focus is on a class of non-differentiable operators. Linear convergence in norm is obtained. The method can be applied to quasilinear elliptic boundary value problems with Neumann boundary conditions.

1. Introduction. The abstract version of the gradient method has undergone extensive development since it was applied by Kantorovich in Hilbert space to linear equations via minimizing the quadratic functional ([6], [7]). Modifications, including the conjugate gradient method, were soon developed for linear equations and extended to non-linear equations with uniformly positive derivatives (see e.g. [1], [2], [7] and the references there). The gradient method also extends to non-differentiable operators ([8]).

The abstract results give rise to numerical methods for quasilinear elliptic boundary value problems. Namely, the gradient method reduces the quasilinear Dirichlet problem to linear Poisson equations, which can be solved by any suitable well-known method. The use of finite difference method or finite element method to these auxiliary equations has been elaborated in [5] and [4], respectively.

We quote the following theorem which yields the optimal linear convergence of the abstract gradient method:

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Theorem 1 (cf. [8], [9]). *Let H be a real Hilbert space and $A : H \to H$ have the following properties*:

(i) *A is Gateaux differentiable*;

(ii) *for any $u, k, w, h \in H$ the mapping $s, t \mapsto A'(u + sk + tw)h$ is continuous from $\mathbb{R}^2$ to H*;

(iii) *for any $u \in H$ the operator $A'(u)$ is self-adjoint*;

(iv) *there are constants $M \geq m > 0$ such that for all $u, h \in H$,*

$$m\|h\|^2 \leq \langle A'(u)h, h\rangle \leq M\|h\|^2.$$

Then for any $b \in H$ the equation

$$A(u) = b$$

has a unique solution $u^* \in H$ and for any $u_0 \in H$ the sequence

$$u_{n+1} := u_n - \frac{2}{M+m}(A(u_n) - b) \qquad (n \in \mathbb{N})$$

converges to u^* according to the linear estimate

$$\|u_n - u^*\| \leq \frac{1}{m}\|A(u_0) - b\|\left(\frac{M-m}{M+m}\right)^n \qquad (n \in \mathbb{N}).$$

All the versions of the gradient method assume that the non-linear operator studied is one-to-one; moreover, uniform ellipticity is usually assumed (as is in the quoted theorem) for the derivative of the operator or its transform in order to achieve linear convergence to the unique solution.

The aim of this paper is to extend the gradient method to a class of operators that are not one-to-one. We investigate the case when the loss of injectivity is caused by a kind of translation invariance of the non-linear operator, and uniform ellipticity is preserved on the orthocomplement of the corresponding subspace. Using the resulting factor space, first the straightforward extension of the gradient method is given when the operator is Gateaux differentiable. Then the method is extended to certain non-differentiable operators by means of a suitable energy space. Linear convergence is proved to an appropriately determined element of the set of solutions.

It is shown that the method obtained in Hilbert space can be applied to quasilinear elliptic boundary value problems with Neumann boundary conditions. The quasilinear Neumann problem is reduced to auxiliary linear Poisson problems with Neumann boundary conditions. (Similarly to the quoted results on the Dirichlet problem, the solution of the auxiliary equations might be achieved by any suitably chosen well-known method. The numerical study of such a coupling of methods is out of the scope and length of this paper.)

2. The gradient method for translation invariant operators. The following notion will be fundamental in this paper:

Definition. Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{H}_0 \subset \mathcal{H}$ a closed subspace. A non-linear operator A in $\mathcal{H}$ is called *translation invariant with respect to* $\mathcal{H}_0$ if for any $u \in D(A)$ and $h \in \mathcal{H}_0$ we have $u + h \in D(A)$ and $A(u + h) = A(u)$.

Remark 2.1. If A is Gateaux differentiable and translation invariant with respect to $\mathcal{H}_0$ then for any $u \in D(A)$ we have $\mathcal{H}_0 \subset \ker A'(u)$.

First the straightforward construction of the gradient method for Gateaux differentiable translation invariant operators is given. We consider the equation

$$A(u) = b \tag{2.1}$$

in $\mathcal{H}$.

Theorem 2.1. *Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Let $A : \mathcal{H} \to \mathcal{H}$ be a non-linear operator which is translation invariant with respect to some $\mathcal{H}_0 \subset \mathcal{H}$. Assume that A has the following properties*:

(i) *A is Gateaux differentiable*;
(ii) *for any $u, k, w, h \in \mathcal{H}$ the mapping $s, t \mapsto A'(u + sk + tw)h$ is continuous from $\mathbb{R}^2$ to $\mathcal{H}$*;
(iii) *for any $u \in \mathcal{H}$ the operator $A'(u)$ is self-adjoint*;
(iv) *there are constants $M \ge m > 0$ such that for all $u \in \mathcal{H}$ and $v \in \mathcal{H}_0^\perp$,*

$$m\|v\|^2 \le \langle A'(u)v, v\rangle \le M\|v\|^2.$$

(*In contrast*, *by Remark* 2.1, *we have $\langle A'(u)v, v\rangle = 0$ for $v \in \mathcal{H}_0$.*) *Then the following assertions hold.*

(1) $R(A) = A(0) + \mathcal{H}_0^\perp$.
(2) *For any $b \in R(A)$ there exists a unique $u^* \in \mathcal{H}_0^\perp$ such that for any $h \in \mathcal{H}_0$,*

$$A(u^* + h) = b \tag{2.2}$$

and thus all solutions of (2.1) *are obtained.*
(3) *For any $u_0 \in \mathcal{H}_0^\perp$ the sequence*

$$u_{n+1} = u_n - \frac{2}{M+m}(A(u_n) - b) \qquad (n \in \mathbb{N}) \tag{2.3}$$

converges to u^* according to the linear estimate

$$\|u_n - u^*\| \le \frac{1}{m}\|A(u_0) - b\| \left(\frac{M-m}{M+m}\right)^n \qquad (n \in \mathbb{N}). \tag{2.4}$$

P r o o f. Let $\mathcal{F} = \mathcal{H}/\mathcal{H}_0$. For any $v \in \mathcal{H}$ denote by $[v] = v + \mathcal{H}_0$ the equivalence class of v. For any $U \in \mathcal{F}$ denote by $U_\perp$ the unique vector $u \in U \cap \mathcal{H}_0^\perp$. Then the scalar product on $\mathcal{F}$ is defined by

$$\langle U, V\rangle_{\mathcal{F}} \equiv \langle U_\perp, V_\perp\rangle \quad (U, V \in \mathcal{F}).$$

We introduce the operator $\mathcal{A} : \mathcal{F} \to \mathcal{F}$,

$$\mathcal{A}(U) = [A(U_\perp)].$$

It is clear that $\mathcal{A}$ inherits properties (i)–(iii) of A and the analogue of (iv) holds on the whole space $\mathcal{F}$, which means that $\mathcal{A}$ satisfies the conditions of Theorem 1 on $\mathcal{F}$. This enables us to prove assertions (1)–(3) of our theorem.

(1) For any $u \in \mathcal{H}$, $h \in \mathcal{H}_0$ we have $\langle A(u) - A(0), h\rangle = \langle A'(\theta u)u, h\rangle = \langle u, A'(\theta u)h\rangle = 0$ (from $\mathcal{H}_0 \subset \ker A'(\theta u)$), i.e. $A(u) - A(0) \in \mathcal{H}_0^\perp$. Thus

$$R(A) \subset A(0) + \mathcal{H}_0^\perp.$$

Now let $b \in A(0) + \mathcal{H}_0^\perp$. Theorem 1 on $\mathcal{A}$ yields that $R(\mathcal{A}) = \mathcal{F}$, hence there exists $U \in \mathcal{F}$ such that $\mathcal{A}(U) = [b]$, i.e. $A(u) = b + h$ with suitable $u \in \mathcal{H}$, $h \in \mathcal{H}_0$. Here both $A(u)$ and b belong to $A(0) + \mathcal{H}_0^\perp$, hence $h \in \mathcal{H}_0^\perp$, i.e. $h = 0$. Thus $b \in R(A)$, i.e.

$$A(0) + \mathcal{H}_0^\perp \subset R(A).$$

(2) Let $b \in R(A)$. By Theorem 1(1) on $\mathcal{A}$ the solution $U^* \in \mathcal{F}$ of the equation $\mathcal{A}(U) = [b]$ is unique. Let $u^* = (U^*)_\perp$. Then the following assertions are equivalent: $A(u) = b \Leftrightarrow \mathcal{A}([u]) = [b] \Leftrightarrow [u] = [u^*] \Leftrightarrow u = u^* + h$ with suitable $h \in \mathcal{H}_0$.

(3) Let $b \in R(A)$, $B = [b]$, $u_0 \in \mathcal{H}_0^\perp$, $U_0 = [u_0]$,

$$U_{n+1} = U_n - \frac{2}{M+m}(\mathcal{A}(U_n) - B) \quad (n \in \mathbb{N}). \tag{2.5}$$

By Theorem 1 we have

$$\|U_n - U^*\|_{\mathcal{F}} \le \frac{1}{m}\|\mathcal{A}(U_0) - B\|_{\mathcal{F}} \left(\frac{M-m}{M+m}\right)^n \quad (n \in \mathbb{N}). \tag{2.6}$$

It is easy to check by induction that the sequence (2.3) satisfies

$$u_n = (U_n)_\perp.$$

Hence $\|U_n - U^*\|_{\mathcal{F}} = \|u_n - u^*\|$ and $\|\mathcal{A}(U_0) - B\|_{\mathcal{F}} = \|A(u_0) - b\|$, i.e. (2.4) and (2.6) coincide. ∎

(The proof might also have been done by transforming A to be invariant on $\mathcal{H}_0^\perp$, which is clearly the same idea in a different setting.)

Remark 2.2. If $A(0) = 0$ then $A(u) = b$ has a solution if and only if $\langle b, v\rangle = 0$ $(v \in \mathcal{H}_0)$.

Now the gradient method obtained is extended to a class of non-differentiable operators. This requires a notion of energy space for semidefinite operators.

DEFINITION. Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, $D \subset \mathcal{H}$ a dense subspace, and $B : D \to \mathcal{H}$ a symmetric, positive-semidefinite linear operator whose kernel is closed in $\mathcal{H}$. For any $u \in \mathcal{H}$ we use the notation

$$u = u_0 + u_\perp \quad (u_0 \in \ker B,\ u_\perp \in (\ker B)^\perp).$$

Then the *energy space* of B is the completion of D under the scalar product

$$\langle u, v \rangle_B \equiv \langle Bu, v \rangle + \langle u_0, v_0 \rangle \quad (u, v \in D) \tag{2.7}$$

and is denoted by $\mathcal{H}_B$.

REMARK 2.3. Let $u \in D$. Then $u \perp \ker B$ in $\mathcal{H}_B$ if and only if $u \perp \ker B$ in $\mathcal{H}$. Hence $D \cap (\ker B)^\perp$ is the same in $\mathcal{H}_B$ as in $\mathcal{H}$.

Proof. For all $u \in D$, $v \in \ker B$ we have $\langle u_\perp, v \rangle = 0$ and $v = v_0$. Hence $\langle u, v \rangle_B = \langle u, Bv \rangle + \langle u_0, v \rangle = \langle u, v \rangle$. ■

THEOREM 2.2. *Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, $D \subset \mathcal{H}$ a dense subspace, $T : D \to \mathcal{H}$ a non-linear operator which is translation invariant with respect to some $\mathcal{H}_0 \subset \mathcal{H}$ and satisfies $T(0) = 0$. Let $B : D \to \mathcal{H}$ be a symmetric, positive-semidefinite linear operator with the following properties*:

(i) $R(B) \supset R(T)$;

(ii) $\ker B = \mathcal{H}_0$;

(iii) *there exists a constant $p > 0$ such that $\langle Bu, u \rangle \ge p\|u\|^2$ $(u \in D \cap \mathcal{H}_0^\perp)$*;

(iv) *there exists a non-linear operator $A : \mathcal{H}_B \to \mathcal{H}_B$ that satisfies conditions* (i)–(iv) *of Theorem* 2.1 *on $\mathcal{H}_B$*;

(v) $BA|_D = T$.

Consider the equation

$$T(u) = g \tag{2.8}$$

in $\mathcal{H}$, where the right side $g \in \mathcal{H}$ satisfies

$$\langle g, v \rangle = 0 \quad (v \in \mathcal{H}_0). \tag{2.9}$$

Then the following assertions hold:

(1) *There exists a unique $u^* \in \mathcal{H}_B$ such that $u^* \perp \mathcal{H}_0$ in $\mathcal{H}_B$ and the set of generalized solutions of* (2.8) *is $\{u^* + h : h \in \mathcal{H}_0\}$, i.e. for any $h \in \mathcal{H}_0$,*

$$\langle A(u^* + h), v \rangle_B = \langle g, v \rangle \quad (v \in \mathcal{H}_B). \tag{2.10}$$

If $g \in R(T)$ then (2.10) *is equivalent to*

$$T(u^* + h) = g. \tag{2.11}$$

(*If condition* (2.9) *fails to hold then* (2.8) *has no generalized solution.*)

(2) *Let* $u_0 \in D \cap \mathcal{H}_0^\perp$ *and, for all* $n \in \mathbb{N}$,

$$u_{n+1} = u_n - \frac{2}{M+m} z_n \quad \text{where } Bz_n = T(u_n) - g, \; z_n \in D \cap \mathcal{H}_0^\perp. \tag{2.12}$$

Then the sequence (u_n) *is uniquely defined and converges to* u^* *according to the linear estimate*

$$\|u_n - u^*\|_B \le \frac{1}{mp^{1/2}} \|T(u_0) - g\| \left(\frac{M-m}{M+m} \right)^n \qquad (n \in \mathbb{N}). \tag{2.13}$$

The proof is preceded by auxiliary lemmas under the hypotheses of the theorem. All the time $D \cap \mathcal{H}_0^\perp$ is the same in $\mathcal{H}_B$ as in $\mathcal{H}$ (by Remark 2.3).

Lemma 2.1. *We have*

$$A|_D = (B|_{D \cap \mathcal{H}_0^\perp})^{-1} T. \tag{2.14}$$

Proof. $\ker B = \mathcal{H}_0$ implies that $B|_{D \cap \mathcal{H}_0^\perp}$ is injective and $R(B) = R(B|_{D \cap \mathcal{H}_0^\perp})$. Hence $(B|_{D \cap \mathcal{H}_0^\perp})^{-1}$ can be applied to $BA|_D = T$ (condition (v) of Theorem 2.2) to obtain (2.14). ∎

Lemma 2.2. $R(B) \subset \mathcal{H}_0^\perp$ *in* $\mathcal{H}$.

Proof. Let $u \in D$, $h \in \mathcal{H}_0$. Then $\langle Bu, h \rangle = \langle u, Bh \rangle = 0$.

Lemma 2.3. $R(A|_D) \subset D \cap \mathcal{H}_0^\perp$.

Proof. Assumption (v) of Theorem 2.2 includes $R(A|_D) \subset D(B) = D$. Further, $T(0) = 0$ implies $A(0) = 0$, thus assertion (1) of Theorem 2.1 yields that $R(A) = \mathcal{H}_0^\perp$. ∎

Lemma 2.4. *For any* $u, v \in D$,

$$\langle T(u), v \rangle = \langle A(u), v \rangle_B.$$

Proof. Let $u, v \in D$. Lemma 2.3 yields $A(u)_0 = 0$, thus $\langle A(u), v \rangle_B = \langle BA(u), v \rangle = \langle T(u), v \rangle$. ∎

Lemma 2.5. *For any* $w \in D \cap \mathcal{H}_0^\perp$,

$$\|w\|_B \le p^{-1/2} \|Bw\|. \tag{2.15}$$

Proof. Let $w \in D \cap \mathcal{H}_0^\perp$. Then

$$\|w\|_B^2 = \langle Bw, w \rangle \ge p \|w\|^2 \tag{2.16}$$

by assumption (iii), hence

$$\|w\|_B^2 = \langle Bw, w\rangle \le \|Bw\| \cdot \|w\| \le p^{-1/2}\|Bw\| \cdot \|w\|_B,$$

which implies (2.15).

Proof of Theorem 2.2. Let $b = (B|_{D\cap\mathcal{H}_0^\perp})^{-1}g$. Then $b \in D \cap \mathcal{H}_0^\perp$ implies

$$\langle b, v\rangle_B = \langle Bb, v\rangle + \langle b_0, v_0\rangle = \langle Bb, v\rangle = \langle g, v\rangle \quad (v \in D). \tag{2.17}$$

Hence $\langle A(u), v\rangle_B = \langle g, v\rangle$ $(v \in \mathcal{H}_B)$ is equivalent to $A(u) = b$ in $\mathcal{H}_B$ since D is dense in $\mathcal{H}_B$.

We are going to apply Theorem 2.1 to the equation

$$A(u) = b$$

in $\mathcal{H}_B$. This can be done since, due to Lemma 2.1, A inherits translation invariance from T and, by assumption (iv), A satisfies the other conditions of Theorem 2.1 in $\mathcal{H}_B$. The assertions of Theorem 2.2 will now be verified.

(1) Theorem 2.1 yields that $R(A) = \mathcal{H}_0^\perp$ in $\mathcal{H}_B$, thus $b \in R(A)$. Hence (2.2) holds, which is equivalent to (2.10).

Now let $g \in R(T)$. If $z \in D$ such that $T(z) = g$, $z = z_\perp + z_0$ $(z_\perp \in \mathcal{H}_0^\perp$, $z_0 \in \mathcal{H}_0)$ and $h \in \mathcal{H}_0$ is arbitrary, then by Lemma 2.4 and translation invariance we have

$$\langle g, v\rangle = \langle T(z), v\rangle = \langle A(z_\perp + z_0), v\rangle_B = \langle A(z_\perp + h), v\rangle_B$$

for all $v \in D$, hence for all $v \in \mathcal{H}_B$ by density. Thus the classical solution is a generalized solution; further, the uniqueness of u^* in $\mathcal{H}_0^\perp$ means that $z_\perp = u^*$ corresponding to g, i.e. all generalized solutions are classical solutions.

(Due to Remark 2.2, (2.17) and density of D in $\mathcal{H}_B$, a generalized solution exists if and only if $\langle b, v\rangle_B = \langle g, v\rangle = 0$ $(v \in D)$. Hence (2.9) is necessary for the existence.)

(2) Let $u_0 \in D \cap \mathcal{H}_0^\perp$ and, for all $n \in \mathbb{N}$,

$$u_{n+1} = u_n - \frac{2}{M+m}(A(u_n) - b) \quad (n \in \mathbb{N}). \tag{2.18}$$

Then, by induction, $u_n \in D\cap\mathcal{H}_0^\perp$ $(n \in \mathbb{N})$. Namely, if $u_n \in D\cap\mathcal{H}_0^\perp$ for some $n \in \mathbb{N}$ then $A(u_n) \in D \cap \mathcal{H}_0^\perp$ (Lemma 2.3) and $b \in D \cap \mathcal{H}_0^\perp$ by definition, hence $u_{n+1} \in D \cap \mathcal{H}_0^\perp$.

Consequently, Lemma 2.1 yields that for all $n \in \mathbb{N}$,

$$u_{n+1} = u_n - \frac{2}{M+m}(B|_{D\cap\mathcal{H}_0^\perp})^{-1}(T(u_n) - g),$$

i.e. (2.12) and (2.18) coincide.

Finally, Theorem 2.1 yields for (2.18) the estimate

$$\|u_n - u^*\|_B \le \frac{1}{m}\|A(u_0) - b\|_B \left(\frac{M-m}{M+m}\right)^n. \tag{2.19}$$

By Lemma 2.5, $\|A(u_0) - b\|_B \le p^{-1/2}\|BA(u_0) - Bb\| = p^{-1/2}\|T(u_0) - g\|$, hence (2.13) is proved. ■

Remark 2.4. Inequality (2.16) yields

$$\|u_n - u^*\| \le \frac{1}{mp}\|T(u_0) - g\| \left(\frac{M-m}{M+m}\right)^n,$$

i.e. the sequence (u_n) converges to u^* in the original norm of $\mathcal{H}$ as well.

Remark 2.5. If assumption (iii) of Theorem 2.2 is omitted, then the linear convergence obtained in (2.19) is preserved in the norm of $\mathcal{H}_B$, but (2.19) cannot be transformed to (2.13), and moreover, convergence in the original norm is lost.

3. Quasilinear elliptic problems with Neumann boundary conditions. The following boundary value problem is considered on a bounded domain $\Omega \subset \mathbb{R}^N$:

$$T(u) \equiv -\mathrm{div}(f(x, \nabla u)\nabla u) = g(x), \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} = 0, \tag{3.1}$$

satisfying the conditions

(C1) $\partial\Omega \in C^{2,\gamma}$ with some $0 < \gamma < 1$.

(C2) $f \in C^{1,\gamma}(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^+)$ and $g \in C^{0,\gamma}(\overline{\Omega})$ are real scalar-valued functions.

(C3) Let $\Phi : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ be defined by $\Phi(x,p) = f(x,p)p$. Then there exist constants $0 < m \le M$ such that for any $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$ the matrix $\frac{\partial \Phi}{\partial p}(x,p)$ is symmetric and has eigenvalues between m and M.

Remark 3.1. As a special case, condition (C3) is satisfied if there exists a function $b \in C^{1,\gamma}(\mathbb{R}^+)$ such that $f(x,p) = b(|p|^2)$, i.e. we have

$$T(u) \equiv -\mathrm{div}(b(|\nabla u|^2)\nabla u)$$

where $0 < m \le b(y) \le m'$ and $0 \le yb(y) \le \mu$ hold with suitable constants $m' \ge m > 0$ and $\mu > 0$. This type of operator arises e.g. in plasticity theory or in the study of magnetic state.

The domain of T is defined as

$$D(T) \equiv D = \left\{ u \in C^{2,\gamma}(\overline{\Omega}) : \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} = 0 \right\}. \tag{3.2}$$

We introduce the real Hilbert space $\mathcal{H} = L^2(\Omega)$ with scalar product $\langle u, v\rangle \equiv \int_\Omega uv\,dx$, and further, the real Sobolev space $H^1(\Omega)$ with scalar product

$$\langle u, v\rangle_{\mathrm{m}} \equiv \int_\Omega \nabla u \cdot \nabla v\,dx + \frac{1}{|\Omega|^2}\Big(\int_\Omega u\,dx\Big)\Big(\int_\Omega v\,dx\Big) \tag{3.3}$$

(where $|\Omega|$ stands for the Lebesgue measure of Ω), i.e. the mean values of u and v are used instead of the usual scalar product

$$\langle u, v\rangle_1 \equiv \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega uv\,dx.$$

Remark 3.2. The norms $\| \ \|_{\mathrm{m}}$ and $\| \ \|_1$ (corresponding to the above scalar products) are equivalent. Namely, for any $u \in H^1(\Omega)$ the Poincaré inequality ([10])

$$\int_\Omega u^2\,dx \le \beta_\Omega\Big[\int_\Omega |\nabla u|^2\,dx + \Big(\int_\Omega u\,dx\Big)^2\Big]$$

(with suitable constant $\beta_\Omega > 0$) implies

$$\int_\Omega u^2\,dx \le K_\Omega \|u\|_{\mathrm{m}}^2 \tag{3.4}$$

with $K_\Omega = \beta_\Omega \max\{1, |\Omega|^2\}$, hence

$$\|u\|_1^2 \le c_\Omega^2 \|u\|_{\mathrm{m}}^2 \tag{3.5}$$

with $c_\Omega = (1+K_\Omega)^{1/2}$, depending only on Ω. The converse is obvious from the Cauchy–Schwarz inequality.

A *weak solution* of (3.1) is defined in the usual way as a function $u^* \in H^1(\Omega)$ satisfying

$$\int_\Omega f(x, \nabla u^*)\nabla u^* \cdot \nabla v\,dx = \int_\Omega gv\,dx \qquad (v \in H^1(\Omega)). \tag{3.6}$$

Remark 3.3. Let $u \in D$. Then u is a weak solution if and only if it is a classical solution of (3.1). This follows from (3.1) by applying the divergence theorem: for any $v \in C^1(\overline{\Omega})$,

$$\int_\Omega T(u)v\,dx = \int_\Omega f(x, \nabla u)\nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} f(x, \nabla u)\frac{\partial u}{\partial \nu} v\,d\sigma.$$

We define

$$\mathcal{H}_0 = \{u \in L^2(\Omega) : u(x) \equiv \text{const on } \Omega\}. \tag{3.7}$$

Remark 3.4. (a) We have, both in $L^2(\Omega)$ and in $H^1(\Omega)$,

$$u \in \mathcal{H}_0^\perp \Leftrightarrow \int_\Omega u\,dx = 0.$$

(b) If $u \in H^1(\Omega)$ and $u \in \mathcal{H}_0^\perp$ then $\|u\|_{\mathrm{m}}^2 = \int_\Omega |\nabla u|^2\,dx$.

Lemma 3.1. *The formula*

$$\langle A(u), v\rangle_{\mathrm{m}} = \int_\Omega f(x, \nabla u)\nabla u \cdot \nabla v\,dx \qquad (v \in H^1(\Omega)) \tag{3.8}$$

defines an operator $A : H^1(\Omega) \to H^1(\Omega)$ *which satisfies conditions* (i)–(iv) *of Theorem* 2.1.

Proof. Assumption (C3) implies that for all $i, j = 1, \ldots, N$ and $(x, p) \in \overline{\Omega} \times \mathbb{R}^N$,

$$\left|\frac{\partial \Phi_i}{\partial p_j}(x, p)\right| \le M.$$

Hence Lagrange's inequality yields that for all $(x, p) \in \overline{\Omega} \times \mathbb{R}^N$ we have

$$|\Phi(x, p)| \le |\Phi(x, 0)| + MN^{1/2}|p|.$$

Now let $u, v \in H^1(\Omega)$. Then

$$\begin{aligned}\Big|\int_\Omega & f(x, \nabla u)\nabla u \cdot \nabla v\,dx\Big| \\ &= \Big|\int_\Omega \Phi(x, \nabla u)\nabla v\,dx\Big| \le \int_\Omega (|\Phi(x, 0)| + MN^{1/2}|\nabla u|)|\nabla v|\,dx \\ &\le (\|\Phi(x, 0)\|_\infty |\Omega|^{1/2} + MN^{1/2}\|\nabla u\|_{L^2(\Omega)})\|\nabla v\|_{L^2(\Omega)},\end{aligned}$$

hence for all fixed $u \in H^1(\Omega)$ Riesz's theorem yields the existence of $A(u) \in H^1(\Omega)$ such that (3.8) holds. Now we prove that the conditions (i)–(iv) of Theorem 2.1 are satisfied for A in $H^1(\Omega)$.

(i) For any $u \in H^1(\Omega)$ let $S(u) \in B(H^1(\Omega))$ be the bounded linear operator defined by

$$\langle S(u)h, v\rangle_{\mathrm{m}} \equiv \int_\Omega \frac{\partial \Phi}{\partial p}(x, \nabla u)\nabla h \cdot \nabla v\,dx \qquad (h, v \in H^1(\Omega)).$$

The existence of $S(u)$ is provided by Riesz's theorem, now using the estimate $M\|\nabla h\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$ for the right side integral. We will prove that

$$A'(u) = S(u) \qquad (u \in H^1(\Omega))$$

in the Gateaux sense. Let $u, h \in H^1(\Omega)$ and $\mathcal{E} := \{v \in H^1(\Omega) : \|v\|_{\mathrm{m}} = 1\}$. Then

$$\begin{aligned}
K_{u,h}(t) &\equiv \frac{1}{t}\|A(u+th)-A(u)-tS(u)h\|_{\mathrm{m}}\\
&= \frac{1}{t}\sup_{v\in\mathcal{E}}\langle A(u+th)-A(u)-tS(u)h,v\rangle_{\mathrm{m}}\\
&= \frac{1}{t}\sup_{v\in\mathcal{E}}\int_\Omega\left[\Phi(x,\nabla u+t\nabla h)-\Phi(x,\nabla u)-t\frac{\partial\Phi}{\partial p}(x,\nabla u)\nabla h\right]\cdot\nabla v\,dx\\
&= \sup_{v\in\mathcal{E}}\int_\Omega\left[\frac{\partial\Phi}{\partial p}(x,\nabla u+t\theta(x,t)\nabla h)-\frac{\partial\Phi}{\partial p}(x,\nabla u)\right]\nabla h\cdot\nabla v\,dx\\
&= \sup_{v\in\mathcal{E}}\bigg\|\bigg(\frac{\partial\Phi}{\partial p}(x,\nabla u+t\theta(x,t)\nabla h)\\
&\qquad -\frac{\partial\Phi}{\partial p}(x,\nabla u)\bigg)\nabla h\bigg\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.
\end{aligned}$$

Here $\|\nabla v\|_{L^2(\Omega)}\le\|v\|_{\mathrm{m}}\le 1$. Further, $|t\theta(x,t)\nabla h(x)|\le|t\nabla h(x)|\to 0$ (as $t\to 0$) almost everywhere on Ω, hence the continuity of $\frac{\partial\Phi}{\partial p}:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}^{N\times N}$ implies that the function in the first $L^2(\Omega)$-norm term converges a.e. to 0 as $t\to 0$. Since the integrand is majorized by $(2M|\nabla h|)^2$ (which belongs to $L^1(\Omega)$), Lebesgue's theorem yields that the resulting expression tends to 0 as $t\to 0$, thus

$$\lim_{t\to 0}K_{u,h}(t)=0.$$

(ii) We can prove similarly to (i) that for fixed functions $u,k,w,h\in H^1(\Omega)$ the mapping $s,t\mapsto A'(u+sk+tw)h$ is continuous from $\mathbb{R}^2$ to $H^1(\Omega)$. Namely,

$$\begin{aligned}
\omega_{u,k,w,h}(s,t) &:= \|A'(u+sk+tw)h-A'(u)h\|_{\mathrm{m}}\\
&= \sup_{v\in\mathcal{E}}\langle A'(u+sk+tw)h-A'(u)h,v\rangle_{\mathrm{m}}\\
&= \sup_{v\in\mathcal{E}}\int_\Omega\left[\frac{\partial\Phi}{\partial p}(x,\nabla u+s\nabla k+t\nabla w)-\frac{\partial\Phi}{\partial p}(x,\nabla u)\right]\nabla h\cdot\nabla v\,dx.
\end{aligned}$$

Using the continuity of the function $\frac{\partial\Phi}{\partial p}$ and Lebesgue's theorem, we conclude just as above that

$$\lim_{s,t\to 0}\omega_{u,k,w,h}(s,t)=0.$$

(iii) It follows immediately from the assumed symmetry of the matrix $\frac{\partial\Phi}{\partial p}$ that $A'(u)$ is self-adjoint.

(iv) For any $u,v\in H^1(\Omega)$,

$$\langle A'(u),v,v\rangle_{\mathrm{m}}=\int_\Omega\frac{\partial\Phi}{\partial p}(x,\nabla u)\nabla v\cdot\nabla v\,dx.$$

Hence from assumption (C3) we have, for any $u \in H^1(\Omega), v \in \mathcal{H}_0^\perp$,

$$m\|v\|_{\mathrm{m}}^2 = m\|\nabla v\|_{L^2(\Omega)}^2 \le \langle A'(u)v, v\rangle_{\mathrm{m}} \le M\|\nabla v\|_{L^2(\Omega)}^2 = M\|v\|_{\mathrm{m}}^2$$

(using Remark 3.4(b)). ∎

COROLLARY 3.1. $R(A) \subset \mathcal{H}_0^\perp$.

This follows from Theorem 2.1 since $A(0) = 0$.

Now we are in a position to apply the gradient method to problem (3.1).

THEOREM 3.1. *Let the conditions* (C1)–(C3) *be satisfied and assume that* $\int_\Omega g\,dx = 0$. *Then*:

(1) *Problem* (3.1) *has a unique weak solution* $u^* \in H^1(\Omega)$ *such that* $\int_\Omega u^*\,dx = 0$. *The set of solutions is* $\{u^* + c : c \in \mathbb{R}\}$. (*If assumption* $\int_\Omega g\,dx = 0$ *fails then there exists no weak solution.*)

(2) *Let* $u_0 \in D$, $\int_\Omega u_0\,dx = 0$. *For any* $n \in \mathbb{N}$ *let*

(3.9) $u_{n+1} = u_n - \frac{2}{M+m} z_n$, *where* $z_n \in C^{2,\gamma}(\overline{\Omega})$ *is the* (*unique*) *solution of equation* $-\Delta z_n = T(u_n) - g$, $\frac{\partial z_n}{\partial \nu}\big|_{\partial\Omega} = 0$, $\int_\Omega z_n\,dx = 0$.

Then (u_n) *converges to* u^* *according to the linear estimate*

$$\|\nabla u_n - \nabla u^*\|_{L^2(\Omega)} \le \frac{1}{mp^{1/2}}\|T(u_0) - g\|_{L^2(\Omega)}\left(\frac{M-m}{M+m}\right)^n \quad (n \in \mathbb{N}) \tag{3.10}$$

(*where* p *is the smallest positive eigenvalue of* $-\Delta$ *on* D).

REMARK 3.5. In the usual $H^1(\Omega)$ norm we have (from (3.5)) the estimate

$$\|u_n - u^*\|_1 \le \frac{c_\Omega}{mp^{1/2}}\|T(u_0) - g\|_{L^2(\Omega)}\left(\frac{M-m}{M+m}\right)^n \quad (n \in \mathbb{N}),$$

since for $u_n - u^* \in \mathcal{H}_0^\perp$ the left side of (3.10) equals $\|u_n - u^*\|_{\mathrm{m}}$.

Proof (of Theorem 3.1). We will apply Theorem 2.2 in $\mathcal{H} = L^2(\Omega)$. The subspace D from (3.2) is dense in $L^2(\Omega)$ and T is translation invariant with respect to $\mathcal{H}_0$ (cf. (3.7)). Let the operator B be defined in $\mathcal{H}$ by

$$B \equiv -\Delta \quad \text{on } D(B) \equiv D.$$

Then $\langle Bu, v\rangle = \int_\Omega \nabla u \cdot \nabla v\,dx$ $(u, v \in D)$, hence B is symmetric and positive semidefinite. Further, $\ker B = \mathcal{H}_0$ since $\Delta u = 0$ and $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$ imply that u is constant. To construct $\mathcal{H}_B$, we note that for any $u \in D$ the function u_0 introduced in (2.7) is given by

$$u_0 = \frac{1}{|\Omega|}\int_\Omega u\,dx.$$

Hence we have

$$\langle u, v\rangle_B = \langle u, v\rangle_{\mathrm{m}} \quad (u, v \in D).$$

Since D is dense in $H^1(\Omega)$, this yields that $\mathcal{H}_B = H^1(\Omega)$ with scalar product (3.3). It remains to check conditions (i)–(v) of Theorem 2.2 for B.

(i) $R(T) \subset C^{0,\gamma}(\overline{\Omega})$ since condition (C2) implies that for any $u \in C^{2,\gamma}(\overline{\Omega})$ we have $T(u) \in C^{0,\gamma}(\overline{\Omega})$. Further, $R(B) = C^{0,\gamma}(\overline{\Omega})$ since the Schauder estimate yields that for any $g \in C^{0,\gamma}(\overline{\Omega})$ the weak solutions of $-\Delta u = g$, $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$ belong to $C^{2,\gamma}(\overline{\Omega})$ (cf. [3]).

(ii) We have seen at the beginning of the proof that $\ker B = \mathcal{H}_0$.

(iii) For any $u \in D \cap \mathcal{H}_0^{\perp}$, using Remark 3.4(b) and (3.4), we have

$$\langle Bu, u\rangle = \int_\Omega |\nabla u|^2 = \|u\|_{\mathrm{m}}^2 \geq p\|u\|_{L^2(\Omega)}^2$$

with $p = 1/K_\Omega$. (The p is the smallest positive eigenvalue of $-\Delta$ on D.)

(iv) The operator A, introduced in Lemma 3.1, satisfies the conditions of Theorem 2.1 in $\mathcal{H}_B = H^1(\Omega)$ and $\mathcal{H}_0$ defined in (3.7).

(v) Let $u, v \in D$. Then

$$\langle T(u), v\rangle = \int_\Omega f(x, \nabla u)\,\nabla u \cdot \nabla v\,dx = \langle A(u), v\rangle_{\mathrm{m}} = \langle A(u), v\rangle_B.$$

Corollary 3.1 implies $\langle A(u), v\rangle_B = \langle BA(u), v\rangle$ (since $A(u)_0 = 0$), which in turn implies $T(u) = BA(u)$ since D is dense in $H^1(\Omega)$.

Consequently, we can apply Theorem 2.2 to problem (3.1). Here (2.10) coincides with (3.6), hence assertion (1) of our theorem is proved. Further, (2.12) coincides with (3.9) since

$$D \cap \mathcal{H}_0^{\perp} = \left\{ u \in C^{2,\gamma}(\overline{\Omega}) : \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} = 0,\ \int_\Omega u\,dx = 0 \right\}$$

in $H^1(\Omega)$. Hence assertion (2) of Theorem 2.2 yields

$$\|u_n - u^*\|_{\mathrm{m}} \leq \frac{1}{mp^{1/2}}\|T(u_0) - g\|_{L^2(\Omega)}\left(\frac{M-m}{M+m}\right)^n \qquad (n \in \mathbb{N}).$$

(Here p is the lower bound of B on $\mathcal{H}_0^{\perp}$ which, as mentioned in (iii), equals the smallest positive eigenvalue of B on D.) By Remark 3.4(b), this is just the desired estimate (3.10). ■

Remark 3.6. If $\partial\Omega$ fails to have the smoothness $C^{2,\gamma}$ in (C1) then we can introduce

$$D \equiv \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} = 0 \text{ in trace sense} \right\}$$

as the domain of T. Then in (C2) it is enough to assume $g \in L^2(\Omega)$ and $f \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ with the extra condition $\left|\frac{\partial f}{\partial x_i}(x,p)\right| \leq \mathrm{const} \cdot |p|$ to ensure that T maps into $L^2(\Omega)$. In this setting Theorem 3.1 can be proved in the same way as above.

REMARK 3.7. From the point of view of numerical realization, the essence of the algorithm obtained is that the original non-linear problem (3.1) is reduced to a sequence of auxiliary linear Poisson equations. This kind of reduction is analogous to the case of Dirichlet problems, mentioned in the introduction. The auxiliary equations can be solved by any suitable well-known method, e.g. Fourier series or (as for the Dirichlet problem in [5] and [4], resp.) finite difference method or finite element method.

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