

An identity for formal derivatives in a commutative algebra

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Abstract. We present a generalization of an identity for formal derivatives in a commutative algebra. Special cases of the formula play a key role in deriving Markov type inequalities. In addition we give a short proof of the specialization to ordinary higher order derivatives and list some of its applications.

1. Introduction. Recently, Baran et al. [6] published the interesting identity

$$(1.1) \quad \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} g^k (fg^{n-k})^{(n)} = f(g')^n$$

($n = 0, 1, 2, \dots$), for sufficiently differentiable functions. This identity is valid in each complex commutative algebra \mathcal{A} with unity and for each derivation operator $D : \mathcal{A} \rightarrow \mathcal{A}$ (a linear operator having the property $D(fg) = gD(f) + fD(g)$) with the notation $D(f) = f'$ and $D^{(k)}(f) = f^{(k)}$. A proof of identity (1.1) is to be given in the forthcoming paper [7]. A first version with $f = 1$ was found by B. Milówka [12, 13] in 2005, for polynomials g . In his PhD dissertation P. Ozorka discovered a version of (1.1) with $f = 1$, for a commutative algebra with unity, and gave a proof which is an adaptation of Milówka's proof from [12]. In 2012, P. Ozorka proved it for arbitrary functions g .

In Milówka's proof the crucial role is played by a formula for $(P^k)^{(n)}$, which was extended to the case of commutative algebras and derivation operators. As was observed by Ozorka, in the algebra of polynomials in N variables the derivative operator is a linear differential operator, that is, $D(P) = Q_1 \frac{\partial P}{\partial x_1} + \dots + Q_N \frac{\partial P}{\partial x_N}$ with some polynomials Q_1, \dots, Q_N . We mention that M. Karaś [10] gave a characterization of a class of nilpotent derivations. The identity (1.1) (also in a version for derivations) was discov-

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ered later by M. Baran and P. Ozorka and the proof that is different than that of this paper will be published soon.

The formula

$$\frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} a^k D^{(n)}(a^{n-k}) = n! D(a)^n$$

(this is (3.4) below with $a_1 = a_2 = 1, b = a, p = 0$) and its special case for the derivatives of polynomials of one variable was discovered by P. Ozorka and B. Milówka independently of earlier results by E. A. Gorin, Yu. Yu. Kochetkov and B. S. Mityagin [8] and M. Klimek [11]. It is worth noticing that those papers were not cited by anybody during a period of more than thirty years.

Such identities play a key role in deriving Markov type inequalities. Let us note some deep consequences of Milówka’s identity.

- (i) B. Milówka [13] showed that Markov’s property for the k th derivative is equivalent to the relevant property for the derivatives of the first order. Moreover, repeating the estimate for the first order derivative we obtain an exact exponent in the bound for the derivative of order k : if $\|P'\|_E \leq M(\deg P)^m \|P\|_E$ then there exist constants M_k such that $\|P^{(k)}\|_E \leq M_k(\deg P)^{km} \|P\|_E$.

- (ii) In 2013, M. Baran and L. Bialas-Ciez [2] published their earlier observation that if for a planar compact set E one has the inequality $\|P^{(k)}\|_E \leq M_k(\deg P)^{km} \|P\|_E$, for a polynomial P , then

$$(1.2) \quad M_k \geq A^k (1/k!)^{m-1}.$$

(Here m, M_k, A denote constants independent of the degree of the polynomial.)

- (iii) As a consequence of (1.2), the idea arises to consider sets E with the property

$$(1.3) \quad \|P^{(k)}\|_E \leq A^k (1/k!)^{m-1} (\deg P)^{km} \|P\|_E,$$

now called *Vladimir Markov’s sets*. V. A. Markov’s famous inequality establishes (1.3) in the case of $E = [-1, 1]$. It is a surprising fact, proved in [2, 3] (see also [4]), that condition (1.3) is equivalent to the Hölder type continuity (with exponent $1/m$) of the complex Green function V_E . The same remains true in higher dimensions if we replace (1.3) by a similar condition involving partial derivatives (see [2, 3]).

- (iv) In [6], there is an interesting observation that A. Markov’s property for a subset of \mathbb{R}^N is equivalent to a suitable bound for the Laplace operator.

- (v) An important fact was proved in [5, Theorem 24]: minimality of Markov’s exponent for spectral norms ($\|P^k\| = \|P\|^k$) in the class of norms possessing a generalized Nikolskii property (with respect to this norm).

The purpose of this note is to give a short transparent proof of a generalization of identity (1.1). In the next section we restrict ourselves to ordinary differentiation of functions on real or complex domains. One special instance is the symmetric form

$$(1.4) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} (f_1 g^k)^{(p)} (f_2 g^{n-k})^{(n-p)} = (-1)^p n! f_1 f_2 (g')^n$$

for $p = 0, \dots, n$. Note that (1.1) is the case $p = 0$, $f_1 = 1$, $f_2 = f$. Identity (1.4) can be extended to several functions (see Theorem 2.1). We present a number of further special cases and applications.

We mention the similarity of (1.4) to another identity for derivatives,

$$(1.5) \quad x \sum_{k=0}^n \binom{n}{k} (x^k f(x))^{(k)} (x^{n-k} g(x))^{(n-k)} = (x^{n+1} f(x) g(x))^{(n)},$$

which is a consequence of a generalization of the Leibniz rule [1] giving a closed form expression for the sum

$$\sum_{k=0}^n \binom{n}{k} (h^k f)^{(k)} (h^{n-k} g)^{(n-k)}.$$

In the last section we prove the identity (1.4), for each derivation operator, in a commutative algebra with unity over a field of characteristic zero.

2. The identity for ordinary derivatives. Our notation uses multi-indices. For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, denote $|\mathbf{k}| = k_1 + \dots + k_r$. For positive integers n , the binomial coefficient is defined by $\binom{n}{\mathbf{k}} = \binom{n}{k_1, \dots, k_r} := \frac{n!}{k_1! \dots k_r! (n - |\mathbf{k}|)!}$. Throughout the paper $\sum_{|\mathbf{k}|=n}$ means summation over all $\mathbf{k} \in \mathbb{N}_0^r$ satisfying $|\mathbf{k}| = n$.

The following theorem contains the most general formula.

THEOREM 2.1. *Let $n \in \mathbb{N}_0$, $r \in \mathbb{N}$, and let f_i, g_i ($i = 1, \dots, r$) be functions which have a derivative of order n . Suppose that $\sum_{i=1}^r g_i = 0$. Then, for $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}_0^r$,*

$$\sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r (f_i g_i^{k_i})^{(s_i)} = \begin{cases} 0 & (|s| < n), \\ n! (\prod_{i=1}^r f_i) \prod_{i=1}^r (g_i')^{s_i} & (|s| = n). \end{cases}$$

Proof. The proof is based on the observation that, by Faà di Bruno's formula, $(\frac{d}{dx})^s f^n(x) = 0$ for $s = 0, \dots, n - 1$, and $(\frac{d}{dx})^n f^n(x) = n! (f'(x))^n$ if $f(x) = 0$. Let f_i, g_i ($i = 1, \dots, r$) be functions which have a derivative of order n at a certain point $x \in \mathbb{R}$. Furthermore, define $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{R}^r$

and $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^r$. We have

$$\begin{aligned} \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r (f_i g_i^{k_i})^{(s_i)}(x) &= \left(\frac{\partial^{|\mathbf{s}|}}{\partial x_1^{s_1} \dots \partial x_r^{s_r}} \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r (f_i g_i^{k_i})(x_i) \right) \Big|_{\mathbf{x}=\mathbf{1}x} \\ &= \left(\frac{\partial^{|\mathbf{s}|}}{\partial x_1^{s_1} \dots \partial x_r^{s_r}} \left[\left(\prod_{i=1}^r f_i(x_i) \right) \left(\sum_{i=1}^r g_i(x_i) \right)^n \right] \right) \Big|_{\mathbf{x}=\mathbf{1}x}, \end{aligned}$$

where we have applied the binomial theorem. Note that $\sum_{i=1}^r g_i(x) = 0$ implies that

$$\frac{\partial^{|\mathbf{s}|}}{\partial x_1^{s_1} \dots \partial x_r^{s_r}} \left(\sum_{i=1}^r g_i(x_i) \right)^n \Big|_{\mathbf{x}=\mathbf{1}x} = \begin{cases} 0 & (|\mathbf{s}| < n), \\ n! \prod_{i=1}^r (g'_i(x))^{s_i} & (|\mathbf{s}| = n). \end{cases}$$

Now the desired formula follows. ■

Now we consider the particular case where the functions g_i differ only by constant factors, i.e., $g_i = c_i g$ with a function g and $c_i \in \mathbb{R}$ ($i = 1, \dots, r$).

COROLLARY 2.2. *Let $n \in \mathbb{N}_0$, $r \in \mathbb{N}$, and let f_i ($i = 1, \dots, r$) and g be functions which have a derivative of order n . Suppose that $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{R}^r$ satisfies $|\mathbf{c}| = 0$. Let $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}_0^r$. Then*

$$\sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r c_i^{k_i} (f_i g^{k_i})^{(s_i)} = \begin{cases} 0 & (|\mathbf{s}| < n), \\ n! (\prod_{i=1}^r c_i^{s_i}) (\prod_{i=1}^r f_i)(g')^{|\mathbf{s}|} & (|\mathbf{s}| = n). \end{cases}$$

REMARK. In the special case $r = 2$, with $c_1 = -1 = -c_2$, we have

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} (f_1 g^k)^{(s_1)} (f_2 g^{n-k})^{(s_2)} &= \begin{cases} 0 & (s_1 + s_2 < n), \\ (-1)^{s_1} n! f_1 f_2 (g')^n & (s_1 + s_2 = n). \end{cases} \end{aligned}$$

We close this section with some direct consequences.

EXAMPLE 2.3. Set $f_i(x) = x^{\alpha_i}$, $g_i(x) = x^\beta$ ($i = 1, \dots, r$). Corollary 2.2 yields, for $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}_0^r$ with $|\mathbf{s}| = n$ and $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{R}^r$ with $|\mathbf{c}| = 0$, the identity

$$(2.1) \quad \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r \binom{k_i}{s_i} \binom{\alpha_i + k_i \beta}{s_i} = \binom{n}{\mathbf{s}} \beta^n \prod_{i=1}^r c_i^{s_i}.$$

Identity (2.1) can be found in Gould's collection of binomial identities

[9, (6.41)]. In the special case $r = 2$ we obtain, for $p = 0, \dots, n$,

$$(2.2) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{\alpha_1 + k\beta}{p} \binom{\alpha_2 + (n-k)\beta}{n-p} = (-1)^p \binom{n}{p} \beta^n.$$

Formula (2.2) is an easy consequence of the fact that $\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^i = 0$ ($i = 0, \dots, n - 1$), $\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^n = n!$ and $\binom{\alpha_1 + k\beta}{p} \binom{\alpha_2 + (n-k)\beta}{n-p}$ is a polynomial in k of degree n with leading term

$$(-1)^{n-p} \beta^n k^n / (p!(n-p)!).$$

EXAMPLE 2.4. If $f_i(x) = e^{\alpha_i x}$ and $g_i(x) = e^{\beta x}$ ($i = 1, \dots, r$), we obtain

$$(2.3) \quad \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r (c_i^{k_i} (\alpha_i + k_i \beta)^{s_i}) = \binom{n}{\mathbf{s}} \beta^n \prod_{i=1}^r c_i^{s_i}.$$

In the special case $r = 2$ we obtain, for $p = 0, \dots, n$,

$$(2.4) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha_1 + k\beta)^p (\alpha_2 + (n-k)\beta)^{n-p} = (-1)^p n! \beta^n.$$

Just as (2.2) in Example 2.3, formula (2.4) is obvious because

$$(\alpha_1 + k\beta)^p (\alpha_2 + (n-k)\beta)^{n-p}$$

is a polynomial in k of degree n with leading term $(-1)^{n-p} \beta^n k^n$.

3. The identity in commutative algebras. We close this note by giving a proof of the identity in commutative algebras. Throughout this section let \mathcal{A} be a commutative algebra with unity $1_{\mathcal{A}}$ over a field of characteristic zero. The zero element of \mathcal{A} is denoted by $0_{\mathcal{A}}$. For $a \in \mathcal{A}$, define $a^0 = 1_{\mathcal{A}}$ and $a^n = a \cdot a^{n-1}$ ($n \in \mathbb{N}$). Furthermore, let $D : \mathcal{A} \rightarrow \mathcal{A}$ be a derivation operator, i.e., a linear operator satisfying Leibniz's law $D(ab) = aD(b) + bD(a)$. Direct consequences are

$$(3.1) \quad D^{(n)}(ab) = \sum_{j=0}^n \binom{n}{j} D^{(j)}(a) \cdot D^{(n-j)}(b)$$

and

$$(3.2) \quad D^{(n)}(a^k) = \sum_{j=0}^k \frac{k!}{(k-j)!} a^{k-j} \cdot \Delta^{(n,j)}(a)$$

with

$$(3.3) \quad \Delta^{(n,j)}(a) = n! \sum \frac{1}{i_1! i_2! i_3! \dots} \left(\frac{D^{(1)}(a)}{1!} \right)^{i_1} \left(\frac{D^{(2)}(a)}{2!} \right)^{i_2} \left(\frac{D^{(3)}(a)}{3!} \right)^{i_3} \dots,$$

where the summation runs over all non-negative integers i_1, i_2, i_3, \dots satisfying $i_1 + i_2 + i_3 + \dots = j$ and $i_1 + 2i_2 + 3i_3 + \dots = n$. Note that $\Delta^{(n,j)}(a) = 0_{\mathcal{A}}$ if $j > n$, because then the sum is empty. Furthermore, we have $\Delta^{(j,j)}(a) = D(a)^j$.

The following theorem is the main result.

THEOREM 3.1. *Let $n \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $a_i, b_i \in \mathcal{A}$ ($i = 1, \dots, r$). Suppose that $\sum_{i=1}^r b_i = 0_{\mathcal{A}}$. Then, for $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}_0^r$,*

$$\sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(a_i b_i^{k_i}) = \begin{cases} 0_{\mathcal{A}} & (|\mathbf{s}| < n), \\ n! (\prod_{i=1}^r a_i) \prod_{i=1}^r D(b_i)^{s_i} & (|\mathbf{s}| = n). \end{cases}$$

The special case $r = 2$, $\mathbf{s} = (p, n - p)$ with $0 \leq p \leq n$ reads

$$(3.4) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} D^{(p)}(a_1 b^k) D^{(n-p)}(a_2 b^{n-k}) = (-1)^p n! a_1 a_2 D(b)^n.$$

As a direct consequence, we obtain the identity in the special case $a_i = 1_{\mathcal{A}}$ ($i = 1, \dots, r$).

COROLLARY 3.2. *Let $n \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $b_i \in \mathcal{A}$ ($i = 1, \dots, r$). Suppose that $\sum_{i=1}^r b_i = 0_{\mathcal{A}}$. Then, for $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}_0^r$,*

$$(3.5) \quad \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(b_i^{k_i}) = \begin{cases} 0_{\mathcal{A}} & (|\mathbf{s}| < n), \\ n! \prod_{i=1}^r D(b_i)^{s_i} & (|\mathbf{s}| = n). \end{cases}$$

Proof of Theorem 3.1. As a first step we prove (3.5), which is the desired identity in the special case $a_i = 1_{\mathcal{A}}$ ($i = 1, \dots, r$). By (3.2), we have

$$\begin{aligned} & \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(b_i^{k_i}) \\ &= \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r \sum_{\ell_i=0}^{k_i} \frac{k_i!}{(k_i - \ell_i)!} b_i^{k_i - \ell_i} \cdot \Delta^{(s_i, \ell_i)}(b_i) \\ &= n! \sum_{|\ell| \leq n} \sum_{\substack{|\mathbf{k}|=n \\ k_i \geq \ell_i \ (i=1, \dots, r)}} \prod_{i=1}^r \frac{1}{(k_i - \ell_i)!} b_i^{k_i - \ell_i} \cdot \Delta^{(s_i, \ell_i)}(b_i) \\ &= \sum_{|\ell| \leq n} \frac{n!}{(n - |\ell|)!} \sum_{\substack{|\mathbf{k}|=n-|\ell| \\ k_i \geq 0 \ (i=1, \dots, r)}} \binom{n - |\ell|}{\mathbf{k}} \prod_{i=1}^r (b_i^{k_i} \cdot \Delta^{(s_i, \ell_i)}(b_i)) \\ &= \sum_{|\ell| \leq n} \frac{n!}{(n - |\ell|)!} \left(\sum_{i=1}^r b_i \right)^{n-|\ell|} \prod_{i=1}^r \Delta^{(s_i, \ell_i)}(b_i). \end{aligned}$$

The assumption $\sum_{i=1}^r b_i = 0_{\mathcal{A}}$ implies that

$$\sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(b_i^{k_i}) = n! \sum_{|\ell|=n} \prod_{i=1}^r \Delta^{(s_i, \ell_i)}(b_i).$$

Because $\Delta^{(s_i, \ell_i)}(b_i) = 0_{\mathcal{A}}$ if $\ell_i > s_i$ ($i = 1, \dots, r$), the latter expression is equal to $0_{\mathcal{A}}$ if $\ell_i \leq s_i$ is not satisfied, for all $i = 1, \dots, r$. Otherwise, we have $|\mathbf{s}| \geq |\ell| = n$. This is possible only if $|\mathbf{s}| \geq n$. In the case $|\mathbf{s}| = n$ we conclude that

$$\sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(b_i^{k_i}) = n! \sum_{|\ell|=n} \prod_{i=1}^r \Delta^{(s_i, s_i)}(b_i) = n! \sum_{|\ell|=n} \prod_{i=1}^r D(b_i)^{s_i}$$

which proves (3.5). By (3.1), we obtain

$$\begin{aligned} & \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(a_i b_i^{k_i}) \\ &= \sum_{j_1=0}^{s_1} \dots \sum_{j_r=0}^{s_r} \left[\prod_{i=1}^r \binom{s_i}{j_i} D^{(j_i)}(a_i) \right] \cdot \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i - j_i)}(b_i^{k_i}). \end{aligned}$$

By (3.5), the latter sum is equal to $0_{\mathcal{A}}$ if $|\mathbf{s}| - |\mathbf{j}| \leq n - 1$, which is valid both if $|\mathbf{s}| \leq n - 1$ and if $|\mathbf{s}| = n$ and $|\mathbf{j}| > 0$. Hence,

$$\sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(a_i b_i^{k_i}) = \left(\prod_{i=1}^r a_i \right) \cdot \sum_{|\mathbf{k}|=n} \binom{n}{\mathbf{k}} \prod_{i=1}^r D^{(s_i)}(b_i^{k_i}),$$

and the identity follows by (3.5). ■

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