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WITOLD ROTER (1932–2015)

BY

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Witold Roter was born on September 20th, 1932 and died on June 19th, 2015. He authored or co-authored 40 papers in differential geometry, published between 1961 and 2010. His nine Ph.D. advisees, listed here along with the year of receiving the degree, are: Czesław Konopka (1972), Edward Głodek (1973), Andrzej Gębarowski (1975), Andrzej Derdziński (1976), Zbigniew Olszak (1978), Ryszard Deszcz (1980), Marian Hotloś (1980), Wiesław Grycak (1984), and Marek Lewkowicz (1989).

Also, for many years, he ran the Wrocław seminar on differential geometry, first started by Władysław Ślebodziński, who had been Witold Roter's Ph.D. advisor. For more biographical information (in Polish), see Zbigniew Olszak's article [44].

This is a brief summary of Witold Roter's selected results, divided into four sections devoted to separate topics. Since he repeatedly returned to questions he had worked on earlier, our presentation is not chronological.

1. Parallel Weyl tensor in the Riemannian case. The curvature tensor R of a given n-dimensional pseudo-Riemannian manifold (M,g) is naturally decomposed into the sum $R = S + E + W$ of its irreducible components [41, p. 47]. The first two correspond to the scalar curvature and Einstein tensor (the traceless part of the Ricci tensor). The third component is the *Weyl tensor* W, also known as the conformal curvature tensor, which is of interest only in dimensions $n \geq 4$ since, for algebraic reasons, $W = 0$ whenever $n \leq 3$.

Viewed as a $(1,3)$ tensor field, so that it sends three vector fields trilinearly to a vector field, W is a *conformal invariant*: it remains unchanged when the metric g is replaced by the product ϕg, where ϕ is any smooth positive function. Thus, $W = 0$ if the metric is conformally flat (meaning that it is locally of the form ϕg, with flat metrics g). Conversely, for $n \geq 4$,

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the condition $W = 0$ is also sufficient for conformal flatness, as shown by Jan Arnoldus Schouten [45] back in 1921.

Pseudo-Riemannian manifolds of dimensions $n \geq 4$ whose Weyl tensor is parallel, that is, invariant under parallel transport (or, equivalently, satisfies the condition $\nabla W = 0$), were first studied in the early 1960s. One necessarily has $\nabla W = 0$ when the metric is conformally flat ($W = 0$), or locally symmetric ($\nabla R = 0$). Most authors discussing manifolds or metrics with $\nabla W = 0$ referred to them as *conformally symmetric*, which may lead to confusion, since the term *conformal symmetry* also appears in the literature with some different meanings. Therefore, following [37], we speak here of manifolds or metrics with *parallel Weyl tensor*, using the acronym "ECS" (short for *essentially conformally symmetric*) for those among them which are neither conformally flat, nor locally symmetric.

Requiring the Weyl tensor W to be parallel is one of the most natural conditions that one may impose on the curvature tensor R of a given pseudo-Riemannian manifold. By comparison, the irreducible component S mentioned above is parallel if and only if the metric has constant scalar curvature, while the equality $\nabla E = 0$ is equivalent, in dimensions $n \geq 3$, to requiring that the Ricci tensor be parallel (which, in the Riemannian case, characterizes Einstein metrics and—locally—their Cartesian products). Finally, it is precisely for locally symmetric manifolds that the irreducible components are all parallel.

Returning to ECS metrics, one encounters two obvious questions: do they exist at all, and can they be Riemannian (that is, positive definite)? Both questions were answered by Witold Roter: the first one in the affirmative, the second in the negative. A detailed presentation of the former result [14, Corollary 3] will be given in Section 2; the latter [19, Theorem 2] can be stated as follows.

THEOREM 1.1. *In a Riemannian manifold of any dimension $n \geq 4$ the Weyl tensor cannot be parallel except in the trivial case, that is, when the metric is conformally flat or locally symmetric.*

In other words, no ECS metric is positive definite.

The next paragraph provides a historical commentary clarifying the issues of both Witold Roter's priority in establishing the above result, and the full credit that he deserves for it.

In the paper [17], presented on Sept. 30, 1975, Witold Roter proved a weaker version of Theorem 1.1 (for $n \geq 5$). The same weaker version was independently obtained by Teturo Miyazawa [43], whose paper was, however, submitted almost a year later, on Sept. 12, 1976.

Witold Roter's proof for the more general case $n \geq 4$ appeared in the paper [19], joint with the author of this note. The latter's contribution, however, dealt just with the Lorentzian case [19, pp. 258–259], and had nothing to do with Theorem 1.1.

2. Local classification of ECS metrics. By ECS manifolds we mean here—just as in Section 1—pseudo-Riemannian manifolds (M, g) of dimensions $n \geq 4$ satisfying, everywhere in M, the relations $\nabla W = 0$ and $W \neq 0$, while having $\nabla R \neq 0$ at some point. We then also call g an ECS metric.

Nonexistence of positive definite ECS metrics is the conclusion of Theorem 1.1, proved by Witold Roter [19, Theorem 2] back in 1976. He also showed [14, Corollary 3], in 1973, that pseudo-Riemannian ECS metrics do exist, in all dimensions $n \geq 4$, and represent all indefinite metric signatures.

This last result is a direct consequence of the construction, described in (1)–(2) below, of metrics with parallel Weyl tensor [14, Theorem 3]. It was the first of two such constructions discovered by Witold Roter; a description of the other one [36, Theorem 21.1], found by him jointly with the present author, will be preceded by some definitions.

Both constructions use a notational convention according to which, if a manifold M is the Cartesian product of some factor manifolds, covariant tensor fields on the factors, including functions, are treated— without changing the notation—as tensor fields on M. The following descriptions are borrowed from [37, formulae (3.1), (4.2)].

The parameters needed for the first construction consist of a C^∞ function $f : I \to \mathbb{R}$ on an open interval $I \subset \mathbb{R}$, a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on a real vector space V of dimension $n - 2 \geq 2$, and a traceless linear operator $A : V \to V$, self-adjoint relative to $\langle\,,\rangle$. With t, s denoting the Cartesian coordinates in $\mathbb{R}^2$, products of differentials standing for their symmetric products, the symbol κ for the function $I \times \mathbb{R} \times V \to \mathbb{R}$ given by $\kappa(t, s, v) = f(t)\langle v, v\rangle + \langle Av, v\rangle$, and δ for the flat "constant" pseudo-Riemannian metric on V corresponding to $\langle\,,\rangle$, the n-dimensional pseudo-Riemannian manifold

$$(M, g) = (I \times \mathbb{R} \times V, \kappa\, dt^2 + dt\, ds + \delta) \tag{1}$$

has parallel Weyl tensor. Its conformal flatness (or local symmetry) is equivalent to the condition $A = 0$ (or, respectively, to constancy of f).

Choosing the above parameters so that

$$A \neq 0 \quad \text{and} \quad f \text{ is nonconstant,} \tag{2}$$

we thus obtain an example of an ECS metric.

Let us now proceed with the definitions mentioned earlier. One calls a connection on a manifold Q *projectively flat* if it is torsion-free and, locally, has the same unparametrized geodesics as some (locally defined) flat connections. By the *Riemann extension* of a connection D on Q we mean the pseudo-Riemannian metric h^{D} on T^*Q defined by requiring that all D-horizontal vectors be null, while $h_x^{\mathrm{D}}(\xi, w) = \xi(d\pi_x w)$ for every point $x \in T^*Q$, every vector $w \in T_x T^*Q$, and every vertical vector $\xi \in \operatorname{Ker} d\pi_x = T^*_{\pi(x)}Q$, where $\pi : T^*Q \to Q$ is the bundle projection.

The parameters for the second construction are: a surface Q with a projectively flat connection D, a nonzero 2-form ζ on Q, parallel relative to D, a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on a real vector space V of dimension $n - 4 \geq 0$, a sign factor $\varepsilon = \pm 1$, and a twice contravariant symmetric smooth tensor field ϕ on Q (that is, a smooth section of $[TQ]^{\odot 2}$), satisfying, along with the Ricci tensor ρ^{D} of D, the differential equation

$$\operatorname{div}^{\mathrm{D}}(\operatorname{div}^{\mathrm{D}}\phi) + (\rho^{\mathrm{D}}, \phi) = \varepsilon \tag{3}$$

(in coordinates: $\phi^{jk}{}_{,jk} + R_{jk}\phi^{jk} = \varepsilon$). The isomorphism $TQ \to T^*Q$ acting on vector fields w via $w \mapsto \zeta(w, \cdot)$ induces an obvious isomorphism $[TQ]^{\odot 2} \to [T^*Q]^{\odot 2}$, which we may use to identify ϕ with a smooth section τ of $[T^*Q]^{\odot 2}$, that is, with a twice *covariant* symmetric smooth tensor field τ on Q. In coordinates, $\tau_{jk} = \zeta_{jl}\zeta_{km}\phi^{lm}$. Denoting by δ the flat "constant" pseudo-Riemannian metric on V corresponding to $\langle\,,\rangle$, and by θ the function $V \to \mathbb{R}$ with $\theta(v) = \langle v, v\rangle$, we now define the n-dimensional pseudo-Riemannian manifold

$$(M, g) = (T^*Q \times V, h^{\mathrm{D}} - 2\tau + \delta - \theta\rho^{\mathrm{D}}) \tag{4}$$

with nonzero parallel Weyl tensor, which is locally symmetric if and only if D has parallel Ricci tensor ($\mathrm{D}\rho^{\mathrm{D}} = 0$). Choosing D so that

$$\mathrm{D}\rho^{\mathrm{D}} \neq 0 \quad \text{somewhere in } Q, \tag{5}$$

we thus obtain another class of examples of ECS metrics.

It is of course a trivial exercise to verify that the two families of examples, (1)–(2) and (4)–(5), actually consist of ECS metrics. These examples are, however, *locally universal*: Witold Roter proved, jointly with the present author, the following classification result [36, Theorem 21.1], [39, Theorem 4.1]:

Every ECS manifold has, locally, the form (1)–(2) *or* (4)–(5)*, with parameters satisfying the conditions listed above.*

This result was obtained in the years 2006–2007. A significant part of it is, however, due to Witold Roter alone. As early as 1973 he showed [14, Theorem 3] that at points of *general position*—where the Ricci tensor and its covariant derivative are both nonzero—an ECS manifold having the additional property of Ricci-recurrency must locally, up to isometry, arise from the construction (1)–(2).

Ricci-recurrent metrics will be discussed in Section 4.

The two families of examples, (1)–(2) and (4)–(5), are easily verified to be mutually disjoint.

3. Compact ECS manifolds. Recall that ECS metrics are the pseudo-Riemannian metrics that have parallel Weyl tensor without being conformally flat or locally symmetric.

Can such a metric exist on a compact manifold? Witold Roter and the present author showed that the answer is 'yes' [40]:

THEOREM 3.1. *In every dimension $n \geq 5$ such that $n \equiv 5 \pmod 3$ there exists a compact ECS manifold with any prescribed indefinite metric signature, diffeomorphic to a nontrivial torus bundle over the circle.*

Note that, just like conformal flatness, the property of having parallel Weyl tensor is not generally preserved by the Cartesian product operation, so that there is no trivial way of extending the above existence result to other dimensions. In particular, the existence problem for $n = 4$ is still open. It is known, however, that the presence of an ECS metric on a given compact manifold M imposes specific restrictions on its fundamental group $\pi_1 M$, Euler characteristic $\chi(M)$, and real Pontryagin classes $p_i(M) \in H^{4i}(M, \mathbb{R})$. Further topological consequences arise in the Lorentzian case. Namely, Witold Roter and the present author proved the following facts [38]:

THEOREM 3.2. *Let (M, g) be a compact ECS manifold. Then the fundamental group $\pi_1 M$ is infinite, $\chi(M) = 0$, and $p_i(M) \in H^{4i}(M, \mathbb{R})$ is zero for all $i \geq 1$.*

THEOREM 3.3. *All four-dimensional Lorentzian-signature ECS manifolds are noncompact.*

THEOREM 3.4. *If (M, g) is a compact Lorentzian ECS manifold, then up to a two-fold covering, M is the total space of a bundle over the circle with a fibre that carries a flat torsion-free connection admitting a nontrivial parallel vector field.*

The last two results show that—at least for the Lorentzian signature—some properties of ECS manifolds whose existence is guaranteed by Theorem 3.1 are not completely accidental.

4. Recurrent metrics and their generalizations. A tensor field H on a manifold with a fixed torsion-free connection ∇ is called *recurrent* if H are $\nabla_v H$ are linearly dependent at every point, for every vector field v. This is equivalent to the relation $\nabla H = \xi \otimes H$ at points where $H \neq 0$, for some 1-form ξ. Similarly, 2-*recurrency* of a tensor field H relative to ∇ means that $\nabla^2 H = \tau \otimes H$ everywhere in the set given by $H \neq 0$, with some twice covariant tensor field τ.

By a *recurrent* or *Ricci-recurrent*, or *conformally recurrent* manifold one means a pseudo-Riemannian manifold (M, g) whose curvature tensor R, or Ricci tensor ρ or, respectively, Weyl tensor W is recurrent relative to the Levi-Civita connection ∇.

Witold Roter's three earliest papers [1]–[3] dealt with manifolds that are recurrent, or have a 2-recurrent curvature tensor. He generalized there some results of Nikolay S. Sinyukov [46] and André Lichnerowicz [42]. It is also worthwhile to mention his six much later papers [16], [25]–[28], [32], devoted to conformally recurrent metrics. They contain, in particular, constructions of nontrivial examples of such metrics [16], [25], and various results on conformal relations between them [26], [28], [32]. In [27] Witold Roter introduced the class of *simple* conformally recurrent metrics, defined to be locally conformal to metrics with nonzero parallel Weyl tensor, gave examples of such metrics which are not recurrent, and proved that—except in the locally symmetric case —their scalar curvature is identically zero.

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