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Cotton tensors on almost coKähler 3-manifolds

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Abstract. Let M^3 be an almost coKähler 3-manifold whose Reeb vector field defines a harmonic map. We prove that if the Cotton tensor of M^3 vanishes, then M^3 is locally isometric to the product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c. We construct some examples illustrating our main results.

1. Introduction. On an n-dimensional pseudo-Riemannian manifold (M^n, g), $n > 3$, the Weyl conformal curvature tensor W is defined by

$$(1.1)\quad \begin{aligned} &W(X,Y)Z \\ &= R(X,Y)Z + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \end{aligned}$$

for any vector fields X, Y and Z, where R, S, Q and r denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively. The Weyl tensor has the property of being invariant under conformal transformations of the metric g. A pseudo-Riemannian manifold M^n is said to be *conformally flat* if the Weyl tensor is vanishing.

Note that the Weyl tensor vanishes on any 3-dimensional pseudo-Riemannian manifold M^3. Therefore, one can consider another conformal invariant of M^3, the $(1,2)$ *Cotton tensor* C, defined by

$$(1.2)\quad C(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \tfrac{1}{4}\{X(r)Y - Y(r)X\}$$

for any vector fields X, Y. A 3-dimensional pseudo-Riemannian manifold is

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said to be *conformally flat* if the Cotton tensor C vanishes; this is equivalent to the property that the Schouten $(1,1)$ tensor defined on M^3 by

$$LX = QX - \tfrac{1}{4}rX \tag{1.3}$$

is of Codazzi type for any vector field X. A pseudo-Riemannian manifold of dimension three is said to be *essentially conformally symmetric* if the Cotton tensor is parallel but the manifold is not conformally flat (see [CL]).

The classification of conformally flat pseudo-Riemannian manifolds has been an interesting and difficult problem for a long time. For such a study on contact metric manifolds, we refer the reader to D. E. Blair [B2, Section 7.6]. Very recently, Cho [Cjt] and Wang [Ww] gave a complete classification of conformally flat coKähler 3-manifolds (see also Dacko [Dp]). Calviño-Louzao et al. [CL] proved that there exist no essentially conformally symmetric Riemannian 3-manifolds. This leads us to consider a new condition weaker than essential conformal symmetry which is named η-parallelism of the Cotton tensor on a coKähler 3-manifold. In fact, in Section 3 we prove that the Cotton tensor of any coKähler 3-manifold is η-parallel.

Conformally flat almost coKähler manifolds of dimension greater than 3 have been extensively investigated by many authors. For related results, we refer the reader to the work of Dacko and Olszak [DO, Oz1, Oz2], Endo [Eh1, Eh2] and Kim and Kim [KK]. As far as we know, the only results regarding 3-dimensional conformally flat almost coKähler manifolds were obtained by Dacko and Olszak [Dp, DO]. In [DO], they constructed a 3-dimensional conformally flat almost coKähler manifold which is neither locally flat nor coKähler. Dacko [Dp] obtained (under some assumptions) a necessary and sufficient condition for a 3-dimensional almost coKähler manifold to be conformally flat. A study of almost coKähler 3-manifolds with harmonic curvature tensors can be found in [Wy].

In Section 4, we consider non-coKähler almost coKähler 3-manifolds M^3 whose Reeb vector field ξ defines a harmonic map into the sphere bundle. We prove that M^3 cannot be conformally flat. Some concrete examples illustrating our main results are also constructed. It is shown that there exists a conformally flat non-coKähler almost coKähler 3-manifold whose Reeb vector field is harmonic but does not procedure a harmonic map into the sphere bundle. Moreover, we show that a simply connected homogeneous non-coKähler almost coKähler 3-manifold is not conformally flat and does not have η-parallel Cotton tensor.

2. Almost coKähler manifolds. By an *almost contact metric structure* defined on a smooth differentiable manifold M^{2n+1} of dimension $2n+1$ we mean a (ϕ, ξ, η, g)-structure satisfying

(2.1) $$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

(2.2) $$\phi^* g = g - \eta \otimes \eta,$$

where ϕ is a $(1,1)$ tensor field, ξ is a vector field called the *Reeb vector field*, η is a 1-form called the *almost contact* 1*-form* and g is a Riemannian metric called *compatible* with the almost contact structure.

In this paper, by an *almost coKähler manifold* we mean an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfying $d\eta = 0$ and $d\Phi = 0$, where the *fundamental* 2*-form* Φ of M^{2n+1} is defined by $\Phi(X,Y) = g(X, \phi Y)$ for any vector fields X and Y.

We consider the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$ and define on it an almost complex structure J by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the Nijenhuis tensor of ϕ. If

$$[\phi, \phi] = -2d\eta \otimes \xi,$$

or equivalently J is integrable, then the almost contact metric structure is said to be *normal*. A normal almost coKähler manifold is called a *coKähler manifold*.

Notice that an (almost) coKähler manifold is nothing but an (almost) cosymplectic manifold defined by D. E. Blair [B1] and studied in the literature [B2, Dp, DO, Eh1, Eh2, KK, Oz1, Oz2, Pd1, Pd2]. H. Li [Lh] pointed out that a coKähler manifold is an odd-dimensional analog of a Kähler manifold. This led some authors to adopt the new terminology in some recent papers (see [CNY, Ww, Wy]).

On an almost coKähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ we set $h := \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ is the Lie differentiation. We consider the Jacobi operator $l = R(\cdot, \xi)\xi$ generated by ξ and define $h' := h \circ \phi$, where R is the Riemannian curvature tensor of g. From [DO, Oz1, Oz2], we know that the three $(1,1)$ tensor fields l, h' and h are symmetric and satisfy $h\xi = 0$, $l\xi = 0$, $\operatorname{tr} h = 0$, $\operatorname{tr} h' = 0$ and $h\phi + \phi h = 0$ and

(2.3) $$\nabla \xi = h'.$$

We denote by $\mathcal{D}$ the distribution $\mathcal{D} = \ker \eta$ which is of dimension $2n$. It is easy to check that each integral manifold of $\mathcal{D}$, equipped with the restriction of ϕ, admits an almost Kähler structure. If this structure is integrable, or equivalently (see [DO])

(2.4) $$(\nabla_X \phi)Y = g(hX, Y)\xi - \eta(Y)hX$$

for any vector fields X, Y, then M^{2n+1} is said to have *Kählerian leaves.* From [B1], we see that an almost coKähler manifold is coKähler if and only if

$$\nabla\phi = 0 \quad (\Leftrightarrow \nabla\Phi = 0). \tag{2.5}$$

Therefore, it follows directly that a 3-dimensional almost coKähler manifold is coKähler if and only if h vanishes (see [Oz1]).

In this paper, all manifolds are assumed to be connected.

3. Cotton tensors on coKähler 3-manifolds. On a coKähler 3-manifold M^3, making use of $h = 0$ in (2.3) we obtain $\nabla\xi = 0$, and hence $R(X, Y)\xi = 0$ for any vector fields X, Y. It follows directly that ξ is an eigenvector field of the Ricci operator Q with eigenvalue zero.

Since the Weyl tensor is vanishing on any 3-dimensional pseudo-Riemannian manifold, from (1.1) we have

$$\begin{aligned} R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X \\ - S(X, Z)Y - \tfrac{1}{2}r\{g(Y, Z)X - g(X, Z)Y\} \end{aligned} \tag{3.1}$$

for any vector fields X, Y, Z. On M^3, setting $Z = \xi$ in (3.1) we get

$$\eta(Y)QX - \eta(X)QY - \tfrac{1}{2}r\{\eta(Y)X - \eta(X)Y\} = 0$$

for any vector fields X, Y. Therefore, replacing Y by ξ in the above relation and using $Q\xi = 0$ we obtain

$$Q = \tfrac{1}{2}r\,\mathrm{id} - \tfrac{1}{2}r\eta \otimes \xi, \tag{3.2}$$

where r denotes the scalar curvature. Applying (3.2) in the well-known formula $2\operatorname{div} Q = \operatorname{grad} r$ we obtain

$$\xi(r) = 0. \tag{3.3}$$

Before giving our main results, let us recall the following proposition (see [Ww, Corollary 4.3 and Theorem 5.1]).

Proposition 3.1 ([Ww]). *On a coKähler 3-manifold, the following three statements are equivalent to each other:*

- *The manifold is conformally flat.*
- *The scalar curvature is a constant.*
- *The manifold is locally isometric to the product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c.*

The above proposition follows from (1.2), (3.2) and (3.3). In this section, we consider a new condition weaker than conformal flatness which is named η-parallelism of the Cotton tensor.

DEFINITION 3.2. The Cotton tensor of an almost contact metric 3-manifold is said to be *η-parallel* if $g((\nabla_{\phi X}C)(\phi Y,\phi Z),\phi W)=0$ for any vector fields X,Y,Z,W.

THEOREM 3.3. *The Cotton tensor of any coKähler* 3*-manifold is η-parallel.*

Proof. Let M^3 be a coKähler 3-manifold. Inserting (3.2) into (1.2) we obtain

$$C(X,Y)=\tfrac{1}{4}X(r)Y-\tfrac{1}{4}Y(r)X$$

for any vector fields X,Y orthogonal to ξ. By the above relation, it is easy to see that $(\nabla_Z C)(X,Y)=0$ for any vector fields X,Y,Z orthogonal to ξ. By Definition 3.2 we find that the Cotton tensor is η-parallel. ∎

Let M^3 be a coKähler 3-manifold whose scalar curvature is not a constant. According to Theorem 3.3 and Proposition 3.1, the Cotton tensor on M^3 is η-parallel but not vanishing.

4. Cotton tensors on almost coKähler 3-manifolds. In this section, we aim to give a local classification of conformally flat almost coKähler 3-manifolds under some restrictions.

A unit vector field V on a Riemannian manifold (M,g) can be regarded as a map from (M,g) into its unit tangent sphere bundle T^1M furnished with the Sasakian metric g_S. The vector field V is said to be *harmonic* if it is a critical point of the *energy function* defined on the set $\mathfrak{X}^1(M)$ of all unit vector fields. The map $V:(M,g)\to(T^1M,g_S)$ is *harmonic* if V is harmonic as a vector field and $\operatorname{tr}\{Y\to R(\nabla_Y V,V)X\}$ vanishes for any vector field X on M.

LEMMA 4.1 ([Pd1, Theorem 4.2]). *On an almost coKähler* 3*-manifold, the Reeb vector field ξ defines a harmonic map into the sphere bundle if and only if ξ is harmonic* ($\Leftrightarrow$ *ξ is an eigenvector field of the Ricci operator*) *and* $\xi(\operatorname{tr}h^2)=0$ *on every dense and open subset of the manifold.*

Next, we construct an example of a non-coKähler almost coKähler 3-manifold whose Reeb vector field is harmonic but does not give a harmonic map into the sphere bundle. This example is in fact a special case of that in [DO, Section 5] for $a=0$.

EXAMPLE 4.2. We set $M^3:=\{(x,y,z)\in\mathbb{R}^3: z>0\}$. On M^3 we define a 3-dimensional almost coKähler structure (ϕ,ξ,η,g) by

$$\phi\frac{\partial}{\partial x}=z^2\frac{\partial}{\partial y},\qquad \phi\frac{\partial}{\partial y}=-\frac{1}{z^2}\frac{\partial}{\partial x},\qquad \phi\frac{\partial}{\partial z}=0,$$

$$\xi=\frac{\partial}{\partial z},\qquad \eta=dz,\qquad g=z^2dx\otimes dx+\frac{1}{z^2}dy\otimes dy+dz\otimes dz.$$

In fact, the fundamental 2-form Φ of M^3 is $\Phi = 2dx \wedge dy$. Thus, it is easy to check that $d\eta = 0$ and $d\Phi = 0$. We set $e := \frac{1}{z}\frac{\partial}{\partial x}$ and hence $\phi e = z\frac{\partial}{\partial y}$. Then $\{\xi, e, \phi e\}$ forms an orthonormal global frame on M^3. By a direct calculation, we have $\nabla_e \xi = (1/z)e$ and $\nabla_{\phi e}\xi = -(1/z)\phi e$, where ∇ is the Levi-Civita connection of g. This implies that M^3 is non-coKähler. Moreover, the Ricci operator of M^3 is given by

$$Q\xi = -\frac{2}{z^2}\xi, \quad Qe = \frac{1}{z^2}e, \quad Q\phi e = -\frac{1}{z^2}\phi e.$$

Because $Q\xi = -(2/z^2)\xi$, we know that ξ is a harmonic vector field. According to (2.3), we see that $h'e = (1/z)e$ and $h'\phi e = -(1/z)\phi e$. Because $\operatorname{tr} h^2 = \operatorname{tr} h'^2 = 2/z^2$, we know that $\operatorname{tr} h^2$ is not invariant along the Reeb vector field. By Lemma 4.1, this means that ξ does not produce a harmonic map.

Let M^3 be an almost coKähler 3-manifold. Let $\mathcal{U}_1$ be the open subset of M^3 defined by $h \neq 0$, and $\mathcal{U}_2$ the open subset defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0$ in a neighborhood of $p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we set $he = \lambda e$ and hence $h\phi e = -\lambda \phi e$, where λ is a positive function on $\mathcal{U}_1$. Note that λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 4.3 ([Pd2, Lemma 2.1]). *On $\mathcal{U}_1$, the Levi-Civita connection is given by*

$$\nabla_\xi \xi = 0, \quad \nabla_\xi e = a\phi e, \quad \nabla_\xi \phi e = -ae, \quad \nabla_e \xi = -\lambda \phi e, \quad \nabla_{\phi e}\xi = -\lambda e,$$
$$\nabla_e e = \frac{1}{2\lambda}\big(\phi e(\lambda) + \sigma(e)\big)\phi e, \qquad \nabla_{\phi e}\phi e = \frac{1}{2\lambda}\big(e(\lambda) + \sigma(\phi e)\big)e,$$
$$\nabla_{\phi e} e = \lambda \xi - \frac{1}{2\lambda}\big(e(\lambda) + \sigma(\phi e)\big)\phi e, \quad \nabla_e \phi e = \lambda \xi - \frac{1}{2\lambda}\big(\phi e(\lambda) + \sigma(e)\big)e,$$

where a is a smooth function and σ is the 1-form defined by $\sigma(\cdot) = S(\cdot\,, \xi)$.

By applying Lemma 4.3, the Ricci operator Q can be expressed (see [Pd2]) on $\mathcal{U}_1$ by

$$\begin{aligned} Q\xi &= -2\lambda^2\xi + \sigma(e)e + \sigma(\phi e)\phi e,\\ Qe &= \sigma(e)\xi + \tfrac{1}{2}(r + 2\lambda^2 - 4\lambda a)e + \xi(\lambda)\phi e, \\ Q\phi e &= \sigma(\phi e)\xi + \xi(\lambda)e + \tfrac{1}{2}(r + 2\lambda^2 + 4\lambda a)\phi e, \end{aligned} \tag{4.1}$$

with respect to the local basis $\{\xi, e, \phi e\}$, where r denotes the scalar curvature.

Since the conformally flat coKähler 3-manifolds have already been completely classified, we next study conformal flatness of non-coKähler almost coKähler 3-manifolds. First, we need the following lemma.

LEMMA 4.4. *Let M^3 be a non-coKähler almost coKähler 3-manifold with harmonic Reeb vector field. If M^3 is conformally flat, then*

$$dr = \xi(r)\eta, \qquad d\lambda = \xi(\lambda)\eta.$$

Proof. Since the Reeb vector field ξ is a harmonic vector field, the set $\mathcal{U}_1$ is non-empty, and hence Lemmas 4.1 and 4.3 are applicable. By Lemma 4.1 we have

$$\sigma(e) = \sigma(\phi e) = 0. \tag{4.2}$$

In this case, making use of (4.1) in (4.2) we obtain

$$\begin{aligned} Q\xi &= -2\lambda^2\xi, \\ Qe &= \big(\tfrac{1}{2}r + \lambda^2 - 2\lambda a\big)e + \xi(\lambda)\phi e, \\ Q\phi e &= \xi(\lambda)e + \big(\tfrac{1}{2}r + \lambda^2 + 2\lambda a\big)\phi e. \end{aligned} \tag{4.3}$$

Applying Lemma 4.3 and (4.3), by a direct calculation we obtain the following nine relations:

$$(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi, \tag{4.4}$$

$$\begin{aligned} (\nabla_\xi Q)e = {}& \big(\tfrac{1}{2}\xi(r) + 2(\lambda - 2a)\xi(\lambda) - 2\lambda\xi(a)\big)e \\ &+ (\xi(\xi(\lambda)) - 4\lambda a^2)\phi e, \end{aligned} \tag{4.5}$$

$$\begin{aligned} (\nabla_\xi Q)\phi e = {}& (\xi(\xi(\lambda)) - 4\lambda a^2)e \\ &+ \big(\tfrac{1}{2}\xi(r) + 2(\lambda + 2a)\xi(\lambda) + 2\lambda\xi(a)\big)\phi e, \end{aligned} \tag{4.6}$$

$$(\nabla_e Q)\xi = -4\lambda e(\lambda)\xi + \lambda\xi(\lambda)e + \lambda\big(\tfrac{1}{2}r + 3\lambda^2 + 2\lambda a\big)\phi e, \tag{4.7}$$

$$\begin{aligned} (\nabla_e Q)e = {}& \lambda\xi(\lambda)\xi + (e(\xi(\lambda)) - 2a\phi e(\lambda))\phi e \\ &+ \left(e\big(\tfrac{1}{2}r + \lambda^2 - 2\lambda a\big) - \frac{1}{\lambda}\xi(\lambda)\phi e(\lambda)\right)e, \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\nabla_e Q)\phi e = {}& \lambda\big(\tfrac{1}{2}r + 3\lambda^2 + 2\lambda a\big)\xi + \big(e(\xi(\lambda)) - 2a\phi e(\lambda)\big)e \\ &+ \left(e\big(\tfrac{1}{2}r + \lambda^2 + 2\lambda a\big) + \frac{1}{\lambda}\xi(\lambda)\phi e(\lambda)\right)\phi e, \end{aligned} \tag{4.9}$$

$$(\nabla_{\phi e} Q)\xi = -4\lambda\phi e(\lambda)\xi + \lambda\big(\tfrac{1}{2}r + 3\lambda^2 - 2\lambda a\big)e + \lambda\xi(\lambda)\phi e, \tag{4.10}$$

$$\begin{aligned} (\nabla_{\phi e} Q)e = {}& \lambda\big(\tfrac{1}{2}r + 3\lambda^2 - 2\lambda a\big)\xi + \big(\phi e(\xi(\lambda)) + 2ae(\lambda)\big)\phi e \\ &+ \left(\phi e\big(\tfrac{1}{2}r + \lambda^2 - 2\lambda a\big) + \frac{1}{\lambda}\xi(\lambda)e(\lambda)\right)e, \end{aligned} \tag{4.11}$$

$$\begin{aligned} (\nabla_{\phi e} Q)\phi e = {}& \lambda\xi(\lambda)\xi + (\phi e(\xi(\lambda)) + 2ae(\lambda))e \\ &+ \left(\phi e\big(\tfrac{1}{2}r + \lambda^2 + 2\lambda a\big) - \frac{1}{\lambda}\xi(\lambda)e(\lambda)\right)\phi e. \end{aligned} \tag{4.12}$$

Setting $X = e$ and $Y = \phi e$ in (1.2) and using (4.9) and (4.11) we obtain

$$\begin{aligned}(4.13)\quad & C(e,\phi e) - 4\lambda^2 a\xi \\ &= \Big(e(\xi(\lambda)) - 2a\phi e(\lambda) + \tfrac{1}{4}\phi e(r) - \phi e\big(\tfrac{1}{2}r + \lambda^2 - 2\lambda a\big) - \frac{1}{\lambda}\xi(\lambda)e(\lambda)\Big)e \\ &+\Big(e\big(\tfrac{1}{2}r + \lambda^2 + 2\lambda a\big) + \frac{1}{\lambda}\xi(\lambda)\phi e(\lambda) - \phi e(\xi(\lambda)) - 2ae(\lambda) - \tfrac{1}{4}e(r)\Big)\phi e.\end{aligned}$$

Similarly, setting $X = \xi$ and $Y = e$ in (1.2) and using (4.5) and (4.7) yields

$$\begin{aligned}(4.14)\quad & C(e,\xi) + \big(4\lambda e(\lambda) + \tfrac{1}{4}e(r)\big)\xi \\ &= \big(2\lambda\xi(a) - \xi\big(\tfrac{1}{2}\lambda^2 + \tfrac{1}{4}r\big)\big)e + \big(\lambda\big(\tfrac{1}{2}r + 4a^2 + 3\lambda^2 + 2\lambda a\big) - \xi(\xi(\lambda))\big)\phi e.\end{aligned}$$

Similarly, setting $X = \xi$ and $Y = \phi e$ in (1.2) and using (4.6) and (4.10) we get

$$\begin{aligned}(4.15)\quad & C(\phi e,\xi) + \big(4\lambda\phi e(\lambda) + \tfrac{1}{4}\phi e(r)\big)\xi \\ &= \big(\lambda\big(\tfrac{1}{2}r + 4a^2 + 3\lambda^2 - 2\lambda a\big) - \xi(\xi(\lambda))\big)e - \big(2\lambda\xi(a) + \xi\big(\tfrac{1}{2}\lambda^2 + \tfrac{1}{4}r\big)\big)\phi e.\end{aligned}$$

As λ is a positive function, from (4.13)–(4.15) we see that M^3 is conformally flat if and only if the following seven relations hold:

$$(4.16)\quad \begin{aligned} & e(\xi(\lambda)) = \tfrac{1}{4}\phi e(r + 4\lambda^2) + \frac{1}{\lambda}\xi(\lambda)e(\lambda), \\ & \phi e(\xi(\lambda)) = \tfrac{1}{4}e(r + 4\lambda^2) + \frac{1}{\lambda}\xi(\lambda)\phi e(\lambda), \\ & \xi(\xi(\lambda)) = \tfrac{1}{2}\lambda(r + 6\lambda^2), \qquad a = 0, \\ & \xi(r) = -4\lambda\xi(\lambda), \qquad e(r) = -16\lambda e(\lambda), \qquad \phi e(r) = -16\lambda\phi e(\lambda).\end{aligned}$$

Plugging the last two formulas of (4.16) into the first two, we see that

$$(4.17)\quad \begin{aligned} e(\xi(\lambda)) &= \frac{1}{\lambda}\xi(\lambda)e(\lambda) - 2\lambda\phi e(\lambda), \\ \phi e(\xi(\lambda)) &= \frac{1}{\lambda}\xi(\lambda)\phi e(\lambda) - 2\lambda e(\lambda).\end{aligned}$$

Applying $a = 0$, from Lemma 4.3 and (4.2) we obtain

$$(4.18)\quad [\xi, e] = \lambda\phi e, \qquad [e,\phi e] = \frac{1}{2\lambda}e(\lambda)\phi e - \frac{1}{2\lambda}\phi e(\lambda)e, \qquad [\phi e,\xi] = -\lambda e.$$

The action of ξ on the sixth formula of (4.16) gives

$$(4.19)\quad 16\xi(\lambda)e(\lambda) + 16\lambda\xi(e(\lambda)) + \xi(e(r)) = 0.$$

By the first formula of (4.18) we obtain $\xi(e(r)) - e(\xi(r)) = \lambda\phi e(r)$. The action of e on the fifth formula of (4.16) gives

$$e(\xi(r)) + 4\xi(\lambda)e(\lambda) + 4\lambda e(\xi(\lambda)) = 0.$$

Applying the previous two relations and the seventh formula of (4.16) in (4.19) we get

$$12\xi(\lambda)e(\lambda) + 16\lambda\xi(e(\lambda)) - 4\lambda e(\xi(\lambda)) - 16\lambda^2\phi e(\lambda) = 0.$$

From the first formula of (4.18) we obtain $\xi(e(\lambda)) - e(\xi(\lambda)) = \lambda\phi e(\lambda)$. Combining this with the above relation yields

$$\xi(\lambda)e(\lambda) = \lambda^2\phi e(\lambda), \tag{4.20}$$

where we have used the first formula of (4.17) and the assumption $\lambda > 0$.

Similarly, the action of ξ on the seventh formula of (4.16) gives

$$16\xi(\lambda)\phi e(\lambda) + 16\lambda\xi(\phi e(\lambda)) + \xi(\phi e(r)) = 0. \tag{4.21}$$

By the third formula of (4.18) we obtain $\xi(\phi e(r)) - \phi e(\xi(r)) = \lambda e(r)$. The action of ϕe on the fifth formula of (4.16) gives $\phi e(\xi(r)) + 4\xi(\lambda)\phi e(\lambda) + 4\lambda\phi e(\xi(\lambda)) = 0$. Applying the previous two relations and the sixth formula of (4.16) in (4.21) we get

$$12\xi(\lambda)\phi e(\lambda) + 16\lambda\xi(\phi e(\lambda)) - 4\lambda\phi e(\xi(\lambda)) - 16\lambda^2 e(\lambda) = 0.$$

From the third formula of (4.18) we obtain $\xi(\phi e(\lambda)) - \phi e(\xi(\lambda)) = \lambda e(\lambda)$. Combining this with the above relation we get

$$\xi(\lambda)\phi e(\lambda) = \lambda^2 e(\lambda), \tag{4.22}$$

where we have used the second formula of (4.17) and the assumption $\lambda > 0$.

Suppose that $e(\lambda) = 0$. Then from (4.20) and $\lambda > 0$ we obtain $\phi e(\lambda) = 0$. This means that $d\lambda = \xi(\lambda)\eta$. Applying this in the last two formulas of (4.16) we obtain $dr = \xi(r)\eta$. Similarly, if $\phi e(\lambda) = 0$, from (4.22) and $\lambda > 0$ we get $e(\lambda) = 0$. This means that $d\lambda = \xi(\lambda)\eta$. Making use of this in the last two formulas of (4.16) we again obtain $dr = \xi(r)\eta$.

Next, we assume that both $e(\lambda) \neq 0$ and $\phi e(\lambda) \neq 0$ on some open subset. Thus, in view of $\lambda > 0$, comparing (4.20) with (4.22) we see that

$$\big(\phi e(\lambda) - e(\lambda)\big)\big(\phi e(\lambda) + e(\lambda)\big) = 0.$$

It follows that either $\phi e(\lambda) = e(\lambda) \neq 0$ or $\phi e(\lambda) = -e(\lambda) \neq 0$.

In the first case, from (4.20) or (4.22) we obtain $\xi(\lambda) = \lambda^2$. In view of $\lambda > 0$, inserting $\xi(\lambda) = \lambda^2$ in the first formula of (4.17) gives $e(\lambda) = -2\phi e(\lambda)$. Obviously, this implies $\phi e(\lambda) = e(\lambda) = 0$, a contradiction.

Finally, we consider the other case $\phi e(\lambda) = -e(\lambda) \neq 0$. Making use of this in (4.20) or (4.22) we obtain $\xi(\lambda) = -\lambda^2$. Inserting this into the first formula of (4.17) we get $e(\lambda) = 2\phi e(\lambda)$ because $\lambda > 0$. Thus, we again arrive at a contradiction since it follows that $\phi e(\lambda) = e(\lambda) = 0$. This completes the proof. ■

From the proof of Lemma 4.4 we obtain

Corollary 4.5. *A non-coKähler almost coKähler* 3*-manifold with harmonic Reeb vector field is conformally flat if and only if on* $\mathcal{U}_1$*,*

$$dr = -4\lambda\xi(\lambda)\eta, \quad d\lambda = \xi(\lambda)\eta, \quad d(\xi(\lambda)) = \tfrac{1}{2}\lambda(r + 6\lambda^2)\eta, \quad a = 0.$$

Our main result is the following.

Theorem 4.6. *A non-coKähler almost coKähler* 3*-manifold with Reeb vector field defining a harmonic map cannot be conformally flat.*

Proof. Suppose that a non-coKähler almost coKähler 3-manifold with ξ defining a harmonic map is conformally flat. From Lemma 4.1, we know that Lemma 4.4 is applicable. This implies that $d\lambda = \xi(\lambda)\eta = 0$, i.e., λ is a positive constant. Also, from the last three formulas of (4.16) we see that the scalar curvature r is a constant. Applying this in the third formula of (4.16) we get $r = -6\lambda^2$. Iserting this and $a = 0$ in (4.4)–(4.12) we see that the Ricci operator is parallel, and hence the manifold is locally symmetric. Perrone [Pd1, Proposition 3.1] proved that a locally symmetric almost coKähler 3-manifold must be coKähler. This gives a contradiction. ∎

The following corollary follows directly from Theorem 4.6 and Proposition 3.1.

Corollary 4.7. *An almost coKähler* 3*-manifold with Reeb vector field defining a harmonic map is conformally flat if and only if it is locally isometric to the product* $\mathbb{R} \times N^2(c)$*, where* $N^2(c)$ *denotes a Kähler surface of constant curvature* c*.*

For the almost coKähler 3-manifold M^3 of Example 4.2, by a direct calculation one observes that M^3 is conformally flat (for more details see also [DO, p. 101]). In this example, the scalar curvature is $r = -2/z^2$. We set $e_1 = (\sqrt{2}/2)(e + \phi e)$, and hence $\phi e_1 = (\sqrt{2}/2)(\phi e - e)$. It follows that $he_1 = (1/z)e_1$ and $h\phi e_1 = -(1/z)\phi e_1$. It is easily seen that both the scalar curvature r and $\operatorname{tr} h^2 = 1/z^2$ are invariant along the distribution $\{\xi\}^\perp$. Thus, Example 4.2 illustrates Lemma 4.4. We also have

Remark 4.8. The assumption $\xi(\operatorname{tr} h^2) = 0$ (which guarantees that ξ is harmonic) in Theorem 4.6 is essential.

Let G be a three-dimensional unimodular Lie group (see Milnor [Mj]) with a left invariant Riemannian metric g and left invariant local orthonormal frame fields $\{u_1, u_2, u_3\}$ satisfying

$$[u_1, u_2] = c_3u_3, \quad [u_2, u_3] = 0, \quad [u_3, u_1] = c_2u_2,$$

where both c_1 and c_2 are constants. We set $\xi := u_1$ and denote by η the dual 1-form $\eta = g(\xi, \cdot)$. We define a $(1, 1)$ tensor field ϕ by $\phi u_2 = u_3$, $\phi u_3 = -u_2$ and $\phi\xi = 0$. One can check that (G, ϕ, ξ, η, g) defines a left invariant almost coKähler structure (for more details see Perrone [Pd1]). By the well-known

Koszul formula, the Levi-Civita connection of g is given by

$$(4.23)\qquad (\nabla_{u_i}u_j)=\begin{pmatrix} 0 & \frac{c_2+c_3}{2}u_3 & -\frac{c_2+c_3}{2}u_2\\ \frac{c_2-c_3}{2}u_3 & 0 & \frac{c_3-c_2}{2}u_1\\ \frac{c_2-c_3}{2}u_2 & \frac{c_3-c_2}{2}u_1 & 0\end{pmatrix}$$

for any $i,j\in\{1,2,3\}$. By a direct calculation we deduce from (4.23) that

$$hu_2=\frac{c_3-c_2}{2}u_2,\qquad hu_3=\frac{c_2-c_3}{2}u_3,$$
$$Q\xi=-\frac{(c_2-c_3)^2}{2}\xi,\qquad Qu_2=\frac{c_2^2-c_3^2}{2}u_2,\qquad Qu_3=\frac{c_3^2-c_2^2}{2}u_3.$$

Thus, from (1.2) and the above relations we get

$$\begin{aligned} C(\xi,u_2)&=\tfrac{1}{2}(c_2-c_3)(c_2^2+c_2c_3+2c_3^2)u_3,\\ C(\xi,u_3)&=\tfrac{1}{2}(c_2-c_3)(2c_2^2+c_2c_3+c_3^2)u_2,\\ C(u_2,u_3)&=\tfrac{1}{2}(c_3^2-c_2^2)(c_3-c_2)\xi.\end{aligned}$$

Applying (4.23) and the above relations we also have

$$\begin{aligned} (\nabla_{u_2}C)(u_2,u_3)&=-\tfrac{1}{4}(c_3-c_2)^2(c_2+3c_3)c_3u_3,\\ (\nabla_{u_3}C)(u_2,u_3)&=\tfrac{1}{4}(c_3-c_2)^2(3c_2+c_3)c_2u_2.\end{aligned}$$

By applying the above relations, we give the following example illustrating our main results.

Example 4.9. A unimodular Lie group G with a left invariant Riemannian metric g and a left invariant local orthonormal frame field $\{u_1,u_2,u_3\}$ satisfying

$$(4.24)\qquad [u_1,u_2]=c_3u_3,\qquad [u_2,u_3]=0,\qquad [u_3,u_1]=c_2u_2,\qquad c_2\neq c_3\in\mathbb{R},$$

admits a non-coKähler almost coKähler structure. For that structure the Reeb vector field defines a harmonic map, but the Cotton tensor is neither vanishing nor η-parallel.

Let G_1 be a 3-dimensional non-unimodular Lie group (see Milnor [Mj]) with a left invariant metric g whose Lie algebra is given by

$$[e_1,e_2]=\alpha e_2,\qquad [e_2,e_3]=0,\qquad [e_1,e_3]=\beta e_2,$$

where $\{e_1,e_2,e_3\}$ is an orthonormal basis with respect to g and $\alpha\neq 0$, $\beta\in\mathbb{R}$. We set $\xi:=e_3$ and define a 1-form η by $\eta=g(\xi,\cdot)$. We define a $(1,1)$ tensor field ϕ by $\phi\xi=0$, $\phi e_1=e_2$ and $\phi e_2=-e_1$. Thus, (G_1,ϕ,ξ,η,g) defines a 3-dimensional left invariant non-coKähler almost coKähler manifold when $\beta\neq 0$, because $he_1=-\frac{1}{2}\beta e_1$ and $he_2=\frac{1}{2}\beta e_2$. Moreover, by the well-known

Koszul formula and (4.24) we have

$$(\nabla_{e_i} e_j) = \begin{pmatrix} 0 & -\frac{1}{2}\beta\xi & \frac{1}{2}\beta e_2 \\ -\alpha e_2 - \frac{1}{2}\beta\xi & \alpha e_1 & \frac{1}{2}\beta e_1 \\ -\frac{1}{2}\beta e_2 & \frac{1}{2}\beta e_1 & 0 \end{pmatrix} \tag{4.25}$$

for any $i, j \in \{1, 2, 3\}$. Using (4.25), one can show that the Ricci operator of G_1 is given by

$$Q\xi = -\tfrac{1}{2}\beta^2\xi - \alpha\beta e_2, \quad Qe_1 = -\big(\alpha^2 + \tfrac{1}{2}\beta^2\big)e_1, \quad Qe_2 = \big(-\alpha^2 + \tfrac{1}{2}\beta^2\big)e_2 - \alpha\beta\xi.$$

Using (4.25) and the above relations we obtain

$$\begin{aligned} C(\xi, e_1) &= \tfrac{1}{2}\beta(2\beta^2 - \alpha^2)e_2 - \tfrac{3}{2}\alpha\beta^2\xi, \\ C(\xi, e_2) &= \tfrac{1}{2}\beta(\alpha^2 + \beta^2)e_1, \\ C(e_1, e_2) &= \tfrac{1}{2}\beta(2\alpha^2 - \beta^2)\xi - \tfrac{3}{2}\alpha\beta^2 e_2. \end{aligned} \tag{4.26}$$

From (4.26) and (4.25) we get

$$\begin{aligned} (\nabla_{e_1} C)(e_1, e_2) &= \tfrac{3}{4}\beta^2(\alpha^2 - \beta^2)e_2 + \tfrac{3}{2}\alpha\beta^3\xi, \\ (\nabla_{e_2} C)(e_1, e_2) &= -\tfrac{3}{4}\alpha^2\beta^2 e_1. \end{aligned}$$

Example 4.10. A non-unimodular Lie group G_1 with a left invariant Riemannian metric g and a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = \beta e_2, \quad 0 \neq \alpha, \beta \in \mathbb{R}, \tag{4.27}$$

admits a non-coKähler almost coKähler structure. For that structure the Reeb vector field defines a harmonic map, but the Cotton tensor is neither vanishing nor η-parallel.

It has been proved by Perrone [Pd1] that a simply connected homogeneous non-coKähler almost coKähler 3-manifold is isometric to a Lie group whose Lie algebra is given by (4.24) or (4.27). Thus, from Examples 4.9 and 4.10 we have

Theorem 4.11. *A simply connected homogeneous non-coKähler almost coKähler* 3*-manifold is not conformally flat and does not have η-parallel Cotton tensor.*

The Cotton tensor on an almost contact metric manifold is said to be *proper η-parallel* if it is η-parallel but not vanishing. As seen in Section 3, there exist proper η-parallel Cotton tensors on coKähler 3-manifolds whose scalar curvature is not a constant. Therefore, taking into account Theorem 4.11, we propose an interesting question given as the following:

Is there a non-coKähler almost coKähler 3*-manifold with proper η-parallel Cotton tensor?*

The existence and classification of almost coKähler 3-manifolds with proper η-parallel Cotton tensors is also an open question.

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