

## Approximation of integration maps of vector measures and limit representations of Banach function spaces

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**Abstract.** We study whether or not the integration maps of vector measures can be computed as pointwise limits of their finite rank Radon–Nikodým derivatives. The positive cases are obtained by using the circle of ideas related to the approximation property for Banach spaces. The negative ones are given by means of an appropriate use of the Daugavet property. As an application, we analyse when the norm in a space  $L^1(m)$  of integrable functions can be computed as a limit of the norms of the spaces of integrable functions with respect to the Radon–Nikodým derivatives of  $m$ .

**1. Introduction.** Let  $X$  be a Banach space and  $m$  be an  $X$ -valued countably additive vector measure. Consider the space  $L^1(m)$  of (scalar) integrable functions with respect to  $m$ . In this paper we are interested in (pointwise) approximation by finite rank operators of the associated integration map  $I_m : L^1(m) \rightarrow X$ . Actually, we will choose nets of finite rank operators of a special class, namely finite rank Radon–Nikodým derivatives of  $m$ .

We will show that this can be done for weakly compact operators with values in a space with the approximation property. This allows us to prove a positive result for a large class of examples, for instance when  $X$  is a reflexive space. However, there are cases where these results cannot be applied; indeed, some of them can even be considered as canonical. Martingale type procedures also allow us to find approximations of integration maps by finite rank operators in some of these situations, for example the identity map (considered as an integration operator) when  $m$  is a (scalar) probability measure without atoms, and so  $L^1(m)$  is a classical Lebesgue space. However,

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we also show that the unconditional pointwise expansion in a series of finite rank operators has some constraints, and cannot be expected in general. To do so, we will use the circle of ideas related to the Daugavet property. A Banach space  $X$  is said to have the *Daugavet property* if every operator  $T : X \rightarrow X$  of rank 1 satisfies the *Daugavet equation*  $\|\text{Id} + T\| = 1 + \|T\|$ . In recent years, great efforts have been made to study the Daugavet property for Banach spaces and Banach lattices, and its natural extension to other operators different from the identity map, called Daugavet centers. The reader can find information about the Daugavet equation for general maps in [5] and the references therein.

The reason why this property is relevant for us is that the Daugavet equation as an estimate of the norm of a sum of operators may be used to establish clear criteria of when a particular map cannot be represented by a pointwise unconditionally convergent series of finite rank operators (see for example [4, Theorem 2.9]). Therefore, to determine some limits to approximation by finite rank maps, we are interested in analysing when the integration operator  $I_m : L^1(m) \rightarrow X$  satisfies the Daugavet equation.

**2. Preliminaries.** The notation that we use is standard. If  $X$  is a Banach space, we write  $B_X$  for its closed unit ball, and  $X^*$  for its dual space. Let  $(\Omega, \Sigma, \mu)$  be a positive finite measure space, and write  $L^0(\mu)$  for the space of all measurable real functions on  $\Omega$  (functions which are equal  $\mu$ -a.e. are identified). We say that a Banach space  $X(\mu)$  of functions in  $L^0(\mu)$  is a *Banach function space* with respect to  $\mu$  if  $\chi_\Omega \in X(\mu)$ , and for each pair of measurable functions  $f, g$ , if  $|f| \leq |g|$  with  $g \in X(\mu)$ , then  $f \in X(\mu)$  and  $\|f\| \leq \|g\|$ . We will write  $X$  instead of  $X(\mu)$  if the measure  $\mu$  is fixed in the context. We say that  $X(\mu)$  is  $\sigma$ -*order continuous* if for every sequence of functions  $f_n \in X(\mu)$  with  $f_n \downarrow 0$  we have  $\|f_n\|_{X(\mu)} \rightarrow 0$ . We will also use the following (non-standard) notation. If  $\mu$  and  $\nu$  are measures on the same measurable space and  $\nu$  is absolutely continuous with respect to  $\mu$ , we will write  $X(\mu) \Subset Y(\nu)$  to indicate that the operator that maps the class of functions  $\mu$ -a.e. equal to  $f$  to the class of functions  $\nu$ -a.e. equal to the same  $f$  is continuous.

Throughout the paper,  $m : \Sigma \rightarrow X$  will be a countably additive vector measure, where  $X$  is a Banach space; the reader can find all the information that is needed—and briefly indicated below—on vector measures and integration in [6, 14]. For each  $x^* \in X^*$  the formula  $\langle m, x^* \rangle(A) := \langle m(A), x^* \rangle$ ,  $A \in \Sigma$ , defines a (countably additive) scalar measure. We write  $|\langle m, x^* \rangle|$  for its variation. The function  $\|m\|$  given on a set  $A \in \Sigma$  by

$$\|m\|(A) = \sup\{|\langle m, x^* \rangle|(A) : x^* \in X^*, \|x^*\| \leq 1\}$$

is called the *semivariation* of  $m$ . It is equivalent to the variation if  $m$  is a scalar measure. A (vector or scalar valued) measure  $m$  is *absolutely contin-*

uous with respect to  $\mu$  if  $\mu(A) = 0$  implies  $\|m\|(A) = 0$ ; in this case we write  $m \ll \mu$ . It is well known that there is always a (non-unique) measure of the form  $|\langle m, x^* \rangle|$  with respect to which  $m$  is absolutely continuous; we will call such a measure a *Rybakov measure* for  $m$ . Let  $\mu$  be a finite measure; we say that a vector measure  $m : \Sigma \rightarrow X$  is *equivalent* to  $\mu$  if for all  $A \in \Sigma$ ,  $\mu(A) = 0$  if and only if  $\|m\|(A) = 0$ .

The space  $L^1(m)$  of integrable functions with respect to  $m$  is a Banach function space over any Rybakov measure; throughout the paper, we will fix one of them, which will usually be denoted by  $\mu$ . The elements of  $L^1(m)$  are (classes of  $\mu$ -a.e. measurable) functions  $f$  that are integrable with respect to each scalar measure  $\langle m, x^* \rangle$ , and for every  $A \in \Sigma$  there is an element  $\int_A f dm \in X$  such that  $\langle \int_A f dm, x^* \rangle = \int_A f d\langle m, x^* \rangle$  for every  $x^* \in X^*$ ; the space  $L^1(m)$  of  $m$ -a.e. equal  $m$ -integrable functions is an order continuous Banach lattice endowed with the norm

$$(2.1) \quad \|f\|_{L^1(m)} := \sup_{x^* \in B_{X^*}} \int |f| d|\langle m, x^* \rangle|, \quad f \in L^1(m),$$

and the  $m$ -a.e. order. The associated integration operator  $I_m : L^1(m) \rightarrow X$  is defined by  $I_m(f) := \int_{\Omega} f dm$ , where  $f \in L^1(m)$ . It is linear and  $\|I_m\| = 1$ .

A continuous operator  $T : X \rightarrow X$  on a Banach space is said to satisfy the *Daugavet equation* if

$$\|\text{Id} + T\| = 1 + \|T\|.$$

A Banach space  $X$  has the *Daugavet property* if this equation is satisfied for every rank 1 operator. It is well known that if this happens, then the same equation holds for each weakly compact or merely Radon–Nikodým operator [9]. Recall that a subset  $A$  of a Banach space is said to have the *Radon–Nikodým property* if every closed convex subset  $B \subseteq A$  is the closed convex hull of its denting points; an operator  $T$  is a *Radon–Nikodým operator* if the closure of  $T(B_X)$  has the Radon–Nikodým property. Every weakly compact operator belongs to this class.

The reader can find a review of the classical results on the Daugavet property in [16]. For the case of Banach lattices of functions, which is particularly important for this paper, see [1, 2] and the references therein. The following generalization of the notion of Daugavet property will be used. Following [4, Definition 1.2], we say that a continuous linear operator  $G : X \rightarrow Y$  between Banach spaces is a *Daugavet center* if  $\|G + T\| = \|G\| + \|T\|$  for every operator  $T : X \rightarrow Y$  of rank 1. For this notion and the main properties which are necessary for this paper, see also [3, 4].

**3. Approximation of weakly compact integration operators of a vector measure by its finite rank Radon–Nikodým derivatives.** Approximation of weakly compact operators by finite rank operators is closely

related to the approximation property of the Banach space  $X$  where the operators are defined, or to this property for its dual space  $X^*$ . A series of papers (e.g., [12]) recently published have shown that  $X^*$  has the approximation property if and only if for every Banach space  $Z$  and every weakly compact operator  $T : Z \rightarrow X$  there is a net of finite rank operators  $T_\alpha : Z \rightarrow X$  of norm  $\|T_\alpha\| \leq \|T\|$  converging to  $T$  pointwise. The problem goes back to the memoir of Grothendieck [7, Chap. I, p. 184], and these developments have provided some useful tools and solutions to some long-standing questions that will be crucial for our arguments. The question is: When and how can the integration map of a vector measure be approximated by nets of finite rank operators? As the reader will see, the result will be used to study when, given a vector measure  $m$ , the norm of each element in the corresponding space of integrable functions can be approximated as the limit of a net of natural finite components of  $I_m$ , which we call finite rank Radon–Nikodým derivative operators of  $m$ .

**3.1. Integration operators and Radon–Nikodým derivatives of vector measures.** Let us start by establishing several results regarding Radon–Nikodým derivatives of scalarizations of a given vector measure. Let  $m : \Sigma \rightarrow E$  be a (countably additive) vector measure.

LEMMA 3.1. *Consider a vector measure  $m : \Sigma \rightarrow X$  and a fixed finite measure  $\mu$  equivalent to  $m$  representing the duality (that is, for  $f \in L^1(m)$  and  $g \in (L^1(m))'$ , the Köthe dual,  $\langle f, g \rangle := \int fg d\mu$ ;  $\mu$  may be a Rybakov measure). Then for every  $x^* \in X^*$ ,*

$$I_m^*(x^*) = \varphi_{x^*}^m := \frac{d\langle m, x^* \rangle}{d\mu} \in (L^1(m))',$$

where  $d\langle m, x^* \rangle/d\mu$  represents the Radon–Nikodým derivative of the measure  $A \mapsto \langle m(A), x^* \rangle$  with respect to  $\mu$ . Moreover,  $\|\varphi_{x^*}^m\|_{(L^1(m))'} \leq \|x^*\|_{X^*}$  for all  $x^* \in X^*$ .

*Proof.* Take  $f \in L^1(m)$  and  $x^* \in X^*$ . Since  $f$  is  $m$ -integrable and  $\langle m, x^* \rangle \ll \mu$ , we have

$$\langle I_m(f), x^* \rangle = \int f d\langle m, x^* \rangle = \int f \frac{d\langle m, x^* \rangle}{d\mu} d\mu = \int f \varphi_{x^*}^m d\mu.$$

Since this is well-defined for each  $f \in L^1(m)$ , we have  $\varphi_{x^*}^m \in (L^1(m))'$ ; more specifically,

$$\left| \int f \frac{d\langle m, x^* \rangle}{d\mu} d\mu \right| \leq \|f\|_{L^1(m)} \cdot \left\| \frac{d\langle m, x^* \rangle}{d\mu} \right\|_{(L^1(m))'}.$$

Moreover,

$$\begin{aligned} \|\varphi_{x^*}^m\|_{(L^1(m))'} &= \sup_{f \in B_{L^1(m)}} \left| \int f \frac{d\langle m, x^* \rangle}{d\mu} d\mu \right| = \sup_{f \in B_{L^1(m)}} \left| \left\langle \int f dm, x^* \right\rangle \right| \\ &\leq \sup_{f \in B_{L^1(m)}} \left\| \int f dm \right\| \cdot \|x^*\|_{X^*} \leq \|x^*\|_{X^*}. \blacksquare \end{aligned}$$

Note, however, that the map  $x^* \mapsto \varphi_{x^*}^m$  need not be injective in general.

### 3.2. Pointwise convergence of nets of norms to the $L^1(m)$ norm.

In this section we prove the main general result concerning convergence of nets of  $L^1(m_\tau)$  norms to  $\|\cdot\|_{L^1(m)}$ .

Fix a finite measure  $\mu$ . Let  $\{X(\mu_\tau) : \tau \in \Lambda\}$  be a net of Banach function spaces, where all the measures  $\mu_\tau$  are absolutely continuous with respect to  $\mu$ . We will study when the norm in a Banach function space  $X(\mu)$  such that  $X(\mu) \subseteq \bigcap_{\tau \in \Lambda} X(\mu_\tau)$  can be computed as  $\|f\|_{X(\mu)} = \lim_\tau \|f\|_{X(\mu_\tau)}$  for all  $f \in X(\mu)$ .

In what follows, we are interested in analysing when, for a net  $\{m_\tau\}$  of vector measures, the  $L^1(m)$  norm is the pointwise limit of the  $L^1(m_\tau)$  norms. We need to introduce some new elements.

Let  $A \in \Sigma$ . We define a function  $h_A$  in  $L^\infty(\mu)$  by  $h_A = \chi_A - \chi_{A^c}$ ; this definition makes sense  $\mu$ -a.e. We will denote by  $\mathcal{H}$  all the (classes of) functions defined in this way.

LEMMA 3.2. *Let  $m : \Sigma \rightarrow X$  be a countably additive vector measure. For all  $f \in L^1(m)$ , we have*

$$\|f\|_{L^1(m)} = \sup_{A \in \Sigma} \left\| \int f h_A dm \right\|_X.$$

*Proof.* This is a direct calculation. We write  $A(x^*)$  for the measurable set, depending on  $x^*$ , where the Radon–Nikodým derivative of  $\langle m, x^* \rangle$  with respect to  $|\langle m, x^* \rangle|$  equals 1. Then

$$\begin{aligned} \|f\|_{L^1(m)} &= \sup_{x^* \in B_{X^*}} \int |f| d|\langle m, x^* \rangle| \\ &= \sup_{x^* \in B_{X^*}} \int f (\chi_{\{\text{sign}(f) > 0\}} - \chi_{\{\text{sign}(f) \leq 0\}}) (\chi_{A(x^*)} - \chi_{A(x^*)^c}) d\langle m, x^* \rangle \\ &= \sup_{x^* \in B_{X^*}} \left[ \int f (\chi_{\{\text{sign}(f) > 0\} \cap A(x^*)} \cup \{\text{sign}(f) \leq 0\} \cap A(x^*)^c}) d\langle m, x^* \rangle \right. \\ &\quad \left. - \int f (\chi_{\{\text{sign}(f) > 0\} \cap A(x^*)^c} \cup \{\text{sign}(f) \leq 0\} \cap A(x^*)}) d\langle m, x^* \rangle \right]. \end{aligned}$$

Since the sets appearing in the characteristic functions above are complementary, and using the fact that the function  $f$  is integrable with respect

to  $m$ , we have

$$\|f\|_{L^1(m)} \leq \sup_{x^* \in B_{X^*}, A \in \Sigma} \left\langle \int fh_A dm, x^* \right\rangle = \sup_{A \in \Sigma} \left\| \int fh_A dm \right\|_X.$$

The converse inequality is an immediate consequence of the fact that  $|f| = |fh_A|$  for all  $A \in \Sigma$ ,  $\|\cdot\|_{L^1(m)}$  is a lattice norm, and the straightforward inequality

$$\left\| \int fh_A dm \right\|_X \leq \|fh_A\|_{L^1(m)} = \|f\|_{L^1(m)} \cdot \blacksquare$$

LEMMA 3.3. *Let  $m, m_1 : \Sigma \rightarrow X$  be vector measures and let  $f \in L^1(m) \cap L^1(m_1)$ . Then*

$$\left| \|f\|_{L^1(m)} - \|f\|_{L^1(m_1)} \right| \leq \sup_{A \in \Sigma} \left\| \int fh_A dm - \int fh_A dm_1 \right\|_X.$$

*Proof.* Consider the linear space VM of all countably additive vector measures from  $\Sigma$  to  $X$ . First fix a simple function  $f$ . The function  $\phi_f : \mathcal{H} \times \text{VM} \rightarrow X$  given by the formula

$$\phi_f(h_A, m) := \int fh_A dm \in X$$

is well-defined. Note that

$$m \mapsto \sup_{A \in \Sigma} \left\| \int fh_A dm \right\|_X$$

is a seminorm on VM. Consequently, if  $m, m_1 \in \text{VM}$ , then

$$\left| \sup_{A \in \Sigma} \left\| \int fh_A dm \right\|_X - \sup_{A \in \Sigma} \left\| \int fh_A dm_1 \right\|_X \right| \leq \sup_{A \in \Sigma} \left\| \int fh_A d(m - m_1) \right\|_X$$

for all simple functions. Thus, as a consequence of Lemma 3.2, we get

$$\left| \|f\|_{L^1(m)} - \|f\|_{L^1(m_1)} \right| \leq \sup_{A \in \Sigma} \left\| \int fh_A dm - \int fh_A dm_1 \right\|_X$$

for all simple functions. Since simple functions are norm dense in both spaces, we get the result for all functions in  $L^1(m) \cap L^1(m_1)$ .  $\blacksquare$

The next theorem provides our first general approximation result, which can be applied to any net of vector measures. Notice that no requirements on  $X$  are imposed. Later on in this section (Theorem 3.7) we will show that there always exists a net as below for every vector measure whenever  $X$  has the approximation property and the integration map is weakly compact. Recall from Lemma 3.1 the definition of the Radon–Nikodým derivatives  $\varphi_{x^*}^m$  of the measure  $m$ .

THEOREM 3.4. *Let  $\Lambda$  be a directed set. Let  $m, m_\eta : \Sigma \rightarrow X$ ,  $\eta \in \Lambda$ , be countably additive vector measures that are absolutely continuous with respect to the scalar measure  $\mu$ . Assume that  $L^1(m) \subseteq \bigcap_{\eta \in \Lambda} L^1(m_\eta)$  and that for every  $\eta \in \Lambda$  the inclusion of  $L^1(m)$  in each  $L^1(m_\eta)$  has norm 1. Suppose that one of the following conditions holds:*

(i) For each  $f \in L^1(m)$ ,

$$\limsup_{\eta} \sup_{A \in \Sigma} \left\| \int f h_A dm - \int f h_A dm_{\eta} \right\| = 0.$$

(ii) For each  $x^* \in X^*$ ,

$$\lim_{\eta} \varphi_{x^*}^{m_{\eta}} = \varphi_{x^*}^m \quad \text{in the weak}^* \text{ topology.}$$

Then for each  $f \in L^1(m)$ ,

$$\lim_{\eta \in A} \|f\|_{L^1(m_{\eta})} = \|f\|_{L^1(m)}.$$

*Proof.* Assume first that (i) holds. Then the result is a consequence of Lemma 3.3. Using the inequality given there and the fact that each  $m$ -integrable function is  $m_{\eta}$ -integrable, we get, for each  $f \in L^1(m)$ ,

$$\left| \|f\|_{L^1(m)} - \|f\|_{L^1(m_{\eta})} \right| \leq \sup_{A \in \Sigma} \left\| \int f h_A dm - \int f h_A dm_{\eta} \right\|_X \xrightarrow{\eta} 0.$$

Suppose now that (ii) holds. Fix  $f \in L^1(m)$  and  $\varepsilon > 0$ . Taking into account Lemma 3.2 we know that there are  $x^* \in B_{X^*}$  and  $A \in \Sigma$  such that

$$\int f h_A d\langle m, x^* \rangle \geq \|f\|_{L^1(m)} - \varepsilon.$$

Then, using the pointwise limit condition, we find that

$$\lim_{\eta} \varphi_{x^*}^{m_{\eta}}(f h_A) = \varphi_{x^*}^m(f h_A) = \int f h_A d\langle m, x^* \rangle \geq \|f\|_{L^1(m)} - \varepsilon.$$

Since  $\int f h_A d\langle m_{\eta}, x^* \rangle \leq \|f\|_{L^1(m_{\eta})}$  for all  $\eta$ , we deduce that there is  $\eta_0 \in A$  such that for all  $\eta \geq \eta_0$ ,

$$\|f\|_{L^1(m_{\eta})} + \varepsilon \geq \|f\|_{L^1(m)} - \varepsilon.$$

On the other hand,  $\|f\|_{L^1(m)} \geq \|f\|_{L^1(m_{\eta})}$ , and since this happens for each  $\varepsilon > 0$ , we obtain

$$\|f\|_{L^1(m)} = \lim_{\eta} \|f\|_{L^1(m_{\eta})}. \quad \blacksquare$$

**3.3. Approximation of weakly compact integration maps.** The arguments in the previous sections lead to the main result of this section. We want to know if it is possible to approximate an integration map by finite rank operators. In fact, we want to know how well an integration map  $I_m : L^1(m) \rightarrow X$  can be approximated using the naturally associated finite rank operators, namely finite rank Radon–Nikodým derivatives of  $m$  to be defined later on. Let us start with an easy case—vector measures with values in a Banach space with a Schauder basis—in order to explain the arguments that we use.

**EXAMPLE 3.5.** Let us show that for vector measures with values in a Banach space with a Schauder basis, nothing is required to obtain pointwise

approximation of the  $L^1(m)$  norm. This case allows a specific treatment using some ad hoc constructed tools.

Let  $X = \ell$  be a Banach space with a normalized monotone Schauder basis  $\{e_i\}_{i=1}^\infty$ ; hence we assume that the basis constant is 1. We define the sequence  $\{e_i^*\}_{i=1}^\infty$  of biorthogonal functionals in  $\ell^*$  as usual: for each  $i, j \in \mathbb{N}$ ,  $e_i^*(e_j) = \delta_{i,j}$ . For a fixed natural number  $n$ , write  $P_n$  for the basis projection on the  $n$ -dimensional space generated by the first  $n$  vectors, that is,

$$P_n(x) := \sum_{i=1}^n \langle x, e_i^* \rangle e_i \in \ell, \quad x \in \ell;$$

we have  $\|P_n\| \leq 1$  by assumption. Consider a countably additive vector measure  $m : \Sigma \rightarrow \ell$  and construct the finite-dimensional components  $m_n$  of  $m$  by

$$m_n(A) := (P_n \circ m)(A) = \sum_{i=1}^n \langle m(A), e_i^* \rangle e_i \in \ell, \quad A \in \Sigma, n \in \mathbb{N}.$$

This is clearly a sequence of countably additive vector measures. Take any  $f \in L^1(m)$ . Then for each fixed  $A \in \Sigma$  we have

$$\left\| \int f h_A dm \right\|_\ell = \lim_n \left\| P_n \left( \int f h_A dm \right) \right\|_\ell = \lim_n \left\| \int f h_A dm_n \right\|_\ell.$$

Thus,

$$\|f\|_{L^1(m)} = \sup_{A \in \Sigma} \left\| \int f h_A dm \right\|_\ell = \sup_{A \in \Sigma} \lim_n \left\| \int f h_A dm_n \right\|_\ell.$$

On the other hand, using  $\|P_n\| \leq 1$ , we obtain

$$\sup_{A \in \Sigma} \left\| \int f h_A dm \right\|_\ell \geq \sup_{A \in \Sigma} \left\| P_n \left( \int f h_A dm \right) \right\|_\ell = \sup_{A \in \Sigma} \left\| \int f h_A dm_n \right\|_\ell$$

for every  $n \in \mathbb{N}$ , and so

$$\|f\|_{L^1(m)} = \sup_{A \in \Sigma} \lim_n \left\| \int f h_A dm_n \right\|_\ell \leq \lim_n \sup_{A \in \Sigma} \left\| \int f h_A dm_n \right\|_\ell \leq \|f\|_{L^1(m)}.$$

This proves the result.

The same arguments yield the following general result.

**PROPOSITION 3.6.** *Let  $m$  be a vector measure. Consider a net  $\{P_\eta\}$  of operators  $P_\eta : X \rightarrow X$  that converges pointwise to the identity map in  $X$ , and such that  $\|P_\eta\| \leq 1$  for all  $\eta$ . Consider the vector measures  $m_\eta := P_\eta \circ m$ . Then  $L^1(m) \subseteq L^1(m_\eta)$  for all  $\eta$ , and for all  $f \in L^1(m)$ ,*

$$\|f\|_{L^1(m)} = \lim_\eta \|f\|_{L^1(m_\eta)}.$$

The natural finite rank maps in the setting of vector measures are the ones that can be written as finite sums of products of vectors in  $X$  by Radon–

Nikodým derivatives of scalarizations of the vector measure, that is, finite rank operators like

$$R_m(\cdot) = \sum_{i=1}^n \varphi_{x_i^*}^m(\cdot)x_i = \sum_{i=1}^n \frac{d\langle m, x_i^* \rangle}{d\mu}(\cdot)x_i.$$

We will call this kind of operator a *finite rank Radon–Nikodým derivative operator* of  $m$ .

**THEOREM 3.7.** *Let  $X$  be a Banach space with the approximation property, and let  $m : \Sigma \rightarrow X$  be a countably additive vector measure such that  $I_m$  is weakly compact. Then there is a net  $\{R_m^\alpha\}_\alpha$  of finite rank Radon–Nikodým derivative operators of norm  $\leq 1$  that converges to  $I_m$  in the strong operator topology. That is, for every  $f \in L^1(m)$ ,*

$$\lim_{\alpha} R_m^\alpha(f) = \int f \, dm = I_m(f).$$

*Proof.* We use [11, Theorem 1.2], which extends the famous Davis–Figiel–Johnson–Pełczyński factorization technique for weakly compact operators. Assume that  $\|I_m\| = 1$ . Since  $X$  has the approximation property, [11, Lemma 1.1] shows that there is a reflexive Banach space  $X_K$  that is embedded in  $X$  with inclusion map  $J_K : X_K \rightarrow X$  such that  $I_m(B_{L^1(m)}) \subseteq B_{X_K}$ . That is, there is a factorization of  $T$  through  $X_K$  as  $J_K \circ I_m^0$ , where  $I_m^0 : L^1(m) \rightarrow X_K$  is the integration map when considered as an operator into  $X_K$ , and  $\|J_K\| \leq 1$ . In [11, proof of Theorem 1.2] it can be seen that there exists a net  $\{A_\alpha\}$  of finite rank operators from  $X_K$  to  $X$  such that  $\sup_\alpha \|A_\alpha\| \leq 1$ , with  $A_\alpha(I_m^0(f)) \rightarrow J_K(I_m^0(f))$  for all  $f \in L^1(m)$ . Define  $R_m^\alpha(f) := A_\alpha \circ I_m^0(f)$ . Each  $A_\alpha$  has the form  $A_\alpha = \sum_{i=1}^n z_i^* \otimes x_i$ . Since  $X_K$  is reflexive and  $J_K$  is injective [11, Lemma 1.1],  $J_K^*$  has dense range. Hence we may assume that  $z_i^* = J_K^* x_i^*$ , and we still have  $\sup_\alpha \|A_\alpha\| \leq 1$  and  $A_\alpha(I_m^0(f)) \rightarrow J_K(I_m^0(f))$  for all  $f \in L^1(m)$ . Now we see that

$$\begin{aligned} A_\alpha(I_m^0(f)) &= \sum_{i=1}^n x_i \left\langle z_i^*, \int f \, dm \right\rangle = \sum_{i=1}^n x_i \left\langle x_i^*, \int f \, dm \right\rangle \\ &= \sum_{i=1}^n \left\langle f, \frac{d\langle m, x_i^* \rangle}{d\mu} \right\rangle x_i; \end{aligned}$$

in the first instance,  $\int f \, dm$  is taken in  $X_K$ , and in the second in  $X$ . ■

The following is the main positive result regarding approximation of the  $L^1(m)$  norm by finite-dimensional components of the integration map  $I_m$ . Recall that, for an operator  $T : L^1(m) \rightarrow X$ , the associated vector measure is given by  $m_T(A) = T(\chi_A)$  for  $A \in \Sigma$ .

**COROLLARY 3.8.** *Let  $X$  be a Banach space with the approximation property, and let  $m : \Sigma \rightarrow X$  be a countably additive vector measure such that  $I_m$*

is weakly compact. Then there is a net  $\{R_m^\alpha\}_\alpha$  of finite rank Radon–Nikodým derivative operators of norm  $\leq 1$  such that

$$\lim_\alpha \|f\|_{L^1(m_\alpha)} = \|f\|_{L^1(m)}, \quad f \in L^1(m),$$

where  $m_\alpha$  is the vector measure associated to  $R_m^\alpha$ .

*Proof.* Consider the net  $\{R_m^\alpha\}_\alpha$  of finite rank operators of norm  $\leq 1$  provided by Theorem 3.7. Note that all the measures  $m_\alpha$  are countably additive, since the  $R_m^\alpha$  are finite rank operators and  $L^1(m)$  is order continuous. Take  $f \in L^1(m)$  and  $\varepsilon > 0$ . Then there is  $A \in \Sigma$  such that

$$\begin{aligned} \|f\|_{L^1(m)} - \varepsilon &< \left\| \int f h_A dm \right\|_X = \lim_\alpha \left\| \int f h_A dm_\alpha \right\|_X \\ &\leq \sup_{A \in \Sigma} \lim_\alpha \left\| \int f h_A dm_\alpha \right\|_X. \end{aligned}$$

Now, since  $\|R_m^\alpha\| \leq 1$ , we obtain

$$\sup_{A \in \Sigma} \left\| \int f h_A dm \right\|_X \geq \sup_{A \in \Sigma} \|R_m^\alpha(f h_A)\|_X = \sup_{A \in \Sigma} \left\| \int f h_A dm_\alpha \right\|_X = \|f\|_{L^1(m_\alpha)}$$

for every  $\alpha$ , which gives

$$\begin{aligned} \|f\|_{L^1(m)} - \varepsilon &< \sup_{A \in \Sigma} \lim_\alpha \left\| \int f h_A dm_\alpha \right\|_X \leq \lim_\alpha \sup_{A \in \Sigma} \left\| \int f h_A dm_\alpha \right\|_X \\ &\leq \lim_\alpha \|f\|_{L^1(m_\alpha)} \leq \|f\|_{L^1(m)}. \quad \blacksquare \end{aligned}$$

This clarifies the situation in a large class of Banach space valued vector measures. For example, if  $X$  is reflexive, we obtain the result directly: each integration map can be approximated by a net of finite rank operators that defines a net  $\{m_\alpha\}_\alpha$  of vector measures, and its norm can be computed as the limit of the  $L^1(m_\alpha)$  norms.

**3.4. Approximation of an integration map by martingale type constructions.** In the previous section we have shown that weakly compact integration operators in spaces with the approximation property allow approximations by finite rank Radon–Nikodým derivative operators. However, this result still excludes the canonical example of an integration map: the identity map in  $L^1[0, 1]$ . The method of approximation that can be used in this case is based on martingale type constructions.

Consider a vector measure  $m : \Sigma \rightarrow X$  and the space  $L^1(m)$ . Let  $\mathbb{P} := \{\mathcal{P}_\eta : \eta \in \Lambda\}$  be the net of (classes of  $\mu$ -a.e. equal) finite measurable partitions of  $\Omega$  endowed with the usual inclusion order. Consider the net of vector measures

$$m_\eta(A) := \sum_{B \in \mathcal{P}_\eta} \frac{\mu(A \cap B)}{\mu(B)} m(B) \in X,$$

where  $\mathcal{P}_\eta \in \mathbb{P}$ .

Let  $I_{m_\eta} : L^1(m_\eta) \rightarrow X$  be the corresponding integration map. It is easy to see that

$$I_{m_\eta}(f) = \sum_{B \in \mathcal{P}_\eta} \frac{\int_B f d\mu}{\mu(B)} m(B) \in X, \quad f \in L^1(m_\eta).$$

A direct computation also shows that  $L^1(m) \subseteq L^1(m_\eta)$ , that is,  $I_{m_\eta}$  is well-defined for all functions in  $L^1(m)$ . The question then is: When can  $I_m$  be pointwise approximated by  $\{I_{m_\eta} : \eta \in \Lambda\}$ ? In other words, is it true that

$$\lim_\eta I_{m_\eta}(f) = I_m(f)$$

for every  $f \in L^1(m)$ ? We will show that in general the answer is negative, although it is positive for  $L^p[0, 1]$ ,  $p \geq 1$ .

EXAMPLE 3.9. Let us check again our canonical example: the identity map in  $L^1[0, 1]$  considered as the integration map for the vector measure  $m(A) := \chi_A$ ,  $A \in \Sigma$ . Fix  $f \in L^1(m)$ . Consider the net  $\mathbb{P}$  as above and fix  $\varepsilon > 0$ . Then there is a simple function  $f_\varepsilon = \sum_{i=1}^n \lambda_i \chi_{B_i}$  such that  $\|f - f_\varepsilon\|_{L^1(m)} \leq \varepsilon$ . We assume, by setting  $\lambda_n = 0$  if necessary, that  $\mathcal{P}_{\eta_0} = \{B_1, \dots, B_n\}$  is a partition. Note that  $I_{m_\eta}(f_\varepsilon) = I_{m_{\eta_0}}(f_\varepsilon) = f_\varepsilon$  for every  $\eta \geq \eta_0$ . Then

$$\begin{aligned} \|I_m(f) - I_{m_\eta}(f)\|_{L^1[0,1]} &\leq \|I_m(f) - I_{m_\eta}(f_\varepsilon)\|_{L^1[0,1]} + \|I_{m_\eta}(f_\varepsilon) - I_{m_\eta}(f)\|_{L^1[0,1]} \\ &\leq \varepsilon + \left\| f_\varepsilon - \sum_{i=1}^n \frac{\int_{B_i} f d\mu}{\mu(B_i)} \chi_{B_i} \right\|_{L^1[0,1]} = \varepsilon + \left\| \sum_{i=1}^n \left( \lambda_i - \frac{\int_{B_i} f d\mu}{\mu(B_i)} \right) \chi_{B_i} \right\|_{L^1[0,1]} \\ &= \varepsilon + \int \sum_{i=1}^n \left| \lambda_i - \frac{\int_{B_i} f d\mu}{\mu(B_i)} \right| \chi_{B_i} d\mu = \varepsilon + \sum_{i=1}^n \left| \lambda_i \mu(B_i) - \int_{B_i} f d\mu \right| \\ &= \varepsilon + \sum_{i=1}^n \left| \int_{B_i} f_\varepsilon d\mu - \int_{B_i} f d\mu \right| \leq \varepsilon + \sum_{i=1}^n \int_{B_i} |f_\varepsilon - f| d\mu \\ &= \varepsilon + \|f_\varepsilon - f\|_{L^1[0,1]} \leq 2\varepsilon. \end{aligned}$$

Therefore  $\lim_\eta I_{m_\eta} = I_m$  pointwise. Moreover, for each  $A \in \Sigma$ , if we replace  $f$  and  $f_\varepsilon$  by  $f h_A$  and  $f_\varepsilon h_A$  respectively in the computations above, we can also prove that for each  $f \in L^1(m)$ ,

$$\lim_\eta \sup_{A \in \Sigma} \left\| \int f h_A dm - \int f h_A dm_\eta \right\| = 0.$$

Thus, by Theorem 3.4(i),  $\lim_\eta \|f\|_{L^1(m_\eta)} = \|f\|_{L^1(m)}$  for each  $f \in L^1(m)$ .

In general, convergence of martingales in Banach function spaces is not guaranteed. It is well known that for the spaces  $L^p[0, 1]$ ,  $1 < p < \infty$ , it does hold, as a consequence of Doob's martingale inequality. Actually, this fact can be extended to Bochner spaces  $L^p(\mu, X)$  over non-atomic probability measures  $\mu$  [15, Theorem 1.5 and Remark 1.7]. However, the arguments that

support these results cannot be transferred to all Banach function spaces. Consider an order continuous Banach function space  $X(\mu)$  over a non-atomic probability measure  $\mu$ , and consider the same vector measure  $\Sigma \ni A \mapsto \chi_A \in X(\mu)$  as in the example above. Note that  $L^1(m) = X(\mu)$  isometrically and  $I_m = \text{Id}$ . In [10], it is studied to what extent the convergence of martingales in Banach function spaces resembles the case of  $L^p[0, 1]$ , and several positive results are shown for spaces like  $L \log L$ . However, a counterexample is also given. It must be noted that in that paper, the definition of Banach function space includes the requirement of having the Fatou property, so the class of Banach function spaces considered there is smaller than the one we are considering.

Under this requirement, [10, Theorem 1] establishes that for an order continuous (and Fatou) Banach function space  $X(\mu)$ , a uniformly integrable martingale defined by a sequence of continuous conditional expectation operators  $E_n : X(\mu) \rightarrow X(\mu)$  when applied to a function  $f \in X(\mu)$  is convergent to  $f$  in the norm of  $X(\mu)$  for each  $f$  if and only if the sequence  $(\|E_n\|)_n$  is uniformly bounded. However, it is also proved that this requirement is not always satisfied. Thus, every sequence of conditional expectation operators is bounded in  $L^p$ , but in a general function space this need not be true. Moreover, [10, Theorem 2] shows that for rearrangement invariant order continuous and Fatou Banach function spaces, convergence of uniformly integrable martingales is guaranteed. Again, there are  $L^1(m)$  spaces that are not rearrangement invariant, so the result does not apply in our setting.

**4. The Daugavet equation for integration maps.** In this section we analyse negative results regarding unconditional pointwise approximation of integration maps. We will show that, even in the simplest examples, the integration map of a vector measure cannot be represented as a pointwise unconditionally convergent series of finite rank operators, or even of weakly compact operators. The main conclusion of this section is the following:

*There are vector measures with associated integration maps of absolutely different nature representing the same space: they may satisfy  $\|f\|_{L^1(m)} = \lim_{\eta} \|f\|_{L^1(m_{\eta})}$  for all  $f \in L^1(m)$ , but  $I_m$  cannot be represented as an unconditionally pointwise convergent series of weakly compact operators.*

The Daugavet property is our main source of examples and counterexamples, together with the following useful result, found in [4, Th. 2.9].

**THEOREM A.** *Let  $G \in L(X, Y)$ . Suppose that the inequality*

$$\|G + T\| \geq C + \|T\|$$

*with  $C > 0$  holds for every operator  $T$  in a subspace  $M \subseteq L(X, Y)$  of operators. Let  $\hat{T} = \sum_{n \in \Gamma} T_n$  be represented as a (maybe uncountable) pointwise unconditionally convergent series of operators  $T_n \in M$ . Then  $\|G - \hat{T}\| \geq C$ .*

Recall that if a Banach space  $X$  has the Daugavet property, then also every weakly compact operator  $T : X \rightarrow X$  satisfies the Daugavet equation  $\|\text{Id} + T\| = 1 + \|T\|$ , so Theorem A applies. For this specific case this theorem can be found in [9, Lemma 2.6].

Let  $(\Omega, \Sigma)$  be a measurable space and  $\mu$  a positive measure without atoms. The main examples of Banach function spaces that have the Daugavet property are  $L^1(\mu)$  and  $L^\infty(\mu)$ . Another example is  $C(K)$  where  $K$  is a compact Hausdorff topological space without isolated points. In this section we study when the integration operator  $I_m$  satisfies the Daugavet equation for a suitable  $L^1(m)$ -valued vector measure, or in a more general sense, when  $\|I_m + I_n\| = \|I_m\| + \|I_n\|$  for vector measures  $m, n : \Sigma \rightarrow X$  such that  $L^1(m) = L^1(n)$ .

Let us start with the main negative example. Let  $m$  be a non-atomic vector measure. It is well known that  $I_m$  is compact if and only if  $m$  has finite variation and admits a Radon–Nikodým derivative, and in this case  $L^1(m) = L^1(|m|)$  (see [14, Ch. 3]). For instance, if  $\mu$  is a Rybakov measure for such a vector measure  $m$  satisfying  $L^1(m) = L^1(\mu)$ , then the integration map  $I_\mu : L^1(\mu) \rightarrow \mathbb{R}$  given by  $f \mapsto \int f d\mu$  produces the same space of integrable functions. We can pretend that this integration map has values in  $L^1(\mu)$  by considering  $f \mapsto \int f d\mu \otimes g$  for a fixed norm 1 function  $g \in L^1(\mu)$ .

However, the operator associated to the vector measure  $m_0(A) := \chi_A \in L^1(\mu)$ , that is, the identity map  $I_{m_0} = \text{Id} : L^1(m_0) \rightarrow L^1(\mu)$ , gives the same space of integrable functions  $L^1(m_0) = L^1(m) = L^1(\mu)$ . In this case, it is clear that the integration map  $I_{m_0} = \text{Id}$  cannot be approximated in the operator norm by a sequence of compact operators.

Theorem A above, together with the Daugavet property of  $L^1[0, 1]$ , implies that the integration map  $I_{m_0} : L^1[0, 1] \rightarrow L^1[0, 1]$  cannot be represented as a pointwise unconditionally convergent series of weakly compact operators. However, if  $m_1(A) := \mu(A) \otimes \chi_{[0,1]}$ , which has a rank 1 integration map, then  $L^1(m_0) = L^1(m_1) = L^1[0, 1]$  and for each  $f \in L^1[0, 1]$  we have  $\|f\|_{L^1(m_0)} = \|f\|_{L^1(m_1)}$ . The results provided in Section 3.4 show that, although we can approximate  $I_{m_0}$  pointwise by a net of finite rank operators, the corresponding series that approximates a function cannot be unconditionally convergent in general.

Let us present another example in this direction.

EXAMPLE 4.1. The previous example gives some ideas regarding spaces  $L^1(m)$  of  $L^1(\mu)$ -valued measures. Let  $(\Omega, \Sigma, \mu)$  be a non-atomic finite measure space. For a continuous linear operator  $T : L^1(\mu) \rightarrow L^1(\mu)$  consider the vector measure defined as  $m_T(A) := T(\chi_A)$  for each measurable set  $A \in \Sigma$ .

Let  $T : L^1(\mu) \rightarrow L^1(\mu)$  be an isomorphism and let  $R : L^1(\mu) \rightarrow L^1(\mu)$  be a Radon–Nikodým operator (for example, a weakly compact operator).

Consider the vector measures  $m_T$  and  $m_{R \circ T}$ . Then the integration operators  $I_{m_T}$  and  $I_{m_{R \circ T}}$  satisfy

$$\|I_{m_T} + I_{m_{R \circ T}}\| = \|I_{m_T}\| + \|I_{m_{R \circ T}}\|.$$

Therefore, for each  $\varepsilon > 0$  there is a function  $f \in L^1(m_T)$  such that

$$\left\| \int f dm_T + \int f dm_{R \circ T} \right\| \geq \|I_{m_T}\| + \|I_{m_{R \circ T}}\| - \varepsilon.$$

In other words, again by Theorem A we cannot approximate pointwise unconditionally the integration map associated to  $T$  by integration maps associated to vector measures constructed using operators like  $R \circ T$ , where  $R$  is a Radon–Nikodým operator.

As in the previous section, in what follows we will describe the validity of the Daugavet equation when integration operators are involved in terms of the Radon–Nikodým derivative operators of their scalarizations. As usual, the restriction of an operator  $T : E \rightarrow X$  to a subset  $Y$  of  $E$  is denoted by  $T|_Y$ .

LEMMA 4.2. *Let  $X, X_0, X_1$  be Banach spaces and consider two vector measures  $m : \Sigma \rightarrow X_0$  and  $m_1 : \Sigma \rightarrow X_1$  that are equivalent to a scalar measure  $\mu$ . Suppose that  $Z = Z(\mu)$  is a Banach function space such that  $Z(\mu) \subseteq L^1(m) \cap L^1(m_1)$  and  $X_0 + X_1 \subseteq X$  with continuous inclusions. Then for every scalar  $\lambda$ ,*

$$\|I_m|_Z + \lambda I_{m_1}|_Z\|_{L(Z(\mu), X)} = \sup_{x^* \in B_{X^*}} \|\varphi_{x^*}^m + \lambda \varphi_{x^*}^{m_1}\|_{Z(\mu)^*}.$$

*Proof.* Note that since  $Z(\mu) \subseteq L^1(m) \cap L^1(m_1)$ , for all  $f \in Z(\mu)$  the integrals  $\int f dm$  and  $\int f dm_1$  make sense. A direct computation gives

$$\begin{aligned} \|I_m|_Z + \lambda I_{m_1}|_Z\|_{L(Z, X)} &= \sup_{f \in B_Z} \|I_m|_Z(f) + \lambda I_{m_1}|_Z(f)\|_X \\ &= \sup_{f \in B_Z} \left\| \int f dm + \lambda \int f dm_1 \right\|_X \\ &= \sup_{f \in B_Z} \sup_{x^* \in B_{X^*}} \left\{ \left\langle \int f dm, x^* \right\rangle + \left\langle \lambda \int f dm_1, x^* \right\rangle \right\} \\ &= \sup_{f \in B_Z, x^* \in B_{X^*}} \left\{ \int f d\langle m, x^* \rangle + \lambda \int f d\langle m_1, x^* \rangle \right\}. \end{aligned}$$

Let  $\varphi_{x^*}^m = d\langle m, x^* \rangle / d\mu$  and  $\varphi_{x^*}^{m_1} = d\langle m_1, x^* \rangle / d\mu$  be the Radon–Nikodým derivatives with respect to the fixed measure  $\mu$ ; note that they belong to  $Z^*$ .

Then

$$\begin{aligned}
 \sup_{f \in B_Z, x^* \in B_{X^*}} & \left\{ \int f d\langle m, x^* \rangle + \lambda \int f d\langle m, x^* \rangle \right\} \\
 &= \sup_{f \in B_Z, x^* \in B_{X^*}} \left\{ \int f \varphi_{x^*}^m d\mu + \int \lambda f \varphi_{x^*}^{m_1} d\mu \right\} \\
 &= \sup_{f \in B_Z, x^* \in B_{X^*}} \left\{ \int f (\varphi_{x^*}^m + \lambda \varphi_{x^*}^{m_1}) d\mu \right\} \\
 &= \sup_{x^* \in B_{X^*}} \|\varphi_{x^*}^m + \lambda \varphi_{x^*}^{m_1}\|_{Z^*}. \blacksquare
 \end{aligned}$$

Let us give some direct applications. Let  $X(\mu)$  be an order continuous Banach function space over a finite measure space  $(\Omega, \Sigma, \mu)$ . We will say that a vector measure  $m : \Sigma \rightarrow X$  represents  $X(\mu)$  if

$$L^1(m) = X(\mu)$$

isometrically and as ordered sets.

We consider first the case when a space  $X(\mu)$  is represented by a vector measure  $m$  having values in the same space  $X(\mu)$ . Let us consider a particular case that will be relevant later on. Suppose that  $m$  and  $m_1$  are vector measures having values in an order continuous Banach function space  $X(\mu)$  such that  $L^1(m) = L^1(m_1) = X(\mu)$  (isometrically) and  $I_{m_1} = \text{Id} : X(\mu) \rightarrow X(\mu)$ ; this happens if  $m_1$  is the vector measure  $\Sigma \ni A \mapsto m_1(A) := \chi_A \in X(\mu)$ , and in this case  $\text{Id}(f) = \int f dm_1 = f$ . Then

$$\|\text{Id} + I_m\| = \sup_{h \in B_{(X(\mu))'}} \|h + \varphi_h^m\|,$$

and consequently the following holds.

**COROLLARY 4.3.** *For each vector measure  $m : \Sigma \rightarrow X(\mu)$  representing  $X(\mu)$ , the Daugavet equation  $\|\text{Id} + I_m\| = 2$  holds if and only if*

$$\sup_{h \in B_{(X(\mu))'}} \left\| h + \frac{d\langle m, h \rangle}{d\mu} \right\| = 2.$$

**EXAMPLE 4.4.** Consider again our canonical example. Take the  $L^1[0, 1]$ -valued vector measures defined as  $m_1(A) := \chi_A$  and  $m(A) := \mu(A) \cdot \chi_{[0,1]}$  for each Lebesgue measurable set  $A$ . Both measures give  $L^1[0, 1]$  as the space of integrable functions isometrically; for them, we have

$$\sup_{h \in B_{L^\infty[0,1]}} \left\| h + \frac{d\langle m, h \rangle}{d\mu} \right\| = 2.$$

Taking into account that the Radon–Nikodým derivative of the measure  $A \mapsto \mu(A) \int \chi_{[0,1]} h \, d\mu = \langle m, h \rangle(A)$  is

$$\frac{d\langle m, h \rangle}{d\mu} = \chi_{[0,1]} \int_{[0,1]} h \, d\mu,$$

we find that in this case, this is equivalent to

$$\sup_{h \in B_{L^\infty[0,1]}} \left\| h + \chi_{[0,1]} \int_{[0,1]} h \, d\mu \right\| = 2;$$

in fact, the same result is true when we replace  $\chi_{[0,1]}$  by any other norm 1 function in  $L^1[0, 1]$ , since all of them define rank 1 operators.

There are other cases of Banach lattices of integrable functions that also have the Daugavet property and are not  $L^1$ -spaces. Let us give some examples. It is well known that if  $\mu$  is a non-atomic probability measure and  $E$  is a Banach space, the space  $L^1(\mu, E)$  of Bochner integrable functions has the Daugavet property [9, p. 858, Example]. Thus, if  $E$  is also a Banach lattice, it is known that  $L^1(\mu, E)$  is a Banach lattice too that has the Daugavet property. Another interesting example of a Banach space having the Daugavet property is the Bochner space  $L^\infty(\mu, E)$ : it has the Daugavet property if  $E$  has it or  $\mu$  is non-atomic [13, Th. 5], and it is also a Banach lattice if  $E$  is. However, since all the  $L^1(m)$  spaces for a vector measure  $m$  are order continuous Banach function spaces, the application of the previously mentioned space as an example of our class is restricted to finite  $\ell^\infty$ -sums of Banach function spaces having the Daugavet property (this case was already considered in [17]). Anyway,  $c_0$ -sums of Banach function spaces with the Daugavet property also have this property, as can be deduced from [9, Proposition 2.16]: take for example a disjoint countable measurable partition  $\{A_i : i \in \mathbb{N}\}$  of  $[0, 1]$ ; the  $c_0$ -sum of  $L^1(\mu|_{A_i})$  has the Daugavet property. This space satisfies the order continuity requirement, and so it can be represented as an  $L^1(m)$  of a vector measure  $m$ .

However, the class of Banach function spaces having the Daugavet property is rather small. The results in [1, 2, 8] show this. Thus, [2, Theorem 3.6] states that for a rearrangement invariant Banach function space  $X(\mu)$  over a finite measure  $\mu$  with the weak Fatou property, the Daugavet property implies that  $X(\mu)$  coincides with either  $L^\infty(\mu)$  or  $L^1(\mu)$  isometrically. Since all the  $L^1(m)$  spaces are order continuous, those with the properties just mentioned are restricted to being isometric to  $L^1$ -spaces of finite measures. In particular, for a non-atomic vector measure  $m$ , if  $L^1(m)$  is isometric to an Orlicz space with the Luxemburg norm that has the Daugavet property, then it must be isometric to  $L^1[0, 1]$  [2, Cor. 4.3].

Therefore we obtain the following result.

PROPOSITION 4.5. *If  $X(\mu)$  is an order continuous Banach function space with the Daugavet property (in particular, if  $X(\mu) = L^1(\mu)$  for a non-atomic measure  $\mu$ ), then:*

- (1) *Let  $m$  be an  $X(\mu)$ -valued countably additive vector measure representing  $X(\mu)$  such that  $I_m$  is a Radon–Nikodým operator. Then*

$$\|\text{Id} + I_m\| = \sup_{h \in B_{(X(\mu))'}} \|h + \varphi_h^m\|_{(X(\mu))'} = 2.$$

- (2) *There is a vector measure  $m$  representing  $X(\mu)$  such that its integration map  $I_m$  is not representable as a pointwise unconditionally convergent series (maybe uncountable) of Radon–Nikodým operators. In fact, the same result holds if  $X(\mu)$  is isomorphic to a space with the Daugavet property.*
- (3) *Let  $\mathcal{M}$  be the normed space of all linear combinations of finite rank Radon–Nikodým derivative operators associated to vector measures representing  $X(\mu)$ . Then the integration map associated to the vector measure  $\Sigma \ni A \mapsto \chi_A \in X(\mu)$  representing  $X(\mu)$ , which is the identity map, cannot be represented as a (maybe uncountable) pointwise unconditionally convergent series of operators in  $\mathcal{M}$ .*

*Proof.* The first statement is a direct consequence of the Daugavet property of  $X(\mu)$ . The second one can be proved as a direct application of Theorem A, taking into account that again by the Daugavet property of  $X(\mu)$ , Radon–Nikodým operators satisfy the Daugavet equation. The third statement follows by considering in particular those operators that are finite sums of finite rank integration maps representing  $X(\mu)$ . ■

REMARK 4.6. A similar version of Proposition 4.5 can be obtained if we replace the operator  $\text{Id}$  by any integration operator  $I_m$  that is a Daugavet center (see [3, 4]), without the assumption that  $X(\mu)$  has the Daugavet property. In fact, using [4, Remark 2.10], more can be said. If a Banach function space  $X(\mu)$  can be represented by an integration map—maybe defined in a different Banach space  $F$ —that is a Daugavet center, then  $X(\mu)$  cannot have an unconditional basis (countable or uncountable) or be represented as an unconditional sum of reflexive spaces. The reason is that then  $I_m$  cannot be written as a pointwise unconditionally convergent series of weakly compact operators; in particular, of course,  $I_m$  cannot be a weakly compact operator.

As a consequence of the previous results and comments, the structure of  $L^1(m)$  spaces with  $I_m$  being a Daugavet center is close to being isomorphic to  $L^1$  spaces of scalar measures. In fact, in [4] it is proved that every Daugavet center fixes a copy of  $\ell^1$ , and so  $L^1(m)$  cannot be a reflexive space.

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### References

- [1] M. D. Acosta, A. Kamińska, and M. Mastyło, *The Daugavet property and weak neighborhoods in Banach lattices*, J. Convex Anal. 19 (2012), 875–912.
- [2] M. D. Acosta, A. Kamińska, and M. Mastyło, *The Daugavet property in rearrangement invariant spaces*, Trans. Amer. Math. Soc. 367 (2015), 4061–4078.
- [3] T. V. Bosenko, *Daugavet centers and direct sums of Banach spaces*, Cent. Eur. J. Math. 8 (2010), 346–356.
- [4] T. Bosenko and V. Kadets, *Daugavet centers*, J. Math. Phys. Anal. Geom. 6 (2010), 3–20.
- [5] S. Brach, E. A. Sánchez Pérez, and D. Werner, *The Daugavet equation for bounded vector valued functions*, Rocky Mountain J. Math., to appear; arXiv:1507.04185 (2015).
- [6] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.
- [7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
- [8] V. Kadets, M. Martín, J. Merí, and D. Werner, *Lushness, numerical index 1 and the Daugavet property in rearrangement invariant spaces*, Canad. J. Math. 65 (2013), 331–348.
- [9] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. 352 (2000), 855–873.
- [10] M. Kikuchi, *A note on the convergence of martingales in Banach function spaces*, Anal. Math. 25 (1999), 265–276.
- [11] Å. Lima, O. Nygaard, and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. 119 (2000), 325–348.
- [12] Å. Lima and E. Oja, *Ideals of operators, approximability in the strong operator topology, and the approximation property*, Michigan Math. J. 52 (2004), 253–265.
- [13] M. Martín and A. R. Villena, *Numerical index and the Daugavet property for  $L_\infty(\mu, X)$* , Proc. Edinburgh Math. Soc. (2) 46 (2003), 415–420.
- [14] S. Okada, W. Ricker, and E. A. Sánchez Pérez, *Optimal Domain and Integral Extension of Operators Acting in Function Spaces*, Operator Theory Adv. Appl. 180, Birkhäuser, Basel, 2008.
- [15] G. Pisier, *Martingales in Banach Spaces*, Cambridge Univ. Press, Cambridge, 2016.
- [16] D. Werner, *Recent progress on the Daugavet property*, Irish Math. Soc. Bull. 46 (2001), 77–97.
- [17] P. Wojtaszczyk, *Some remarks on the Daugavet equation*, Proc. Amer. Math. Soc. 115 (1992), 1047–1052.

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