

Copolar convexity

ALEXANDER RASHKOVSKII (Stavanger)

Abstract. We introduce a new operation, *copolar addition*, on unbounded convex subsets of the positive orthant of \mathbb{R}^n and establish convexity of the covolumes of the corresponding convex combinations. The proof is based on a technique of geodesics of plurisubharmonic functions. As an application, we show that there are no relative extremal functions inside a nonconstant geodesic curve between two toric relative extremal functions.

1. Introduction. Recall that the volumes of the Minkowski convex combinations $P_t := (1-t)P_0 + tP_1$ of two convex bodies $P_j \subset \mathbb{R}^n$ satisfy the classical Brunn–Minkowski inequality

$$\text{Vol}(P_t)^{1/n} \geq (1-t)\text{Vol}(P_0)^{1/n} + t\text{Vol}(P_1)^{1/n}.$$

For its various aspects and consequences, see [6]. We would like to mention here a few variants of the inequality. In the first one (proved under certain symmetry conditions), the Minkowski combinations P_t are replaced by certain log-Minkowski combinations [3], [19], [21]. Another version considers, instead of the combinations P_t , balls in the norms of complex Calderón interpolated spaces [4]. Finally, in [12], convex bodies are replaced by *coconvex* ones, that is, bounded subsets of a closed strictly convex cone $C \subset \mathbb{R}^n$ whose complements to C are convex, the operation on the coconvex set being taking the complement to the Minkowski sum of their complements. Consideration of such sets is motivated by their relations (mostly by means of the Newton polyhedra [13]) to singularity theory, commutative algebra, and complex analysis [11], [15], [16], [17], [20]. In such a setting, the *reversed* Brunn–Minkowski inequality for coconvex sets was established in [12]; in

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terms of their complements P_t , it can be written as

$$(1.1) \quad \text{Covol}(P_t)^{1/n} \leq (1-t) \text{Covol}(P_0)^{1/n} + t \text{Covol}(P_1)^{1/n},$$

where

$$(1.2) \quad \text{Covol}(P) = \text{Vol}(C \setminus P).$$

In this note, we will deal (for the sake of simplicity) with proper convex subsets of the positive orthant \mathbb{R}_+^n of \mathbb{R}^n with bounded complements, and Minkowski addition will be replaced with a different operation, \oplus , which we will call *copolar addition* (see (3.1) and (3.2) for the definition), also motivated by consideration of Newton polyhedra and thus relevant to the aforementioned application areas. In a sense, such an operation is more natural than Minkowski addition; for example, the copolar sum of cosimplices (complements of simplices) is still a cosimplex, while their Minkowski sum is a more complicated polyhedron. In Theorem 4.1, we prove the inequality

$$(1.3) \quad \text{Covol}(P_t^\oplus) \leq (1-t) \text{Covol}(P_0) + t \text{Covol}(P_1),$$

which is strict unless $P_0 = P_1$. We give a simple example (see the Remark after the statement of Theorem 4.1) for which (1.3) is sharper than (1.1).

We prove (1.3) by extending recent results on geodesics of plurisubharmonic functions [18] to relative extremal functions in a toric setting. In particular, we prove that, as in the case of compact manifolds, the Legendre transform of the convex image of any toric geodesic u_t on a bounded Reinhardt domain is an affine function of t (Theorem 5.1).

As follows from the definition of copolar combinations (see Sections 3 and 4), (1.3) can be equivalently described as *convexity* of the Monge–Ampère capacity of multiplicative combinations K_t^\times of logarithmically convex Reinhardt compacts of \mathbb{C}^n . Such combinations appear in the toric case of complex Calderón interpolated spaces; the corresponding version of the Brunn–Minkowski inequality [4], [5] expresses however *logarithmic concavity* of the volumes of K_t^\times (for convex K_j).

The results are applied to plurisubharmonic functions and their singularities. In particular, we show that there is no relative extremal function inside a nonconstant geodesic curve between two toric relative extremal functions (Corollary 6.2).

2. Newton sets and Monge–Ampère masses. Here we recall some results on the Monge–Ampère masses of plurisubharmonic (psh) functions, related to the notion of Newton polyhedron. They can be viewed as a development of the fact that the multiplicity of a generic holomorphic mapping at the origin of \mathbb{C}^n equals $n!$ times the covolume of its Newton polyhedron [13].

2.1. The first set of results concerns characteristics of singularities of psh functions, obtained in [14]–[17]. Let φ be a plurisubharmonic function on a neighborhood of $0 \in \mathbb{C}^n$ such that $\varphi(0) = -\infty$, and let Ψ_φ be its indicator, i.e.,

$$(2.1) \quad \Psi_\varphi(z) = \limsup_{y \rightarrow z} \lim_{m \rightarrow \infty} m^{-1} \varphi(y_1^m, \dots, y_n^m).$$

It is known that Ψ_φ is a *toric* (that is, independent of the arguments of the variables) nonpositive plurisubharmonic function in the unit polydisk \mathbb{D}^n , satisfying $\Psi_\varphi(|z_1|^c, \dots, |z_n|^c) = c\Psi_\varphi(z)$ for any $c > 0$, and $\varphi \leq \Psi_\varphi + O(1)$ near 0. If the Monge–Ampère current $(dd^c\varphi)^n$ is well defined near 0 (for example, this is so if 0 is an isolated singularity of φ), then the Monge–Ampère measure of Ψ_φ is concentrated at 0:

$$(2.2) \quad (dd^c\Psi_\varphi)^n = N_\varphi \delta_0$$

for some $N_\varphi \geq 0$ (the *Newton number* of φ at 0), and $(dd^c\varphi)^n(0) \geq N_\varphi$.

The Newton number can be computed as follows. Denote

$$\mathbb{R}_+^n = \{a \in \mathbb{R}^n : a_j > 0, 1 \leq j \leq n\}, \quad \mathbb{R}_-^n = -\mathbb{R}_+^n,$$

and let the mappings $\text{Exp} : \mathbb{R}_-^n \rightarrow \mathbb{D}^n$ and $\text{Log} : \mathbb{D}^n \cap (\mathbb{C}_*)^n \rightarrow \mathbb{R}_-^n$ be defined as

$$\text{Exp } s = (e^{s_1}, \dots, e^{s_n}), \quad \text{Log } z = (\log |z_1|, \dots, \log |z_n|).$$

The *convex image* $\psi_\varphi = \text{Exp}^* \Psi_\varphi$ of the indicator Ψ_φ is a convex, positive homogeneous function on \mathbb{R}_-^n , i.e., $\psi_\varphi(cs) = c\psi_\varphi(s)$ for any $c > 0$ and all $s \in \mathbb{R}_-^n$. Therefore, ψ_φ is the restriction of the support function of a convex set $\Gamma_\varphi \subset \mathbb{R}_+^n$ to \mathbb{R}_-^n :

$$\psi_\varphi(s) = \sup_{a \in \Gamma_\varphi} \langle a, s \rangle, \quad s \in \mathbb{R}_-^n$$

(note that the support function of Γ_φ equals $+\infty$ outside $\overline{\mathbb{R}_+^n}$). By relating the complex and real Monge–Ampère operators, one then computes

$$(2.3) \quad N_\varphi = n! \text{Covol}(\Gamma_\varphi),$$

where covolume is defined by (1.2) with $C = \mathbb{R}_+^n$. Moreover, the *generalized Lelong number* (in the sense of Demailly) $\nu(u, \varphi) := dd^c u \wedge (dd^c \Psi_\varphi)^{n-1}(0)$ of any psh function u near 0 with respect to Ψ_φ can be represented as a mixed covolume of the corresponding convex sets and computed as

$$\nu(u, \varphi) = n! \int_{L_\varphi} |\psi_u(s)| d\gamma_\varphi(s),$$

where γ_φ is a positive measure supported on the set of extreme points of the convex set

$$(2.4) \quad L_\varphi = \{s \in \mathbb{R}_-^n : \psi_\varphi(s) \leq -1\}.$$

For a holomorphic mapping F , the set $\Gamma_{\log|F|}$ is the *Newton polyhedron of F at 0* in the sense of [13], that is, the convex hull of the set

$$\bigcup_{k \in A} (k + \mathbb{R}_+^n),$$

where A is the collection of all multi-indices k such that z^k has a nonzero coefficient in the Taylor expansion of at least one of the components of F .

2.2. A slightly different setting studied in [1] concerns relative Monge–Ampère capacities $\text{Cap}(K)$ of subsets K of bounded hyperconvex domains $\Omega \subset \mathbb{C}^n$. We recall that $\text{Cap}(K)$ equals the total Monge–Ampère mass of the *relative extremal function*

$$(2.5) \quad \omega_K(z) = \limsup_{y \rightarrow z} \sup\{u(y) : u \in \text{PSH}(\Omega), u < 0, u|_K \leq -1\};$$

namely,

$$\text{Cap}(K) = (dd^c \omega_K)^n(\Omega) = (dd^c \omega_K)^n(K).$$

Let Ω be the unit polydisk \mathbb{D}^n , $D \Subset \mathbb{D}^n$ be a nonempty complete logarithmically convex Reinhardt domain, and $L = L_K \subset \mathbb{R}_-^n$ be the logarithmic image of $K = \overline{D}$, that is, $L = \text{Log}(K \cap (\mathbb{C}_*)^n)$. Then, by [1, Lemma 3],

$$(2.6) \quad \omega_K(z) = \sup_{a \in \mathbb{R}_+^n} \frac{\sum a_k \log |z_k|}{|h_L(a)|}, \quad z \in \mathbb{D}^n \setminus D,$$

and, by [1, Theorem 7],

$$(2.7) \quad \text{Cap}(K) = n! \text{Covol}(\Gamma_L),$$

where

$$(2.8) \quad \Gamma_L = \{a \in \mathbb{R}_+^n : h_L(a) \leq -1\}$$

and $h_L(a)$ is the restriction of the support function of the (convex) set L to \mathbb{R}_+^n :

$$h_L(a) = \sup_{s \in L} \langle a, s \rangle, \quad s \in \mathbb{R}_+^n$$

(and h_L equals $+\infty$ outside $\overline{\mathbb{R}_+^n}$). Note that since the right hand side of (2.6) is homogeneous in a , it coincides with $h_{\Gamma_L}(\text{Log } z)$, so (2.6) can be written as

$$(2.9) \quad \omega_K(z) = \max\{h_{\Gamma_L}(\text{Log } z), -1\}.$$

In both the examples, one deals with pairs $\{L, \Gamma\}$ of unbounded convex subsets of \mathbb{R}_+^n and \mathbb{R}_-^n , and the Monge–Ampère masses are computed as covolumes of the corresponding subsets of \mathbb{R}_+^n . The difference here is that in the first setting one starts with a set in \mathbb{R}_+^n and arrives at a set in \mathbb{R}_-^n , while in the second case the things go the other way round.

3. Copolars. Now let us for a moment strip away the plurisubharmonic content of the above settings and concentrate on its convex counterpart.

Let C_+ denote the collection of closed convex subsets Γ of $\overline{\mathbb{R}_+^n}$ that are complete in the sense that $\Gamma + \mathbb{R}_+^n \subset \Gamma$. Similarly, we introduce the family $C_- = -C_+$ of closed convex complete subsets of $\overline{\mathbb{R}_-^n}$.

Given a set $A \in C_+$ or $A \in C_-$, we introduce its *copolar* A° as

$$(3.1) \quad A^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq -1 \ \forall y \in A\}.$$

Equivalently, the copolars can be described by means of the *support functions*

$$h_A(x) = \sup_{y \in A} \langle x, y \rangle, \quad x \in \mathbb{R}^n,$$

namely, $A^\circ = \{x \in \mathbb{R}^n : h_A(x) \leq -1\}$.

PROPOSITION 3.1.

- (1) If $A \in C_+$, then $A^\circ \subset \overline{\mathbb{R}_-^n}$.
- (2) If $A \in C_-$, then $A^\circ \subset \overline{\mathbb{R}_+^n}$.
- (3) $(C_+)^\circ = C_-$ and $(C_-)^\circ = C_+$.
- (4) If A belongs either to C_+ or to C_- , then $(A^\circ)^\circ = A$.

Proof. The first two assertions follow from the completeness of A , and the third one follows from its convexity.

To prove the last one, assume that $A \in C_-$, consider the support functions h_A and h_{A° and set $B = \{y \in \mathbb{R}_-^n : h_{A^\circ}(y) \leq -1\}$. We claim that $B = A$.

First, let $a \in A$ then $h_A(x) \geq \langle x, a \rangle$ for any $x \in \mathbb{R}_+^n$. Denote

$$\delta A^\circ = \overline{\partial A^\circ \cap \mathbb{R}_+^n} = \{x \in \mathbb{R}^n : h_A(x) = -1\}.$$

Then

$$h_{A^\circ}(a) = \sup\{\langle x, a \rangle : x \in \delta A^\circ\} \leq \sup\{h_A(x) : x \in \delta A^\circ\} = -1,$$

which shows $A \subset B$.

To prove the reverse inclusion, take any $a \in \mathbb{R}_-^n \setminus A$. Since the set A is complete, there exists a point $x^* \in \delta A^\circ$ with $\langle x^*, a \rangle > h_L(a^*)$. We then have

$$h_{A^\circ}(a) = \sup\{\langle x, a \rangle : x \in \delta A^\circ\} \geq \langle x^*, a \rangle > h_L(a^*) = -1,$$

so $a \notin B$, and thus $B \subset A$. ■

So far, we have had no essential difference between the collections C_+ and C_- . From now on, we will be considering a subclass of sets in C_+ whose copolars will not belong to the corresponding subclass of C_- . Namely, we will restrict ourselves to the class CC_+ of *cobounded* sets in C_+ (those with bounded complements in \mathbb{R}_+^n). It is easy to see that $P \in CC_+$ if and only

if $P^\circ \subset s^* + \mathbb{R}_+^n$ for some $s^* \in \mathbb{R}_+^n$; we denote the latter subclass of C_- by UC_- , so $(CC_+)^\circ = UC_-$ and $(UC_-)^\circ = CC_+$.

Let $\mathcal{L}[f]$ denote the *Legendre transform* of a function f :

$$\mathcal{L}[f](y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - f(x) \}.$$

We recall that $\mathcal{L}[f]$ is always a convex, lower semicontinuous (lsc) function, and $\mathcal{L}[\mathcal{L}[f]]$ is the largest convex lsc minorant of f .

In particular, the support function h_A of a closed convex set A is the Legendre transform of the indicator function \mathbb{I}_A of A (which equals 0 on A and $+\infty$ on $\mathbb{R}^n \setminus A$), and the following formula (to be used later on) for cutoff functions expresses copolarity as Legendre duality.

PROPOSITION 3.2. *Let $L \in UC_-$. Then*

$$\mathcal{L}[\max\{h_{L^\circ}, -1\}] = \max\{h_L + 1, 0\}.$$

Proof. As is known,

$$\mathcal{L}[\min\{f, g\}] = \max\{\mathcal{L}[f], \mathcal{L}[g]\}.$$

Applying this to $f = \mathbb{I}_{L^\circ}$ and $g = \mathbb{I}_{\mathbb{R}_+^n} + 1$, we conclude that the Legendre transform of $\max\{h_{L^\circ}, -1\}$ is the largest convex lsc minorant of the function equal to 0 on L° and to 1 on $\mathbb{R}_+^n \setminus L^\circ$. In particular, its restriction to any segment l connecting the origin to an arbitrary $p \in \delta L^\circ$ does not exceed the affine function F_l on l defined by $F_l(0) = 1$ and $F_l(p) = 0$.

Since the function $h_L + 1$ satisfies the bounds, and its restriction to l coincides with F_l for any l , this completes the proof. ■

We introduce *copolar addition* on CC_+ as

$$(3.2) \quad P \oplus Q = (P^\circ + Q^\circ)^\circ.$$

Evidently, $P \oplus Q \in CC_+$ if $P, Q \in CC_+$.

EXAMPLE 3.3. A set $L \in CC_+$ of the form $L = \{a \in \mathbb{R}_+^n : \langle a, b_L \rangle \geq 1\}$ for some vector $b_L \in \mathbb{R}_+^n$ will be called a *cosimplex*. It is easy to see that if P and Q are cosimplices in \mathbb{R}_+^n , then $P \oplus Q$ is a cosimplex as well, and its reference vector $b = b_{P \oplus Q}$ is the one whose components b_j satisfy

$$b_j^{-1} = b_{P_j}^{-1} + b_{Q_j}^{-1}, \quad 1 \leq j \leq n.$$

4. Convexity of covolumes. Given $P_0, P_1 \in CC_+$, we form the collection of *copolar combinations*

$$(4.1) \quad P_t^\oplus = ((1-t)P_0^\circ + tP_1^\circ)^\circ, \quad 0 < t < 1.$$

THEOREM 4.1. *Copolar combinations of sets $P_0, P_1 \in CC_+ \setminus \{\overline{\mathbb{R}_+^n}\}$ satisfy*

$$(4.2) \quad \text{Covol}(P_t^\oplus) \leq (1-t) \text{Covol}(P_0) + t \text{Covol}(P_1), \quad 0 < t < 1;$$

equality occurs in (4.2) for some $t \in (0, 1)$ if and only if $P_0 = P_1$.

REMARK. Unlike standard versions of the Brunn–Minkowski type inequalities, there are no $1/n$ exponents in (4.2). As was shown in [12] (in a more general setting), the covolumes of sets from CC_+ satisfy the reversed Aleksandrov–Fenchel inequality and, as a consequence, the reversed Brunn–Minkowski inequality (1.1) with respect to the standard Minkowski combinations $P_t = (1 - t)P_0 + tP_1$. None of the inclusions $P_t^\oplus \subset P_t$ and $P_t \subset P_t^\oplus$ is generally true. As an example, consider

$$P_0 = \{a \in \mathbb{R}_+^2 : a_1/3 + a_2 \geq 1\}, \quad P_1 = \{a \in \mathbb{R}_+^2 : a_1 + a_2/3 \geq 1\}.$$

Then $P_{1/2}$ is the polyhedron in CC_+ with vertices at $(2, 0)$, $(1/2, 1/2)$ and $(0, 2)$, while $P_{1/2}^\oplus = \{a \in \mathbb{R}_+^2 : a_1 + a_2 \geq 3/2\}$. Since $\text{Covol}(P_{1/2}) = 1$, $\text{Covol}(P_{1/2}^\oplus) = 9/8$ and $\text{Covol}(P_0) = \text{Covol}(P_1)$, in this example (4.2) takes the form $9/8 < 3/2$, while (1.1) is equivalent to $1 < 3/2$.

We do not know if a corresponding inequality for copolar addition involving $\text{Covol}^{1/n}$ is true.

Proof of Theorem 4.1. To prove (4.2), we recall the motivation examples from Section 2 and run them the opposite way. Let $P \in CC_+ \setminus \{\overline{\mathbb{R}_+^n}\}$; then $L := P^\circ \in (CC_+)^\circ$. In the terminology of Section 3, the set Γ_L defined by (2.8) is exactly the copolar L° to L , that is, $\Gamma_L = P$. The set $K := \overline{\text{Exp } L}$ is the closure of a complete logarithmically convex Reinhardt subdomain of the unit polydisk \mathbb{D}^n . By (2.7), we then have

$$(4.3) \quad \text{Cap}(K) = n! \text{Covol}(P).$$

Starting with $P_0, P_1 \in CC_+ \setminus \{\overline{\mathbb{R}_+^n}\}$, we get the family $P_t^\oplus \subset CC_+$ of their copolar combinations, and then, as described above, the corresponding Reinhardt compact subsets K_t^\times of \mathbb{D}^n that are actually multiplicative combinations of K_0 and K_1 :

$$\text{Log } K_t^\times = (1 - t) \text{Log } K_0 + t \text{Log } K_1.$$

Now we use [18, Thm. 4.3] stating that, in this setting,

$$(4.4) \quad \text{Cap}(K^\times) \leq (1 - t) \text{Cap}(K_0) + t \text{Cap}(K_1).$$

By (4.3) and the definition of copolar addition, we get the desired inequality.

The equality statement will be proved in Section 6 (Corollary 6.3). ■

REMARK. The multiplicative combinations K_t^\times of the convex Reinhardt bodies are the toric version of balls in the norms of complex Calderón interpolated spaces, mentioned in the introduction. As was proved in [4], their *volumes* satisfy the (*nonreversed*) Brunn–Minkowski inequality

$$(4.5) \quad \text{Vol}(K_t^\times) \geq \text{Vol}(K_0)^{1-t} \text{Vol}(K_1)^t,$$

which is a bound on the size of K_t^\times opposite to (4.4). Note also that (4.4) means *convexity* of the capacities for the *logarithmically convex* Reinhardt

bodies, while (4.5) expresses *logarithmic concavity* of the volumes for *convex Reinhardt* bodies.

In the next section, we explain where the crucial inequality (4.4) comes from.

5. Representation of toric geodesics. The proof of Theorem 4.1 is based on a technique of geodesics, developed first for metrics on compact Kähler manifolds (see [8] and bibliography therein) and then applied to plurisubharmonic functions on domains of \mathbb{C}^n in [18], [10]. In the latter setting, plurisubharmonic functions on a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ with zero boundary values on $\partial\Omega$ are considered. A *subgeodesic* of two such functions, u_0 and u_1 , is a family of functions v_t , $0 < t < 1$, such that $v(z, \zeta) := v_{\log|\zeta|}(z)$ belongs to the class $W(u_0, u_1)$ of nonpositive plurisubharmonic functions on the product $\Omega \times \{1 < |\zeta| < e\}$ whose boundary values on $\Omega \times \{\log|\zeta| = j\}$ do not exceed u_j . The *geodesic* of u_0 and u_1 is their largest subgeodesic.

It was shown in [18] that for u_j from Cegrell's energy class $\mathcal{F}_1(\Omega)$, the geodesic u_t attains u_j as its boundary values (as uniform limits if u_j are bounded, and in capacity in the general case) as $t \rightarrow j$, and furthermore the energy functional

$$\mathbf{E}(u) = \int_{\Omega} u(dd^c u)^n$$

is affine on u_t . If ω_K is the relative extremal function (2.5) of a compact subset K of Ω , then

$$\mathbf{E}(\omega_K) = -\text{Cap}(K),$$

and this is where the Monge–Ampère capacities come into play.

For toric plurisubharmonic functions, this can be translated to the language of convex functions on \mathbb{R}^n_- (as was done in [2] for a global setting of \mathbb{R}^n), and the geodesics give rise to the multiplicative convex combinations K_t^\times as in the proof of Theorem 4.1. Moreover, such geodesics can be described in a way which is a bit similar to copolar addition. Any toric plurisubharmonic function u on \mathbb{D}^n with zero boundary values on $\partial\mathbb{D}^n$ can be identified with its convex image

$$\check{u} = \text{Exp}^* u,$$

which is a convex function on \mathbb{R}^n_- , increasing in each variable and equal to 0 on $\partial\mathbb{R}^n_-$. Let u_0 and u_1 be two toric plurisubharmonic functions on \mathbb{D}^n with zero boundary values, and let u_t , $0 < t < 1$, be the corresponding geodesic. Geodesics on the space of Kähler metrics on toric Kähler manifolds can be characterized as those whose Legendre transforms are affine in t ([7], see also [2], [9]), which can be proved by differentiating the defining equation for the

Legendre transform of the geodesic. A similar fact is true in the local setting as well; since we have a nonsmooth situation, we give an independent proof.

THEOREM 5.1. *Let u_t be the geodesic of two toric plurisubharmonic functions u_0 and u_1 on \mathbb{D}^n with zero boundary values. Then its convex image \check{u}_t has the representation*

$$(5.1) \quad \check{u}_t = \mathcal{L}[(1-t)\mathcal{L}[\check{u}_0] + t\mathcal{L}[\check{u}_1]].$$

REMARK. Observe resemblance between (5.1) and (4.1).

Proof of Theorem 5.1. Note that $\check{u}(t, s) := \check{u}_t(s)$ is convex in $n+1$ variables, although $\mathcal{L}[\check{u}]$ is not. To avoid this difficulty, we consider the $(n+1)$ -dimensional Legendre transform

$$\mathcal{L}^*[f](y, r) = \sup\{\langle x, y \rangle + t \cdot r - f(x, t) : x \in \mathbb{R}^n, r \in \mathbb{R}\}$$

of convex functions $f(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$ and apply it to the function \check{u} ; to get \check{u} defined on the whole $\mathbb{R}^n \times \mathbb{R}$, we set it equal to $+\infty$ when $s \in \mathbb{R}^n \setminus \mathbb{R}^n$ or $t \in \mathbb{R} \setminus [0, 1]$. We will still write $\mathcal{L}[u_t]$ for the Legendre transform of $u_t(s)$ in s .

First assume that the functions u_j are bounded: $u_j \geq -M_j$. Take any bounded toric subgeodesic v_t connecting u_0 and u_1 . Replacing it, if necessary, with $\max\{v_t, u_0 - M_1 t, u_1 - M_0(1-t)\}$, we can assume that v_t converges uniformly to u_j as $t \rightarrow j \in \{0, 1\}$. It follows from the definition of the Legendre transform that

$$\mathcal{L}^*[\check{v}](a, r) = \sup_{0 \leq t \leq 1} \{\mathcal{L}[\check{v}_t](a) + t \cdot r\}$$

for the convex image \check{v} of v_t . Therefore,

$$\mathcal{L}^*[\check{v}](a, r) \geq (\mathcal{L}[\check{v}_t](a) + t \cdot r)|_{t=0} = \mathcal{L}[\check{u}_0](a)$$

and

$$\mathcal{L}^*[\check{v}](a, r) \geq (\mathcal{L}[\check{v}_t](a) + t \cdot r)|_{t=1} = \mathcal{L}[\check{u}_1](a) + r,$$

so

$$(5.2) \quad \mathcal{L}^*[\check{v}](a, r) \geq \max\{\mathcal{L}[\check{u}_0](a), \mathcal{L}[\check{u}_1](a) + r\}.$$

Let now $\check{w}_t(s)$ be defined by the right hand side of (5.1). Evidently, it is convex in (s, t) . Since $\mathcal{L}[\check{w}_t] \rightarrow \mathcal{L}[\check{u}_j]$ uniformly as $t \rightarrow j$, its plurisubharmonic image w is in $W(u_0, u_1)$. Furthermore,

$$\begin{aligned} \mathcal{L}^*[\check{w}](a, r) &= \sup_{0 \leq t \leq 1} \{(1-t)\mathcal{L}[\check{u}_0](a) + t\mathcal{L}[\check{u}_1] + t \cdot r\} \\ &= \sup_{0 \leq t \leq 1} \{t[r - \mathcal{L}[\check{u}_0](a) + \mathcal{L}[\check{u}_1](a)] + \mathcal{L}[u_0]\} \\ &= \max\{\mathcal{L}[\check{u}_0](a), \mathcal{L}[\check{u}_1](a) + r\}. \end{aligned}$$

Comparing this with (5.2), we conclude $\mathcal{L}^*[\check{w}] \leq \mathcal{L}^*[\check{v}]$, and so $\check{w} \geq \check{v}$ for any subgeodesic v_t . Therefore, w_t is the geodesic.

In the general case of unbounded u_j , we apply what has already been proved to the cutoff functions $u_{j,N} = \max\{u_j, -N\}$. The convex images of their geodesics are

$$\check{u}_{t,N} = \mathcal{L}[(1-t)\mathcal{L}[\check{u}_{0,N}] + t\mathcal{L}[\check{u}_{1,N}]],$$

and letting $N \rightarrow \infty$, we get (5.1). ■

REMARK. The same proof works for toric plurisubharmonic functions on arbitrary bounded hyperconvex Reinhardt domains.

6. Applications. Here we apply the results to geodesics of toric relative extremal functions and to toric singularities. Next, the uniqueness part of Theorem 4.1 is proved.

6.1. By (2.9), the relative extremal function ω_K can be expressed in terms of the support function of the polar L° (denoted by Γ_L there) of the logarithmic image $L = \text{Log } K$ of K :

$$\omega_K = \max\{\text{Log}^* h_{L^\circ}, -1\}, \quad z \in \mathbb{D}^n.$$

Let $K_0, K_1 \Subset \mathbb{D}^n$ be the closures of two complete logarithmically convex Reinhardt domains, and let $u_j = \omega_{K_j}$. Then

$$\check{u}_j = \text{Exp}^* u_j = \max\{h_{L_j^\circ}, -1\},$$

and thus, by Proposition 3.2,

$$\mathcal{L}[\check{u}_j] = \max\{h_{L_j} + 1, 0\}, \quad j = 0, 1.$$

Therefore, (5.1) gives us the following explicit formula for the geodesics of relative extremal functions.

THEOREM 6.1. *Let $K_j \subset \mathbb{D}^n$ be the closures of nonempty complete logarithmically convex Reinhardt domains D_j , $j = 0, 1$. Then the geodesic u_t of $u_j = \omega_{K_j}$ can be represented as*

$$\check{u}_t = \mathcal{L}[(1-t)\max\{h_{L_0} + 1, 0\} + t\max\{h_{L_1} + 1, 0\}], \quad 0 < t < 1,$$

where $L_j \in UC_-$ are the logarithmic images of K_j .

COROLLARY 6.2. *Under the assumptions of Theorem 6.1, u_t for some $t \in (0, 1)$ is the relative extremal function of some $K \subset \mathbb{D}^n$ if and only if $K_0 = K_1$.*

Proof. Assume $u_t = \omega_K$ for some $t \in (0, 1)$, and let $L = \text{Log } K$. Then

$$(6.1) \quad (1-t)\max\{h_{L_0}(a) + 1, 0\} + t\max\{h_{L_1}(a) + 1, 0\} \\ = \max\{h_L(a) + 1, 0\}, \quad a \in \mathbb{R}_+^n.$$

In particular, for all a close to the origin, $h_L(a) = (1-t)h_{L_0}(a) + th_{L_1}(a)$, which implies

$$(6.2) \quad L = (1-t)L_0 + tL_1.$$

On the other hand, (6.1) for a close to $\delta L^\circ = \{x \in \mathbb{R}^n : h_A(x) = -1\}$ implies $L^\circ = L_0^\circ \cap L_1^\circ$, which contradicts (6.2) unless $L_0 = L_1$. ■

COROLLARY 6.3. *The equality statement of Theorem 4.1 is true.*

Proof. Since equality occurs in (4.4) if and only if the geodesic u_t is the relative extremal function of K_t^\times , the result follows from Corollary 6.2. ■

6.2. Theorem 4.1 can be applied to investigation of residual Monge–Ampère masses of toric plurisubharmonic singularities. As was mentioned in Section 2.1, the support function h_P of a set $P \in CC_+$ is the convex image of an indicator $\Psi_P \in \text{PSH}(\mathbb{D}^n)$, and the set $L_\varphi \subset \mathbb{R}_-^n$ defined by (2.4) is the copolar to P . Copolar addition on CC_+ induces copolar addition of the indicators:

$$\Psi_P \oplus \Psi_Q = \Psi_{P \oplus Q};$$

furthermore, it gives rise to the copolar combinations

$$(6.3) \quad \Psi_t^\oplus = \Psi_{P_t^\oplus}$$

of Ψ_{P_j} for $P_0, P_1 \in CC_+$, where the sets P_t^\oplus are defined by (4.1).

For example, the cosimplices $P_j = \{a \in \mathbb{R}_+^n : \langle a, b^j \rangle \geq 1\}$ generate the indicators

$$\Psi_{P_j}(z) = \Phi_{b^j}(z) := \max_{1 \leq k \leq n} \frac{\log |z_k|}{b_k^j}, \quad j = 0, 1,$$

so in this case

$$\Psi_t^\oplus = \Phi_{c(t)},$$

where the components c_k of the weight vector $c(t) \in \mathbb{R}_+^n$ are defined by

$$\frac{1}{c_k} = \frac{1-t}{b_k^0} + \frac{t}{b_k^1}, \quad 1 \leq k \leq n.$$

By (2.2) and (2.3), $(dd^c \Psi_t^\oplus)^n(0) = n! \text{Covol}(P_t^\oplus)$, so Theorem 4.1 and Corollary 6.2 imply

COROLLARY 6.4. *Let Ψ_j be indicators corresponding to complete Reinhardt sets $K_j = \bar{D}_j \subset \mathbb{D}^n$, $j = 0, 1$. Then the Monge–Ampère measures of their copolar combinations (6.3) satisfy*

$$(dd^c \Psi_t^\oplus)^n \leq (1-t)(dd^c \Psi_0)^n + t(dd^c \Psi_1)^n, \quad 0 < t < 1,$$

and the inequality is strict unless $\Psi_0 = \Psi_1$.

REMARK. As follows from [18, Thm. 6.2], no pair Ψ_P, Ψ_Q of different indicators can be connected by a geodesic because it does not exceed $\Psi_{P \cap Q} \leq \min\{\Psi_P, \Psi_Q\}$ (this can also be easily deduced from Theorem 5.1). Moreover, the main result of [10] implies that two toric plurisubharmonic functions φ_0, φ_1 with zero boundary values and isolated singularities at 0

can be connected by a geodesic φ_t (in the sense that $\varphi_t \rightarrow \varphi_j$ in capacity as $t \rightarrow j \in \{0, 1\}$) if and only if their indicators Ψ_{φ_j} as in (2.1) are equal.

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Alexander Rashkovskii
Tek/Nat
University of Stavanger
4036 Stavanger, Norway
E-mail: alexander.rashkovskii@uis.no

