BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 65, No. 2, 2017

MATHEMATICAL LOGIC AND FOUNDATIONS

Definable continuous selections of set-valued maps in o-minimal expansions of the real field

by

Saronsad SOKANTIKA and Athipat THAMRONGTHANYALAK

Presented by Czesław BESSAGA

Summary. Let T be a set-valued map from a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose $(\mathbb{R};+,\cdot,T)$ is o-minimal. We prove that (1) if for every $x \in \mathbb{R}^n$, each connected component of $T(x)$ is convex, then T has a continuous selection if and only if T has a continuous selection definable in $(\mathbb{R};+,\cdot,T)$; (2) if $n = 1$ or $m = 1$, then T has a continuous selection if and only if T has a continuous selection definable in $(\mathbb{R};+,\cdot,T)$.

1. Introduction. Let X and Y be sets. We denote a map T from X to the power set of Y by $T\colon X \rightrightarrows Y$ and call such T a *set-valued map*. For $T\colon X \rightrightarrows Y$, a *continuous selection of* T is a continuous map $f\colon X \to Y$ such that $f(x) \in T(x)$ for every $x \in X$.

Let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$. In [7, 8], E. Michael gives a sufficient condition for the existence of continuous selections of T. However, the given construction may produce a continuous selection that is far more complicated than how T arose (see A. Thamrongthanyalak [11] for a discussion of this problem). Therefore, the following question arises:

> *If T is well behaved in some prescribed sense and we know that T has a continuous selection, is it possible to find a continuous selection that is as well behaved as T?*

Here we employ first-order logic, namely definability in expansions of the real field, to study this question:

2010 *Mathematics Subject Classification*: Primary 03C64; Secondary 03C98, 26B05, 54C65.
Key words and phrases: selections, o-minimal structures.
Received 4 September 2017.
Published online 8 November 2017.

DOI: 10.4064/ba8130-10-2017

If T has a continuous selection, is there a continuous selection of T definable in $(\mathbb{R}; +, \cdot, T)$?

Here $(\mathbb{R}; +, \cdot, T)$ is the expansion of the real field by T, and "definable" means "definable possibly with parameters." (We refer to L. van den Dries and C. Miller [13] for more background.) Informally, if T has a continuous selection, is there a continuous selection of T that can be defined by *only* $+$, $\cdot$ and T?

The answer is "no". Let M be the double helix defined in [4] and define $T\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ by $T(z) = \{(x, y) \in \mathbb{R}^2 : (x, y, z) \in M\}$. Then a continuous selection of T must be interdefinable with a connected component of M, which is not definable in $(\mathbb{R}; +, \cdot, T)$.

On the other hand, there are some cases where the answer is "yes". Let $T\colon E \rightrightarrows \mathbb{R}^m$ where $E \subseteq \mathbb{R}^n$. If $(\mathbb{R}; +, \cdot, T)$ defines $\mathbb{Z}$, then every Borel set is definable in $(\mathbb{R}; +, \cdot, T)$ (see, e.g., A. Kechris [6, Exercise 37.6]). Therefore, if E is Borel and $(\mathbb{R}; +, \cdot, T)$ defines $\mathbb{Z}$, then the answer is yes.

From now on, let $\mathfrak{R}$ be an expansion of the real field, and let "definable" mean "definable in $\mathfrak{R}$ possibly with parameters". We say that $\mathfrak{R}$ is *o-minimal* if every unary definable set is a finite union of points and open intervals (see [12] for more information).

Let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$. The following are our main results:

THEOREM A. *Let $\mathfrak{R}$ be o-minimal. Suppose for each $x \in E$, each connected component of $T(x)$ is convex, T is definable and has a continuous selection. Then T has a definable continuous selection.*

THEOREM B. *Let $\mathfrak{R}$ be o-minimal. Suppose $T\colon E \rightrightarrows \mathbb{R}^m$ is definable and either $n = 1$ or $m = 1$. Then T has a continuous selection if and only if T has a definable continuous selection.*

Conventions and notation. Throughout this paper, m, n and l will range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.

We denote the Euclidean norm on $\mathbb{R}^n$ by $\|\ \|$. For $x \in \mathbb{R}^n$, $S \subseteq \mathbb{R}^n$ and $\epsilon > 0$, let $B_n(x, \epsilon) := \{y \in \mathbb{R}^n : \|x - y\| < \epsilon\}$ and $d(x, S) := \inf\{\|x - y\| : y \in S\}$.

Let $S \subseteq \mathbb{R}^n$. We denote by $\operatorname{int} S$ the interior, $\operatorname{cl} S$ the closure, and $\operatorname{fr} S := \operatorname{cl} S \setminus S$ the frontier of S. Let $\operatorname{Aff} S := \{\sum_{i=0}^n \alpha_i x_i : x_i \in S, \alpha_i \in \mathbb{R}, \sum_{i=0}^n \alpha_i = 1\}$ be the affine hull of S, and $\operatorname{int}_{\operatorname{Aff} S} S$ the relative interior of S in $\operatorname{Aff} S$.

For notational simplicity, set-valued maps are also treated as sets, that is, we denote a set-valued map and its graph by the same symbol.

2. Generalization of the definable Michael selection theorem. Throughout this section, assume $\mathfrak{R}$ is o-minimal and let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$.

We say that T is *lower semicontinuous* if for all $x_0 \in E$ and $y_0 \in T(x_0)$ and any neighborhood V of y_0, there is a neighborhood U of x_0 such that $T(x) \cap V \neq \emptyset$ for every $x \in U$. In [7], Michael studied the question of the existence of continuous selections of set-valued maps and proved Michael's selection theorem: *if T is lower semicontinuous and each $T(x)$ is nonempty closed and convex, then T has a continuous selection.* This theorem has applications in various fields of mathematics (see [5], [9], [10]).

M. Aschenbrenner and A. Thamrongthanyalak [1] proved a definable version of Michael's selection theorem. Here is a minor variant (the proof is essentially the same).

2.1 (cf. [1, Theorem 2.2.1]). *Let T be definable and lower semicontinuous. Suppose $T(x)$ is nonempty, closed and convex for every $x \in E$. Then T has a definable continuous selection.*

The main goal of this section is to prove the following generalization of 2.1:

2.2. *Let T be definable and lower semicontinuous. Suppose $T(x)$ is nonempty and convex for every $x \in E$. Then T has a definable continuous selection.*

Now we begin working towards the proof of 2.2. First we recall several facts about o-minimal structures.

2.3 (L. van den Dries, [12, Lemma 3.5, p. 101]). *Let E_0, E_1 be disjoint definable closed subsets of a definable set E. Then there exist disjoint definable open subsets U_0, U_1 of E with $E_i \subseteq U_i$, $i = 0, 1$.*

2.4 (L. van den Dries, [12, Lemma 3.8, p. 102]). *Let E_0, E_1 be disjoint definable closed subsets of a definable set E. Then there is a definable continuous function $f\colon E \to [0,1]$ such that $f^{-1}(0) = E_0$ and $f^{-1}(1) = E_1$.*

2.5 (L. van den Dries, [12, Corollary 3.10, p. 138]). *Let S be a closed subset of E and $f\colon S \to \mathbb{R}^m$ be a definable continuous map. Then there is a definable continuous map from E to $\mathbb{R}^m$ extending f.*

Next, we will prove the following special case of 2.2.

2.6. *Suppose T is definable and lower semicontinuous, and $T(x)$ is a non-empty convex subset of $B_m(0,1)$ for every $x \in E$. Then T has a definable continuous selection.*

Proof. We proceed by induction on the dimension of E. If $\dim E = 0$, then E is finite and the result holds trivially. Suppose $\dim E > 0$ and the result holds for every set of dimension less than $\dim E$. Define $T_0\colon E \rightrightarrows \mathbb{R}^m$ by $T_0(x) := \operatorname{int}_{\operatorname{Aff} T(x)} T(x)$. Obviously, T_0 is definable and $T_0(x) \subseteq T(x)$ for every $x \in E$. By the Definable Choice and the Cell Decomposition Theorem, there exist $S \subseteq E$ open in E and a definable map $f\colon E \to \mathbb{R}^m$ such that $\dim E \setminus S < \dim E$, $f{\restriction}S$ is continuous and $f(x) \in T_0(x)$ for every $x \in T_0(x)$. Inductively, let $g_0\colon E\setminus S \to \mathbb{R}^m$ be a definable continuous selection of $T{\restriction}E\setminus S$. Define $T_1\colon E \rightrightarrows \mathbb{R}^m$ by

$$T_1(x) := \begin{cases} \operatorname{cl} T(x) & \text{if } x \in S, \\ \{g_0(x)\} & \text{if } x \in E \setminus S. \end{cases}$$

It is routine to check that T_1 is lower semicontinuous and definable, and $T_1(x)$ is closed and convex for every $x \in E$. By 2.1, there is a definable continuous selection g_1 of T_1. By 2.3 and 2.4, there exist $U \subseteq E$ definable open and $t\colon E \to [0,1]$ definable continuous such that $E \setminus S \subseteq U \subsetneq E$, $t^{-1}(0) = E \setminus S$, and $t^{-1}(1) = E \setminus U$.

Next, we define $h\colon E \to \mathbb{R}^m$ by

$$h(x) := (1 - t(x))g_1(x) + t(x)f(x).$$

Obviously, h is definable. To prove that h is a selection of T, it suffices to show that if $C \subseteq \mathbb{R}^m$ is convex, $a \in \operatorname{int}_{\operatorname{Aff} C} C$, $b \in \operatorname{cl} C$, and $0 < t < 1$, then $(1-t)a + tb \in \operatorname{int}_{\operatorname{Aff} C} C$.

Let $c = (1-t)a + tb$. First, assume $b \in \operatorname{int}_{\operatorname{Aff} C} C$. Since $a, b \in \operatorname{int}_{\operatorname{Aff} C} C$, there is $\epsilon > 0$ such that $B_m(a,\epsilon) \cap \operatorname{Aff} C \subseteq C$ and $B_m(b,\epsilon) \cap \operatorname{Aff} C \subseteq C$.

We will show that $B_m(c,\epsilon) \cap \operatorname{Aff} C \subseteq C$. Let $d \in B_m(c,\epsilon) \cap \operatorname{Aff} C$. Then $a + d - c \in B_m(a,\epsilon) \cap \operatorname{Aff} C$ and $b + d - c \in B_m(b,\epsilon) \cap \operatorname{Aff} C$. Therefore, $a+d-c, b+d-c \in C$. Since C is convex and $d = (1-t)(a+d-c) + t(b+d-c)$, we have $d \in C$.

Now let $b \in \operatorname{cl} C$. Then there is $u \in B_m(b, (1-t)\epsilon/t) \cap \operatorname{int}_{\operatorname{Aff} C} C$. Since $a \in \operatorname{int}_{\operatorname{Aff} C} C$, there is $\epsilon > 0$ such that $B_m(a,\epsilon) \cap \operatorname{Aff} C \subseteq C$. Let $v := t(b-u)/(1-t) + a \in \operatorname{Aff} C$. Then $\|v - a\| < \epsilon$. Hence, $v \in \operatorname{int}_{\operatorname{Aff} C} C$. Since $tu + (1-t)v = ta + (1-t)b = c$, by the previous paragraph, $c \in \operatorname{int}_{\operatorname{Aff} C} C$.

Therefore, it remains to prove that h is continuous. Since $E \setminus S$ is closed in E, it suffices to show that h is continuous at $x_0 \in E \setminus S$. Let $x_0 \in E \setminus S$ and $\epsilon > 0$. Since g_1 is continuous at x_0, there is $\delta > 0$ such that for all $x \in B_n(x_0, \delta)$, $\|g_1(x_0) - g_1(x)\| < \epsilon/2$ and $\|t(x)\| = \|t(x_0) - t(x)\| < \epsilon/4$. Let $x \in B_n(x_0,\delta)$. Hence,

$$\begin{aligned} \|h(x_0) - h(x)\| &\le \|g_1(x_0) - g_1(x)\| + \|t(x)\| \cdot \|g_1(x)\| + \|t(x)\| \cdot \|f(x)\| \\ &< \epsilon/2 + 2\epsilon/4 = \epsilon. \end{aligned}$$

Therefore, h is continuous at x_0. ∎

Now one may wish to apply semialgebraic homeomorphisms from $\mathbb{R}^m$ to $B_m(0,1)$ and the above result to prove 2.2. However, such homeomorphisms do not preserve convexity. Therefore, we propose the following:

Proof of 2.2. We proceed by induction on $\dim E$. The case $\dim E = 0$ is trivial. Suppose that $\dim E > 0$ and the result holds for sets of dimension less than $\dim E$. Suppose T is definable and lower semicontinuous such that $T(x)$ is nonempty and convex for every $x \in E$. Similar to the proof of 2.6, there exist a definable selection f of T and $S \subseteq E$ definable such that $E \setminus S$ is closed in E, $\dim E \setminus S < \dim E$, and $f{\restriction}S$ is continuous. By the inductive hypothesis and 2.5, let $g\colon E \to \mathbb{R}^m$ be definable and continuous such that $g(x) \in T(x)$ for every $x \in E \setminus S$. Define $T - g\colon E \rightrightarrows \mathbb{R}^m$ by

$$T - g(x) := \{y - g(x) : y \in T(x)\}.$$

Observe that $T - g$ is definable and lower semicontinuous, and $T - g(x)$ is nonempty and convex for every $x \in E$. Note that if h is a definable continuous selection of $T - g$, then $h + g$ is a definable continuous selection of T. It suffices to assume that $g = 0$.

Let $W = \{x \in E : B_m(0,1) \cap T(x) \neq \emptyset\}$. Then $E \setminus S \subseteq W$ and W is open in E since T is lower semicontinuous. By 2.3, there are definable sets $U, V \subseteq E$ such that U, V are open in E and $E \setminus S \subseteq V \subseteq E \cap \operatorname{cl} V \subseteq U \subseteq E \cap \operatorname{cl} U \subseteq W$.

Define $T_0\colon E \cap \operatorname{cl} U \rightrightarrows \mathbb{R}^m$ by $T_0(x) = B_m(0,1) \cap T(x)$. It is routine to check that T_0 is definable and lower semicontinuous. By 2.6 and 2.5, there is a definable continuous map $s\colon E \to \mathbb{R}^m$ such that $s(x) \in T_0(x)$ for every $x \in E \cap \operatorname{cl} U$. Since $E \cap \operatorname{cl} V \subseteq U$, there is a definable continuous function $t\colon E \to [0,1]$ such that $t^{-1}(0) = E \cap \operatorname{cl} V$ and $t^{-1}(1) = E \setminus U$. Define $h\colon E \to \mathbb{R}^n$ by

$$h(x) = (1 - t(x))s(x) + t(x)f(x).$$

Note that h is definable, $h{\restriction}V = s{\restriction}V$ and $h{\restriction}E \setminus U = f{\restriction}E \setminus U$. Since $s(x), f(x) \in T(x)$ for every $x \in U \setminus V$, h is a selection of T.

It remains to show that h is continuous. Since s, f and t are continuous on S, $h{\restriction}S$ is continuous. The restriction $h{\restriction}V$ is continuous because it equals $s{\restriction}V$. Since V, S are open and $V \cup S = E$, h is continuous. ■

3. Proofs of Theorems A and B. Let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$. The *Glaeser refinement* of T is the set-valued map $T'\colon E \rightrightarrows \mathbb{R}^m$ given by

$$T'(x_0) := \{y \in T(x_0) : d(y, T(x)) \to 0 \text{ as } E \ni x \to x_0\} \quad \text{for } x_0 \in E.$$

Obviously, if T is definable, then so is T'. Since $T' \subseteq T$, selections of T' are selections of T. Moreover:

3.1. *Suppose $T(x_0)$ is convex for every $x_0 \in E$. Then $T'(x_0)$ is convex for every $x_0 \in E$.*

Proof. Let $x_0 \in E$. If $T'(x_0) = \emptyset$, then we are done. Suppose $T'(x_0) \neq \emptyset$. Let $a, b \in T'(x_0)$, and $t \in [0,1]$. We will show that

$$c = (1-t)a + tb \in T'(x_0).$$

Let $\epsilon > 0$. Then there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta)$ there exist $a_x \in B_m(a, \epsilon) \cap T(x)$ and $b_x \in B_m(b, \epsilon) \cap T(x)$. Hence, for each $x \in B_n(x_0, \delta)$,

$$\begin{aligned} \|((1-t)a + tb) - ((1-t)a_x + tb_x)\| &= \|(1-t)(a - a_x) + t(b - b_x)\| \\ &\leq (1-t)\|(a - a_x)\| + t\|(b - b_x)\| \\ &< (1-t)\epsilon + t\epsilon = \epsilon. \end{aligned}$$

Hence, $ta + (1-t)b \in T'(x_0)$. ∎

We have the following two straightforward results whose proofs are left to the reader:

3.2. *If T is lower semicontinuous, then $T' = T$.*

3.3. *If $U \subseteq E$ is open in E, then $T' \restriction U = (T \restriction U)'$.*

For the rest of this section, assume that $\mathfrak{R}$ is o-minimal and let $E \subseteq \mathbb{R}^n$ and $T\colon E \rightrightarrows \mathbb{R}^m$ be definable. Then we have:

3.4 (M. Aschenbrenner, A. Thamrongthanyalak [1])**.** *There is a cell decomposition $\mathcal{C}$ of E such that $T \restriction C$ is lower semicontinuous for every $C \in \mathcal{C}$.*

Next, we inductively define a sequence $(T^{(l)})_{l \in \mathbb{N}}$ of set-valued maps by $T^{(0)} := T$ and $T^{(l+1)} := (T^{(l)})'$ for every $l \in \mathbb{N}$.

3.5. *For all $l \geq \dim E$, $T^{(l)} = T^{(\dim E)}$.*

Proof. It is enough to prove that for each $d \leq \dim E$, there exists $U \subseteq E$ definable and open in E such that $\dim(E \setminus U) < \dim E - d$ and $T^{(d)} \restriction U$ is lower semicontinuous. We proceed by induction on d. The case $d = 0$ is immediate from 3.4. Suppose the result holds for d. Let $V \subseteq E$ be definable and open in E such that $\dim(E \setminus V) < \dim E - d$ and $T^{(d)} \restriction V$ is lower semicontinuous. Then $T^{(d)}(x) = T^{(d+1)}(x)$ for every $x \in V$. By 3.4 and the Cell Decomposition Theorem, let $\mathcal{C}$ be a cell decomposition of $E \setminus V$ such that for each $C \in \mathcal{C}$, $T^{(d+1)} \restriction C$ is lower semicontinuous and $\operatorname{fr} C$ is a finite union of cells in $\mathcal{C}$.

Let $U = V \cup \bigcup\{C \in \mathcal{C} : \dim C = \dim(E \setminus V)\}$. It is routine to check that U is definable and open in E, and $\dim(E \setminus U) < \dim E - d - 1$.

It remains to prove that $T^{(d+1)} \restriction U$ is lower semicontinuous. Note that $T^{(d)} \restriction V$ is lower semicontinuous. Let $x_0 \in U \setminus V$ and $y \in T^{(d+1)}(x_0)$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $B_m(y, \epsilon) \cap T^{(d)}(x) \neq \emptyset$ for every $x \in B_n(x_0, \delta)$. Since $T^{(d)}(x) = T^{(d+1)}(x)$ for every $x \in B_n(x_0, \delta) \cap V$, we have $B_m(y, \epsilon) \cap T^{(d+1)}(x) \neq \emptyset$ for every such x. Observe that $T^{(d+1)} \restriction U \setminus V$ is lower semicontinuous. Hence, there exists $\delta \geq \delta' > 0$ such that

$B_m(y,\epsilon) \cap T^{(d+1)}(x) \neq \emptyset$ for every $x \in B_n(x_0,\delta') \cap (U \setminus V)$. Thus, $T^{(d+1)}\restriction U$ is lower semicontinuous. ∎

We say that T is *stable* (*under Glaeser refinement*) if $T' = T$. As a consequence of 3.5, $T^{(\dim E)}$ is stable. Let $T^{(*)} = T^{(\dim E)}$. Observe that $T^{(*)}$ is definable and lower semicontinuous.

3.6. *Suppose $T(x)$ is convex for every $x \in E$. If T has a continuous selection, then T has a definable continuous selection.*

Proof. Suppose that $T(x)$ is convex for every $x \in E$ and T has a continuous selection. By 3.1, $T^{(*)}(x)$ is convex and nonempty for every $x \in E$. Since $T^{(*)}$ is stable, it is lower semicontinuous. Therefore, by 2.2, it has a definable continuous selection, which is also a definable selection of T. ∎

Proof of Theorem A. Let $T\colon E \rightrightarrows \mathbb{R}^n$ be definable such that for each $x \in E$, all connected components of $T(x)$ are convex, and let $f\colon E \to \mathbb{R}^n$ be a continuous selection of T. By the Trivialization Theorem and the Cell Decomposition Theorem, there exist a cell decomposition $\mathcal{C}$ of E and $(A_C)_{C\in\mathcal{C}}$ and $(h_C)_{C\in\mathcal{C}}$ such that for each $C \in \mathcal{C}$, $A_C \subseteq \mathbb{R}^n$ is definable and $h_C\colon T\restriction C \to C \times A_C$ is a definable homeomorphism, and $h_C(\{x\} \times T(x)) = \{x\} \times A_C$ for each $x \in C$.

Let $C \in \mathcal{C}$. Since $f\restriction C$ is connected, $h_C(f\restriction C)$ is contained in a unique connected component of $C \times A_C$. Every such component is of the form $C \times X$ where X is a connected component of A_C. For each $C \in \mathcal{C}$, let X_C be the connected component of A_C such that $h_C(f\restriction C) \subseteq C \times X_C$.

Let $T_0 = \bigcup\{h_C^{-1}(C \times X_C) : C \in \mathcal{C}\}$. Then $T_0 \subseteq T$ is a definable set-valued map $E \rightrightarrows \mathbb{R}^n$ such that $T_0(x)$ is convex for every $x \in E$. Note that f is a continuous selection of T_0. By 3.6, T_0 has a definable continuous selection; therefore, so does T. ∎

Let $T\colon E \rightrightarrows \mathbb{R}$ be definable. Since every connected subset of $\mathbb{R}$ is convex, T has a continuous selection if and only if it has a definable continuous selection. To finish the proof of Theorem B, it suffices to consider the case $n = 1$.

M. Czapla and W. Pawłucki [3] proved

3.7 (M. Czapla, W. Pawłucki [3])**.** *Let $T : E \rightrightarrows \mathbb{R}^n$ be a definable and lower semicontinuous set-valued map with nonempty connected values and* $\dim E = 1$. *Then T has a definable continuous selection.*

We are now ready to prove a slight modification of Theorem B.

3.8. *Suppose* $\dim E = 1$ *and T is definable. If T has a continuous selection, then T has a definable continuous selection.*

Proof. Let $f\colon E \to \mathbb{R}^m$ be a continuous selection of T. By the same argument as in the proof of Theorem A, there exists a definable set-valued map $T_0\colon E \rightrightarrows \mathbb{R}^m$ such that $f \subseteq T_0$ and $T_0(x)$ is connected for every $x \in E$. By 3.4, there is a cell decomposition $\mathcal{C}$ of E such that $T_0\restriction C$ is lower semicontinuous for every $C \in \mathcal{C}$. Let $E_0 = \bigcup\{C \in \mathcal{C} : \dim C = 0\}$. Then E_0 is finite. Define $T_1\colon E \rightrightarrows \mathbb{R}^m$ by

$$T_1(x) := \begin{cases} \{f(x)\} & \text{if } x \in E_0, \\ T_0(x) & \text{if } x \notin E_0. \end{cases}$$

It is routine to prove that T_1 is lower semicontinuous. By 3.7, this completes the proof. ∎

4. Concluding remarks

4.1. Readers may ask a similar question to the main question in the expansion of the real ordered group:

If T has a continuous selection, is there a continuous selection of T definable in $(\mathbb{R}; <, +, T)$?

However, the answer is no. See [2] for an example of a semilinear set-valued map that has continuous selections but no semilinear continuous selections.

4.2. All results in Section 2 hold when $\mathfrak{R}$ is an expansion of real closed ordered fields. However, Theorems A and B do not necessarily hold in such expansions. Let R be the set of real algebraic numbers and $\mathfrak{M} := (R; +, \cdot)$. Let $T\colon R \rightrightarrows R$ be defined by

$$T(x) := \begin{cases} \{0\} & \text{if } x < 0, \\ \{0, 1\} & \text{if } 0 \leq x \leq 4, \\ \{1\} & \text{if } x > 4. \end{cases}$$

The structure $\mathfrak{M}$ is o-minimal and T is definable in $\mathfrak{M}$. The map $f\colon R \to R$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x < \pi, \\ 1 & \text{if } x > \pi, \end{cases}$$

is a continuous selection of T. However, by o-minimality, T has no continuous selection definable in $\mathfrak{M}$.

4.3. It is natural to ask whether the above technique can be improved to solve the main question given in the introduction. Observe that the proofs of Theorems A and B rely on the definable Michael Selection Theorem. Therefore, the first step towards the improvement should be to generalize that theorem. However, there is a lower semicontinuous o-minimal set-valued map $T\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ that has no continuous selection: Let $D := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and define $T\colon D \rightrightarrows D$ by $T(x) = D \setminus \{x\}$. By Brouwer's Fixed-Point Theorem, T has no continuous selections.

Note that this argument does not give a negative answer to the main question because T has no continuous selections at all. To completely answer the question, we think new ideas are needed.

Acknowledgements. The authors thank Professor Matthias Aschenbrenner for introducing us to this question. The first author was supported by a scholarship from Development and Promotion of Science and Technology Talents Project (DPST). This paper was partially written when the second author was supported by Zassenhaus Assistant Professorship from the Department of Mathematics, the Ohio State University.

References

[1] M. Aschenbrenner and A. Thamrongthanyalak, *Whitney's extension problem in o-minimal structures*, preprint, www.math.ucla.edu/~matthias/pdf/Whitney.pdf.

[2] M. Aschenbrenner and A. Thamrongthanyalak, *Michael's selection theorem in a semilinear context*, Adv. Geom. 15 (2015), 293–313.

[3] M. Czapla and W. Pawłucki, *Michael's selection theorem for a mapping definable in an o-minimal structure defined on a set of dimension 1*, Topol. Methods Nonlinear Anal. 49 (2017), 377–380.

[4] A. Fornasiero, *Definably connected nonconnected sets*, Math. Logic Quart. 58 (2012), 125–126.

[5] Z. Han, X. Cai, and J. Huang, *Theory of Control Systems Described by Differential Inclusions*, Springer Tracts in Mechanical Engineering, Shanghai Jiaotong Univ. Press, Shanghai, and Springer, Berlin, 2016.

[6] A. Kechris, *Classical Descriptive Set Theory*, Grad. Texts Math. 156, Springer, New York, 1995.

[7] E. Michael, *Continuous selections. I*, Ann. of Math. (2) 63 (1956), 361–382.

[8] E. Michael, *Selected selection theorems*, Amer. Math. Monthly 63 (1956), 233–238.

[9] S. Park, *Applications of Michael's selection theorems to fixed point theory*, Topology Appl. 155 (2008), 861–870.

[10] D. Samet, *Continuous selections for vector measures*, Math. Oper. Res. 12 (1987), 536–543.

[11] A. Thamrongthanyalak, *Michael's selection theorem in d-minimal expansions of the real field*, preprint, pioneer.netserv.chula.ac.th/~tathipa1/dminselection.pdf.

[12] L. van den Dries, *Tame Topology and o-Minimal Structures*, London Math. Soc. Lecture Note Ser. 248, Cambridge Univ. Press, Cambridge, 1998.

[13] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. 84 (1996), 497–540.

Saronsad Sokantika, Athipat Thamrongthanyalak
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Bangkok 10330, Thailand
E-mail: s.sokantika@gmail.com
athipat.th@chula.ac.th