COLLOQUIUM MATHEMATICUM

VOL. 150 2017 NO. 2

COFINALITIES OF MARCZEWSKI-LIKE IDEALS

BY

JÖRG BRENDLE (Kobe), YURII KHOMSKII (Hamburg)
and WOLFGANG WOHOFSKY (Hamburg)

Abstract. We show in ZFC that the cofinalities of both the Miller ideal m^0 (the σ-ideal naturally related to Miller forcing $\mathbb{M}$) and the Laver ideal ℓ^0 (related to Laver forcing $\mathbb{L}$) are larger than the size $\mathfrak{c}$ of the continuum.

1. Introduction. The purpose of this note is to prove (in ZFC) that the ideals naturally related to Laver forcing $\mathbb{L}$ and to Miller forcing $\mathbb{M}$, the *Laver ideal* ℓ^0 and the *Miller ideal* m^0, have cofinalities strictly larger than $\mathfrak{c}$, the size of the continuum (Corollary 18 below). We will phrase our result in a more general framework and show that $\mathsf{cof}(t^0) > \mathfrak{c}$ for all tree ideals t^0 derived from tree forcings $\mathbb{T}$ satisfying a certain property (Theorem 13 in Section 3). This was known previously for the Marczewski ideal s^0 [JMS] and the nowhere Ramsey ideal r^0 [Ma], but it is unclear whether the method of proof for these two ideals works for ℓ^0 and m^0 (see the discussion in Section 2), and our approach is more general. We emphasize that while there is a close connection with the corresponding forcing notions, our results are in ZFC and no knowledge of forcing theory is required for understanding them.

For $n \le \omega$ let $\omega^n = \{f : n \to \omega\}$ be the collection of all functions from n to ω, or equivalently, the collection of all sequences of length n of natural numbers. In particular, ω^ω is the *Baire space*, and $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$ is the collection of finite sequences of natural numbers. A subset T of $\omega^{<\omega}$ is called a *tree* (or a *subtree* of $\omega^{<\omega}$) if it is closed under initial segments, that is, for all $s \in T$ and $n \in \omega$ we have $s{\restriction}n \in T$. For a tree T, $[T] = \{x \in \omega^\omega : x{\restriction}n \in T$ for all $n \in \omega\}$ denotes the *set of branches* through T.

Definition 1 (Combinatorial tree forcing). A collection $\mathbb{T}$ of subtrees of $\omega^{<\omega}$ is a *combinatorial tree forcing* if

---

2010 *Mathematics Subject Classification*: Primary 03E17; Secondary 03E40, 03E50.
Key words and phrases: Laver forcing, Miller forcing, tree forcing, Marczewski ideal, nowhere Ramsey ideal, Laver ideal, Miller ideal, tree ideal, cofinality of an ideal.
Received 12 October 2016; revised 1 May 2017.
Published online 29 September 2017.

DOI: 10.4064/cm7113-5-2017

(1) $\omega^{<\omega} \in \mathbb{T}$,
(2) (closure under subtrees) if $T \in \mathbb{T}$ and $s \in T$, then the tree $T_s = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$ also belongs to $\mathbb{T}$,
(3) (large disjoint antichains) there is a continuous function $f : \omega^\omega \to 2^\omega$ such that for all $x \in 2^\omega$, $f^{-1}(\{x\})$ is the set of branches of a tree in $\mathbb{T}$,
(4) (homogeneity) if $T \in \mathbb{T}$, then there is an order-preserving injection $i : \omega^{<\omega} \to T$ such that the map $g : \omega^\omega \to [T]$ given by $g(x) = \bigcup\{i(x{\restriction}n) : n \in \omega\}$ is a homeomorphism and for any subtree $S \subseteq \omega^{<\omega}$, $S \in \mathbb{T}$ iff the closure of $i(S)$ under initial segments belongs to $\mathbb{T}$.

$\mathbb{T}$ is partially ordered by inclusion: for $S, T \in \mathbb{T}$, $S \leq T$ if $S \subseteq T$.

Homogeneity says that the partial order looks the same below each element. In view of homogeneity, "large disjoint antichains" implies that

(5) each $T \in \mathbb{T}$ splits into continuum many trees with pairwise disjoint sets of branches, that is, there are $T_\alpha \in \mathbb{T}$, $\alpha < \mathfrak{c}$, with $T_\alpha \subseteq T$ such that $[T_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$.

Recall here that $S, T \in \mathbb{T}$ are *incompatible* if there is no $U \in \mathbb{T}$ with $U \leq S, T$. A set $\mathcal{T} \subseteq \mathbb{T}$ is an *antichain* if any two distinct elements of $\mathcal{T}$ are incompatible. An antichain $\mathcal{T}$ is *maximal* if for each $S \in \mathbb{T}$ there is $T \in \mathcal{T}$ compatible with S. A partial order $\mathbb{T}$ satisfies the *countable chain condition* (*ccc* for short) if every antichain in $\mathbb{T}$ is at most countable. Clause (5) then says that there are $\mathfrak{c}$-sized antichains in a combinatorial tree forcing $\mathbb{T}$ so that $\mathbb{T}$ is not ccc and some classical forcing notions like Cohen forcing and random forcing do not fit into this framework.

For partial orders whose elements are subtrees of $2^{<\omega}$ like Sacks forcing $\mathbb{S}$, an analogous definition applies, with $\omega^{<\omega}$ and ω^ω replaced by $2^{<\omega}$ and the *Cantor space* 2^ω, respectively.

Definition 2 (Tree ideal). The *tree ideal* t^0 associated with the combinatorial tree forcing $\mathbb{T}$ consists of all $X \subseteq \omega^\omega$ such that for all $T \in \mathbb{T}$ there is $S \leq T$ with $X \cap [S] = \emptyset$.

For a tree T (in $\omega^{<\omega}$ or $2^{<\omega}$) and $t \in T$, $\mathrm{succ}_T(t) = \{n \in \omega : t\hat{\ }n \in T\}$ is the *set of successors* of t in T. A node $s \in T$ is called a *splitting node* of T if $\mathrm{succ}_T(s)$ has at least two elements. The stem of T, $\mathrm{stem}(T)$, is the smallest splitting node. A subtree $T \subseteq 2^{<\omega}$ is a *Sacks tree* (or *perfect tree*) if for each $t \in T$ there is a splitting node $s \supseteq t$ in T. *Sacks forcing* $\mathbb{S}$, the collection of all Sacks trees, is a combinatorial tree forcing: "large disjoint antichains" is witnessed by $f : 2^\omega \to 2^\omega$ given by $f(x)(n) = x(2n)$ for $x \in 2^\omega$ and $n \in \omega$. The *Marczewski ideal* s^0 is the corresponding tree ideal. A subtree $T \subseteq \omega^{<\omega}$ is a *Laver tree* [La] if for all $t \in T$ containing $\mathrm{stem}(T)$, $\mathrm{succ}_T(t)$ is infinite. Further, $T \subseteq \omega^{<\omega}$ is a *Miller tree* [Mi] (or *superperfect tree*) if for all $t \in T$ there is $s \supseteq t$ in T such that $\mathrm{succ}_T(s)$ is infinite. *Laver forcing* $\mathbb{L}$

(*Miller forcing* $\mathbb{M}$, respectively) is the collection of all Laver trees (Miller trees, respectively). Both are combinatorial tree forcings in the above sense because $f : \omega^\omega \to 2^\omega$ given by $f(x)(n) = x(n) \bmod 2$ for all $x \in \omega^\omega$ and $n \in \omega$ witnesses "large disjoint antichains". The *Laver* and *Miller ideals* ℓ^0 and m^0 are the corresponding tree ideals. For basic facts about such tree ideals, like non-inclusion between different ideals, see e.g. [Br].

Definition 3 (Cofinality of an ideal). Given an ideal $\mathcal{I}$ on ω^ω or 2^ω, its *cofinality* $\mathsf{cof}(\mathcal{I})$ is the smallest cardinality of a family $\mathcal{J} \subseteq \mathcal{I}$ such that every member of $\mathcal{I}$ is contained in a member of $\mathcal{J}$.

A family like $\mathcal{J}$ in this definition is said to be a *basis* of $\mathcal{I}$ (or *cofinal* in $\mathcal{I}$).

While the topic of our work are cofinalities of tree ideals, we note that other cardinal invariants of tree ideals t^0, such as the *additivity* $\mathsf{add}(t^0)$ (the least size of a subfamily $\mathcal{J} \subseteq t^0$ whose union is not in t^0) and the *covering number* $\mathsf{cov}(t^0)$ (the least size of a subfamily $\mathcal{J} \subseteq t^0$ whose union is ω^ω) have been studied as well. If there is a fusion argument for $\mathbb{T}$, then t^0 is a σ-ideal, and one has $\omega_1 \le \mathsf{add}(t^0) \le \mathsf{cov}(t^0) \le \mathfrak{c}$, while the exact value of these two cardinals depends on the model of set theory. Furthermore, by "large disjoint antichains", the *uniformity* $\mathsf{non}(t^0)$ of a tree ideal t^0 (the smallest cardinality of a subset of ω^ω not belonging to t^0) is always equal to $\mathfrak{c}$. Since $\mathsf{cof}(\mathcal{I}) \ge \mathsf{non}(\mathcal{I})$ for any non-trivial ideal $\mathcal{I}$, $\mathsf{cof}(t^0) \ge \mathfrak{c}$ follows, and the main problem about cofinalities of tree ideals is whether they can be equal to $\mathfrak{c}$ or must be strictly above $\mathfrak{c}$.

The question whether $\mathsf{cof}(\ell^0)$ and $\mathsf{cof}(m^0)$ are larger than $\mathfrak{c}$ was discussed in private communication with M. Dečo and M. Repický, and Repický [Re] in the meantime used our result to obtain a characterization of $\mathsf{cof}(\ell^0)$ as $\mathfrak{d}((\ell^0)^{\mathfrak{c}})$.

2. The disjoint maximal antichain property

Definition 4. Let $\mathbb{T}$ be a combinatorial tree forcing. Then $\mathbb{T}$ has the *disjoint maximal antichain property* if there is a maximal antichain $(T_\alpha : \alpha < \mathfrak{c})$ in $\mathbb{T}$ such that $[T_\alpha] \cap [T_\beta] = \emptyset$ for all $\alpha \ne \beta$.

The following has been known for some time (see also [Re, Theorem 1.2]).

Proposition 5. *Assume* $\mathbb{T}$ *has the disjoint maximal antichain property. Then* $\mathrm{cf}(\mathsf{cof}(t^0)) > \mathfrak{c}$.

Proof. Let $(T_\alpha : \alpha < \mathfrak{c})$ be a disjoint maximal antichain in $\mathbb{T}$. Also let $\kappa = \mathrm{cf}(\mathsf{cof}(t^0))$ and assume $\kappa \le \mathfrak{c}$. We shall derive a contradiction. Assume $\mathcal{X}_\alpha \subseteq t^0$, $\alpha < \kappa$, are of size $< \mathsf{cof}(t^0)$. We shall show that $\mathcal{X} = \bigcup\{\mathcal{X}_\alpha : \alpha < \kappa\}$ is not cofinal in t^0. By homogeneity of the tree forcing $\mathbb{T}$, we know that $\mathcal{X}_\alpha$ is not cofinal below T_α, that is, there is $X_\alpha \in t^0$, $X_\alpha \subseteq [T_\alpha]$, such that $X_\alpha \not\subseteq Y$

for all $Y \in \mathcal{X}_\alpha$. Let $X = \bigcup\{X_\alpha : \alpha < \kappa\}$. By disjointness of the maximal antichain, we see that $X \in t^0$. Obviously $X \not\subseteq Y$ for all $Y \in \mathcal{X}$, and we are done. ∎

Note that for only showing $\mathsf{cof}(t^0) > \mathfrak{c}$, the homogeneity of the forcing is not needed (that is, properties (1), (2), and (5) of Definition 1 are enough).

DEFINITION 6. Let $\mathbb{T}$ be a combinatorial tree forcing. Then $\mathbb{T}$ has the *incompatibility shrinking property* if for any $T \in \mathbb{T}$ and any family $(S_\alpha : \alpha < \mu)$, $\mu < \mathfrak{c}$, in $\mathbb{T}$ such that S_α is incompatible with T for all α, one can find $T' \leq T$ such that $[T']$ is disjoint from all the $[S_\alpha]$.

For the next proof recall that a set $\mathcal{D}$ in a partial order $\mathbb{T}$ is *dense* if for all $S \in \mathbb{T}$ there is $T \leq S$ belonging to $\mathcal{D}$. If $\mathcal{T}$ is a maximal antichain in $\mathbb{T}$, then the set $\mathcal{D} = \{S \in \mathbb{T} : S \leq T \text{ for some } T \in \mathcal{T}\}$ is easily seen to be dense.

PROPOSITION 7. *Let $\mathbb{T}$ be a combinatorial tree forcing. The incompatibility shrinking property for $\mathbb{T}$ implies the disjoint maximal antichain property for $\mathbb{T}$. In fact, it implies that any maximal antichain can be refined to a disjoint maximal antichain.*

Proof. Let $(T_\alpha : \alpha < \mathfrak{c})$ be a dense set of trees in $\mathbb{T}$ all of which lie below a given maximal antichain of size $\mathfrak{c}$. We construct $A \subseteq \mathfrak{c}$ of size $\mathfrak{c}$ and $\{S_\alpha : \alpha \in A\} \subseteq \mathbb{T}$ such that

- $S_\alpha \leq T_\alpha$ for $\alpha \in A$,
- if $\alpha \notin A$, then T_α is compatible with some S_β for $\beta < \alpha$ with $\beta \in A$,
- $[S_\alpha] \cap [S_\beta] = \emptyset$ for $\alpha \neq \beta$ from A.

Clearly, these conditions imply that $(S_\alpha : \alpha \in A)$ is a disjoint maximal antichain. Also, A must necessarily have size $\mathfrak{c}$.

Suppose we are at stage $\alpha < \mathfrak{c}$ of the construction. If T_α is compatible with some S_β where $\beta < \alpha$, $\beta \in A$, let $\alpha \notin A$, and we are done. If this is not the case, let $\alpha \in A$. By the incompatibility shrinking property we find $T' = S_\alpha$ as required. ∎

Say that a Laver tree $T \subseteq \omega^{<\omega}$ is a *Mathias tree* if stem(T) is a strictly increasing sequence and there is an infinite $A \subseteq \omega$ such that for all $t \in T$ containing stem(T), $\mathrm{succ}_T(t) = A \setminus (t(|t|-1)+1)$. *Mathias forcing* $\mathbb{R}$ is the collection of Mathias trees, and the ideal r^0 of *nowhere Ramsey* sets is the corresponding tree ideal. A Sacks tree $T \subseteq 2^{<\omega}$ is a *Silver tree* if there are an infinite $B \subseteq \omega$ and $g : \omega \setminus B \to 2$ such that $t \in T$ iff $t{\restriction}(\omega \setminus B) = g{\restriction}|t|$ (in particular, $t \in T$ is a splitting node iff $|t| \in B$). *Silver forcing* $\mathbb{V}$ is the collection of Silver trees, and v^0 is the corresponding *Silver ideal*. Both $\mathbb{R}$ and $\mathbb{V}$ are combinatorial tree forcings (in the case of $\mathbb{R}$, in Definition 1, use the collection of strictly increasing finite sequences and the space of strictly increasing functions instead of $\omega^{<\omega}$ and the Baire space ω^ω, respectively).

EXAMPLE 8. Sacks forcing $\mathbb{S}$, Mathias forcing $\mathbb{R}$, and Silver forcing $\mathbb{V}$ have the incompatibility shrinking property, and thus also the disjoint maximal antichain property.

To see this, simply use the fact that for any incompatible $S, T \in \mathbb{S}$, $[S] \cap [T]$ is at most countable, while in the case of $\mathbb{V}$ this intersection is finite, and for $\mathbb{R}$ even empty.

From this we deduce that $\mathrm{cf}(\mathsf{cof}(s^0)) > \mathfrak{c}$ [JMS, Theorem 1.3], that $\mathrm{cf}(\mathsf{cof}(r^0)) > \mathfrak{c}$ [Ma], and that $\mathrm{cf}(\mathsf{cof}(v^0)) > \mathfrak{c}$.

We also note that if t^0 is a σ-ideal (which is the case if there is a fusion argument for $\mathbb{T}$), then the continuum hypothesis CH implies the incompatibility shrinking property and thus also the disjoint maximal antichain property. For Laver forcing and Miller forcing, a weaker hypothesis is sufficient.

PROPOSITION 9. *Assume $\mathfrak{b} = \mathfrak{c}$. Then Laver forcing $\mathbb{L}$ has the incompatibility shrinking property and thus also the disjoint maximal antichain property.*

Recall here that the *unbounding number* $\mathfrak{b}$ is the least size of an $\mathcal{F} \subseteq \omega^\omega$ that is unbounded in the partial order $(\omega^\omega, \leq^*)$, where $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. The *dominating number* $\mathfrak{d}$ is the least size of an $\mathcal{F} \subseteq \omega^\omega$ that is cofinal in $(\omega^\omega, \leq^*)$. It is well-known and easy to see that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Proof of Proposition 9. Fix $T \in \mathbb{L}$, $\mu < \mathfrak{c}$, and any family $(S_\alpha : \alpha < \mu)$ in $\mathbb{L}$ such that S_α is incompatible with T for all α. Since $T \cap S_\alpha$ does not contain a Laver tree, by [GRSS, Lemma 2.3], there is a function $g_\alpha : \omega^{<\omega} \to \omega$ such that if $x \in [T] \cap [S_\alpha]$, then there are infinitely many n with $x(n) < g_\alpha(x{\restriction}n)$. By $\mathfrak{b} = \mathfrak{c}$, there is $f : \omega^{<\omega} \to \omega$ eventually dominating all g_α. Let $T' = \{s \in T : s(n) > f(s{\restriction}n) \text{ for all } n \in \mathrm{dom}(s) \text{ beyond the stem of } T\}$. Clearly T' is still a Laver tree with the same stem as T. Furthermore, $[T'] \cap [S_\alpha] = \emptyset$, for if x belonged to the intersection, we would have $x(n) < g_\alpha(x{\restriction}n)$ for infinitely many n and $x(n) > f(x{\restriction}n)$ for all n beyond the stem of T', a contradiction. ∎

A similar argument which we leave to the reader shows:

PROPOSITION 10. *Assume $\mathfrak{d} = \mathfrak{c}$. Then Miller forcing $\mathbb{M}$ has the incompatibility shrinking property and thus also the disjoint maximal antichain property.*

QUESTION 11. *Do $\mathbb{L}$ or $\mathbb{M}$ have the disjoint maximal antichain property in ZFC?*

3. The selective disjoint antichain property. We now consider a property, weaker than the disjoint maximal antichain property, which is sufficient to show that the cofinalities of the Laver ideal ℓ^0 and the Miller ideal m^0 are larger than $\mathfrak{c}$ in ZFC.

Definition 12. Let $\mathbb{T}$ be a combinatorial tree forcing. Then $\mathbb{T}$ has the *selective disjoint antichain property* if there is an antichain $(T_\alpha : \alpha < \mathfrak{c})$ in $\mathbb{T}$ such that

- $[T_\alpha] \cap [T_\beta] = \emptyset$ for all $\alpha \neq \beta$,
- for all $T \in \mathbb{T}$ there is $S \leq T$ such that either
  - $S \leq T_\alpha$ for some $\alpha < \mathfrak{c}$, or
  - $|[S] \cap [T_\alpha]| \leq 1$ for all $\alpha < \mathfrak{c}$.

For our applications, it would be enough to have $|[S] \cap [T_\alpha]| \leq \aleph_0$ in the last clause.

Theorem 13. *Assume $\mathbb{T}$ has the selective disjoint antichain property. Then* $\mathrm{cf}(\mathsf{cof}(t^0)) > \mathfrak{c}$.

Proof. Let $(T_\alpha : \alpha < \mathfrak{c})$ be a selective disjoint antichain in $\mathbb{T}$. Also assume that $(S_\beta : \beta < \mathfrak{c})$ is a list of all trees S in $\mathbb{T}$ such that $|[S] \cap [T_\alpha]| \leq 1$ for all $\alpha < \mathfrak{c}$. Set $\kappa = \mathrm{cf}(\mathsf{cof}(t^0))$ and assume $\kappa \leq \mathfrak{c}$. Also assume $\mathcal{X}_\alpha \subseteq t^0$, $\alpha < \kappa$, are of size $< \mathsf{cof}(t^0)$. As in the proof of Proposition 5, we shall show that $\mathcal{X} = \bigcup\{\mathcal{X}_\alpha : \alpha < \kappa\}$ is not cofinal in t^0.

By "large disjoint antichains", we find $T'_\alpha \leq T_\alpha$ with $[T'_\alpha] \cap \bigcup_{\beta<\alpha}[S_\beta] = \emptyset$. By homogeneity, there is $X_\alpha \in t^0$ with $X_\alpha \subseteq [T'_\alpha]$ such that $X_\alpha \not\subseteq Y$ for all $Y \in \mathcal{X}_\alpha$. Let $X = \bigcup\{X_\alpha : \alpha < \kappa\}$. Obviously $X \not\subseteq Y$ for all $Y \in \mathcal{X}$. We need to show that X belongs to t^0. To this end, let $T \in \mathbb{T}$.

First assume there is $S \leq T$ such that $S \leq T_\alpha$ for some $\alpha < \mathfrak{c}$. Then $[S] \cap X \subseteq X_\alpha$. Since $X_\alpha \in t^0$, there is $S' \leq S$ such that $[S'] \cap X_\alpha = \emptyset$, and $[S'] \cap X = \emptyset$ follows.

Next assume there is $S \leq T$ such that $|[S] \cap [T_\alpha]| \leq 1$ for all $\alpha < \mathfrak{c}$. Then $S = S_\beta$ for some $\beta < \mathfrak{c}$. By construction, we know that $X_\alpha \cap [S_\beta] = \emptyset$ for $\alpha > \beta$. Hence $[S_\beta] \cap X \subseteq \bigcup_{\alpha\leq\beta}[S_\beta] \cap [T_\alpha]$ and therefore $|[S_\beta] \cap X| < \mathfrak{c}$. Using "large disjoint antichains" again, we see that there is $S' \leq S_\beta$ such that $[S'] \cap X = \emptyset$, as required. ∎

Once more note that for only showing $\mathsf{cof}(t^0) > \mathfrak{c}$, the homogeneity of the forcing is not needed (that is, properties (1), (2), and (5) of Definition 1 are enough).

The next property of a combinatorial tree forcing $\mathbb{T}$ implies that $\mathbb{T}$ adds a minimal real and, in fact, standard proofs of minimality go via this property.

Definition 14. Let $\mathbb{T}$ be a combinatorial tree forcing. Then $\mathbb{T}$ has the *constant or one-to-one property* if for all $T \in \mathbb{T}$ and all continuous $f : [T] \to 2^\omega$, there is $S \leq T$ such that $f{\restriction}[S]$ is either constant or one-to-one.

It is known that both Miller forcing and Laver forcing have the constant or one-to-one property. For the former, this is implicit in work of Miller [Mi, Section 2], for the latter, in work of Gray [Gra] (see also [Gro, Theorems 2 and 7] for similar arguments). These results are formulated in terms of minimality. For completeness' sake, we include a proof of the more difficult case of Laver forcing in our formulation. Note also that the result for $\mathbb{M}$ is a trivial consequence of the result for $\mathbb{L}$.

THEOREM 15 (Miller). *Miller forcing* $\mathbb{M}$ *has the constant or one-to-one property.*

THEOREM 16 (Gray). *Laver forcing* $\mathbb{L}$ *has the constant or one-to-one property.*

Proof. Fix f and T. The pure decision property of Laver forcing (see, e.g., [BJ, Lemma 7.3.32]) implies:

CLAIM 16.1. *Let* $n \in \omega$ *and* $\tau \in T$ *with* $\mathrm{stem}(T) \subseteq \tau$. *There are* $T' \leq_0 T_\tau$ *and* $s \in 2^n$ *such that* $[T'] \subseteq f^{-1}([s])$.

Here, $T' \leq_0 T$ if $T' \leq T$ and $\mathrm{stem}(T) = \mathrm{stem}(T')$.

CLAIM 16.2. *Let* $\tau \in T$ *with* $\mathrm{stem}(T) \subseteq \tau$. *There are* $T' \leq_0 T_\tau$ *and* $x = x_\tau \in 2^\omega$ *such that if* $(k_\tau^n : n \in \omega)$ *is the increasing enumeration of* $\mathrm{succ}_{T'}(\tau)$ *then* $[T'_{\tau\hat{}k_\tau^n}] \subseteq f^{-1}([x{\restriction}(|\tau|+n)])$.

Proof. Using Claim 16.1, construct a $\leq_0$-decreasing sequence $(S^n : n \in \omega)$ with $S^0 \leq_0 T_\tau$ and a $\subset$-increasing sequence $(s^n \in 2^{n+|\tau|} : n \in \omega)$ such that $[S^n] \subseteq f^{-1}([s^n])$ for all n. Let $k_\tau^n = \min(\mathrm{succ}_{S^n}(\tau) \setminus (k_\tau^{n-1}+1))$ where we set $k_\tau^{-1} = -1$. Let T' be such that $\mathrm{succ}_{T'}(\tau) = \{k_\tau^n : n \in \omega\}$ and $T'_{\tau\hat{}k_\tau^n} = S^n_{\tau\hat{}k_\tau^n}$. Also let $x = \bigcup_n s^n \in 2^\omega$. Then $T' \leq_0 T_\tau$ and $[T'_{\tau\hat{}k_\tau^n}] = [S^n_{\tau\hat{}k_\tau^n}] \subseteq [S^n] \subseteq f^{-1}([s^n]) = f^{-1}([x{\restriction}(|\tau|+n)])$. ∎

By Claim 16.2 and a fusion argument we obtain:

CLAIM 16.3. *There are* $T' \leq_0 T$, $(x_\tau : \tau \in T', \mathrm{stem}(T) \subseteq \tau)$, *and* $((k_\tau^n : n \in \omega) : \tau \in T', \mathrm{stem}(T) \subseteq \tau)$ *such that* $(k_\tau^n : n \in \omega)$ *is the increasing enumeration of* $\mathrm{succ}_{T'}(\tau)$ *for all* τ *and* $[T'_{\tau\hat{}k_\tau^n}] \subseteq f^{-1}([x_\tau{\restriction}(|\tau|+n)])$ *for all* n *and all* τ. *In particular* $[T'_\tau] \subseteq f^{-1}([x_\tau{\restriction}|\tau|])$ *for all* τ.

The properties of the x_τ imply in particular that $x_{\tau\hat{}k_\tau^n}$ converges to x_τ as n goes to infinity. Now define a rank function for $\tau \in T'$ as follows:

- $\rho(\tau) = 0 \Leftrightarrow \exists^\infty k \in \mathrm{succ}_{T'}(\tau)$ such that $x_{\tau\hat{}k} \neq x_\tau$,
- for $\alpha > 0$, $\rho(\tau) = \alpha \Leftrightarrow \neg\rho(\tau) < \alpha \wedge \exists^\infty k \in \mathrm{succ}_{T'}(\tau)\, (\rho(\tau\hat{}k) < \alpha)$.

By the convergence property of the x_τ, we see that $\rho(\tau) = 0$ implies in particular that the set $\{x_{\tau\hat{}k_\tau^n} : n \in \omega\}$ is infinite.

CASE 1: $\rho(\tau) = \infty$ *for some* $\tau \in T'$ (i.e., the rank is undefined). Then we can easily construct a Laver tree $S \leq T'$ such that $\mathrm{stem}(S) = \tau$ and $x_\sigma = x_\tau$

for all $\sigma \in S$ with $\sigma \supseteq \tau$. We claim that $f\restriction[S]$ is constant with value x_τ. Indeed, let $y \in [S]$. Fix $k \geq |\tau|$. By construction $y \in [S_{y\restriction k}] \subseteq f^{-1}([x_\tau\restriction k])$. Since this holds for all k, we have $f(y) = x_\tau$, and we are done.

Case 2: *$\rho(\tau)$ is defined for all $\tau \in T'$.* Recall that $F \subseteq T'$ is a *front* if for all $y \in [T']$ there is a unique n with $y\restriction n \in F$. We build a subtree S of T' by specifying fronts F_n, $n \in \omega$, such that for every $\sigma \in F_{n+1}$ there is a (necessarily unique) $\tau \in F_n$ with $\tau \subset \sigma$. That is, S will be the tree generated by the fronts: $\sigma \in S$ iff there are $n \in \omega$ and $\tau \in F_n$ with $\sigma \subseteq \tau$. Additionally, we shall guarantee that there are $s_\tau \subseteq x_\tau$ for $\tau \in \bigcup_n F_n$ such that

- if $\sigma \neq \sigma'$ both are in F_n then $[s_\sigma] \cap [s_{\sigma'}] = \emptyset$,
- $[S_\tau] \subseteq f^{-1}([s_\tau])$ for $\tau \in \bigcup_n F_n$,
- if $\sigma \subset \tau$ with $\sigma \in F_n$ and $\tau \in F_{n+1}$ then $s_\sigma \subset s_\tau$,
- if $\sigma \subset \tau$ with $\sigma \in F_n$ and $\tau \in F_{n+1}$ then for every k with $|\sigma| \leq k < |\tau|$, $x_{\tau\restriction k} = x_\sigma$ and $\rho(\sigma) > \rho(\tau\restriction|\sigma|+1) > \cdots > \rho(\tau\restriction|\tau|-1) = 0$.

We first verify that this is enough to guarantee that $f\restriction[S]$ is one-to-one. If $y, y' \in [S]$ are distinct, then there are $n, i, i' \in \omega$ such that $y\restriction i$ and $y'\restriction i'$ are distinct elements of F_n. Then $y \in f^{-1}([s_{y\restriction i}])$, $y' \in f^{-1}([s_{y'\restriction i'}])$ by the second clause, and $[s_{y\restriction i}]$ and $[s_{y'\restriction i'}]$ are disjoint by the first clause. Hence $f(y) \neq f(y')$ as required. Thus it suffices to construct the F_n and s_τ.

For $n = 0$, we let $F_0 = \{\mathrm{stem}(T')\} = \{\mathrm{stem}(S)\}$. Also let $s_{\mathrm{stem}(S)} = x_{\mathrm{stem}(S)}\restriction|\mathrm{stem}(S)|$.

Suppose F_n and s_σ for $\sigma \in F_n$ have been constructed. We shall construct F_{n+1} and s_σ for $\sigma \in F_{n+1}$, as well as the part of the tree S between F_n and F_{n+1}. Fix $\sigma \in F_n$. We denote by A^n_σ the part of S between σ and F_{n+1}, that is, $A^n_\sigma = \{\tau \in S : \sigma \subseteq \tau \text{ and } \tau \subset \upsilon \text{ for some } \upsilon \text{ in } F_{n+1}\}$. We will construct A^n_σ recursively so that it satisfies the fourth clause above.

Put σ into A^n_σ. Suppose some $\tau \supseteq \sigma$ has been put into A^n_σ, $x_\tau = x_\sigma$ and, in case $\tau \supset \sigma$, $\rho(\sigma) > \rho(\tau)$. In case $\rho(\tau) = 0$, no successor of τ will be in A^n_σ and the successors of τ will belong to F_{n+1}, as explained below. If $\rho(\tau) > 0$, then $x_{\tau\hat{}k} = x_\tau$ for almost all $k \in \mathrm{succ}_{T'}(\tau)$ and $\rho(\tau\hat{}k) < \rho(\tau)$ for infinitely many $k \in \mathrm{succ}_{T'}(\tau)$. Hence we can prune the successor level of τ to $\mathrm{succ}_S(\tau)$ so that $x_{\tau\hat{}k} = x_\tau$ and $\rho(\tau\hat{}k) < \rho(\tau)$ for all $k \in \mathrm{succ}_S(\tau)$. The fourth clause is clearly satisfied. This completes the construction of A^n_σ.

Now fix $\tau \in A^n_\sigma$ with $\rho(\tau) = 0$. By pruning $\mathrm{succ}_{T'}(\tau)$ if necessary, we may assume without loss of generality that the $x_{\tau\hat{}k^m_\tau}$, $m \in \omega$, are all pairwise distinct and converge to $x_\tau = x_\sigma$, and that, in fact, there is a strictly increasing sequence $(i^m_\tau : m \in \omega)$ such that $i^m_\tau = \min\{i : x_{\tau\hat{}k^m_\tau}(i) \neq x_\tau(i)\}$. Unfixing τ, we may additionally assume that if $\tau \neq \tau'$ are both in A^n_σ of rank 0 and $m, m' \in \omega$, then $i^m_\tau \neq i^{m'}_{\tau'}$. Finally we may assume that all such i^m_τ are larger than $|s_\sigma|$. This means in particular that $s_\sigma \subseteq x_{\tau\hat{}k^m_\tau}$ for all τ and m because $s_\sigma \subseteq x_\sigma = x_\tau$. Now choose $s_{\tau\hat{}k^m_\tau} \subseteq x_{\tau\hat{}k^m_\tau}$ such that $|s_{\tau\hat{}k^m_\tau}| > i^m_\tau$. Then

$s_\sigma \subset s_{\tau\hat{}k_\tau^m}$ and the $s_{\tau\hat{}k_\tau^m}$ for distinct pairs (τ, m) with $\tau \in A_\sigma^n$ of rank 0 and $m \in \omega$ are pairwise incompatible.

Unfix $\sigma \in F_n$. Let $F_{n+1} = \{\tau\hat{}k_\tau^m : \tau \in A_\sigma^n$ for some $\sigma \in F_n$, $\rho(\tau) = 0$, and $m \in \omega\}$. The third clause is immediate. To see the first clause, take distinct $\tau, \tau' \in F_{n+1}$. There are $\sigma, \sigma' \in F_n$ such that $\sigma \subset \tau$ and $\sigma' \subset \tau'$. If $\sigma \neq \sigma'$, then $[s_\tau] \cap [s_{\tau'}] = \emptyset$ because $[s_\sigma] \cap [s_{\sigma'}] = \emptyset$ and $s_\sigma \subset s_\tau$ and $s_{\sigma'} \subset s_{\tau'}$. If $\sigma = \sigma'$, then $[s_\tau] \cap [s_{\tau'}] = \emptyset$ by the construction in the previous paragraph. Finally, to see the second clause, by pruning T'_τ for $\tau \in F_{n+1}$ if necessary, we may assume $[T'_\tau] \subseteq f^{-1}([s_\tau])$ (see Claim 16.3). This completes the recursive construction and the proof of the theorem. ∎

PROPOSITION 17. *Assume $\mathbb{T}$ is a combinatorial tree forcing with the constant or one-to-one property. Then $\mathbb{T}$ has the selective disjoint antichain property.*

Proof. Let $f : \omega^\omega \to 2^\omega$ be a continuous function witnessing "large disjoint antichains" of $\mathbb{T}$. Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of 2^ω. Let $T_\alpha \in \mathbb{T}$ be such that $[T_\alpha] = f^{-1}(\{x_\alpha\})$. We check that $(T_\alpha : \alpha < \mathfrak{c})$ witnesses the selective disjoint antichain property. Clearly $[T_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$. Given $T \in \mathbb{T}$, find $S \leq T$ such that $f\restriction[S]$ is constant or one-to-one. In the first case, $S \leq T_\alpha$ for some α, and in the second case, $|[S] \cap [T_\alpha]| \leq 1$ for all α, and we are done. ∎

We are finally ready to complete the proof of the main result of this note.

COROLLARY 18. $\mathrm{cf}(\mathsf{cof}(\ell^0)) > \mathfrak{c}$ *and* $\mathrm{cf}(\mathsf{cof}(m^0)) > \mathfrak{c}$.

Proof. This follows from Theorem 16, Theorem 15, Proposition 17, and Theorem 13. ∎

4. Problems. For some natural tree forcings, we still do not know whether the cofinality of the corresponding tree ideal is larger than $\mathfrak{c}$ in ZFC. A Miller tree $T \subseteq \omega^{<\omega}$ is a *full splitting Miller tree* if whenever $s \in T$ is a splitting node then $s\hat{}n \in T$ for all $n \in \omega$. *Full splitting Miller forcing* $\mathbb{FM}$, originally introduced by [NR] (see also [KL]), consists of all full splitting Miller trees, and fm^0 is the *full splitting Miller ideal*. Observe that $\mathbb{FM}$ is also a combinatorial tree forcing.

QUESTION 19. *Is* $\mathsf{cof}(fm^0) > \mathfrak{c}$*?*

By the discussion in Section 2 (before Proposition 9), we know this is true under CH.

More generally, one may ask:

QUESTION 20. *Are there combinatorial tree forcings $\mathbb{T}$ which consistently fail to have the disjoint maximal antichain property? Which consistently fail to satisfy* $\mathsf{cof}(t^0) > \mathfrak{c}$*? For which t^0 consistently has a Borel basis?*

Note that the existence of a Borel basis implies $\mathsf{cof}(t^0) = \mathfrak{c}$. By the above comment, fm^0 has no Borel basis under CH, but this is open in ZFC. Question 20 is also of interest for tree forcings which do not necessarily satisfy all the clauses of Definition 1, e.g., for non-homogeneous forcing notions.

By [JMS, Theorems 1.4 and 1.5], we know that $\mathsf{cof}(s^0)$ can consistently assume arbitrary values $\leq 2^{\mathfrak{c}}$ whose cofinality is larger than $\mathfrak{c}$, and it is easy to see that the same arguments work for other tree ideals like m^0 and ℓ^0. (In these models CH holds.)

QUESTION 21. *Can we consistently separate the cofinalities of different tree ideals? For example, are* $\mathsf{cof}(s^0) < \mathsf{cof}(m^0)$ *or* $\mathsf{cof}(m^0) < \mathsf{cof}(s^0)$ *consistent?*

Added to the revised version: Shelah and Spinas [SS] recently proved the consistency of, e.g., $\mathsf{cof}(m^0) < \mathsf{cof}(s^0)$ and $\mathsf{cof}(\ell^0) < \mathsf{cof}(s^0)$. The consistency of $\mathsf{cof}(s^0) < \mathsf{cof}(m^0)$, however, remains open.

Acknowledgments. This research was started in February 2015 while the first author was visiting the other two authors at the KGRC of the University of Vienna and at the Vienna University of Technology. He thanks the JSPS for its support and the universities in Vienna for their hospitality. Our work was continued in November 2015 at the University of East Anglia (first and second authors) and at the Isaac Newton Institute for Mathematical Sciences (INI) (first and third authors). We thank the INI for support and both institutions for hospitality. The third author also thanks Otmar Spinas for a helpful conversation about the topic of this paper.

The first author was partially supported by Grants-in-Aid for Scientific Research (C) 24540126 and (C) 15K04977, Japan Society for the Promotion of Science, by JSPS and FWF under the Japan–Austria Research Cooperative Program *New developments regarding forcing in set theory*, and by the Isaac Newton Institute for Mathematical Sciences in the programme *Mathematical, Foundational and Computational Aspects of the Higher Infinite* (HIF) funded by EPSRC grant EP/K032208/1.

The second and third authors were partially supported by the Isaac Newton Institute for Mathematical Sciences in the programme *Mathematical, Foundational and Computational Aspects of the Higher Infinite* (HIF) funded by EPSRC grant EP/K032208/1.

REFERENCES

[BJ] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the Real Line*, A K Peters, Wellesley, MA, 1995.

[Br] J. Brendle, *Strolling through paradise*, Fund. Math. 148 (1995), 1–25.

[GRSS] M. Goldstern, M. Repický, S. Shelah, and O. Spinas, *On tree ideals*, Proc. Amer. Math. Soc. 123 (1995), 1573–1581.

[Gra] C. Gray, *Iterated forcing from the strategic point of view*, Ph.D. thesis, Univ. of California, Berkeley, CA, 1980.

[Gro] M. Groszek, *Combinatorics on ideals and forcing with trees*, J. Symbolic Logic 52 (1987), 582–593.

[JMS] H. Judah, A. Miller, and S. Shelah, *Sacks forcing, Laver forcing, and Martin's axiom*, Arch. Math. Logic 31 (1992), 145–161.

[KL] Y. Khomskii and G. Laguzzi, *Full-splitting Miller trees and infinitely often equal reals*, Ann. Pure Appl. Logic 168 (2017), 1491–1506.

[La] R. Laver, *On the consistency of Borel's conjecture*, Acta Math. 137 (1976), 151–169.

[Ma] P. Matet, unpublished.

[Mi] A. Miller, *Rational perfect set forcing*, in: Contemp. Math. 31, Amer. Math. Soc., Providence, RI, 1984, 143–159.

[NR] L. Newelski and A. Rosłanowski, *The ideal determined by the unsymmetric game*, Proc. Amer. Math. Soc. 117 (1993), 823–831.

[Re] M. Repický, *Cofinality of the Laver ideal*, Arch. Math. Logic 55 (2016), 1025–1036.

[SS] S. Shelah and O. Spinas, *Different cofinalities of tree ideals*, preprint.

Jörg Brendle
Graduate School of System Informatics
Kobe University
Rokko-dai 1-1, Nada-ku
Kobe 657-8501, Japan
E-mail: brendle@kobe-u.ac.jp

Yurii Khomskii, Wolfgang Wohofsky
Hamburg University
Bundesstraße 55 (Geomatikum)
20146 Hamburg, Germany
E-mail: yurii@deds.nl
wolfgang.wohofsky@gmx.at