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Uniform continuity and normality of metric spaces in **ZF**

by

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Summary. Let $\mathbf{X} = (X, d)$ and $\mathbf{Y} = (Y, \rho)$ be two metric spaces.

(a) We show in **ZF** that:

(i) If $\mathbf{X}$ is separable and $f : \mathbf{X} \to \mathbf{Y}$ is a continuous function then f is uniformly continuous iff for any $A, B \subseteq X$ with $d(A, B) = 0$, $\rho(f(A), f(B)) = 0$. But it is relatively consistent with **ZF** that there exist metric spaces $\mathbf{X}$, $\mathbf{Y}$ and a continuous, non-uniformly continuous function $f : \mathbf{X} \to \mathbf{Y}$ such that for any $A, B \subseteq X$ with $d(A, B) = 0$, $\rho(f(A), f(B)) = 0$.

(ii) If S is a dense subset of $\mathbf{X}$, $\mathbf{Y}$ is Cantor complete and $f : \mathbf{S} \to \mathbf{Y}$ a uniformly continuous function, then there is a unique uniformly continuous function $F : \mathbf{X} \to \mathbf{Y}$ extending f. But it is relatively consistent with **ZF** that there exist a metric space $\mathbf{X}$, a complete metric space $\mathbf{Y}$, a dense subset S of $\mathbf{X}$ and a uniformly continuous function $f : \mathbf{S} \to \mathbf{Y}$ that does not extend to a uniformly continuous function on $\mathbf{X}$.

(iii) $\mathbf{X}$ is complete iff for any Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$, if $\overline{\{x_n : n \in \mathbb{N}\}} \cap \overline{\{y_n : n \in \mathbb{N}\}} = \emptyset$ then $d(\{x_n : n \in \mathbb{N}\}, \{y_n : n \in \mathbb{N}\}) > 0$.

(b) We show in **ZF+CAC** that if $f : \mathbf{X} \to \mathbf{Y}$ is a continuous function, then f is uniformly continuous iff for any $A, B \subseteq X$ with $d(A, B) = 0$, $\rho(f(A), f(B)) = 0$.

1. Notation and terminology. Let $\mathbf{X} = (X, d)$ be a metric space, $x \in X$ and $\varepsilon > 0$. As usual, $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ denotes the open ball in $\mathbf{X}$ with center x and radius ε. Given a non-empty $B \subseteq X$, $\delta(B) = \sup\{d(x, y) : x, y \in B\} \in \mathbb{R}_+ \cup \{\infty\}$ will denote the *diameter* of B.

$\mathbf{X}$ is *bounded* iff $\delta(X) < \infty$.

Let $\mathcal{U}$ be an open cover of $\mathbf{X}$. We say that $\mathcal{U}$ has a *Lebesgue number* $\delta > 0$ iff for every $A \subseteq X$ with $\delta(A) < \delta$ there exists $U \in \mathcal{U}$ with $A \subseteq U$.

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$\mathbf{X}$ is *Lebesgue* (resp. *countably Lebesgue*) iff every open cover (resp. countable open cover) of $\mathbf{X}$ has a Lebesgue number.

$\mathbf{X}$ is *complete* iff every Cauchy sequence in $\mathbf{X}$ converges to some element of X.

$\mathbf{X}$ is *Cantor complete* iff $\bigcap\{G_n : n \in \omega\} \neq \emptyset$ for every descending family $\{G_n : n \in \omega\}$ of non-empty closed subsets of $\mathbf{X}$ with $\lim_{n\to\infty} \delta(G_n) = 0$.

$\mathbf{X}$ is *UC* (resp. *strongly UC*) iff every continuous real valued function on X is uniformly continuous (resp. for every metric space $\mathbf{Y} = (Y, \rho)$, every continuous function $f : \mathbf{X} \to \mathbf{Y}$ is uniformly continuous).

$\mathbf{X}$ is *normal* iff the distance of any two disjoint, non-empty closed subsets of $\mathbf{X}$ is strictly positive.

$\mathbf{X}$ is *sequentially normal* (resp. *weakly sequentially normal*) iff for any two sequences (resp. Cauchy sequences) $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ such that $\overline{\{x_n : n \in \mathbb{N}\}} \cap \overline{\{y_n : n \in \mathbb{N}\}} = \emptyset$, we have $d(\{x_n : n \in \mathbb{N}\}, \{y_n : n \in \mathbb{N}\}) > 0$.

$\mathbf{X}$ is *totally bounded* iff for every $\varepsilon > 0$, there exists an ε-net of $\mathbf{X}$, i.e., a finite subset $\{x_i : i \leq n\}$ of X such that $\bigcup\{B(x_i, \varepsilon) : i \leq n\} = X$.

An infinite set X is *Dedekind-infinite,* denoted by $\mathbf{DI}(X)$*,* iff X contains a countably infinite set. Otherwise it is *Dedekind-finite.* By universal quantifying over X, $\mathbf{DI}(X)$ gives rise to the choice principle **IDI**: $\forall X$ (X infinite $\to \mathbf{DI}(X)$), that is, "every infinite set is Dedekind-infinite" [5, Form 9].

Below we list some weak forms of the axiom of choice we shall deal with.

- **IDI($\mathbb{R}$)** [5, Form 13]: $\forall X \in \mathcal{P}(\mathbb{R})$ (X infinite $\to \mathbf{DI}(X)$).
- **CAC** [5, Form 8]: For every countable family $\mathcal{A}$ of non-empty sets there exists a function f such that $f(x) \in x$ for all $x \in \mathcal{A}$.
- $\mathbf{CAC}_{\text{fin}}$ [5, Form 10]: **CAC** restricted to countable families of non-empty finite sets. Equivalently (see [5, Form [10 O]], every infinite well ordered family of non-empty finite sets has an infinite subfamily with a choice set.
- $\mathbf{PKW}(\aleph_0, \geq 2, \infty)$ [5, Form 167], Partial Kinna–Wagner principle: Every disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ with $|A_i| \geq 2$ for all $i \in \omega$ has a partial Kinna–Wagner choice, i.e., there exists an infinite subfamily $\mathcal{B} = \{A_{k_i} : i \in \omega\}$ of $\mathcal{A}$ and a family $\mathcal{F} = \{F_i : i \in \omega\}$ of non-empty sets such that $F_i \subsetneq A_{k_i}$ for all $i \in \omega$.
- **UCE**: For every metric space $\mathbf{X} = (X, d)$ and every complete metric space $\mathbf{Y} = (Y, \rho)$, if S is a dense subset of $\mathbf{X}$ and $f : \mathbf{S} \to \mathbf{Y}$ is a uniformly continuous function then there exists a unique uniformly continuous extension $F : \mathbf{X} \to \mathbf{Y}$ of f, i.e. $F(s) = f(s)$ for all $s \in S$.

It is known (see e.g. [5]) that in **ZF** the ε-δ definition of continuity is stronger than the sequential one, and a sequentially closed set may not be closed. We stress that in this paper, continuity will always mean ε-δ continuity, and a set is closed iff its complement is open.

2. Introduction and some preliminary results. Our intended context for reasoning and statements of theorems will be the Zermelo–Fraenkel set theory **ZF** without the axiom of choice **AC**. To stress that a result is proved in **ZF** (resp. **ZF** + **CAC**) we shall write (**ZF**) (resp. (**ZF** + **CAC**)) at the beginning of the statement.

S. Mrówka [9] has established

THEOREM 1. (**ZF** + **CAC**) *A metric space* $\mathbf{X} = (X, \rho)$ *is normal iff it is* UC.

In view of Theorem 1 one may ask whether **CAC** is needed for the proof of "normal $\leftrightarrow$ UC". The following was proved in [6]:

THEOREM 2 ([6]).

(a) (**ZF**) *Let* $\mathbf{X} = (X, d)$ *be a metric space. Each of the following statements implies the one beneath it:*
   - (i) $\mathbf{X}$ *is Lebesgue;*
   - (ii) $\mathbf{X}$ *is countably Lebesgue;*
   - (iii) $\mathbf{X}$ *is* UC*;*
   - (iv) $\mathbf{X}$ *is normal.*

(b) (**ZF** + **CAC**) (i)–(iv) *of* (a) *are equivalent.*

Hence we see that **CAC** is not needed for the proof of "$UC \to$ normal". So, one may ask:

QUESTION 1. Is **CAC** needed for the proof of "normal $\to UC$"?

We show in Theorem 11 that the answer to Question 1 is also negative.

The following well-known characterization of uniform continuity in **ZF** + **CAC** has been established in [2].

THEOREM 3 ([2]). (**ZF** + **CAC**) *Let* $\mathbf{X} = (X, d)$ *and* $\mathbf{Y} = (Y, \rho)$ *be metric spaces and* $f : X \to Y$ *a function. The following are equivalent:*

- (i) f *is uniformly continuous.*
- (ii) *For any subsets* A *and* B *of* X*, if* $d(A, B) = 0$ *then* $\rho(f(A), f(B)) = 0$.
- (iii) *For any countable subsets* A *and* B *of* X*, if* $d(A, B) = 0$ *then* $\rho(f(A), f(B)) = 0$.

The next result shows that some parts of the proof of Theorem 3 can be given in **ZF**.

THEOREM 4. (**ZF**) *Let* $\mathbf{X} = (X, d)$ *and* $\mathbf{Y} = (Y, \rho)$ *be metric spaces and* $f : \mathbf{X} \to \mathbf{Y}$ *a function.*

- (i) *If for any* $A, B \subseteq X$, $d(A, B) = 0$ *implies* $\rho(f(A), f(B)) = 0$*, then* f *is continuous. The converse fails.*

(ii) *If* f *is uniformly continuous then for any* $A, B \subseteq X$, $d(A,B) = 0$ *implies* $\rho(f(A), f(B)) = 0$.

(iii) *Assume that every continuous real valued function* g *on* $\mathbf{X}$ *satisfies: For any* $A, B \subset X$, $d(A,B) = 0$ *implies* $\rho(g(A), g(B)) = 0$, *where* ρ *is the usual metric on* $\mathbb{R}$. *Then* $\mathbf{X}$ *is sequentially normal.*

Proof. The proof of (i) and (ii) is left as an easy exercise for the reader.

(iii) Assume the contrary and fix injective sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ such that $\overline{\{x_n : n \in \mathbb{N}\}} \cap \overline{\{y_n : n \in \mathbb{N}\}} = \emptyset$ and $d(\{x_n : n \in \mathbb{N}\}, \{y_n : n \in \mathbb{N}\}) = 0$. Let $h : \mathbf{Y} \to \mathbb{R}$, where $Y = A \cup B$, $A = \overline{\{x_n : n \in \mathbb{N}\}}$, $B = \overline{\{y_n : n \in \mathbb{N}\}}$, be the function given by

$$h(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in B. \end{cases}$$

Clearly, h is continuous. Hence, by the Tietze extension theorem which holds in **ZF** (see e.g. [3, Lemma 2]), h extends to a continuous real valued g on $\mathbf{X}$. By our hypothesis, $\rho(g(A), g(B)) = 0$. However, $\rho(g(A), g(B)) = \rho(h(A), h(B)) = 1$, a contradiction. ∎

The following two theorems characterize, in **ZF**, the classes of all normal and sequentially normal metric spaces respectively.

THEOREM 5. (**ZF**) *Let* $\mathbf{X} = (X, \rho)$ *be a metric space. Then the following are equivalent:*

(i) $\mathbf{X}$ *is normal.*

(ii) *For every metric space* $\mathbf{Y} = (Y, d)$, *every continuous function* $f : \mathbf{X} \to \mathbf{Y}$, *and any* $A, B \subseteq X$, *if* $\rho(A,B) = 0$ *then* $d(f(A), f(B)) = 0$.

(iii) *For every continuous real valued function* f *on* $\mathbf{X}$ *and any* $A, B \subseteq X$, *if* $\rho(A,B) = 0$ *then* $d(f(A), f(B)) = 0$ *where* d *is the usual metric on* $\mathbb{R}$.

Proof. (i)→(ii) Let $f : \mathbf{X} \to \mathbf{Y}$ be continuous and fix $A, B \subseteq X$ with $\rho(A,B) = 0$. Since $\rho(\overline{A}, \overline{B}) = 0$, it follows by our hypothesis that $\overline{A} \cap \overline{B} \neq \emptyset$. Hence, $\emptyset \neq f(\overline{A} \cap \overline{B}) \subseteq f(\overline{A}) \cap f(\overline{B})$. By the continuity of f we have $f(\overline{A}) \subseteq \overline{f(A)}$ and $f(\overline{B}) \subseteq \overline{f(B)}$. Hence, $\emptyset \neq \overline{f(A)} \cap \overline{f(B)}$. Therefore, $d(\overline{f(A)}, \overline{f(B)}) = 0$, and consequently $d(f(A), f(B)) = 0$ as required.

(ii)→(iii) This is straightforward.

(iii)→(i) Fix $A, B \subseteq X$ closed and disjoint. We show that $\rho(A,B) > 0$. Assume that $\rho(A,B) = 0$. It is well known that the function $f : \mathbf{X} \to \mathbb{R}$ given by

$$f(x) = \frac{\rho(x,A)}{\rho(x,A) + \rho(x,B)}$$

is continuous. Hence, by hypothesis, $0 = d(f(A), f(B)) = d(\{0\}, \{1\}) = 1$, a contradiction. ∎

Similarly to the proof of Theorem 5 one can establish the following result.

THEOREM 6. (**ZF**) *Let* $\mathbf{X} = (X, \rho)$ *be a metric space. Then the following are equivalent:*

(i) $\mathbf{X}$ *is sequentially normal.*
(ii) *For every metric space* $\mathbf{Y} = (Y, d)$*, every continuous function* $f : \mathbf{X} \to \mathbf{Y}$*, and any countable subsets* $A, B \subseteq X$*, if* $\rho(A, B) = 0$ *then* $d(f(A), f(B)) = 0$.
(iii) *For every continuous real valued function* f *on* X*, and any countable subsets* $A, B \subseteq X$*, if* $\rho(A, B) = 0$ *then* $d(f(A), f(B)) = 0$.

Another well-known theorem of **ZF**+**CAC** regarding uniformly continuous functions states that:

($*$) If $\mathbf{X}$ and $\mathbf{Y}$ are metric spaces, $\mathbf{S}$ is a dense subspace of $\mathbf{X}$, $\mathbf{Y}$ is complete, and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous, then there exists a unique uniformly continuous extension F of f to all of $\mathbf{X}$.

Let us replace in ($*$) the word "complete" with "Cantor complete":

($**$) If $\mathbf{X}$ and $\mathbf{Y}$ are metric spaces, $\mathbf{S}$ is a dense subspace of $\mathbf{X}$, $\mathbf{Y}$ is Cantor complete, and $f : \mathbf{S} \to \mathbf{Y}$ is uniformly continuous, then there exists a unique uniformly continuous extension of f to all of $\mathbf{X}$.

We show in Theorem 15 that ($**$) is a theorem of **ZF**.

The following lemma is stated in [8] without proof and it is attributed to Efremovich. For the reader's convenience, we supply a **ZF** proof.

LEMMA 7 (Efremovich, [8]). *Let* $\mathbf{X} = (X, d)$ *be a metric space and* $\varepsilon > 0$. *Suppose that* $((x_n, y_n))_{n\in\mathbb{N}}$ *is a sequence in* $X \times X$ *satisfying* $d(x_n, y_n) \geq \varepsilon$ *for all* $n \in \mathbb{N}$. *Then there exists a subsequence* $((x_{k_n}, y_{k_n}))_{n\in\mathbb{N}}$ *and a* $\kappa \in \mathbb{N}$ *such that* $d(x_{k_n}, y_{k_l}) \geq \varepsilon/\kappa$ *for all* $n, l \in \mathbb{N}$.

Proof. Let $K = \{x_n : n \in \mathbb{N}\}$ and $F = \{y_n : n \in \mathbb{N}\}$. If K or F is finite then some term of $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ repeats infinitely often. Assume that $(x_{k_n})_{n\in\mathbb{N}}$ is a constant subsequence of $(x_n)_{n\in\mathbb{N}}$. Clearly, $((x_{k_n}, y_{k_n}))_{n\in\mathbb{N}}$ is as required. So, now assume that no term of $(x_n)_{n\in\mathbb{N}}$ or $(y_n)_{n\in\mathbb{N}}$ repeats infinitely often. By passing to subsequences, we may assume that both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are injective. We consider the following cases:

(i) Either K or F has an infinite totally bounded subset. Assume that K is totally bounded. In this case we can construct as in [10, Theorem 3.1] a Cauchy subsequence $(x_{k_n})_{n\in\mathbb{N}}$. Fix $n_0 \in \mathbb{N}$ such that $d(x_{k_n}, x_{k_m}) \leq \varepsilon/2$ for all $n, m \geq n_0$. For all $n, m \geq n_0$ we have

$$\varepsilon \leq d(x_{k_n}, y_{k_n}) \leq d(x_{k_n}, x_{k_m}) + d(x_{k_m}, y_{k_n}) \leq d(x_{k_m}, y_{k_n}) + \varepsilon/2.$$

Therefore, $d(x_{k_m}, y_{k_n}) \geq \varepsilon/2 > \varepsilon/4$.

(ii) Neither K nor F has an infinite totally bounded subset. Since K is not totally bounded, via an easy induction we construct a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that for all distinct $n, m \in \mathbb{N}$, $d(x_{k_n}, x_{k_m}) \geq \delta_1$ for some $\delta_1 > 0$. Since $\{y_{k_n} : n \in \mathbb{N}\}$ is not totally bounded, via a second induction, we construct a subsequence $(y_{k_{t_n}})_{n\in\mathbb{N}}$ such that $d(y_{k_{t_n}}, y_{k_{t_m}}) \geq \delta_2$ for some $\delta_2 > 0$ and all distinct $n, m \in \mathbb{N}$. Fix $\kappa \in \mathbb{N} \setminus \{1\}$ such that $\delta = \varepsilon/\kappa < \min\{\delta_1, \delta_2\}$. For convenience assume that for all $n \in \mathbb{N}$, $x_{k_{t_n}} = x_n$ and $y_{k_{t_n}} = y_n$. Hence, for all distinct $n, m \in \mathbb{N}$, we have $d(x_n, x_m) > \delta$, $d(y_n, y_m) > \delta$ and $d(x_n, y_n) > \delta$.

Claim 1. *For every $n \in \mathbb{N}$, there is at most one $m \in \mathbb{N}$ such that*

$$\min\{d(x_n, y_m), d(x_m, y_n)\} < \delta/2.$$

Proof of Claim 1. Fix $n \in \mathbb{N}$. Assume that there exists $m \in \mathbb{N}$ such that $\min\{d(x_n, y_m), d(x_m, y_n)\} < \delta/2$. We will show that for every $v \in \mathbb{N}\setminus\{n, m\}$, $d(x_n, y_v) \geq \delta/2$ and $d(y_n, x_v) \geq \delta/2$. We consider two cases:

(a) $d(x_n, y_m) < \delta/2$. Since $\delta < d(y_m, y_v) \leq d(x_n, y_m) + d(x_n, y_v)$ and $\delta < d(x_m, x_v) \leq d(y_n, x_m) + d(y_n, x_v)$, it follows that $\delta/2 < \delta - d(x_n, y_m) \leq d(x_n, y_v)$ and $\delta/2 < \delta - d(y_n, x_m) \leq d(y_n, x_v)$.

(b) $d(x_m, y_n) < \delta/2$. Working as in (a) we can show that $d(x_n, y_v) \geq \delta/2$ and $d(y_n, x_v) \geq \delta/2$.

Using Claim 1 and a straightforward induction we construct a subsequence $((x_{k_n}, y_{k_n}))_{n\in\mathbb{N}}$ such that $d(x_{k_n}, y_{k_l}) \geq \delta$ for all $n, l \in \mathbb{N}$, finishing the proof of the lemma. ∎

3. Main results. Our first result in this section shows that the existence of a sequentially normal metric space which is not UC is relatively consistent with **ZF**.

Theorem 8. *The statement: "Every sequentially normal metric space is normal" implies* **IDI**(ℝ).

Proof. Aiming for a contradiction, assume that A is a Dedekind-finite subset of $\mathbb{R}$. Let d be the usual metric on A (that is, $d(x, y) = |x - y|$). N. Brunner [1] has shown that if there is a Dedekind-finite subset of $\mathbb{R}$ then there also exists a dense one. So, we assume that $\overline{A} = \mathbb{R}$ and $A \cap \mathbb{Q} = \emptyset$. Clearly, any two countable subsets F, G of A with $d(F, G) = 0$ intersect. [If $F \cap G = \emptyset$ then $d(F, G)$, being the distance of two finite disjoint sets, is greater than 0.] Hence, $d(f(A), f(B)) = 0$. So, $\mathbf{A}$ is sequentially normal but not normal (for every $q \in \mathbb{Q}$, $H = (-\infty, q] \cap A$ and $K = A \cap [q, \infty)$ are disjoint closed subsets of $\mathbf{A}$ such that $d(H, K) = 0$)—a contradiction. ∎

Next we show that **CAC** implies "every sequentially normal metric space is normal".

PROPOSITION 9. (**CAC**) *A metric space is sequentially normal iff it is normal.*

Proof. Assume $\mathbf{X} = (X, d)$ is a sequentially normal metric space, and fix closed and disjoint subsets A, B of $\mathbf{X}$. Assume for contradiction that $d(A, B) = 0$. Clearly, for every $n \in \mathbb{N}$,

$$W_n = \{(x, y) \in A \times B : d(x, y) < 1/n\} \neq \emptyset.$$

By **CAC**, for every $n \in \mathbb{N}$, fix $(x_n, y_n) \in W_n$. Let $H = \{x_n : n \in \mathbb{N}\}$ and $K = \{y_n : n \in \mathbb{N}\}$. Clearly, $\overline{H} \subseteq A$, $\overline{K} \subseteq B$ and $d(\overline{H}, \overline{K}) = 0$, contradicting the fact that $\mathbf{X}$ is sequentially normal. ∎

THEOREM 10. (**ZF**) *Let $\mathbf{X} = (X, d)$ be a metric space. Then:*

(i) *If $\mathbf{X}$ is Lebesgue then $\mathbf{X}$ is strongly UC.*
(ii) *$\mathbf{X}$ is weakly sequentially normal iff it is complete.*

Proof. (i) See [10, Theorem 7.3]. In the statement of that theorem it is stated that $\mathbf{X}$ is compact but the proof uses only the fact that $\mathbf{X}$ is Lebesgue.

(ii) ($\rightarrow$) Fix a weakly sequentially normal metric space $\mathbf{X} = (X, d)$. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbf{X}$. If $(x_n)_{n\in\mathbb{N}}$ has a limit point then there is nothing to show. So assume that $(x_n)_{n\in\mathbb{N}}$ has no limit points. It follows that any infinite subset of $\{x_n : n \in \mathbb{N}\}$ has no limit points, and consequently it is closed. In particular, $A = \{x_{2n} : n \in \mathbb{N}\}$ and $B = \{x_{2n+1} : n \in \mathbb{N}\}$ are closed and $(x_{2n})_{n\in\mathbb{N}}, (x_{2n+1})_{n\in\mathbb{N}}$ are Cauchy sequences such that $d(A, B) = 0$, contradicting the weak sequential normality of $\mathbf{X}$.

($\leftarrow$) Fix Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$ such that

$$\overline{\{x_n : n \in \mathbb{N}\}} \cap \overline{\{y_n : n \in \mathbb{N}\}} = \emptyset.$$

By the completeness of $\mathbf{X}$, let $\lim x_n = x \in X$ and $\lim y_n = y \in X$. Since $\overline{\{x_n : n \in \mathbb{N}\}} = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ and $\overline{\{y_n : n \in \mathbb{N}\}} = \{y\} \cup \{y_n : n \in \mathbb{N}\}$, it follows that $\overline{\{x_n : n \in \mathbb{N}\}}$ and $\overline{\{y_n : n \in \mathbb{N}\}}$ are compact disjoint subsets of $\mathbf{X}$, and so

$$d(\overline{\{x_n : n \in \mathbb{N}\}}, \overline{\{y_n : n \in \mathbb{N}\}}) > 0$$

as required. ∎

REMARK 1. In **ZF**, every sequentially normal metric space is weakly sequentially normal, but the converse fails. Indeed, $\mathbb{R}^2$ is complete, hence by Theorem 10, weakly sequentially normal. However, it is not sequentially normal: The sets $A = \{(1/n, n) : n \in \mathbb{N}\}$ and $B = \{(0, n) : n \in \mathbb{N}\}$ are closed and disjoint, but their distance is 0.

Next we show that the proof of (ii)$\rightarrow$(i) of Theorem 3 can be given in **ZF** if $\mathbf{Y}$ is separable and in **ZF** + **CAC** if $\mathbf{Y}$ is not separable.

THEOREM 11.

(i) (**ZF**) *Let* $\mathbf{X} = (X, d)$ *and* $\mathbf{Y} = (Y, \rho)$ *be metric spaces and* $f : \mathbf{X} \to \mathbf{Y}$ *a continuous function. If* $\mathbf{Y}$ *is separable, then* f *is uniformly continuous iff for any* $A, B \subseteq X$ *with* $d(A, B) = 0$, $\rho(f(A), f(B)) = 0$. *In particular* $\mathbf{X}$ *is normal iff it is UC.*

(ii) (**ZF + CAC**) *Let* $\mathbf{X} = (X, d)$ *and* $\mathbf{Y} = (Y, \rho)$ *be metric spaces and* $f : \mathbf{X} \to \mathbf{Y}$ *a continuous function. Then* f *is uniformly continuous iff for any* $A, B \subseteq X$ *with* $d(A, B) = 0$, $\rho(f(A), f(B)) = 0$.

Proof. (i) ($\to$) This follows from Theorem 4.

($\leftarrow$) Let $f : \mathbf{X} \to \mathbf{Y}$ be a continuous function such that for any $A, B \subseteq X$, if $d(A, B) = 0$ then $\rho(f(A), f(B)) = 0$. Aiming for a contradiction, assume that f is not uniformly continuous and fix $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exist $x_n, y_n \in X$ such that

$$d(x_n, y_n) < 1/n \quad \text{and} \quad \rho(f(x_n), f(y_n)) \geq 5\varepsilon. \tag{1}$$

Fix a countable dense subset $Q = \{q_n : n \in \mathbb{N}\}$ of $\mathbf{Y}$.

CLAIM 2. *For every* $n \in \mathbb{N}$ *there exists a pair* $(p_n, q_n) \in Q^2$ *with* $\rho(p_n, q_n) \geq 4\varepsilon$ *and* $x, y \in X$ *such that*

$$d(x, y) < 1/n, \quad \rho(f(x), p_n) < \varepsilon/3, \quad \rho(f(y), q_n) < \varepsilon/3.$$

Proof of Claim 2. Fix $n \in \mathbb{N}$. By (1) there exist $x, y \in X$ such that $d(x, y) < 1/n$ and $\rho(f(x), f(y)) \geq 5\varepsilon$. Fix $p_n, q_n \in Q$ such that $\rho(f(x), p_n) < \varepsilon/3$ and $\rho(f(y), q_n) < \varepsilon/3$. By the triangle inequality,

$$\begin{aligned} 5\varepsilon \leq \rho(f(x), f(y)) &\leq \rho(f(x), p_n) + \rho(p_n, q_n) + \rho(f(y), q_n) \\ &< \varepsilon/3 + \rho(p_n, q_n) + \varepsilon/3, \end{aligned}$$

so $\rho(p_n, q_n) \geq 4\varepsilon$ as required.

For every $n \in \mathbb{N}$ fix a pair $(p_n, q_n) \in Q^2$ as in Claim 2. By Lemma 7 there exists a subsequence $(p_{k_n}, q_{k_n})_{n \in \mathbb{N}}$ and $v \in \mathbb{N}$ such that $\rho(p_{k_n}, q_{k_l}) \geq \varepsilon/v$ for all $n, l \in \mathbb{N}$. For convenience assume that $v = 1$. Let

$$A = \{x \in X : \rho(f(x), \{p_{k_n} : n \in \mathbb{N}\}) < \varepsilon/3\}, \tag{2}$$

$$B = \{x \in X : \rho(f(x), \{q_{k_n} : n \in \mathbb{N}\}) < \varepsilon/3\}. \tag{3}$$

By Claim 2 for every $n \in \mathbb{N}$ there exist $x, y \in X$ such that $d(x, y) < 1/k_n$, $\rho(f(x), p_{k_n}) < \varepsilon/3$, and $\rho(f(y), q_{k_n}) < \varepsilon/3$. By (2) and (3) we see that $x \in A$ and $y \in B$. Hence, $d(A, B) = 0$. However, $\rho(f(A), f(B)) \geq \varepsilon/3$. Indeed, for every $x \in A$ and $y \in B$ there exist $n, l \in \mathbb{N}$ such that $\rho(f(x), p_{k_n}) < \varepsilon/3$ and $\rho(f(y), q_{k_l}) < \varepsilon/3$. Hence,

$$\begin{aligned} \varepsilon \leq \rho(q_{k_l}, p_{k_n}) &\leq \rho(f(x), p_{k_n}) + \rho(f(x), f(y)) + \rho(f(y), q_{k_l}) \\ &< \varepsilon/3 + \rho(f(x), f(y)) + \varepsilon/3. \end{aligned}$$

Therefore, $\rho(f(x), f(y)) > \varepsilon/3$ for all $x \in A$ and $y \in B$, and consequently $\rho(f(A), f(B)) \geq \varepsilon/3$ as required. This contradicts our assumption.

The second assertion of (i) follows from the first and Theorem 5.

(ii) It suffices to show ($\leftarrow$), as the other direction follows from Theorem 4. We shall mimic the proof of (i). Towards a contradiction, fix a continuous, non-uniformly continuous function $f : \mathbf{X} \to \mathbf{Y}$ such that for any $A, B \subseteq X$, if $d(A, B) = 0$ then $\rho(f(A), f(B)) = 0$. Let $\varepsilon > 0$ satisfy: For every $n \in \mathbb{N}$, there are $x, y \in X$ with $d(x, y) < 1/n$ and $\rho(f(x), f(y)) \geq \varepsilon$. Equivalently, if we let $a = f(x)$ and $b = f(y)$,

$$\forall n \in \mathbb{N},\ \exists a, b \in \operatorname{Ran}(f), \quad d(f^{-1}(a), f^{-1}(b)) < 1/n \text{ and } \rho(a, b) \geq \varepsilon. \tag{4}$$

By **CAC**, for every $n \in \mathbb{N}$, fix $(a_n, b_n) \in \operatorname{Ran}(f)^2$ with $d(f^{-1}(a_n), f^{-1}(b_n)) < 1/n$ and $\rho(a_n, b_n) \geq \varepsilon$. By Lemma 7 there exists a subsequence $(a_{k_n}, b_{k_n})_{n\in\mathbb{N}}$ and $v \in \mathbb{N}$ such that $\rho(a_{k_n}, b_{k_l}) \geq \varepsilon/v$ for all $n, l \in \mathbb{N}$. Set

$$A = \bigcup\{f^{-1}(a_{k_n}) : n \in \mathbb{N}\} \quad \text{and} \quad B = \bigcup\{f^{-1}(b_{k_n}) : n \in \mathbb{N}\}.$$

Clearly, $d(A, B) = 0$ but $\rho(f(A), f(B)) \geq \varepsilon/v$, a contradiction. ∎

COROLLARY 12. (**ZF**) *Let* $\mathbf{X} = (X, d)$ *be a separable metric space,* $\mathbf{Y} = (Y, \rho)$ *a metric space and* $f : \mathbf{X} \to \mathbf{Y}$ *a function. Then* f *is uniformly continuous iff for any* $A, B \subseteq X$ *with* $d(A, B) = 0$, $\rho(f(A), f(B)) = 0$.

Proof. The proof follows at once from Theorem 11 and the fact that the separability of $\mathbf{X}$ implies the separability of $f(\mathbf{X})$. ∎

THEOREM 13.

(i) *The statement: "Every normal metric space is strongly UC" implies* $\mathbf{PKW}(\aleph_0, \geq 2, \infty)$.
(ii) *The statement: "Every countably Lebesgue metric space is strongly UC" implies* $\mathbf{PKW}(\aleph_0, \geq 2, \infty)$.

Proof. (i) Assume for contradiction that $\mathbf{PKW}(\aleph_0, \geq 2, \infty)$ fails and fix a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty sets such that no infinite subfamily of $\mathcal{A}$ has a Kinna–Wagner choice set. Let $X = \bigcup\{A_n : n \in \mathbb{N}\}$ and $d : X \times X \to \mathbb{R}$ be the metric given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1/n & \text{if } x, y \in A_n, \\ 1 & \text{if } x \in A_n,\ y \in A_m \text{ and } n \neq m. \end{cases} \tag{5}$$

Let ρ be the discrete metric on X. We claim that (X, d) is normal. It suffices to show that there are no non-empty disjoint subsets A, B of X with $d(A, B) = 0$. Assume that A and B are such subsets. Since $d(A, B) = 0$, there is an infinite subset N of $\mathbb{N}$ such that for every $n \in N$, $A \cap A_n \neq \emptyset$

and $B \cap A_n \neq \emptyset$. Hence, $\{A \cap A_n : n \in N\}$ is a partial Kinna–Wagner choice set for $\mathcal{A}$, contradicting our hypothesis. Thus, (X, d) is normal as claimed.

By hypothesis, (X, d) is strongly UC. Let $f : (X, d) \to (X, \rho)$ be given by $f(x) = x$, $x \in X$. Since d induces the discrete topology on X, it follows that f is continuous. Hence, by hypothesis, it is uniformly continuous. Therefore, for $\varepsilon = 1$ there is a $k \in \mathbb{N}$ such that for all $x, y \in X$, if $d(x, y) < 1/k$ then $\rho(f(x), f(y)) < 1$. On the other hand, if $m > k$ and $x, y \in A_m$, $x \neq y$ then $d(x, y) < 1/k$ but $\rho(f(x), f(y)) \geq 1$, a contradiction.

(ii) Let A, X and d be as in the proof of (i). It suffices to show that $\mathbf{X}$ is countably Lebesgue. To see this, fix a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of $\mathbf{X}$. Since $\mathcal{A}$ has no Kinna–Wagner choice set, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $A_n \subseteq U$ for some $U \in \mathcal{U}$. We claim that $\delta = 1/n_0$ is a Lebesgue number of $\mathcal{U}$. Indeed, if $B \subseteq X$ and $\delta(B) < \delta$ then B is a singleton or $B \subseteq A_k$ for some $k > n_0$. In both cases there is $U \in \mathcal{U}$ with $B \subseteq U$. Hence, $\mathbf{X}$ is countably Lebesgue as required. ∎

Remark 2. The metric space $\mathbf{X} = (X, d)$ in the proof of (i) of Theorem 13 is not Lebesgue. Indeed, the open cover $\mathcal{U} = \{\{x\} : x \in X\}$ of X has no Lebesgue number. [For every $\delta > 0$ pick $k \in \mathbb{N}$ with $1/k < \delta$. Then $\delta(A_k) < \delta$ but A_k is included in no member of $\mathcal{U}$.] Thus, in view of the proof of part (ii) of Theorem 13, $\mathbf{X}$ is countably Lebesgue but not Lebesgue.

Theorem 14. **CAC** *implies* **UCE**, *and* **UCE** *implies* $\mathbf{CAC_{fin}}$.

Proof. **CAC** $\to$ **UCE**. The usual proof of **AC** $\to$ **UCE** applies with **CAC** in place of **AC**.

UCE $\to$ $\mathbf{CAC_{fin}}$. Assume, towards a contradiction, that $\mathbf{CAC_{fin}}$ fails, and fix a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty finite sets having no infinite subfamily with a choice function. Let $S = \bigcup\{A_i : i \in \omega\}$ and $X = S \cup \{\infty\}$ where $\infty \notin S$. Let $d : X \times X \to \mathbb{R}$ be the metric given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1/n & \text{if } x \in A_n,\ y \in A_m \text{ and } n \leq m, \\ 1/n & \text{if } x \in A_n,\ y = \infty. \end{cases}$$

We claim that the subspace $\mathbf{S}$ of $\mathbf{X}$ is complete. To see this, fix a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{S}$. Since $\mathcal{A}$ has no infinite subfamily with a choice set and its members are finite, it follows that $\{x_n : n \in \mathbb{N}\}$ is a finite set. Thus, some term of $(x_n)_{n\in\mathbb{N}}$ repeats infinitely often. Hence, $(x_n)_{n\in\mathbb{N}}$ has a subsequence converging to some $x \in S$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to x, and $\mathbf{S}$ is complete as claimed.

Let $f : \mathbf{S} \to \mathbf{S}$ be the identity function. Since f is uniformly continuous, by **UCE** there exists a uniformly continuous extension $F : \mathbf{X} \to \mathbf{S}$ of f.

Assume $F(\infty) = x_\infty \in A_k$. By the uniform continuity of F, there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta$, $d(F(x), F(y)) < 1/k$. Fix $m \in \mathbb{N}$ such that $m > k$ and $1/m < \delta$, and let $y \in A_m$. We have $d(y, \infty) = 1/m < \delta$ but $d(F(y), F(\infty)) = d(y, x_\infty) = 1/k$, a contradiction. ∎

THEOREM 15. *Let $\mathbf{X} = (X, d)$ be a metric space, S a dense subset of $\mathbf{X}$, $\mathbf{Y} = (Y, \rho)$ a Cantor complete metric space and $f : \mathbf{S} \to \mathbf{Y}$ a uniformly continuous function. Then there exists a unique uniformly continuous function $F : \mathbf{X} \to \mathbf{Y}$ extending f.*

Proof. From the uniform continuity of f on S it follows that

$$(6) \qquad \forall n \in \mathbb{N}, \exists \delta_n > 0, \forall x, y \in S, \quad d(x,y) < \delta_n \to \rho(f(x), f(y)) < 1/(6n).$$

Fix a sequence $(\delta_n)_{n\in\mathbb{N}}$ of rationals satisfying (6). For every $x \in X$ let $\mathcal{F}_x = \{B(x, \delta_n/2) \cap S : n \in \mathbb{N}\}$ and $\mathcal{G}_x = \{G_{xn} : n \in \mathbb{N}\}$ where $G_{xn} = f(B(x, \delta_n/2) \cap S)$ for every $n \in \mathbb{N}$. Clearly, $\mathcal{G}_x$ is a descending family of subsets of Y such that $\delta(G_{xn}) < 1/n$ for every $n \in \mathbb{N}$. Hence, by hypothesis, $\bigcap\{\overline{G_{xn}} : n \in \mathbb{N}\}$ is a singleton of Y, say $\{\ell_x\}$. Define $F : \mathbf{X} \to \mathbf{Y}$ by letting $F(x) = \ell_x$ for every $x \in X$.

We claim that $F(s) = f(s)$ for every $s \in S$. Indeed, for every $n \in \mathbb{N}$, we have $s \in B(s, \delta_n/2) \cap S$. Hence, $f(s) \in G_{sn}$ for all $n \in \mathbb{N}$ and consequently $f(s) \in \{\ell_s\}$, meaning that $f(s) = F(s)$ as required.

We next show that F is uniformly continuous. To see this, fix $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ satisfy $1/n_0 < \varepsilon$. We will show that $\rho(F(x), F(y)) < 1/n_0$ for any $x, y \in X$ with $d(x,y) < \delta_{n_0}/3$. Fix $x, y \in X$ with $d(x,y) < \delta_{n_0}/3$. We consider the following cases:

(i) $x, y \in S$. In this case the conclusion follows from (6) and the fact that $f(x) = F(x)$ and $f(y) = F(y)$.

(ii) $x \in S$, $y \notin S$. Fix $x_1 \in B(y, \delta_{n_0}/2) \cap S$. Since $f(x_1) \in G_{yn_0}$ and $F(y) \in \overline{G_{yn_0}}$, it follows that $\rho(f(x_1), F(y)) < 1/(3n_0)$. [If $z \in G_{yn_0}$ with $z = f(t)$ and $t \in B(x, \delta_n/2) \cap S$ satisfies $\rho(z, F(y)) < 1/(6n_0)$ then $\rho(f(x_1), F(y)) \leq \rho(f(x_1), z) + \rho(z, F(y)) < 1/(6n_0) + 1/(6n_0) = 1/(3n_0)$.] We have

$$d(x, x_1) \leq d(x,y) + d(y, x_1) < \delta_{n_0}/3 + \delta_{n_0}/2 < \delta_{n_0}.$$

Hence, $\rho(f(x), f(x_1)) < 1/(6n_0)$, and consequently

$$\rho(f(x), F(y)) \leq \rho(f(x), f(x_1)) + \rho(f(x_1), F(y)) < 1/(6n_0) + 1/(3n_0) < 1/n_0.$$

(iii) $x \notin S$, $y \in S$. This case can be treated as in (ii).

(iv) $x \notin S$, $y \notin S$. Fix $x_1 \in B(x, \delta_{n_0}/3) \cap S$ and $x_2 \in B(y, \delta_{n_0}/3) \cap S$. As in case (ii), $\rho(f(x_1), F(x)) < 1/(3n_0)$ and $\rho(f(x_2), F(y)) < 1/(3n_0)$. Clearly,

$$d(x_1, x_2) \leq d(x_1, x) + d(x,y) + d(y, x_2) < \delta_{n_0}/3 + \delta_{n_0}/3 + \delta_{n_0}/3 = \delta_{n_0}.$$

Hence, by (6), $\rho(f(x_1), f(x_2)) < 1/(6n_0)$. We have

$$\begin{aligned}\rho(F(x), F(y)) &\leq \rho(F(x), f(x_1)) + \rho(f(x_1), f(x_2)) + \rho(F(y), f(x_2)) \\ &< 5/(6n_0) < 1/n_0.\end{aligned}$$

Hence, F is uniformly continuous as claimed.

Finally, we show that F is unique. Assume that $h : \mathbf{X} \to \mathbf{Y}$ is a uniformly continuous function extending f. Suppose there exists $x \in X$ such that $h(x) \neq F(x)$. Then there exists $n \in \mathbb{N}$ such that $h(x) \notin \overline{G_{xn}}$. Hence, there is a $k \in \mathbb{N}$ such that $B(h(x), 1/k) \cap G_{xn} = \emptyset$. Fix $m > \max\{6n, k\}$. Since h is continuous and $\overline{S} = X$, there exists $s \in S \cap h^{-1}(B(h(x), 1/m))$ such that $d(s, x) < \delta_m/2$. Hence, $f(s) = h(s) \in B(h(x), 1/m)$ and $f(s) \in G_{xm}$, meaning that $B(h(x), 1/m) \cap G_{xm} \neq \emptyset$. Thus,

$$\emptyset \neq B(h(x), 1/m) \cap G_{xm} \subseteq B(h(x), 1/k) \cap G_{xn} = \emptyset,$$

a contradiction. ■

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