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GROUPS WHOSE PROPER SUBGROUPS HAVE RESTRICTED INFINITE CONJUGACY CLASSES

BY

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Abstract. A group G is said to have the AFC-property if for each element x of G at least one of the indices $|G : C_G(x)|$ and $|C_G(x) : \langle x\rangle|$ is finite. The class of AFC-groups, which generalize FC-groups, has been studied by De Falco et al. (2017) and Shalev (1994). Here the structure of groups whose proper subgroups have the AFC-property is investigated.

1. Introduction. If G is a group, then the set of all elements of G admitting only finitely many conjugates is a characteristic subgroup, called the *FC-centre* of G, and G is said to be an *FC-group* if it coincides with its FC-centre. Obvious examples of groups with the FC-property are abelian groups and finite groups, while the FC-centre of the infinite dihedral group has index 2, so that infinite groups with trivial centre may have a large FC-centre.

A group G is called an *AFC-group* if for each element x of G at least one of the indices $|G : C_G(x)|$ and $|C_G(x) : \langle x\rangle|$ is finite, which of course means that elements with infinitely many conjugates must have small centralizers. Thus groups with the AFC-property generalize FC-groups. Clearly, *Tarski groups* (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) have the AFC-property, but it was proved by A. Shalev [12] that the FC-centre of any locally finite AFC-group has finite index. Moreover, it has been shown in [4] that locally (soluble-by-finite) AFC-groups have finite rank over the FC-centre (here a group G is said to have *finite rank* if there exists a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements).

If $\mathfrak{X}$ is a class of groups, a group G is said to be *minimal non-$\mathfrak{X}$* if G is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. Minimal non-$\mathfrak{X}$ groups have been investigated for several different choices of $\mathfrak{X}$. In particular,

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V. V. Belyaev and N. F. Sesekin [1] completely described the structure of a minimal non-FC group G, under the assumption that G admits a non-trivial homomorphic image which is either finite or abelian. It seems to be unknown whether perfect locally finite minimal non-FC groups exist, but it was proved that such a group should be a p-group for some prime number p (see [9]).

The aim of this paper is to study minimal non-AFC groups in the framework of locally (soluble-by-finite) groups. We will prove that in this universe any minimal non-AFC group admitting proper subgroups of finite index has a large FC-centre, like AFC-groups themselves. Although imperfect minimal non-AFC groups need not contain proper subgroups of finite index, they have a restricted structure, and in particular the FC-property holds for their commutator subgroups. Locally nilpotent minimal non-AFC groups will be studied in the last section.

Most of our notation is standard and can be found in [11].

2. Preliminaries. It is easy to show that the class of minimal non-FC groups and that of minimal non-AFC groups are incomparable. In fact, as every proper subgroup of the locally dihedral 2-group $D(2^\infty)$ is either finite or abelian, $D(2^\infty)$ is a minimal non-FC group, and it is also clear that $D(2^\infty)$ has the AFC-property. Take now a prime number p, and let

$$A = \langle a_n \mid n \in \mathbb{N}_0 \rangle \quad \text{and} \quad B = \langle b_n \mid n \in \mathbb{N}_0 \rangle$$

be two groups of type p^∞ with the usual relations

$$a_0 = b_0 = 1, \qquad a_{n+1}^p = a_n, \qquad b_{n+1}^p = b_n.$$

Set $M = A \times B$, and consider the automorphism x of M defined by $a_n^x = b_n$ and $b_n^x = a_n$ for all n. Consider the semidirect product $G = \langle x \rangle \ltimes M$. Then

$$C = \{a_n b_n \mid n \in \mathbb{N}_0\}$$

is a central subgroup of type p^∞ of G and $C_G(x) = C \times \langle x \rangle$, so that both $|G : C_G(x)|$ and $|C_G(x) : \langle x \rangle|$ are infinite. Therefore G is not an AFC-group. On the other hand, if X is any infinite proper non-abelian subgroup of G, then $X \cap M$ is an infinite proper subgroup of M, and so it contains a subgroup P of type p^∞ which has finite index. Clearly, P also has finite index in X, so that every infinite subgroup of X has finite index, and hence the AFC-property holds for X. Therefore the group G is minimal non-AFC. Moreover, G is not minimal non-FC, because

$$U = \{a_n^{-1} b_n \mid n \in \mathbb{N}_0\}$$

is a normal subgroup of type p^∞ of G, and $\langle U, x \rangle$ is a proper subgroup of G which is not FC.

Recall that the *Baer radical* of a group G is the subgroup generated by all cyclic subnormal subgroups of G, and G is a *Baer group* if it coincides with its Baer radical, or equivalently if all its cyclic subgroups are subnormal. Of course, the Baer radical is locally nilpotent and contains the Fitting subgroup of G. It turns out that the Baer radical of any AFC-group is contained in the FC-centre, and in particular any Baer group with the AFC-property is an FC-group [4, Theorem 3.2]. As mentioned in the introduction, Shalev proved that any locally finite AFC-group is finite over the FC-centre. Our next two lemmas show that all locally (soluble-by-finite) AFC-groups have a large FC-centre.

LEMMA 2.1. *Let G be an infinite locally (soluble-by-finite) AFC-group. Then the FC-centre of G is infinite.*

Proof. Assume for a contradiction that the FC-centre F of G is finite. Then the group G is soluble-by-finite [4, Theorem 3.7], and so its Baer radical B is infinite, which is impossible, since B is contained in F [4, Theorem 3.2]. ∎

LEMMA 2.2. *Let G be a locally (soluble-by-finite) AFC-group with no proper subgroups of finite index. Then G is abelian.*

Proof. Assume for a contradiction that G is not abelian, so that $G/Z(G)$ must be infinite. On the other hand, the centre and the FC-centre of G coincide, because G has no proper subgroups of finite index, and hence it follows from Shalev's result that G is not periodic. Let x be any element of infinite order of G. If x has infinitely many conjugates in G, then $|C_G(x) : \langle x\rangle|$ is finite. In particular, $Z(G)/\langle x\rangle \cap Z(G)$ is finite, and hence $\langle x\rangle \cap Z(G) \neq \{1\}$, because $Z(G)$ is infinite by Lemma 2.1. It follows that $Z(G)$ is not periodic, so that $G/Z(G)$ is periodic [4, Lemma 2.2], and so even locally finite. But $G/Z(G)$ has the AFC-property [4, Theorem 2.5], so that another application of Shalev's theorem shows that $G/Z(G)$ is nilpotent. Then G is nilpotent, so it coincides with its FC-centre, and hence is abelian. This contradiction completes the proof. ∎

The following result shows that the class of AFC-groups, like that of FC-groups, is *countably recognizable*.

LEMMA 2.3. *Let G be a group whose countable subgroups all have the AFC-property. Then G is an AFC-group.*

Proof. Assume for a contradiction that G contains an element x such that both $C_G(x)/\langle x\rangle$ and the conjugacy class of x in G are infinite. Consider a subgroup H of $C_G(x)$ such that $x \in H$ and $H/\langle x\rangle$ is countably infinite, and let W be a countably infinite subset of G such that $yC_G(x) \neq zC_G(x)$ whenever $y, z \in W$ and $y \neq z$. Then $K = \langle H, W\rangle$ is a countable subgroup of G such that $|K : C_K(x)|$ and $|C_K(x) : \langle x\rangle|$ are both infinite, so that K is not an FC-group. This contradiction proves that G has the AFC-property. ∎

COROLLARY 2.4. *Let G be a minimal non-AFC-group. Then G is countable.*

As infinite simple groups have trivial centre and no proper subgroups of finite index, the following lemma holds.

LEMMA 2.5. *A simple group G has the AFC-property if and only if $|C_G(x) : \langle x\rangle|$ is finite for each non-trivial element x of G. In particular, all abelian subgroups of a simple AFC-group are finitely generated.*

Notice that the combination of Lemma 2.5 with the well-known theorem of Hall–Kulatilaka and Kargapolov shows that any simple locally finite AFC-group must be finite.

Notice that Tarski groups are minimal non-FC, but they obviously have the AFC-property. An example of simple minimal non-AFC group can be constructed in the following way. Consider two odd prime numbers p and q such that q divides $p-1$, and the semidirect product $U = \langle y\rangle \ltimes A$, where A is a group of type p^∞ and y is an element of order q acting fixed-point-freely on A. It follows from a result of V. N. Obratzsov [10] that U can be embedded in a simple 2-generator group G in which every proper non-cyclic subgroup is contained in some conjugate of U. Since U has the AFC-property, the group G is minimal non-AFC by Lemma 2.5. Observe also that U is not an FC-group, and so G is not minimal non-FC.

The problem of the existence of simple locally finite minimal non-AFC groups seems to be more complicated, although it can of course be reduced to the countable case by Corollary 2.4. On the other hand, it is known that if G is any countably infinite simple locally finite group, then there exists a strictly ascending chain $G_1 < G_2 < \cdots$ of finite subgroups such that $G = \bigcup_{n\in\mathbb{N}} G_n$, and for each positive integer n the group G_n contains a maximal normal subgroup M_n for which $G_n \cap M_{n+1} = \{1\}$ [8, Theorem 4.5]. The collection $(G_n, M_n)_{n\in\mathbb{N}}$ is called a *Kegel sequence* of G.

By using the properties of centralizers of elements in locally finite groups described in [6] and [2], and repeating the arguments developed in [9], the following result can be proved.

THEOREM 2.6. *Let G be a simple locally finite minimal non-AFC group. Then G cannot admit a Kegel sequence $(G_n, M_n)_{n\in\mathbb{N}}$ such that $M_n \le Z(G_n)$ for all n.*

This result rules out "most" of the simple locally finite groups; in fact, simple linear groups, the Hall universal group and simple homogeneous symmetric locally finite groups admit Kegel sequences satisfying the condition of Theorem 2.6. On the other hand, there exist simple locally finite groups which cannot have such a Kegel sequence, like for instance the groups constructed by Hickin [7] and by Zalesskiĭ and Serezhkin [14].

Finally, we point out that, as in [9, proof of Lemma 1], it can be shown that a simple locally finite minimal non-AFC group G cannot admit a Kegel sequence $(G_n, M_n)_{n\in\mathbb{N}}$ such that $\langle G_n, M_{n+1}, M_{n+2}, \ldots\rangle$ is an FC-group for all n.

3. Minimal non-AFC groups. Recall that the *finite residual* of a group G is the intersection of all (normal) subgroups of finite index of G, and G is *residually finite* if its finite residual is trivial.

LEMMA 3.1. *Let G be a minimal non-AFC-group. Then there exists a prime number p such that every finite homomorphic image of G is a cyclic p-group. Moreover, if J is the finite residual of G, the factor group G/J is either finite or torsion-free abelian.*

Proof. Let N be a normal subgroup of finite index of G. If $x \in G$ is such that $|G : C_G(x)|$ and $|C_G(x) : \langle x\rangle|$ are both infinite, then x has infinitely many conjugates in $\langle N, x\rangle$, so that $G = \langle N, x\rangle$ and hence G/N is cyclic. Assume for a contradiction that the order of G/N is divisible by two different prime numbers p and q. Then $\langle N, x^p\rangle$ and $\langle N, x^q\rangle$ are proper subgroups of G, so that x^p and x^q have finitely many conjugates in G, and hence also the conjugacy class of x in G is finite. This contradiction shows that $|G/N|$ is a power of a prime number. As the intersection of two subgroups of finite index also has finite index, the first part of the statement is proved.

Let T/J be the subgroup consisting of all elements of finite order of the abelian group G/J. As G/J is residually finite, it follows from the first part of the proof that T/J is a cyclic p-group for some prime number p. Thus there exists a subgroup K/J of G/J such that

$$G/J = T/J \times K/J.$$

On the other hand, every finite homomorphic image of G is cyclic of prime-power order, and hence the subgroups T/J and K/J cannot be both non-trivial. Therefore either $T = J$, and G/J is a torsion-free abelian group, or $K = J$ and G/J is a cyclic p-group. ∎

COROLLARY 3.2. *Let G be a residually finite group whose proper subgroups have the AFC-property. Then G is an AFC-group.*

We are now in a position to describe locally (soluble-by-finite) minimal non-AFC groups in which the finite residual is a proper subgroup.

THEOREM 3.3. *Let G be a locally* (*soluble-by-finite*) *minimal non-AFC group admitting a proper subgroup of finite index. Then the FC-centre F of G is abelian and G/F has prime order.*

Proof. Let N be a proper normal subgroup of finite index of G, and let x be an element of G such that $|G : C_G(x)|$ and $|C_G(x) : \langle x\rangle|$ are both infinite. Of course, x does not belong to N. If k is any positive integer such that $\langle x^k, N\rangle$ is properly contained in G, then x^k has finitely many conjugates in $\langle x^k, N\rangle$, and so also in G, because $\langle x^k, N\rangle$ is an AFC-group and $C_G(x) \leq C_G(x^k)$. This conclusion holds in particular when x^k lies in N.

Assume for a contradiction that G/F is infinite. Obviously, the finite residual J of G is a proper subgroup, and so it has the AFC-property. If G/J is finite, the subgroup J has no proper subgroups of finite index, and so it is abelian by Lemma 2.2, which is impossible, since G/F is infinite. Therefore the factor group G/J is infinite, and so torsion-free abelian by Lemma 3.1. It follows that x has infinite order, and so the FC-centre $F(N)$ of N is not periodic. Then the infinite group $N/F(N)$ is periodic [4, Lemma 2.2]. On the other hand, every abelian subgroup of N has torsion-free rank 1 [4, Lemma 3.4], and hence G has torsion-free rank 1. Thus the torsion-free group G/J is locally cyclic and J is an infinite locally finite subgroup. In particular, J is finite over its FC-centre $F(J)$.

Let $y \in F(J)$. Then $C_G(y)$ is infinite and so y lies in the FC-centre of any subgroup of finite index of G. It follows that $F(J) \subseteq F$, and so $J/F \cap J$ is finite. But $F \cap J$ lies obviously in the centre of J, so that $J/Z(J)$ is finite, and hence J' is finite by the theorem of Schur. Then also G/J' is a minimal non-AFC-group [4, Lemma 2.4] and F/J' is the FC-centre of G/J'. The replacement of G by G/J' allows us to assume without loss of generality that J is abelian. Then J lies in F, and so the periodic group G/F is locally cyclic. Moreover, J is even contained in $Z(F)$, so that F is abelian, since G/J is locally cyclic. By Lemma 3.1 the order of every finite homomorphic image of G is a power of a fixed prime number p, and so

$$G/F = P/F \times Q/F,$$

where P/F is a p-group and Q/F is a divisible p'-group. Then the residually finite group $Q/C_Q(F)$ is trivial, so that $F \subseteq Z(Q)$, and hence Q is abelian. It follows that P/F is infinite, and so it is a group of type p^∞, and again $P/C_P(F)$ is trivial. Therefore $F \subseteq Z(G)$, and so G is abelian.

This contradiction shows that G/F is finite, and so it is a cyclic q-group for some prime number q by Lemma 3.1. In particular, $\langle x^q, F\rangle$ is a proper subgroup of G, and the first part of the proof can be applied to conclude that $x^q \in F$. Therefore G/F has order q, and the proof is complete. ∎

It is well-known that any minimal non-FC group with no proper subgroups of finite index must be perfect. Although the situation is more complicated in the case of minimal non-AFC groups, it turns out that their abelianizations cannot be large.

THEOREM 3.4. *Let G be a minimal non-AFC group which has no proper subgroups of finite index. Then either $G = G'$, or G/G' is a group of type p^∞ for some prime number p.*

Proof. Suppose that $G' \neq G$, and assume for a contradiction that the divisible abelian group G/G' is not of type p^∞ for any prime p. Let $x \in G$ be such that $|G : C_G(x)|$ and $|C_G(x) : \langle x\rangle|$ are both infinite. Since x has infinitely many conjugates in G, the cyclic subgroup $\langle x\rangle$ has infinite index in its normal closure, and in particular $\langle x\rangle$ has infinite index in the proper normal subgroup $L = \langle x, G'\rangle$. The divisible abelian group G/L cannot be of type p^∞ for any prime p, and so there exist proper subgroups U and V of G such that $G = UV$ and $L \leq U \cap V$. But x does not belong to the FC-centre of G, and hence $|U : C_U(x)|$ and $|V : C_V(x)|$ cannot both be finite. On the other hand, U and V are AFC-groups, and so at least one of $|C_U(x) : \langle x\rangle|$ and $|C_V(x) : \langle x\rangle|$ must be finite. It follows that $\langle x\rangle$ has finite index in $C_L(x)$, and so $|C_G(x) : C_L(x)|$ is infinite. Then $C_G(x)L/L$ is an infinite subgroup of G/L, and hence there exists an infinite proper subgroup H/L of G/L such that $H \leq C_G(x)L$. Clearly, $C_G(x)H = C_G(x)L$, and hence

$$H = L(C_G(x) \cap H) = LC_H(x).$$

Therefore the group $C_H(x)/C_L(x) \simeq H/L$ is infinite, and so in particular $|C_H(x) : \langle x\rangle|$ is infinite. But H is an AFC-group, so that $C_H(x)$ has finite index in H, and hence

$$|L : \langle x\rangle| = |L : C_L(x)| \cdot |C_L(x) : \langle x\rangle|$$

is also finite. This contradiction proves the statement. ■

LEMMA 3.5. *Let Q be a group of type p^∞ for some prime number p, and let A be an infinite irreducible Q-module. Then every extension of A by Q is a minimal non-AFC group.*

Proof. Let

$$A \rightarrowtail G \twoheadrightarrow Q$$

be any extension of A by Q, inducing on A the given structure of Q-module, so that A is a minimal normal subgroup of G. Let X be any proper non-abelian subgroup of G. Then $A \cap X$ is a non-trivial normal subgroup of AX, so that $AX \neq G$ and A has finite index in AX. It follows that A is contained in the FC-centre of AX. Moreover, if y is any element of AX, then the centralizer $C_A(y) = C_A(\langle y, A\rangle)$ is normal in G, so that either $C_A(y) = A$ or $C_A(y) = \{1\}$, and hence either $C_{AX}(y)$ has finite index in AX or it is finite. Therefore all proper subgroups of G have the AFC-property, and G is a minimal non-AFC group. ■

The above lemma shows in particular that certain metabelian periodic groups constructed by V. S. Charin [3] are minimal non-AFC and have no proper subgroups of finite index. In fact, let p be a prime number, and let $\mathcal{K}$ be the algebraic closure of the field with p elements. Then the multiplicative group $\mathcal{K}^*$ of $\mathcal{K}$ is a direct product of groups of type q^∞, one for each prime $q \neq p$. Consider a set π of prime numbers other than p, and let Q be the π-component of $\mathcal{K}^*$. If $\mathcal{K}_1$ is subfield of $\mathcal{K}$ generated by Q, and A is the additive group of $\mathcal{K}_1$, we may construct the semidirect product

$$G(p,\pi) = Q \ltimes A,$$

where the action of an element x of Q on A is multiplication by x (in $\mathcal{K}_1$). Note that A is the unique minimal normal subgroup of $G(p,\pi)$. The group $G(p,\pi)$ is called the *Charin group* of type (p,π). If the set π consists of a single prime $q \neq p$, the group $G(p,\pi)$ will be denoted by $G(p,q)$. It follows from Lemma 3.5 that $G(p,q)$ is a minimal non-AFC group for each pair (p,q) of different primes.

THEOREM 3.6. *Let G be a locally (soluble-by-finite) minimal non-AFC group such that $G' \neq G$. Then the commutator subgroup G' is an FC-group.*

Proof. If the group G contains a proper subgroup of finite index, then the commutator subgroup G' lies in the FC-centre of G by Theorem 3.3, and the statement is obvious. Suppose now that G has no proper subgroups of finite index, so that Theorem 3.4 shows that G/G' is a group of type p^∞ for some prime number p. Let F be the FC-centre of G'. Since G' has the AFC-property, the factor group G'/F is soluble-by-finite and has finite rank [4, Theorem 3.7]. If S/F is the largest soluble normal subgroup of G'/F, the group G'/S is finite, and so it is contained in the centre of G/S, because G has no proper subgroups of finite index. It follows that G'/F is soluble. Let T/FG'' be the subgroup of all elements of finite order of the abelian group G'/FG''. Then G'/T is a torsion-free abelian group of finite rank, and so all its periodic groups of automorphisms are finite (see for instance [13, Theorem 9.33]). Thus G'/T lies in the centre of G/T, and so G/T is abelian. It follows that $T = G'$, so that G'/FG'' is a periodic abelian group of finite rank, and hence any periodic group of automorphisms of a primary component of G'/FG'' is finite. Therefore G'/FG'' is contained in the centre of G/FG'', so G/FG'' is abelian, and hence $G' = FG''$. On the other hand, G'/F is soluble, and so $G' = F$ is an FC-group. ∎

LEMMA 3.7. *Let G be a group containing a finite normal subgroup N such that G/N is a minimal non-AFC group. If G has no proper subgroups of finite index, then it is minimal non-AFC.*

Proof. Let X be any proper subgroup of G. Then $|G : X|$ is infinite, so that $H = XN$ is a proper subgroup of G, and hence H/N has the AFC-property. Let $h \in H$ admit infinitely many conjugates in H. Then the coset hN has infinitely many conjugates in H/N, and so the cyclic subgroup $\langle hN\rangle$ has finite index in the centralizer $C_{H/N}(hN)$. It follows that $\langle h\rangle$ has finite index in $C_H(x)$, so that H is an AFC-group, and hence the AFC-property holds also for X. Therefore all proper subgroups of G are AFC-groups, and G is minimal non-AFC. ∎

Theorem 3.6 shows that the commutator subgroup of a soluble minimal non-AFC group cannot be too large. Actually, we have no examples in which the commutator subgroup is not abelian. The construction of a group with this property would certainly be possible if we could find a metabelian minimal non-AFC group Q with no proper subgroups of finite index such that the Schur multiplicator $M(Q)$ is not divisible. In fact, if p is a prime number such that $M(Q)^p \neq M(Q)$ and C is a group of order p, then there exists a non-split central extension

$$C \rightarrowtail G \twoheadrightarrow Q$$

and it follows from Lemma 3.7 that G is a soluble minimal non-AFC group with G'' of order p. In particular, we leave as an open question whether the Schur multiplicator of the Charin group $G(p,q)$ has a homomorphic image of order p.

4. The locally nilpotent case. It will be proved that the study of locally nilpotent minimal non-AFC groups reduces either to the study of minimal non-FC groups or to groups containing a proper subgroup of finite index.

THEOREM 4.1. *Let G be a locally nilpotent minimal non-AFC-group. Then G is either hypercentral or minimal non-FC.*

Proof. Assume that G is not hypercentral. As any locally nilpotent AFC-group is hypercentral [4, Corollary 3.10], G is a minimal non-hypercentral group. Then G is locally finite, its centre $Z(G)$ is the last term of the upper central series, and $HK \neq G$ whenever H is a proper subgroup and K is a proper normal subgroup of G [5, Theorem 3.8]. Moreover, Theorem 3.3 implies that G cannot have proper subgroups of finite index.

Suppose first that $G' \neq G$, so that G/G' is a group of type p^∞ for some prime p by Theorem 3.4. For each positive integer n, let U_n/G' be the subgroup of order p^n of G/G'. Assume that there exists m such that the proper normal subgroup $U = U_m Z(G)$ is not an FC-group. Since U has the AFC-property, it contains some element with finite centralizer, and in particular $Z(U)$ must be finite. But G has no proper subgroups of finite index, so that $Z(U)$ lies in $Z(G)$, and hence $Z(U) = Z(G)$. Moreover, $U/Z(G)$ cannot have the FC-property, so that its centre is likewise finite, and hence is contained in $Z(G/Z(G)) = \{1\}$. Thus $Z(U) = Z_2(U)$, a contradiction, because U is hypercentral. It follows that U_n is an FC-group for all positive integers n. If X is any proper subgroup of G, then XG' is likewise properly contained in G, so that $XG' = U_n$ for some n, and hence X is an FC-group. Therefore all proper subgroups of G have the FC-property, and G is minimal non-FC.

Suppose now that $G' = G$, and assume that G properly contains its Fitting subgroup F. Let N/F be any proper normal subgroup of G/F. As N is a locally finite AFC-group, it is nilpotent-by-finite, so that its Fitting subgroup is a nilpotent normal subgroup of G. Thus F is the Fitting subgroup of N, and hence N/F is finite. On the other hand, G is clearly generated by its proper normal subgroups, and so G/F is an FC-group, a contradiction since G is perfect and locally nilpotent. Therefore $G = F$ is generated by its nilpotent normal subgroups, and so all its subgroups have the same property. As every nilpotent normal subgroup of an AFC-group lies in the FC-centre, it follows that all proper subgroups of G have the FC-property, and hence G is minimal non-FC also in this case. ∎

Corollary 4.2. *A perfect locally nilpotent group G is minimal non-AFC if and only if it is minimal non-FC.*

Proof. It is well-known that any hypercentral group have a proper commutator subgroup, so that G cannot be hypercentral. Thus it follows from Theorem 4.1 that if G is minimal non-AFC, then it is also minimal non-FC. Conversely, if G is a minimal non-FC group, then G is minimal non-AFC, as all locally nilpotent AFC-groups are hypercentral. ∎

Corollary 4.3. *Let G be a locally nilpotent minimal non-AFC group such that $G' \neq G$. Then G contains a proper subgroup of finite index.*

Proof. Assume for a contradiction that G has no proper subgroups of finite index. Then G cannot be minimal non-FC, and hence it is hypercentral by Theorem 4.1. It follows that the subgroup T of all elements of finite order of G is contained in the centre $Z(G)$ [11, Part 2, Theorem 9.23]. Let X/T be any proper subgroup of G/T. Then X/T is a torsion-free locally nilpotent AFC-group, so that it is abelian-by-finite [4, Corollary 4.2], and hence even abelian. It follows that G/T is abelian, and so G is nilpotent. Therefore all proper subgroups of G have the FC-property, and G is minimal non-FC. This contradiction completes the proof. ∎

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