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ON CONVERGENCE OF POWER SERIES OF L_p CONTRACTIONS

GUY COHEN

Department of Electrical Engineering, Ben-Gurion University
E-mail: guycohen@bgu.ac.il

CHRISTOPHE CUNY

Equipe ISEA, University of New-Caledonia
E-mail: christophe.cuny@univ-nc.nc

MICHAEL LIN

Department of Mathematics, Ben-Gurion University
E-mail: lin@math.bgu.ac.il

Dedicated to the memory of Jaroslav Zemánek

Abstract. Let T be a power-bounded operator on a (real or complex) Banach space. We study the convergence of the power series $\sum_{k=0}^{\infty} \beta_k T^k x$ when $\{\beta_k\}$ is a Kaluza sequence with divergent sum such that $\beta_k \to 0$ and $\sum_{k=0}^{\infty} \beta_k z^k$ converges in the open unit disk. We prove that weak and strong convergence are equivalent, and in a reflexive space also $\sup_n \|\sum_{k=0}^{n} \beta_k T^k x\| < \infty$ is equivalent to the convergence of the series. The last assertion is proved also when T is a mean ergodic contraction of L_1.

For normal operators on a Hilbert space we obtain a spectral characterization of the convergence of $\sum_{n=0}^{\infty} \beta_n T^n x$, and a sufficient condition expressed in terms of norms of the ergodic averages, which in some cases is also necessary.

For T Dunford–Schwartz of a σ-finite measure space or a positive contraction of L_p, $1 < p < \infty$, we prove that when $\{\beta_k\}$ is also completely monotone (i.e. a Hausdorff moment sequence) and $\beta_k = O(1/k)$, the norm convergence of $\sum_{k=0}^{\infty} \beta_k T^k f$ implies a.e. convergence.

For T a positive contraction of L_p, $p > 1$, $f \in L_p$ and $\beta \in \mathbb{R}$, we show that if the series $\sum_{n=0}^{\infty} \frac{(\log(n+1))^\beta}{(n+1)^{1-1/r}} T^n f$ converges in L_p-norm for some $r \in (\frac{p}{p-1}, \infty]$, then it converges a.e.

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1. Introduction. Izumi [32] raised the question of the a.e. convergence of the one-sided ergodic Hilbert transform (EHT) $\sum_{k=1}^{\infty}\frac{f\circ\theta^k}{k}$ associated to a probability preserving ergodic transformation θ and centered functions in $L_2(S,\Sigma,\mu)$ (which, by Kronecker's lemma, would be a strengthening of Birkhoff's pointwise ergodic theorem). Halmos [29] proved that for every ergodic probability preserving transformation on a non-atomic space there always exists a centered $f\in L_2$ such that the one-sided EHT fails to converge in L_2-norm. Dowker and Erdős [20] (see also Del Junco and Rosenblatt [33]) obtained even the existence of $f\in L^\infty(X)$, centered, such that $\sup_{n\ge 1}|\sum_{k=1}^{n}\frac{f\circ\theta^k}{k}|=+\infty$ a.s.; see [4] for additional background and references.

For T unitary on a complex Hilbert space H, Gaposhkin [25] obtained a spectral characterization of the norm convergence of the one-sided EHT $\sum_{k=1}^{\infty}\frac{T^k f}{k}$, and gave an example of a unitary T on L_2 for which the one-sided EHT converges in norm, but not a.e.; he then asked if for the one-sided EHT of an invertible measure preserving θ, L_2-norm convergence implies a.e. convergence.

Recently, Cuny [13] proved that for T a Dunford–Schwartz operator on a σ-finite measure space or a positive contraction of an L_p space, $1<p<\infty$, norm convergence of $\sum_{k=1}^{\infty}\frac{T^k f}{k}$ implies a.e. convergence. For additional results and references concerning pointwise convergence see [25] and [4].

In [9] we proved that for any power-bounded T on a Banach space X, norm convergence of the one-sided EHT is equivalent to its weak convergence. The result of [9] was proved independently, at about the same time, by Haase and Tomilov [28], who looked at more general power series of power-bounded operators.

In this paper we consider a power-bounded operator T on a Banach space X (which is therefore a contraction in an equivalent norm), and for $x\in X$ we study the convergence of the series

$$\sum_{k=0}^{\infty}\beta_k T^k x \tag{1}$$

where $\{\beta_k\}_{k\ge 0}$ is a Kaluza sequence with divergent sums, such that the series $\sum_{k\ge 0}\beta_k z^k$ converges for z in the open unit disk D.

We assume that $\beta_k\to 0$, and extend the method of [9] to recover some of the results of [28] without using spectral theory (so the field of scalars can be also $\mathbb{R}$). The method is then applied to prove that for a mean ergodic contraction T in L_1 the series $\sum_{k=1}^{\infty}\beta_k T^k f$ converges if (and only if) $\sup_n\big\|\sum_{k=1}^{n}\beta_k T^k f\big\|_1<\infty$. This solves a problem left open in [9] (for the one-sided EHT, which corresponds to $\beta_0=1$ and $\beta_k=1/2k$ for $k\ge 1$).

We then prove for normal contractions in a Hilbert space $\mathcal{H}$ a spectral characterization of the convergence of $\sum_n \beta_n T^n x$. It extends the characterizations of [18] for fractional coboundaries and of [10] for the one-sided EHT. We also obtain a sufficient condition, expressed in terms of norms of the ergodic averages, which in some cases is also necessary. Applications are made to general contractions in $\mathcal{H}$. As a corollary, for the one-sided EHT we extend a result obtained in [10] for normal contractions, and show that for *any* contraction T,

$$\sum_{n=1}^{\infty}\frac{T^n x}{n}\ \text{converges}\quad\Longleftrightarrow\quad\sum_{n=1}^{\infty}\frac{\langle T^n x,x\rangle\log n}{n}\ \text{converges.}$$

Finally, we study the case of Dunford–Schwartz operators on L_1 of a σ-finite measure space or positive contractions on L_p, $1 < p < \infty$, or positive mean ergodic contractions of L_1, and prove that if $\{\beta_n\}$ is a Hausdorff moment sequence with $\beta_n = O(1/n)$, then norm convergence of $\sum_{k=1}^{\infty} \beta_k T^k f$ implies a.e. convergence.

For the case $\beta_n \asymp b(n) n^{1-1/r}$ with $1 < r \le \infty$ and $b(n)$ slowly varying, the same is shown for positive contractions of L_p when $p > r/(r-1)$. As a consequence, we infer that for T a positive contraction of L_p, $p > 1$ and $f \in L_p$, if the series $\sum_{n=0}^{\infty} \frac{(\log(n+1))^{\beta}}{(n+1)^{1-1/r}} T^n f$ converges in L_p-norm for some $r \in (\frac{p}{p-1}, \infty]$, $\beta \in \mathbb{R}$, then it converges a.e.

2. On some operator power series. Motivated by conditions for the central limit theorem for stationary ergodic Markov chains, Derriennic and Lin [18] defined for $0 < \alpha < 1$ and T power-bounded on X the operator $(I - T)^{\alpha}$ by the series $(I - T)^{\alpha} = I - \sum_{n=1}^{\infty} a_n^{(\alpha)} T^n$, where the coefficients are those of the expansion of $(1 - t)^{\alpha}$ in the interval $[-1, 1]$, which satisfy $a_n^{(\alpha)} > 0$ and $\sum_{n=1}^{\infty} a_n^{(\alpha)} = 1$. It is not difficult to show that $(I - T)^{\alpha} X \subset \overline{(I - T)X}$. It is proved in [18] that when T is mean-ergodic, $x \in (I - T)^{\alpha} X$ if and only if the series $\sum_{n=1}^{\infty} \frac{T^n x}{n^{1-\alpha}}$ converges strongly. The proof uses the series representation of $1/(1 - t)^{\alpha}$, which converges for $|t| < 1$. This suggested the idea in [13] that in order to study the one-sided EHT, we try to connect it with the inverse of an analytic function on the open unit disk D; this idea was further developed in [9].

In this section we use the method of [9] to study the domain of definition of some operators of the form $H(T)x := \sum_{n \ge 0} \beta_n T^n x$, where $\{\beta_n\}$ is a bounded sequence of positive numbers (with divergent sum) and T is a power-bounded operator on X.

Throughout this paper $\{\beta_n\}_{n \ge 0}$ will be a bounded sequence of positive real numbers with $\beta_0 = 1$. The series $H(z) := \sum_{n \ge 0} \beta_n z^n$ then converges on $D := \{z \in \mathbb{C} : |z| < 1\}$, and does not vanish around 0. When $H(z)$ does not vanish in D, the function $G(z) = 1/H(z)$ is analytic in D, and its power series $G(z) = \sum_{n \ge 0} \alpha_n z^n$ converges in all of D.

By definition $\alpha_0 = G(0) = 1$, and the identity $G(z)H(z) = 1$ yields

$$\beta_n + \sum_{k=1}^{n} \alpha_k \beta_{n-k} = \beta_n + \sum_{k=0}^{n-1} \beta_k \alpha_{n-k} = 0 \qquad \text{for all } n \ge 1. \tag{2}$$

Since β_n are real numbers, so are the α_n.

The following proposition is a slight extension of [28, Lemma 4.5] (see Example 1 below).

Proposition 2.1. *Let $\{\beta_n\}$ be a bounded sequence of positive real numbers with $\beta_0 = 1$, such that $H(z) := \sum_{n \ge 0} \beta_n z^n$ does not vanish on D. Let $G(z) := 1/H(z)$, with $G(z) = \sum_{n \ge 0} \alpha_n z^n$, and assume that α_n is eventually non-positive. Then $\sum_{n \ge 0} |\alpha_n| < \infty$. In particular, G extends to a continuous function on $\overline{D}$.*

Proof. By assumption, there exists $n_0 \ge 0$ such that $\alpha_n \le 0$ for every $n > n_0$. For $0 \le t < 1$ we have

$$\frac{1}{H(t)} = G(t) = 1 + \sum_{n \ge 1} \alpha_n t^n = 1 + \sum_{n=1}^{n_0} \alpha_n t^n + \sum_{n > n_0} \alpha_n t^n.$$

Hence, by a theorem of Abel,

$$\sum_{n>n_0} |\alpha_n| t^n = -\sum_{n>n_0} \alpha_n t^n = 1 + \sum_{n=1}^{n_0} \alpha_n t^n - \frac{1}{H(t)} \underset{t\uparrow 1}{\longrightarrow} 1 + \sum_{n=1}^{n_0} \alpha_n - \frac{1}{\sum_{n\ge 0} \beta_n},$$

which shows that $\sum_{n>n_0} |\alpha_n| = \lim_{t\uparrow 1} \sum_{n>n_0} |\alpha_n| t^n < \infty$. ∎

Remark. If, under the assumptions of the proposition, $\sum_{n\ge 0} \beta_n = \infty$, then we must have $G(1) = 0$, which means that $\sum_{n\ge 0} \alpha_n = 0$, so $\alpha_0 = 1$ yields $1 = |\sum_{n\ge 1} \alpha_n| \le \sum_{n\ge 1} |\alpha_n|$.

Now take $\{\beta_n\}$ and $\{\alpha_n\}$ as in the proposition above and fix a power-bounded operator T on a (real or complex) Banach space X. Since the series $\sum_{n\ge 0} |\alpha_n|$ converges and the coefficients are real, the operator series $\sum_{n\ge 0} \alpha_n T^n$ converges in operator norm, and defines a bounded operator on X, denoted by $G(T)$. For $n \ge 1$ we define

$$H_n(T) := \sum_{k=0}^{n} \beta_k T^k \tag{3}$$

and put $H(T)x = \lim_{n\to\infty} H_n(T)x$ whenever the limit exists in norm. If $\sum_{n\ge 0} \beta_n < \infty$ the operator $H(T)$ is a bounded operator defined everywhere, with $H(T)G(T) = I$, so we will be interested only in the case $\sum_{n\ge 0} \beta_n = \infty$.

Proposition 2.2. *Let $\{\beta_n\}_{n\ge 0}$ and $\{\alpha_n\}_{n\ge 0}$ be as in Proposition 2.1. Then there exists $C > 0$ such that for any power-bounded operator T on a Banach space X we have*

$$\sup_{n\ge 1} \|H_n(T)G(T)\| \le C \sup_{n\ge 0} \|T^n\|. \tag{4}$$

Proof. Let $n \ge 0$, $t \in (0,1)$. Write $H_n(t) = \sum_{k=0}^{n} \beta_k t^k$. We have

$$\begin{aligned} H_n(t)G(t) &= \sum_{k=0}^{n} \beta_k t^k + \sum_{k=0}^{n} \beta_k \sum_{m\ge 1} \alpha_m t^{m+k} = \sum_{k=0}^{n} \beta_k t^k + \sum_{k=0}^{n} \beta_k \sum_{m\ge k+1} \alpha_{m-k} t^m \\ &= \sum_{k=0}^{n} \beta_k t^k + \sum_{m=1}^{n} \Big(\sum_{k=0}^{m-1} \beta_k \alpha_{m-k} \Big) t^m + \sum_{m\ge n+1} \Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k} \Big) t^m \\ &= 1 + \sum_{m\ge n+1} \Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k} \Big) t^m, \end{aligned} \tag{5}$$

where we used $\beta_0 = 1$ and (2) for the last equality. By assumption, there exists $n_0 \ge 0$ such that $\alpha_n \le 0$, for every $n > n_0$. Since $H_n(t)G(t) \ge 0$, for $n > n_0$ we have

$$\begin{aligned} 1 &+ \sum_{m=n+1}^{n+n_0} \Big(\sum_{k=n-n_0+1}^{n} \beta_k \alpha_{m-k} \Big) t^m \\ &\ge -\sum_{m=n+1}^{n+n_0} \Big(\sum_{k=0}^{n-n_0} \beta_k \alpha_{m-k} \Big) t^m - \sum_{m\ge n+n_0+1} \Big(\sum_{k=0}^{n} \beta_k \alpha_{m-k} \Big) t^m. \end{aligned}$$

The LHS is bounded by $K := 1 + n_0^2(\sup_{k\geq 0}\beta_k)(\sup_{0\leq l\leq 2n_0}|\alpha_l|)$, and the RHS contains only non-negative terms; hence, letting $t\uparrow 1$, we obtain

$$\sum_{m=n+1}^{n+n_0}\Big(\sum_{k=0}^{n-n_0}\beta_k|\alpha_{m-k}|\Big) + \sum_{m\geq n+n_0+1}\Big(\sum_{k=0}^{n}\beta_k|\alpha_{m-k}|\Big) \leq K. \tag{6}$$

Putting T instead of t in (5), for $n > n_0$ we obtain

$$\begin{aligned}H_n(T)G(T) &= I + \sum_{m\geq n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m\\ &= I + \sum_{m=n+1}^{n+n_0}\Big(\sum_{k=n-n_0+1}^{n}\beta_k\alpha_{m-k}\Big)T^m + \sum_{m=n+1}^{n+n_0}\Big(\sum_{k=0}^{n-n_0}\beta_k\alpha_{m-k}\Big)T^m\\ &\qquad + \sum_{m\geq n+n_0+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m.\end{aligned} \tag{7}$$

Hence

$$\begin{aligned}\big\|H_n(T)G(T)\big\| &\leq 1 + \sum_{m=n+1}^{n+n_0}\Big(\sum_{k=n-n_0+1}^{n}\beta_k|\alpha_{m-k}|\Big)\sup_j\|T^j\|\\ &\quad + \Big(\sum_{m=n+1}^{m+n_0}\sum_{k=0}^{n-n_0}\beta_k|\alpha_{m-k}| + \sum_{m\geq n+n_0+1}\sum_{k=0}^{n}\beta_k|\alpha_{m-k}|\Big)\sup_j\|T^j\|,\end{aligned}$$

and the result follows by (6). ∎

Remarks.

1. Equation (6) implies

$$\sum_{k=0}^{n}\beta_k\sum_{m\geq n+n_0+1}|\alpha_m| \leq \sum_{k=0}^{n}\beta_k\sum_{m\geq n+n_0+1}|\alpha_{m-k}| \leq K, \tag{8}$$

which yields another proof of Proposition 2.1.

2. Equation (6) and the equation preceding it yield that in the expansion $H_n(t)G(t) = 1 + \sum_{k\geq n+1}\gamma_k^{(n)}t^k$ we have $\sup_n\sum_{k\geq n+1}|\gamma_k^{(n)}| < \infty$.

The next proposition provides the main tool for our results.

Proposition 2.3. *Let $\{\beta_n\}_{n\geq 0}$ and $\{\alpha_n\}_{n\geq 0}$ be as in Proposition 2.1, and in addition assume that $\sum_{n\geq 0}|\beta_n - \beta_{n+1}| < \infty$ and $\beta_n \to 0$. Then for T power-bounded on X and any $x \in \overline{(I-T)X}$ we have*

$$\lim_{n\to\infty}\|x - H_n(T)G(T)x\| = 0.$$

Proof. By the previous proposition, it is enough to prove the convergence for $x \in (I-T)X$. By (7), the assertion is that for $x \in (I-T)X$

$$\Big\|\sum_{m\geq n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m x\Big\| \underset{n\to+\infty}{\longrightarrow} 0.$$

We define $M := \sup_{n\ge 0} \|T^n\|$. For $u \in X$ we have

$$\begin{aligned}
&\sum_{m\ge n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m(u-Tu)\\
&\qquad= \sum_{m\ge n+1}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k}\Big)T^m u - \sum_{m\ge n+2}\Big(\sum_{k=0}^{n}\beta_k\alpha_{m-k-1}\Big)T^m u\\
&\qquad= \sum_{m\ge n+2}\Big(\sum_{k=1}^{n}(\beta_k-\beta_{k-1})\alpha_{m-k}\Big)T^m u + \Big(\sum_{k=0}^{n}\beta_k\alpha_{n+1-k}\Big)T^{n+1}u\\
&\qquad\qquad+\sum_{m\ge n+1}\alpha_m T^m u - \sum_{m\ge n+2}\beta_n\alpha_{m-n-1}T^m u. \qquad (9)
\end{aligned}$$

We estimate the norms of the four terms above. The third term converges to zero as a tail of a convergent series. The norm of the fourth term is bounded by $\beta_n M\|u\|\sum_{k=1}^{\infty}|\alpha_k|$, which tends to 0 since $\beta_n \to 0$. By (2), the norm of the second term is bounded by $|\sum_{k=0}^{n}\beta_k\alpha_{n+1-k}|M\|u\| = M\|u\|\beta_{n+1}\to 0$.

It remains to deal with the norm of the first term in (9). Splitting the inner sum according to $k \le [n/2]$ we obtain

$$\begin{aligned}
&\Big\|\sum_{m\ge n+2}\Big(\sum_{k=1}^{n}(\beta_k-\beta_{k-1})\alpha_{m-k}\Big)T^m u\Big\| \le M\|u\|\sum_{m\ge n+2}\sum_{k=1}^{n}|\beta_k-\beta_{k-1}|\cdot|\alpha_{m-k}|\\
&\quad= M\|u\|\Big\{\sum_{k=1}^{[n/2]}|\beta_{k-1}-\beta_k|\sum_{m\ge n+2}|\alpha_{m-k}| + \sum_{k=[n/2]+1}^{n}|\beta_{k-1}-\beta_k|\sum_{m\ge n+2}|\alpha_{m-k}|\Big\}\\
&\quad\le M\|u\|\Big\{\sum_{k=1}^{[n/2]}|\beta_{k-1}-\beta_k|\sum_{m>[n/2]}|\alpha_m| + \sum_{k=[n/2]+1}^{n}|\beta_{k-1}-\beta_k|\sum_{m\ge 2}|\alpha_m|\Big\}\\
&\quad\le M\|u\|\Big\{\sum_{k=1}^{\infty}|\beta_{k-1}-\beta_k|\sum_{m>[n/2]}|\alpha_m| + \sum_{k=[n/2]+1}^{\infty}|\beta_{k-1}-\beta_k|\sum_{m\ge 2}|\alpha_m|\Big\} \to 0. \quad\blacksquare
\end{aligned}$$

REMARKS.

1. It is easy to show that the assumptions imply convergence of $H_n(T)(I-T)u$ for any $u \in X$, so the convergence of $G(T)H_n(T)$ on $\overline{(I-T)X}$ follows from Proposition 2.2. Our proof yields also the limit.

2. In Proposition 2.3 we cannot omit the assumption $\beta_n \to 0$. Indeed, take $\beta_n = 1$ for $n \ge 0$, so $H(z) = \sum_{n\ge 0} z^n$ and $G(z) = 1-z$. Clearly $H_n(T)G(T) = I - T^{n+1}$ and $x - H_n(T)G(T)x = T^{n+1}x$ does not converge to zero in general (for $x \in \overline{(I-T)X}$), e.g., for T isometry. Also weak convergence need not hold, even if T has no unimodular eigenvalues different from 1 — take a weakly mixing dynamical system which is not mixing.

EXAMPLE 1 (The one-sided ergodic Hilbert transform). Let $\beta_0 = 1$, $\beta_n = \frac{1}{n}$ for $n \ge 1$. Then $\sum_{n\ge 0}\beta_n z^n = 1 - \log(1-z) = \log\big(\frac{e}{1-z}\big)$. Computations by (2) yield $\alpha_1 = -1$, $\alpha_2 = 1/2$, $\alpha_3 = -1/3$, and $\alpha_4 = 1/6$. However, the asymptotic value of the coefficients of $\log\left(\frac{e}{1-z}\right)^{-1}$ is $\alpha_n \simeq -1/n(\log n)^2$, by [44, V.2.34, p. 192], so the previous proposition applies (this was the approach in [9]).

Similarly, $\left[\log\left(\frac{e}{1-z}\right)\right]^m = \sum_{n\geq 0} \beta_n z^n$, with $m \geq 2$ an integer, has positive coefficients, and we can apply the above cited formula from [44] to obtain that α_n is eventually negative.

Example 2 (Kaluza sequences). Let $\{\beta_n\}$ be a strictly positive sequence with $\beta_0 = 1$ such that

$$\frac{\beta_{n+1}}{\beta_n} \geq \frac{\beta_n}{\beta_{n-1}} \quad \forall n \geq 1.$$

Such sequences are called *Kaluza sequences* (and sometimes *log-convex* sequences). We assume that $H(z) := \sum_{n\geq 0} \beta_n z^n$ converges in D; this is equivalent to $\lim \beta_{n+1}/\beta_n \leq 1$, and thus equivalent to boundedness of the Kaluza sequence, implying that $\{\beta_n\}$ is monotone non-increasing. The theorem of Kaluza [35] (see [30, Ch. IV, Theorem 22]) yields that $H(z) \neq 0$ in D, and $\alpha_n \leq 0$ for $n \geq 1$ with $\sum_{n\geq 1} |\alpha_n| \leq 1$. When $\sum_{n\geq 0} \beta_n = \infty$, we have $\sum_{n\geq 1} |\alpha_n| = 1$. Thus, bounded Kaluza sequences provide examples for Propositions 2.1 and 2.2. If in addition $\beta_n \to 0$, we have examples for the application of Proposition 2.3. Particular examples, with $\beta_n \to 0$, are:

(i) $\beta_0 = 1$ and $\beta_n = 1/2n$ for $n \geq 1$, which generate the EHT.
(ii) $\beta_n = 1/(n+1)$ for $n \geq 0$ (used in [28] for studying the EHT).
(iii) $\beta_n = c/(n+1)\log(n+2)$, with $c = \log 2$.
(iv) $\beta_n = c\log(n+8)/(n+8)$, where $c = 8/\log 8$.
(v) $\beta_0 = 1$ and $\beta_n = 1/2n^{1-\gamma}$ for $n \geq 1$ (with $0 < \gamma < 1$), used in [18].
(vi) $\beta_n = \int_0^1 t^n \, d\nu(t)$ for a Borel probability ν on [0,1] with $\nu(\{1\}) = 0$ decreases to 0 (and is Kaluza by the Cauchy–Schwartz inequality). The sequence $\{\beta_n\}$ is called the *Hausdorff moment sequence* of ν. Note that $\sum_n \beta_n = \infty$ if and only if $\int_0^1 (1-t)^{-1} \, d\nu(t) = \infty$. The special case $\frac{1}{n+1} = \int_0^1 t^n \, dt$ yields (ii).

Example 3 (Strengthening of weighted averages). As noted in the introduction, convergence of the one-sided EHT is a strengthening of the ergodic theorem. Let $\{w_n\}_{n\geq 0}$ be a positive non-increasing sequence with $W_n := \sum_{j=0}^n w_j \to \infty$. Then whenever $\frac{1}{n}\sum_{k=0}^n T^k f$ converges, also $\frac{1}{W_n}\sum_{k=0}^n w_k T^k f$ converges [36, p. 258]. In order to study the convergence of $\sum_{n\geq 0} \frac{w_n}{W_n} T^n f$, we put $\beta_n = w_n/W_n$ for $n \geq 0$, and strengthen the assumption of w_n non-increasing to w_{n+1}/w_n non-decreasing with limit 1. Then $\{\beta_n\}$ is a Kaluza sequence, and satisfies the assumptions of Proposition 2.3, with $n\beta_n \leq 1$. Kronecker's lemma yields that $\sum_{n\geq 0} \beta_n = \infty$.

We assume that $\{\beta_n\}$ and $\{\alpha_n\}$ satisfy all the assumptions of Proposition 2.3, and also that $\sum_{n\geq 0} \beta_n = \infty$. In this case, $\sum_{n\geq 0} \alpha_n = G(1) = 0$, so $G(T) = I + \sum_{n\geq 1} \alpha_n T^n = \sum_{n\geq 1} \alpha_n (T^n - I)$; hence $G(T)X \subset \overline{(I-T)X}$. Now the following results of [9] hold, with the same proofs.

Lemma 2.4. *Let T be a power-bounded operator on a Banach space X and let $x \in X$. If $\liminf_{n\to\infty} \left\|\sum_{k=1}^n \beta_k T^k x\right\| < \infty$, then $x \in \overline{(I-T)X}$.*

THEOREM 2.5. *Let T be a power-bounded operator on a Banach space X and let $x \in X$. Then the following are equivalent*:

(i) *There is an increasing $\{n_j\}$ such that $\sum_{k=1}^{n_j} \beta_k T^k x$ converges weakly.*
(ii) *The series $\sum_{k=1}^{\infty} \beta_k T^k x$ converges weakly.*
(iii) *The series $\sum_{k=1}^{\infty} \beta_k T^k x$ converges in norm.*

PROPOSITION 2.6. *Let T be power-bounded on X.*

(i) *The series $\sum_{k=1}^{\infty} \beta_k T^k x$ converges if and only if $x \in G(T)[\overline{(I-T)X}]$.*
(ii) *If T is mean ergodic, $\sum_{k=1}^{\infty} \beta_k T^k x$ converges if and only if $x \in G(T)X$.*

Proof. When T is mean ergodic, $G(T)[\overline{(I-T)X}] = G(T)X$ by the ergodic decomposition, since $G(1) = 0$ implies $G(T)y = 0$ whenever $Ty = y$. ■

COROLLARY 2.7. *Let T be power-bounded on X. Then*

$$(I-T)X \subset G(T)[\overline{(I-T)X}] \subset G(T)X \subset \overline{(I-T)X}.$$

PROPOSITION 2.8. *Let $\{\beta_n\}$ and its associated sequence $\{\alpha_n\}$ satisfy the hypotheses of Proposition* 2.3, *and assume that $\sum_{n=1}^{\infty} \beta_n = \infty$. Then the operator*

$$H(T)x = \lim_{n\to\infty} H_n(T)x$$

is a closed operator, with domain dense in $\overline{(I-T)X}$.

Proof. The domain of $H(T)$ is dense in $\overline{(I-T)X}$ by Proposition 2.6 and the previous corollary.

Let $\{x_k\} \in G(T)\overline{(I-T)X}$ with $x_k \to x$ and $H(T)x_k \to y$. By Proposition 2.3 $x_k = H(T)G(T)x_k = G(T)H(T)x_k$, so $x = \lim_k x_k = G(T)y$ by continuity of $G(T)$. Since $y \in \overline{(I-T)X}$, we conclude that x is in the domain of $H(T)$, and then $H(T)x = H(T)G(T)y = y$, which yields that $H(T)$ is closed. ■

LEMMA 2.9. *Let $\{\beta_n\}_{n\ge 0}$ be a monotone non-increasing sequence of positive numbers with $\sum_{n=0}^{\infty} \beta_n = \infty$ and put $H(z) = \sum_{k=0}^{\infty} \beta_k z^k$ for $|z| < 1$. Then*

$$H\Big(1 - \frac{1}{n}\Big) \asymp \sum_{k=0}^{n} \beta_k. \tag{10}$$

Proof. In one direction, for $n \ge 2$ we have

$$\sum_{k=0}^{\infty} \beta_k \Big(1-\frac{1}{n}\Big)^k \ge \sum_{k=0}^{n} \beta_k \Big(\frac{n-1}{n}\Big)^k \ge \frac{n-1}{n}\Big(\frac{n-1}{n}\Big)^{n-1} \sum_{k=0}^{n} \beta_k \ge \frac{1}{2e} \sum_{k=0}^{n} \beta_k.$$

Above we have used only positivity of $\{\beta_k\}$. For the converse we use also the monotonicity.

$$\sum_{k=n}^{\infty} \beta_k \Big(1-\frac{1}{n}\Big)^k \le \beta_n \sum_{k=n}^{\infty} \Big(1-\frac{1}{n}\Big)^k = \beta_n n \Big(1-\frac{1}{n}\Big)^n \le n\beta_n.$$

Hence $H(1-\frac{1}{n}) \le \sum_{k=0}^{n-1} \beta_k + n\beta_n \le 2\sum_{k=0}^{n-1} \beta_k$. ■

Using the previous lemma, we obtain the following restatement of a result of Gomilko, Haase and Tomilov [27].

THEOREM 2.10. *Let $\{\beta_n\}_{n\geq 0}$ be a Kaluza sequence with $\sum_{n=0}^\infty \beta_n = \infty$ and $\beta_n \to 0$. Let T be a power-bounded operator on a Banach space X. If for $x \in X$ the series $\sum_{k=1}^n \beta_k T^k x$ converges weakly, then $\big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\big\| = o(1/\sum_{k=0}^n \beta_k)$.*

REMARKS.

1. Since $\beta_n \downarrow 0$, strong convergence of $\sum_{n=0}^\infty \beta_n T^n x$ implies, by Kronecker's lemma, that $\big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\big\| = o(1/n\beta_n)$. This yields no rate when the EHT converges ($\beta_n = 1/(n+1)$), while Theorem 2.10 yields a rate of $1/\log n$. On the other hand, when $\beta_n = 1/(n+1)^\gamma$ with $0 < \gamma < 1$, Kronecker's lemma and Theorem 2.10 yield the same rate.

2. Without the assumption that $\beta_n \to 0$ (which yields the condition $\frac{1}{n}\sum_{k=0}^n \beta_k \to 0$ needed in [27]), it is proved there that

$$\Big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\Big\| \leq \frac{2\mathrm{e}\sup_j \|T^j\|}{H(1-1/n)}\|y\| \leq \frac{4\mathrm{e}^2 \sup_j \|T^j\|}{\sum_{k=0}^n \beta_k}\|y\|. \tag{11}$$

The following theorem was proved in [10] for the one-sided EHT, using the connection with the semi-group $\{(I-T)^r\}_{r\geq 0}$ restricted to $\overline{(I-T)X}$ [18]. The proof below for the general case uses Theorem 2.10.

THEOREM 2.11. *Let $\{\beta_n\}_{n\geq 0}$ be a Kaluza sequence with $\sum_{n=0}^\infty \beta_n = \infty$ and $\beta_n \to 0$. Then the following are equivalent for a power-bounded operator T on a Banach space X:*

(i) *$(I-T)X$ is closed in X.*
(ii) *For $S = T_{|\overline{(I-T)X}}$ the series $\sum_{n=0}^\infty \beta_n S^n$ converges (uniformly) in operator norm.*
(iii) *$\sum_{n=0}^\infty \beta_n T^n x$ converges in norm for every $x \in \overline{(I-T)X}$.*
(iv) *$\sum_{n=0}^\infty \beta_n T^n x$ converges weakly for every $x \in \overline{(I-T)X}$.*

Proof. (i) implies (ii) by simple computation, using monotonicity of $\{\beta_n\}$. The implications from (ii) to (iv) are obvious.

Assume (iv). Let $Y := \overline{(I-T)X}$. For $x \in Y$ let

$$y = weak - \lim_{n\to\infty} \sum_{k=0}^n \beta_k S^k x = weak - \lim_n H_n(T)x$$

which exists by (iv). By Proposition 2.3 and weak continuity of $G(T)$ we have $x = \lim_n G(T)H_n(T)x = G(T)y$. By (11) we have

$$\Big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\Big\| \leq \frac{2\mathrm{e}\sup_j \|T^j\|}{H(1-1/n)}\,\|y\|.$$

This holds for every $x \in Y$, so by the Banach–Steinhaus theorem

$$\sup_n H(1-1/n)\Big\|\frac{1}{n}\sum_{k=0}^{n-1} S^k\Big\| < \infty.$$

Since $\lim_n H(1-1/n) = \sum_n \beta_n = \infty$, we obtain $\big\|\frac{1}{n}\sum_{k=0}^{n-1} S^k\big\| \to 0$. Hence for large n we have $I_Y - \frac{1}{n}\sum_{k=0}^{n-1} S^k$ invertible on Y, so $Y = (I_Y - S)Y \subset (I-T)X \subset Y$. Hence $(I-T)X = Y$ is closed. ∎

REMARKS.

1. The condition $(I-T)X$ closed is equivalent to uniform (operator norm) convergence of $\frac{1}{n}\sum_{k=0}^{n-1} T^k$ [38].

2. It seems that the proof of [27] is valid also if $\{\alpha_n\}$ are only eventually negative, and then the previous theorem will be true also under our general assumptions on $\{\beta_n\}$ (with divergent sum) as in Proposition 2.3.

For the next result, which shows the optimality of the rate in Theorem 2.10, we need more precise estimates on the relations between $B_n := \sum_{k=0}^n \beta_k$, $H_n(z) := \sum_{k=0}^n \beta_k z^k$, and the series $H(z) = \sum_{k=0}^\infty \beta_k z^k$ (which converges for $|z| \le 1$, $z \ne 1$ when $\beta_n \to 0$ monotonically). As before, we put $G(z) = \frac{1}{H(z)} = 1 + \sum_{n=1}^\infty \alpha_n z^n$. We assume $\{\beta_n\}$ to be a Kaluza sequence, so $\alpha_n \le 0$ for $n \ge 1$.

For every $x > 0$ define the functions

$$\psi(x) = \sum_{n=0}^{[1/x]} \beta_n = B_{[1/x]} \tag{12}$$

$$\chi(x) = x\bigg(\sum_{n=0}^{[1/x]} n|\alpha_n|\bigg) + 2\sum_{n>1/x} |\alpha_n|. \tag{13}$$

LEMMA 2.12. *Let $\{\beta_n\}_{n\ge0}$ be a Kaluza sequence with $B_n \to \infty$ and $\beta_n \to 0$. Then there exists $C > 1$ such that for every $z \in \overline{D} \setminus \{1\}$ we have*

$$\frac{1}{\chi(|1-z|)} \le |H(z)| \le C\psi(|1-z|). \tag{14}$$

Proof. Let $z \in \overline{D} \setminus \{1\}$. Write $S_n = S_n(z) = \sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}$. Since $\beta_n \to 0$ and $|S_n(z)| \le 2/|1-z|$, Abel summation by parts yields

$$\begin{aligned}
|H(z)| \le \sum_{0\le n\le 1/|1-z|} \beta_n &+ \Big|\sum_{n>1/|1-z|} \beta_n(S_{n+1} - S_n)\Big| \\
&\le \psi(|1-z|) + \frac{2\beta_{[1/|1-z|]}}{|1-z|} + \Big|\sum_{n>1/|1-z|} (\beta_n - \beta_{n+1})S_{n+1}\Big| \\
&\le \psi(|1-z|) + \frac{4\beta_{[1/|1-z|]}}{|1-z|} \le C\psi(|1-z|),
\end{aligned}$$

where we used the fact that $\beta_n \to 0$ monotonically and $(n+1)\beta_n \le B_n$.

Let us prove the second inequality. Since $B_n \to \infty$, we have $\sum_{n=1}^\infty \alpha_n = -1$, and

$$\begin{aligned}
|G(z)| &= \Big|\sum_{n\ge1} \alpha_n(z^n - 1)\Big| \\
&\le |1-z| \sum_{1\le n\le 1/|1-z|} n|\alpha_n| + 2\sum_{n>1/|1-z|} |\alpha_n| = \chi(|1-z|). \ \blacksquare
\end{aligned}$$

LEMMA 2.13. *Let $\Delta \subset \overline{D} \setminus \{1\}$ be such that 1 is an accumulation point of Δ. Then there exist $\delta > 0$ and an increasing sequence of integers $\{n_k\}$ such that for every $k \ge 1$ there exists $z_k \in \Delta$ satisfying $\frac{1}{n_k+1} < |1-z_k| \le \frac{1}{n_k}$ and $|1 - z_k^{n_k}| \ge \delta$.*

Proof. By assumption, there exists an increasing sequence of integers $\{n_k\}$ and a sequence $\{z_k\} \subset \Delta$ such that $\frac{1}{n_k+1} < |1 - z_k| \le \frac{1}{n_k}$. Write $z_k = r_k e^{i\theta_k}$, with $0 < r_k \le 1$ and $-\pi \le \theta_k \le \pi$. Actually we may and do assume that $r_k \ge \frac{1}{2}$ and $-\pi/2 \le \theta_k \le \pi/2$.

Assume that $1 - r_k \ge \frac{1}{4n_k}$. Since $1 - r_k \le |1 - z_k| \le \frac{1}{n_k}$, it follows from Lemma 5.1 of Gomilko–Haase–Tomilov [27] that

$$|1 - z_k^{n_k}| \ge 1 - |z_k|^{n_k} \ge \frac{1}{4e}\,.$$

Assume now that $0 \le 1 - r_k < \frac{1}{4n_k}$. We have

$$1 - 2r_k \cos\theta_k + r_k^2 = |1 - z_k|^2 \ge \frac{1}{(n_k+1)^2} \ge \frac{1}{4n_k^2}\,.$$

But,

$$1 - 2r_k \cos\theta_k + r_k^2 = 2r_k(1 - \cos\theta_k) + (1 - r_k)^2 \le 2(1 - \cos\theta_k) + \frac{1}{16n_k^2}\,.$$

Hence,

$$1 - \cos\theta_k \ge \frac{3}{32n_k^2}\,.$$

On the other hand, we have

$$\frac{|\sin\theta_k|}{2} \le r_k|\sin\theta_k| = |\mathrm{Im}(1 - z_k)| \le \frac{1}{n_k}\,.$$

Using $1 - \cos t \le \frac{t^2}{2}$, $\forall t \in \mathbb{R}$, and $|\sin t| \ge \frac{2|t|}{\pi}$, $\forall t \in [-\pi/2, \pi/2]$, we obtain

$$\frac{\sqrt{3}}{4n_k} \le |\theta_k| \le \frac{\pi}{n_k}\,.$$

Hence

$$\frac{\sqrt{3}}{4} \le n_k|\theta_k| \le \pi,$$

and

$$|1 - z_k^{n_k}| \ge \inf_{r\in[0,1],|\theta|\in[\frac{\sqrt{3}}{4},\pi]} |1 - re^{i\theta}| > 0,$$

which finishes the proof. ∎

THEOREM 2.14. *Let (β_n) be a Kaluza sequence with divergent sum and $\beta_n \to 0$. Let T be a power-bounded operator on a Banach space X, such that 1 is an accumulation point of $\sigma(T)$. Then for every sequence $\{\varepsilon_n\}$ tending to 0, there exists $x \in X$ such that $\sum_{n=0}^\infty \beta_n T^n x$ converges and $\big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\big\| \ne O(\varepsilon_n/\sum_{k=0}^n \beta_k)$.*

Proof. Let $\Delta = \sigma(T) \setminus \{1\}$ and $\{n_k\}$ and $\{z_k\}$ be as obtained in the previous lemma. Put $A_n(z) = \frac{1}{n}\sum_{k=0}^{n-1} z^k$. By the "spectral inclusion theorem" of [27] (see also [21, Lemma 2.28]), $\{A_n(z)G(z) : z \in \sigma(T)\} \subset \sigma(A_n(T)G(T))$. Hence

$$\begin{aligned}\|A_{n_k}(T)G(T)\| &\ge \sup\{|z| : z \in \sigma(A_{n_k}(T)G(T)\}\\ &\ge \sup_{z\in\sigma(T)} |A_{n_k}(z)G(z)| \ge |A_{n_k}(z_k)|\,|G(z_k)| = \frac{|(1 - z_k^{n_k})G(z_k)|}{n_k|1 - z_k|} \ge \delta|G(z_k)|.\end{aligned}$$

Now, by the right-hand side inequality of (14), we have

$$|G(z_k)| = \frac{1}{|H(z_k)|} \geq \frac{1}{C\psi(|1-z_k|)} \geq \frac{1}{C\psi(\frac{1}{n_k+1})} = \frac{1}{CB_{n_k+1}} \geq \frac{1}{2CB_{n_k}},$$

using the simple inequality $B_{n+1} = B_n + \beta_{n+1} \leq B_n + 1 \leq 2B_n$. Hence for every sequence $\{\varepsilon_n\}$ decreasing to 0,

$$\sup_{n\geq 1} \frac{B_n}{\varepsilon_n} \|A_n(T)G(T)\| = +\infty,$$

and we conclude by the Banach–Steinhaus theorem. ∎

Remarks.

1. The result was proved in [27] assuming a "non-tangential" approach to 1 in $\sigma(T)$. Our proof is along the same lines.

2. If $(I-T)X$ is closed, then for every $x \in (I-T)X$ the series $\sum_n \beta_n T^n x$ converges, but we have the rate $\|A_n(T)x\| = O(\varepsilon_n/B_n)$ with $\varepsilon_n = \frac{1}{n}B_n \to 0$ (assuming $\beta_n \to 0$). This is because 1 is not only isolated in $\sigma(T)$, but is not in the spectrum of the restriction of T to $\overline{(I-T)X}$. For $\beta_n = 1/(n+1)^{1-\alpha}$, $\alpha \in (0,1)$, we have $B_n \sim Cn^\alpha$, and with $\varepsilon_n = \frac{1}{n^\gamma} B_n$, $\gamma \in (\alpha, 1)$, we obtain even $\|A_n(T)x\| = o(\varepsilon_n/B_n)$.

3. Convergence of $\sum_n \beta_n T^n x$ yields $n\beta_n\|A_n(T)x\| \to 0$, by Kronecker's lemma. For $\beta_n = 1/(n+1)^{1-\alpha}$, $\alpha \in (0,1)$, we obtain the rate of [18, Corollary 2.15]; by Theorem 2.14 this rate is optimal. For $\beta_n = 1/(n+1)$ Theorem 2.10 gives the rate $1/\log n$, while $n\beta_n \to 1$ yields no rate.

Proposition 2.15. *Let $X = Y^*$ be a dual Banach space and let $T = S^*$ be a power-bounded dual operator on X. If*

$$\liminf_{n\to\infty} \left\| \sum_{k=1}^{n} \beta_k T^k x \right\| < \infty, \tag{15}$$

then $x \in G(T)X$. If in addition T is mean ergodic, then $\sum_{k=1}^{\infty} \beta_k T^k x$ converges in norm.

The proof is similar to that in [9] for the one-sided EHT.

Corollary 2.16. *Let T be a power-bounded operator on a Banach space X such that T^{**} is mean ergodic on X^{**}. Then $\sum_{k=1}^{\infty} \beta_k T^k x$ converges if and only if*

$$\liminf_{n\to\infty} \left\| \sum_{k=1}^{n} \beta_k T^k x \right\| < \infty.$$

Corollary 2.17. *Let T be a power-bounded operator on a reflexive Banach space. Then $\sum_{k=1}^{\infty} \beta_k T^k x$ converges in norm if and only if*

$$\liminf_{n\to\infty} \left\| \sum_{k=1}^{n} \beta_k T^k x \right\| < \infty.$$

3. Convergence of some power series of L_1 contractions. Lin and Sine [39] proved that for T a contraction in $L_1(S, \Sigma, \mu)$ of a σ-finite measure space, a function f is in $(I-T)L_1$ if (and obviously only if) $\sup_n \|\sum_{k=1}^{n} T^k f\|_1 < \infty$. We use their method to extend Corollary 2.17 to contractions of L_1. Of course the problem is only when $\sum_{n\geq 0}\beta_n = \infty$.

Recall that for a complete finite measure space, $L_\infty^* = L_1^{**}$ is identified with the space $ba(S,\Sigma,\mu)$ of bounded finitely additive (signed) measures, called charges (see [22, IV.8.16]), and by the canonical embedding L_1 is identified with the space $M(S,\Sigma,\mu)$ of countably additive signed measures absolutely continuous with respect to μ. A charge $\eta \in ba(S,\Sigma,\mu)$ is called a "pure charge" if $|\eta|$ does not bound any non-negative measure, and then $\|\eta+\nu\| = \|\eta\|+\|\nu\|$ for any countably additive ν. Every $\eta \in L_1^{**}$ can be decomposed as $\eta = \eta_1 + \eta_2$ with η_1 countably additive and η_2 a pure charge [43].

Recall that when $\sum_{n\geq 0}\beta_n = \infty$ we have, under the assumptions of Proposition 2.1, $\sum_{n\geq 1}|\alpha_n| \geq -\sum_{n\geq 1}\alpha_n = 1$.

Theorem 3.1. *Let $\{\beta_n\}$, $\{\alpha_n\}$ be as in Proposition 2.3, and assume that $\sum_{n\geq 0}\beta_n = \infty$ and $\alpha_n \leq 0$ for $n \geq 1$. Let T be a contraction of $L_1(S,\Sigma,\mu)$. Then $f \in G(T)L_1$ if and only if $\liminf_n \|\sum_{k=0}^n \beta_k T^k f\|_1 < \infty$.*

When T is also mean ergodic, the series $\sum_{k=0}^\infty \beta_k T^k f$ converges in norm if and only if $\liminf_n \|\sum_{k=0}^n \beta_k T^k f\|_1 < \infty$.

Proof. If $f \in G(T)L_1$, then $\sup_n \|H_n(T)f\|_1 < \infty$ by Proposition 2.2.

Assume now that $\liminf_n \|\sum_{k=0}^n \beta_k T^k f\|_1 < \infty$. Put $F(z) := \sum_{n=1}^\infty \alpha_n z^n$, hence $G(T) = I + F(T)$, and $\|F(T)\| \leq 1$ by the assumptions. We identify $f \in L_1$ with the measure it defines. We apply Proposition 2.15 to T^{**} and obtain an element $\eta \in L_1^{**}$ with $G(T^{**})\eta = f$. We decompose $\eta = \eta_1+\eta_2$ with η_1 countably additive and η_2 a pure charge. Then

$$f = G(T^{**})\eta = [I + F(T^{**})](\eta_1+\eta_2) = \eta_1 + F(T^{**})\eta_1 + \eta_2 + F(T^{**})\eta_2.$$

Since $F(T^{**})\eta_1 = F(T)\eta_1 \in M(S,\Sigma,\mu)$, we deduce that $\nu_1 := \eta_2 + F(T^{**})\eta_2$ is countably additive. Using $\|F(T^{**})\| \leq \sum_{n\geq 1}|\alpha_n| = 1$ and $\nu_1 \perp \eta_2$ we obtain

$$\|\eta_2\| \geq \|F(T^{**})\eta_2\| = \|\nu_1 - \eta_2\| = \|\nu_1\| + \|\eta_2\|.$$

Hence $\nu_1 = 0$ and $f = G(T^{**})\eta_1 = G(T)\eta_1$ and $\eta_1 << \mu$, i.e., $f = G(T)g$ with $g = \frac{d\eta_1}{d\mu}$.

If T is also mean ergodic, $G(T)L_1 = G(T)\overline{(I-T)L_1}$ and Proposition 2.6 yields the convergence of the series when the partial sums are bounded. ∎

Corollary 3.2. *Let $\{\beta_n\}$ be a Kaluza sequence with $\beta_n \to 0$. If T is a mean ergodic contraction on L_1, then $\sum_{k=0}^\infty \beta_k T^k f$ converges in norm if and only if*

$$\liminf_n \Big\|\sum_{k=0}^n \beta_k T^k f\Big\|_1 < \infty.$$

When applied to $\{\beta_n\}$ of Example 2(i), we obtain positive answer to a question posed in [9]: for T a mean ergodic contraction of L_1

$$\sum_{n\geq 1}\frac{T^n f}{n} \quad \text{converges in norm} \quad \Longleftrightarrow \quad \liminf_n \Big\|\sum_{k=0}^n \frac{T^k f}{k}\Big\|_1 < \infty.$$

Remark. It is crucial in the proof that $\alpha_n \leq 0$ for *every* $n \geq 1$, since the proof depends on $\|F(T)\| \leq 1$.

4. Power series of Hilbert space contractions. In this section we give a spectral characterization for the convergence of $\sum_{n=0}^\infty \beta_n T^n x$ when T is a normal contraction in a Hilbert space $\mathcal{H}$ and $\{\beta_n\}$ is a Kaluza sequence, as defined in Example 2. We then apply the results to general contractions in a Hilbert space, using their unitary dilations. To avoid trivialities, we assume that $\sum_{n=0}^\infty \beta_n = \infty$.

Let $\{\beta_n\}_{n\ge 0}$ be a positive sequence decreasing monotonically to 0. By boundedness of $\{\beta_n\}$, the function $H(z) = \sum_{n=0}^\infty \beta_n z^n$ is defined for $|z| < 1$. The monotone convergence to 0 of β_n yields, by Abel summation, that also for $z \ne 1$ with $|z| = 1$ the series $H(z) = \sum_{n=0}^\infty \beta_n z^n$ converges [44, Vol. I, p. 182]. The same holds if $\{\beta_n\}$ is of bounded variation, with $\beta_n \to 0$. At $z = 1$ we have $H(1) = \infty$ by assumption.

Lemma 4.1. *Let $\{\beta_n\}$ be a Kaluza sequence with divergent sum and $\beta_n \to 0$. Then $H(z) = \sum_{n=0}^\infty \beta_n z^n$ is defined for $|z| \le 1$ with $z \ne 1$, and there exists $C > 0$ such that*

$$\sup_{n\ge 0}\Big|\sum_{k=0}^n \beta_k z^k\Big| \le C|H(z)| < \infty \quad (|z|\le 1,\ z\ne 1). \tag{16}$$

Proof. Since $\{\beta_n\}$ is a bounded Kaluza sequence, it is monotone, so the assumption $\beta_n \to 0$ yields that $\sum_{n=0}^\infty \beta_n z^n$ converges for $|z| = 1$, $z \ne 1$, with the limit denoted by $H(z)$. As noted in Example 2, $\{\beta_n\}$ satisfies the hypotheses of Proposition 2.3, and $G(z) = 1/H(z)$ extends to the boundary $|z| = 1$. By continuity we see that $G(z) \ne 0$ for $z \ne 1$.

For $|z| \le 1$ we apply Proposition 2.2 to the multiplication by z in $\mathbb{C}$ and obtain $\sup_{n\ge 0}\big|G(z)\sum_{k=0}^n \beta_k z^k\big| \le C$. Since $G(z) = \frac{1}{H(z)} \ne 0$ for $z \ne 1$, the assertion follows. ∎

Theorem 4.2. *Let $\{\beta_n\}$ be a Kaluza sequence with $\sum_{n=0}^\infty \beta_n = \infty$ and $\beta_n \to 0$. Let T be a normal contraction on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ with scalar spectral measure σ_x. Then*

$$\sum_{n=0}^\infty \beta_n T^n x \quad \textit{converges} \quad \iff \quad \int_{\sigma(T)} |H(z)|^2\, d\sigma_x(z) < \infty, \tag{17}$$

$$\sum_{n=0}^\infty \beta_n T^n x \quad \textit{converges} \quad \iff \quad \sum_{n=0}^\infty \beta_n T^{*n} x \quad \textit{converges.} \tag{18}$$

Proof. By the previous lemma, $H(z) = \sum_{n=0}^\infty \beta_n z^n$ converges for $z \ne 1$ with $|z| \le 1$.

Let T be a normal contraction in $\mathcal{H}$, and $x \in \mathcal{H}$.

Assume that $\sum_{n=0}^\infty \beta_n T^n x$ converges. Then by Lemma 2.4, $x \in \overline{(I-T)\mathcal{H}}$, so $\sigma_x(\{1\}) = 0$, and thus $H(z)$ is finite σ_x-a.e. on $\sigma(T)$. By Fatou's lemma and the spectral theorem,

$$\int_{\sigma(T)} |H(z)|^2\, d\sigma_x(z) \le \liminf_{n\to\infty} \int_{\sigma(T)} \Big|\sum_{k=0}^n \beta_k z^k\Big|^2 d\sigma_x(z) = \liminf_{n\to\infty} \Big\|\sum_{k=0}^n \beta_k T^k x\Big\|^2 < \infty.$$

Assume now that $\int_{\sigma(T)} |H(z)|^2\, d\sigma_x(z) < \infty$. Since $H(1) = \sum_{n=0}^\infty \beta_n = \infty$ by assumption, we have $\sigma_x(\{1\}) = 0$. By (16) we have $\sup_n \big|\sum_{k=0}^n \beta_k z^k\big| \le C|H(z)|$ for every $|z| \le 1$, with $|H(z)| < \infty$ for $z \neq 1$. Hence

$$\sup_n \Big\|\sum_{k=0}^n \beta_k T^k x\Big\|^2 = \sup_n \int_{\sigma(T)} \Big|\sum_{k=0}^n \beta_k z^k\Big|^2 \, d\sigma_x(z)$$
$$\le \int_{\sigma(T)} \sup_n \Big|\sum_{k=0}^n \beta_k z^k\Big|^2 \, d\sigma_x(z) \le C \int_{\sigma(T)} |H(z)|^2 \, d\sigma_x(z) < \infty.$$

By Corollary 2.17 the series $\sum_{n=0}^\infty \beta_n T^n x$ converges in norm.

By normality $\big\|\sum_{n=k}^m \beta_n T^n x\big\| = \big\|\sum_{n=k}^m \beta_n T^{*n} x\big\|$, which yields (18). ∎

Remarks.

1. The theorem applies also to $\{\beta_n\}$ satisfying the assumptions of Proposition 2.3.
2. By Proposition 2.6, (18) is equivalent to $G(T)H = G(T^*)H$.

Proposition 4.3. *Let $\{\beta_n\}$ be a Kaluza sequence with $B_n := \sum_{k=0}^n \beta_k \to \infty$ and $\beta_n \to 0$. Let T be a normal contraction on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ with scalar spectral measure σ_x. If*

$$\sum_{n=1}^\infty \frac{\beta_n B_n \|\sum_{k=1}^n T^k x\|^2}{n^2} < \infty \tag{19}$$

then $\sum_{n=0}^\infty \beta_n T^n x$ converges.

Proof. We shall use the spectral condition of (17). Following [10], for $n \ge 1$ put

$$D_n := \Big\{z = re^{2\pi i\theta} \in \overline{D} : 1 - \frac{1}{n} < r \le 1,\ |\theta| \le \frac{1}{2n}\Big\}.$$

Then $\{D_n\}$ is decreasing, $D_1 \cup \{0\} = \overline{D}$, and $\bigcup_{n\ge1}(D_n \setminus D_{n+1}) = \overline{D} \setminus \{1\}$. It is shown in the proof of [10, Proposition 3.2] that for T and x we have $\sigma_x(D_n) \le \frac{36}{n^2}\big\|\sum_{k=1}^n T^k x\big\|^2$.

The function $|H(z)|$ is continuous on $\overline{D_1 \setminus D_2}$, so it is bounded. Next, $|1-z| \ge 1/(n+1)$ for $n > 1$ and $z \in D_n \setminus D_{n+1}$, so by (14) $|H(z)| \le C\psi(|1-z|) \le C\psi(1/(n+1)) = CB_{n+1}$. (19) implies that $\frac{1}{n}\sum_{k=1}^n T^k x \to 0$, since $\sum_{n=0}^\infty \beta_n B_n = \infty$. Hence $\sigma_x(\{1\}) = 0$, and

$$\int_{\sigma(T)} |H(z)|^2 \, d\sigma_x(z) = \int_{\overline{D}\setminus\{1\}} |H(z)|^2 \, d\sigma_x(z) = \sum_{n=1}^\infty \int_{D_n \setminus D_{n+1}} |H(z)|^2 \, d\sigma_x(z)$$
$$\le \max_{z \in D_1 \setminus D_2} |H(z)|^2 \|x\|^2 + C \sum_{n=2}^\infty B_{n+1}^2 \big(\sigma_x(D_n) - \sigma_x(D_{n+1})\big).$$

We prove the convergence of the last series, which will prove the assertion.

$$\sum_{n=1}^N B_{n+1}^2 \big(\sigma_x(D_n) - \sigma_x(D_{n+1})\big)$$
$$= B_2^2 \sigma_x(D_1) + \sum_{n=2}^{N-1} \sigma_x(D_n)(B_{n+1}^2 - B_n^2) - B_{N+1}\sigma_x(D_{N+1})$$
$$\le B_2^2 \|x\|^2 + \sum_{n=2}^{N-1} 2B_{n+1}\beta_{n+1}\sigma_x(D_n) \le B_2^2\|x\|^2 + \sum_{n=2}^{N-1} 2B_{n+1}\beta_{n+1}\frac{36}{n^2}\Big\|\sum_{k=1}^n T^k x\Big\|^2$$

using the estimate from [10], and the last series converges by (19). ∎

For the next results we need additional estimates on the relations between $B_n := \sum_{k=0}^n \beta_k$, $H_n(z) := \sum_{k=0}^n \beta_k z^k$, and the series $H(z) = \sum_{k=0}^\infty \beta_k z^k$ (which converges for $|z| \le 1$, $z \neq 1$ when $\beta_n \to 0$ monotonically). As before, we put $G(z) = H^{-1}(z) = 1+\sum_{n=1}^\infty \alpha_n z^n$. We assume $\{\beta_n\}$ to be Kaluza, so $\alpha_n \le 0$ for $n \ge 1$. We use the functions $\psi(x)$ and $\chi(x)$ defined in Section 2 (before Lemma 2.12).

Lemma 4.4. *Let $\{\beta_n\}_{n\ge 0}$ be as in Lemma 2.12. Then there exists $K > 1$, such that for every $t \in [0,1)$,*

$$\frac{\psi(1-t)}{K} \le H(t) \le \frac{K}{\chi(1-t)}\,. \tag{20}$$

Proof. It follows from Theorem 5.2 of [27] that there exists $K > 0$ such that

$$H(t) \le \frac{K}{\chi(1-t)}\,. \tag{21}$$

Let us prove the first inequality of (20). Since ψ is monotone and H does not vanish and is continuous, we only need to prove the existence of $K > 0$ such that the left-hand side of (20) is satisfied for t close to 1. Hence, we may and do assume that $t \in [1/2, 1)$. Then for $t \in [1/2, 1)$ we have

$$H(t) \ge t^{1/(1-t)} \sum_{n=0}^{[1/(1-t)]} \beta_n \ge \frac{1}{4}\,\psi(1-t). \ \blacksquare$$

Corollary 4.5. *Let $\{\beta_n\}_{n\ge 0}$ be a Kaluza sequence with $B_n \to \infty$ and $\beta_n \to 0$. Then there exists $L > 1$ such that for every $z \in \overline{D} \setminus \{1\}$*

$$\frac{\psi(|1-z|)}{L} \le |H(z)| \le L\psi(|1-z|). \tag{22}$$

Proof. By (20) we have $\chi(1-t)\psi(1-t) \le K^2$ for $t \in [0,1)$. Hence for $|1-z| < 1$ Lemma 2.12 yields

$$|H(z)| = \frac{1}{|G(z)|} \ge \frac{1}{\chi(|1-z|)} \ge \frac{\psi(|1-z|)}{K^2}\,.$$

For $1 \le |1-z| \le 2$ we have $\psi(|1-z|) \le B_1$, and in that region the non-vanishing $|H(z)|$ has positive minimum.

The other inequality is proved in Lemma 2.12 — see (14). $\blacksquare$

Proposition 4.6. *Let (β_n) be a Kaluza sequence with $\beta_n \to 0$ such that $\sum_{k\ge n} \frac{\beta_k B_k}{k^2} = O(\frac{\beta_n B_n}{n})$. Let T be a contraction on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ be such that $\sum_{n\ge 0} \beta_n T^n x$ converges. Then*

$$\sum_{n=1}^\infty \frac{\beta_n B_n \| \sum_{k=1}^n T^k x\|^2}{n^2} < \infty.$$

Proof. We start by showing that there exists $C > 0$ such that

$$\sum_{n\ge 1} \beta_n B_n \frac{|1+\ldots+z^{n-1}|^2}{n^2} \le C|H(z)|^2 \qquad \forall z \in \overline{D} \setminus \{1\}. \tag{23}$$

Let $z \in \overline{D} \setminus \{1\}$, and $n' = [1/|1-z|]$. Using $n\beta_n \le B_n$ by monotonicity and the estimate (22) we obtain

$$\sum_{n\ge 1} \beta_n B_n \frac{|1+\ldots+z^{n-1}|^2}{n^2} \le \sum_{n=1}^{n'-1} \beta_n B_n + \frac{2}{|1-z|^2} \sum_{n\ge n'} \frac{\beta_n B_n}{n^2}$$
$$\le \psi^2(|1-z|) + \frac{2}{|1-z|^2} C \frac{\beta_{n'} B_{n'}}{n'} \le D\psi^2(|1-z|) \le DL|H(z)|^2.$$

Now let T be a contraction on $\mathcal{H}$ and $x \in \mathcal{H}$ with $\sum_{n=0}^{\infty} \beta_n T^n x$ convergent. By Proposition 2.6, there exists $y \in \mathcal{H}$ such that $x = G(T)y$. Let U be the unitary dilation of T, defined on $\mathcal{H}_1 \supset \mathcal{H}$, with P the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}$. Then $G(T)x = PG(U)x$. Let σ_y be the scalar spectral measure of y with respect to U, defined on the Borel subsets of the unit circle $\mathbb{T}$. Then, by (23),

$$\sum_{n\ge 1} \beta_n B_n \frac{\|\sum_{k=1}^n T^k x\|^2}{n^2} = \sum_{n\ge 1} \beta_n B_n \frac{\|\sum_{k=1}^n T^k G(T)y\|^2}{n^2}$$
$$\le \sum_{n\ge 1} \beta_n B_n \frac{\|\sum_{k=1}^n U^k G(U)y\|^2}{n^2}$$
$$= \int_{\mathbb{T}} \sum_{n\ge 1} \beta_n B_n \frac{|\sum_{k=0}^{n-1} \mathrm{e}^{ik\theta}|^2}{n^2} |G(\mathrm{e}^{i\theta})|^2 \, d\sigma_y(\theta) \le C\|y\|^2. \ \blacksquare$$

REMARKS.

1. For normal contractions the result follows directly from (23) and the spectral theorem.
2. For a Kaluza sequence $\beta_n \le 1$, so $\beta_n B_n \le B_n \le n$.

COROLLARY 4.7. *Let (β_n) be a Kaluza sequence with $\beta_n \to 0$ such that $\sum_{k\ge n} \frac{\beta_k B_k}{k^2} = O(\frac{\beta_n B_n}{n})$. Let T be a normal contraction on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. Then*

$$\sum_{n=0}^{\infty} \beta_n T^n x \quad \textit{converges} \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{\beta_n B_n \|\sum_{k=1}^n T^k x\|^2}{n^2} < \infty. \tag{24}$$

Proof. Combine Propositions 4.3 and 4.6. $\blacksquare$

REMARKS.

1. For the one-sided EHT, to which the corollary applies, the result is in [10].
2. The corollary applies to $\beta_n = 1/(n+1)^{1-\alpha}$ with $\alpha \in (0,1)$; see [16, Section 2] for another approach.
3. If $\{n^\gamma \beta_n B_n\}$ decreases for some $0 \le \gamma < 1$, we have

$$\sum_{k\ge n} \frac{\beta_k B_k}{k^2} \le n^\gamma \beta_n B_n \sum_{k\ge n} \frac{1}{k^{2+\gamma}} \le C \frac{\beta_n B_n}{n}.$$

LEMMA 4.8. *Let $\{\beta_n\}$ be a Kaluza sequence such that $\{\beta_n B_n\}$ decreases to 0. Then $\sum_{n\ge 0} \beta_n B_n \mathrm{e}^{in\theta}$ converges for every $\theta \in (0, 2\pi)$, and there exists $C > 0$ such that*

$$\sup_{n\ge 0} \Big|\sum_{k=0}^n \beta_k B_k \mathrm{e}^{ik\theta}\Big| \le C|H(\mathrm{e}^{i\theta})|^2 \qquad \forall \theta \in (0, 2\pi).$$

Proof. The series $\sum_{n\geq 0}\beta_n B_n \mathrm{e}^{in\theta}$ converges by [44, Theorem I.2.6]. Fix $\theta\in(0,2\pi)$ and put $m=[1/|1-\mathrm{e}^{i\theta}|]$. Let, as before, $S_k=S_k(\mathrm{e}^{ik\theta})=\sum_{j=0}^k \mathrm{e}^{ji\theta}$, so $|S_k|\leq 2(m+1)$. Using Abel summation, monotonicity of $\{\beta_n B_n\}$ and (22), we obtain for $n\geq 0$ (with an empty sum being interpreted as 0)

$$\Big|\sum_{k=0}^{n}\beta_k B_k \mathrm{e}^{ik\theta}\Big| \leq \sum_{k=0}^{m}\beta_k B_k + \Big|\sum_{k=m+1}^{n}\beta_k B_k(S_k-S_{k-1})\Big|$$
$$\leq \psi^2(|1-\mathrm{e}^{i\theta}|)+\Big|\sum_{k=m+1}^{n-1}(\beta_k B_k-\beta_{k+1}B_{k+1})S_k\Big| + \beta_{m+1}B_{m+1}|S_m| + \beta_n B_n|S_n|$$
$$\leq \psi^2(|1-\mathrm{e}^{i\theta}|)+6(m+1)\beta_{m+1}B_{m+1}\leq C|H(\mathrm{e}^{i\theta})|^2.$$

We used $(m+1)\beta_{m+1}B_{m+1}\leq (m+1)\beta_m B_m\leq B_m^2=\psi^2(|1-\mathrm{e}^{i\theta}|)$. ■

Theorem 4.9. *Let $\{\beta_n\}$ be a Kaluza sequence such that $\{\beta_n B_n\}$ decreases to 0. Let T be a contraction of a Hilbert space $\mathcal{H}$ and $x\in\mathcal{H}$. If both series $\sum_{n\geq 0}\beta_n T^n x$ and $\sum_{n\geq 0}\beta_n T^{*n}x$ converge, then $\sum_{n\geq 0}\beta_n B_n\langle T^n x,x\rangle$ converges. In particular, if T is a normal contraction, then the convergence of $\sum_{n\geq 0}\beta_n T^n x$ implies the convergence of $\sum_{n\geq 0}\beta_n B_n\langle T^n x,x\rangle$.*

Proof. By Proposition 2.6, there exist $y,z\in\mathcal{H}$ such that $x=G(T)y$ and $x=G(T^*)z$. Let $q>p>1$. Let U be the unitary dilation of T and σ_y the scalar spectral measure of y, with respect to U. We then have

$$\Big|\sum_{n=p}^{q}\beta_n B_n\langle T^n x,x\rangle\Big|^2 = \Big|\sum_{n=p}^{q}\beta_n B_n\langle T^n G(T)y, G(T^*)z\rangle\Big|^2$$
$$\leq \Big\|\sum_{n=p}^{q}\beta_n B_n T^n G(T)^2 y\Big\|^2\|z\|^2 \leq \Big\|\sum_{n=p}^{q}\beta_n B_n U^n G(U)^2 y\Big\|^2\|z\|^2$$
$$\leq \|z\|^2\int_{\mathbb{T}}\Big|\sum_{n=p}^{q}\beta_n B_n \mathrm{e}^{in\theta}G(\mathrm{e}^{i\theta})^2\Big|^2\,\sigma_y(d\theta),$$

and the latter converges to 0 as $q>p\to\infty$, by Lebesgue's dominated convergence theorem, since by the previous lemma the integrand is bounded by $2C$ for every $q>p$. ■

The next corollary is new for the one-sided EHT (it is proved in [10] for T normal).

Corollary 4.10. *Let T be a contraction in a Hilbert space $\mathcal{H}$ and $x\in\mathcal{H}$. Then*

$$\sum_{n=1}^{\infty}\frac{T^n x}{n}\ \textit{converges}\quad\Longleftrightarrow\quad \sum_{n=1}^{\infty}\frac{\langle T^n x,x\rangle\log n}{n}\ \textit{converges.}$$

Proof. If $\sum_{n=1}^{\infty}\frac{T^n x}{n}$ converges, then also $\sum_{n=0}^{\infty}\frac{T^n x}{n+1}$ converges, and using Campbell's result [7] that the one-sided EHT satisifies (18), we apply the previous theorem to the Kaluza sequence $\{\frac{1}{n+1}\}$. The converse implication is proved in [10, Theorem 4.2]. ■

Proposition 4.11. *Let $\{\beta_n\}$ be a Kaluza sequence with $\{\beta_n B_n\}$ decreasing to 0. For T an isometry of $\mathcal{H}$ and $x\in\mathcal{H}$ (24) holds, and the convergence of $\sum_{n\geq 0}\beta_n T^n x$ implies the convergence of $\sum_{n\geq 0}\beta_n B_n\langle T^n x,x\rangle$.*

Proof. When T is an isometry of $\mathcal{H}$, for every $x \in \mathcal{H}$ the unitary dilation U satisfies $T^n x = U^n x$ for $n \geq 0$. Thus results for unitary operators (not involving the dual) can be transferred to isometries (as done in [10]). The proposition follows by applying Corollary 4.7 and Theorem 4.9 to U. ∎

REMARK. In general, (18) need not hold for contractions which are not normal; examples (with isometries) for $\beta_n = 1/(n+1)^{1-\alpha}$ with $\alpha \geq \frac{1}{2}$ are given in [18, p. 125]. This is why the assumption that $\sum_{n=0}^{\infty} \beta_n T^n x$ converges is not sufficient for our proof of Theorem 4.9. For isometries, the argument in the proof Proposition 4.11 yields that $G(T)\mathcal{H} \subset G(T^*)\mathcal{H}$.

PROPOSITION 4.12. *Let $\{\beta_n\}$ be a Kaluza sequence with divergent sum and assume $\beta_n = O(\frac{1}{n})$. Then for every contraction T in a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$* (18) *holds.*

Proof. Let U be a unitary operator on $\mathcal{H}$ with resolution of the identity $E(dz)$ defined on the Borel subsets of the unit circle $\mathbb{T}$. By the spectral theorem $H_n(U)x = \int_{\mathbb{T}} \sum_{k=0}^{n} \beta_k z^k \, E(dz)x$ and $H_n(U^*)x = \int_{\mathbb{T}} \sum_{k=0}^{n} \beta_k \bar{z}^k \, E(dz)x$. Hence

$$H_n(U)x - H_n(U^*)x = \int_{\mathbb{T}} \sum_{k=0}^{n} \beta_k (z^k - \bar{z}^k) \, E(dz)x.$$

We show that $\{H_n(U)x - H_n(U^*)x\}$ converges. Writing $z = e^{2\pi i\theta}$ with $|\theta| \leq 1/2$ we obtain

$$H_n(U)x - H_n(U^*)x = 2i \int_{\mathbb{T}} \sum_{k=0}^{n} \beta_k \sin(2\pi k\theta) \, E(dz)x.$$

Since $\beta_n \to 0$ monotonely, the integrand converges for every θ [44, Theorem I.2.6], and it is uniformly bounded by boundedness of $n\beta_n$ [44, Vol. I, p. 183]. Hence $H_n(U)x - H_n(U^*)x$ converges for every $x \in \mathcal{H}$.

Now let T be a contraction in $\mathcal{H}$, and U be its unitary dilation, defined on $\mathcal{H}_1 \supset \mathcal{H}$, with P the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}$. Then for every $x \in \mathcal{H}$, $H_n(T)x - H_n(T^*)x = P(H_n(U)x - H_n(U^*)x)$ converges, by continuity of P, which proves (18). ∎

REMARKS.

1. Our proof follows that given for the EHT by Campbell [7].

2. As mentioned above, for $\beta_n = 1/(n+1)^{1-\alpha}$ with $\alpha \geq \frac{1}{2}$ there exists a contraction T for which (18) fails ([18, p. 125]). If U is the unitary dilation of T, then $H_n(U)y - H_n(U^*)y$ does not converge for some y; however, U satisfies (18), by normality.

3. Note that for any $0 < \alpha < 1$ the series $\sum_{n=1}^{\infty} (1/n^{1-\alpha}) \sin(2\pi n\theta)$ behaves near 0 like $C\theta^{-\alpha}$ [44, Formula V.2.1], so for $\beta_n = 1/(n+1)^{1-\alpha}$ the partial sums $\sum_{k=0}^{n} \beta_k \sin(2\pi ik\theta)$ cannot be uniformly bounded. Theorem [44, V.2.31] can be used (with $\alpha = -1$ and $\beta = 2$) to show the same unboundedness of the partial sums for $\beta_n = \log(n+2)/(n+1)$.

5. Almost everywhere convergence. In this section we assume T to be a power-bounded operator on $L_p(S, \Sigma, \mu)$, $p \geq 1$, such that T admits a linear modulus denoted by $\mathbf{T}$, acting on $L_p(\mu)$, which is power-bounded and satisfies the pointwise ergodic theorem in $L_p(\mu)$. The above assumption applies to positive contractions of $L_p(\mu)$ when $1 < p < \infty$ [1], and to contractions with mean ergodic modulus [11] or Dunford–Schwartz operators in $L_1(\mu)$.

For T as above and $h \in L_p$ we define $S_m := \sum_{k=1}^m \mathbf{T}^k|h|$ and $h^* := \sup_{m\geq 1} S_m/m$. By assumption, h^* is finite μ-a.e.

If $\{\beta_n\}$ is a positive sequence with $\sum_{n\geq 0}\beta_n < \infty$, then for T power-bounded on L_p and any $f \in L_p$, the series $\sum_{n\geq 0}\beta_n T^n f$ is a.e. absolutely convergent: for $p > 1$ we can assume (by an appropriate reduction, see [36, p. 189]) that μ is a probability, and then $\sum_{n\geq 0}\beta_n\|T^n f\|_1 < \infty$ and we apply Beppo Levi's theorem. Hence throughout this section we assume $\sum_{n\geq 0}\beta_n = \infty$.

Note that for any bounded positive sequence $\{\beta_n\}$ with $\sum_{n\geq 0}\beta_n = \infty$, for every T induced by an ergodic probability preserving transformation on an atomless (S,Σ,μ) there exists a *bounded* function f with $\int f\,d\mu = 0$ such that $\sum_{n\geq 0}\beta_n T^n f$ diverges a.e. and does not even converge in L_1-norm ([20]). In fact, we can have even $\sup_k |\sum_{n=0}^k \beta_n T^n f| < \infty$ a.e. ([33], [34]). Note that the non-convergence in norm follows also from Theorem 2.11, since the spectrum of T is the whole unit circle.

For $\{\beta_n\}$ and the associated $\{\alpha_n\}$ as in Proposition 2.3, we want to obtain a.e. convergence of $\sum_{k=0}^\infty \beta_k T^k f$ from the L_p-norm convergence of the series.

For $G(T) = \sum_{n\geq 0}\alpha_n T^n$ and $H_n(T) = \sum_{k=0}^n \beta_k T^k$ we obtained in (7)

$$H_n(T)G(T) = I + \sum_{m\geq n+1}\Big(\sum_{k=0}^n \beta_k\alpha_{m-k}\Big)T^m$$

$$= I + \sum_{m=n+1}^{3n}\Big(\sum_{k=0}^n \beta_k\alpha_{m-k}\Big)T^m + \sum_{m\geq 3n+1}\Big(\sum_{k=0}^n \beta_k\alpha_{m-k}\Big)T^m = I + M_n(T) + N_n(T).$$

Identity (7) was used in Proposition 2.2 to show that $\sup_n \|H_n(T)G(T)\| < \infty$. For our particular operator T we want to prove a maximal inequality for the sequence $\{H_n(T)G(T)\}$.

It suffices to deal separately with $\{M_n(T)\}$ and $\{N_n(T)\}$.

Proposition 5.1. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Proposition 2.1 and assume moreover that $\{|\alpha_n|\}$ is eventually non-increasing. Let T be an operator on $L_p(S,\Sigma,\mu)$ as above. Then for every $h \in L_p(\mu)$*

$$\sup_{n\geq 1}|N_n(T)h| \leq C_1 h^* < \infty \qquad \mu\text{-a.e.}$$

Proof. By assumption, h^* is finite μ-a.e.

Recall that $\alpha_n \leq 0$ for $n \geq n_0$; by assumption there exists n_1, and we may assume $n_1 \geq 2n_0$, such that $\{|\alpha_n|\}_{n\geq n_1}$ is non-increasing. It suffices to show that

$$\sup_{n\geq n_1}|N_n(T)h| < \infty \quad \mu\text{-a.e.}$$

For $n \geq n_1$,

$$|N_n(T)h| = \Big|\sum_{m\geq 3n+1}\Big(\sum_{k=0}^n \beta_k\alpha_{m-k}\Big)T^m h\Big| \leq \sum_{m\geq 3n+1}|\alpha_{m-n}|\Big(\sum_{k=0}^n \beta_k\Big)\mathbf{T}^m|h|$$

$$\leq \sum_{k=0}^n \beta_k \sum_{m\geq 3n+1}|\alpha_{m-n}|(S_m - S_{m-1}) \leq \sum_{k=0}^n \beta_k\Big[\sum_{m\geq 3n+1}(|\alpha_{m-n}| - |\alpha_{m+1-n}|)S_m\Big].$$

Hence, using Abel summation again, we obtain

$$|N_n(T)h| \le \sum_{k=0}^{n} \beta_k \Big[\sum_{m\ge 3n+1} (|\alpha_{m-n}| - |\alpha_{m+1-n}|) m h^* \Big]$$
$$\le h^* \sum_{k=0}^{n} \beta_k \Big(4n|\alpha_{2n}| + \sum_{m\ge 2n+1} |\alpha_m| \Big) \le C' h^*,$$

using $n|\alpha_{2n}| \le \frac{n}{n-n_0} \sum_{m=n+n_0+1}^{2n} |\alpha_m| \le 2 \sum_{m=n+n_0+1}^{2n} |\alpha_m|$ and (8). ∎

For the next proposition we need an extra assumption on $\{\beta_n\}$.

Proposition 5.2. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Proposition* 2.1 *and assume moreover that $\beta_n = O(1/n)$. Let T be an operator on $L_p(S, \Sigma, \mu)$ as above. Then for every $h \in L_p(\mu)$ we have*

$$\sup_{n\ge 1} |M_n(T)h| \le C_2 h^* < \infty \qquad \mu\text{-a.e.}$$

Proof. We assume $\beta_n \le C/n$. Using (2), we obtain

$$|M_n(T)h| = \Big| \sum_{m=n+1}^{3n} \beta_m T^m h + \sum_{m=n+1}^{3n} \Big(\sum_{k=n+1}^{m-1} \beta_k \alpha_{m-k} \Big) T^m h \Big|$$
$$\le \sum_{m=n+1}^{3n} \beta_m \mathbf{T}^m |h| + \sum_{m=n+1}^{3n} \Big(\sum_{k=n+1}^{m-1} \beta_k |\alpha_{m-k}| \Big) \mathbf{T}^m |h|$$
$$\le \frac{C}{n} S_{3n} + \sum_{m=n+1}^{3n} \Big(\frac{C}{n} \sum_{k\ge 1} |\alpha_k| \Big) \mathbf{T}^m |h| \le 3C \Big(1 + \sum_{k\ge 1} |\alpha_k| \Big) h^*. \ ∎$$

Theorem 5.3. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Proposition* 5.1. *Assume moreover that $\sum_n |\beta_n - \beta_{n+1}| < \infty$ (e.g. $\{\beta_n\}$ is eventually non-increasing) and $\beta_n = O(1/n)$. Let T on $L_p(\mu)$ be as above, and let $f \in L_p(\mu)$. If $\sum_{k=1}^\infty \beta_k T^k f$ converges in norm, then it converges a.e.*

Proof. Since $\sum_{k=1}^\infty \beta_k T^k f$ converges, then by Proposition 2.6 we conclude that $f \in \overline{G(T)(I-T)L_p(\mu)}$. To prove the result, we show that $H_n(T)G(T)h$ converges a.e. for every $h \in \overline{(I-T)L_p(\mu)}$. According to Proposition 5.1 and Proposition 5.2 and using Banach's principle, it is enough to show that $H_n(T)G(T)h$ converges a.e. for any $h = u - Tu$. The limit was already identified (as h) by Proposition 2.3. Put $v = G(T)u$, then

$$H_n(T)G(T)(h) = H_n(T)G(T)(u - Tu) = H_n(T)(v - Tv)$$
$$= \sum_{k=0}^{n} \beta_k (T^k v - T^{k+1} v) = \beta_n (v - T^{n+1} v) + \sum_{k=0}^{n-1} (\beta_k - \beta_{k+1})(v - T^{k+1} v).$$

By the assumption $\beta_n = O(1/n)$ and the pointwise ergodic theorem for $\mathbf{T}$, we conclude that $\beta_n T^n v \to 0$ a.e. For the remaining series we have absolute convergence, since

$$\sum_{k=0}^{\infty} |\beta_k - \beta_{k+1}| \, \|(v - T^{k+1} v)\|_1 \le 2M \|v\|_1 \sum_{k=0}^{\infty} |\beta_k - \beta_{k+1}| < \infty$$

implies $\sum_{k=0}^\infty |\beta_k - \beta_{k+1}| \cdot |(v - T^{k+1} v)| < \infty$ a.e. ∎

REMARKS.

1. The conditions $\sum_{n\ge 0}|\beta_n-\beta_{n+1}|<\infty$ and $\beta_n\to 0$ ensure that $\sum_{n\ge 0}\beta_n T^n f$ converges in norm when $f=(I-T)g$. For a.e. convergence for such f we must have that $\beta_n T^{n+1}g\to 0$ a.e. If this holds for every Dunford–Schwartz operator T and $g\in L_1$, we must have $\beta_n=O(1/n)$. See below for a specific counter-example that the theorem fails when $\beta_n=O(1/\sqrt{n})$.

2. Instead of assuming $|\alpha_n|$ non-increasing, it is enough to have a non-increasing positive sequence $\{\gamma_n\}$ with $\sum_{n\ge 1}\gamma_n<\infty$ satisfying (8) and $|\alpha_n|\le\gamma_n$ for $n\ge 1$.

COROLLARY 5.4. *Let T be a contraction of $L_1(\mu)$ with mean ergodic modulus, and let $\{\beta_n\}$ be a Kaluza sequence with $\beta_n=O(1/n)$ and $|\alpha_n|$ eventually non-increasing. If $f\in L_1$ satisfies* $\liminf_n\big\|\sum_{k=1}^n\beta_k T^k f\big\|_1<\infty$, *then* $\sum_{k=1}^\infty\beta_k T^k f$ *converges a.s.*

Proof. If the modulus of T is mean ergodic, so is T [11]. Now combine Corollary 3.2 with Theorem 5.3. ■

REMARK. The corollary applies to Dunford–Schwartz operators in probability spaces, which must have mean ergodic modulus.

COROLLARY 5.5. *Let $\{\beta_n\}$ and the corresponding $\{\alpha_n\}$ satisfy the assumptions of Theorem* 5.3. *Let $1<p<\infty$ and let T be a positive contraction of $L_p(\mu)$. If $f\in L_p$ satisfies* $\liminf_n\big\|\sum_{k=1}^n\beta_k T^k f\big\|_p<\infty$, *then* $\sum_{k=1}^\infty\beta_k T^k f$ *converges a.s. and* $\sup_n\big|\sum_{k=1}^n\beta_k T^k f\big|$ *is in L_p.*

Proof. Combine Corollary 2.17, Theorem 5.3 and Propositions 5.1, 5.2. ■

EXAMPLE 4 (Hausdorff moment sequences). For a finite positive Borel measure ν on $[0,1]$, $\{\int_0^1 t^n\,d\nu(t)\}$ is the *Hausdorff moment sequence* of ν. When ν is an absolutely continuous probability on $[0,1]$ with $\phi=\frac{d\nu}{dt}$ bounded, we put $\beta_n=\int_0^1 t^n\phi(t)\,dt$, and assume that $\int_0^1\phi(t)(1-t)^{-1}\,dt=\infty$; we obtain that $\{\beta_n\}$ is a Kaluza sequence with $\beta_n=O(1/n)$ and $\sum_n\beta_n=\infty$. Since $\{\beta_n\}$ is a Hausdorff moment sequence, by [35, Satz 6] also $\{|\alpha_{n+1}|\}_{n\ge 0}$ is a moment sequence (of a finite positive measure), and in particular monotone decreasing. Hence Corollaries 5.4 and 5.5 can be applied to $\{\beta_n\}$. Thus, for $\beta_n=\frac{1}{n+1}=\int_0^1 t^n\,dt$ we obtain a proof of the result of [13] without computing the order of $|\alpha_n|$.

For many Kaluza sequences $\{\beta_n\}$, the condition of Theorem 5.3 most difficult to check is the monotonicity of $\{|\alpha_n|\}_{n\ge 1}$. This problem is solved if $\{\beta_n\}$ is a Hausdorff moment sequence of some probability ν on $[0,1]$, by [35, Satz 6]. Hausdorff characterized moment sequences of finite positive measures on $[0,1]$ as completely monotone sequences (see [42, p. 108] for definitions and proofs). A non-negative function $f(t)$ defined on $(0,\infty)$ is called *completely monotone* (for $t>0$) if all its derivatives exist and satisfy $(-1)^n f^{(n)}(t)\ge 0$ for $t>0$. By [42, Theorem 11d, p. 158], if $f(t)$ is completely monotone for $t>0$, then the sequence $\{f(n+1)\}_{n\ge 0}$ is completely monotone. Leibniz's rule of differentiating products yields that the product of two completely monotone functions is completely monotone. Obviously $1/t$ is completely monotone for $t>0$, and by [40, Theorem 2] $1/\log(1+t)$ is also completely monotone. Hence $\frac{1}{t\log(t+1)}$ is completely monotone, so the sequence $\{\frac{1}{(n+1)\log(n+2)}\}_{n\ge 0}$ is completely monotone. Hence by Hausdorff's characterization there exists a probability measure ν on $[0,1]$ such that $\beta_n := \frac{\log 2}{(n+1)\log(n+2)} = \int_0^1 t^n\, d\nu(t)$. Thus Corollaries 5.4 and 5.5 can be applied to $\{\beta_n\}$.

Since $1/\log(1+t)$ is completely monotone, an application of [40, Theorem 2] yields that

$$f(t) := \frac{1}{\log(1+\log(1+t))} = \frac{1}{\log(\log(\mathrm{e}+\mathrm{e}t))}$$

is completely monotone, hence also $f(\epsilon+\delta t)$ is completely monotone, and with $\epsilon = 3\mathrm{e}^{-1}-1$ and $\delta = \mathrm{e}^{-1}$ we conclude that $1/\log(\log(3+t))$ is completely monotone. Now arguments as above show that Corollaries 5.4 and 5.5 can be applied to the sequences

$$\left\{\frac{\log(\log 4)}{(n+1)\log(\log(n+4))}\right\} \quad \text{and} \quad \left\{\frac{\log 2\log(\log 4)}{(n+1)\log(n+2)\log(\log(n+4))}\right\}.$$

The above discussion shows that we can generate Kaluza sequences $\{\beta_n\}$ with $\{|\alpha_n|\}$ decreasing using completely monotone functions. A tool for constructing such functions on $(0,\infty)$ is provided by Bernstein's theorem [42, Theorem 12b, p. 161], which says that $f(t)$ is completely monotone for $t>0$ if and only if f is the Laplace transform of some positive measure on $[0,\infty)$ (necessarily finite on finite intervals).

In the next propositions, we show that for some specific sequences $\{\beta_n\}$ and the corresponding $\{\alpha_n\}$, we can relax the rate assumption $\beta_n = O(1/n)$ and obtain results in L_p for large enough values of p (see the remark following Theorem 5.3). For simplicity of the formulations we state the results only for Kaluza sequences. We are motivated by the examples

$$(1-z)^{1/r} = \sum_{n\ge 0}\alpha_n z^n, \qquad \frac{1}{(1-z)^{1/r}} = \sum_{n\ge 0}\beta_n z^n$$

with $r>1$, treated in [18], for which it is easily checked that $\{\beta_n\}$ is Kaluza and $\{|\alpha_n|\}$ is decreasing, with $\alpha_n = O(1/n^{1+1/r})$ and $\beta_n = O(1/n^{1-1/r})$.

Notation. Let $\{a_n\}$ and $\{b_n\}$ be two positive sequences. We write $a_n \asymp b_n$ if $0 < \liminf_n (a_n/b_n) < \limsup (a_n/b_n) < \infty$.

Definition. A positive measurable function $L(x)$ on $[0,\infty)$ is called *slowly varying* if $\lim_{x\to\infty}\frac{L(tx)}{L(x)} = 1$ for every $t>0$ (see [24, p. 276]). Bojanic and Seneta [5, pp. 93–94] defined a positive sequence $\{c(n)\}$ to be *slowly varying* if $\lim_{n\to\infty}\frac{c([tn])}{c(n)} = 1$ for every $t>0$. It follows from Theorem 2 of [5] that if $\{c(n)\}$ is a slowly varying sequence, then $L(x) := c([x])$ is a slowly varying function; thus $c(n) = L(n)$ for some slowly varying function $L(x)$ on $[0,\infty)$. On the other hand, if $L(x)$ is a slowly varying function which is eventually monotone, then the inequalities

$$(t-[t])n < tn-1 < [tn] \le tn \qquad n > \frac{1}{[t]}$$

imply $L([tn])/L(n) \to 1$ as $n\to\infty$, so $c(n) := L([n])$ is a slowly varying sequence. Thus a positive sequence $\{c(n)\}$ is an *eventually monotone slowly varying sequence* if and only if it is of the form $c(n) = L(n)$ for some eventually monotone slowly varying function. In the following we will use simple properties of slowly varying functions, see [24, Ch. VIII §9, Lemma], and apply them to the derived slowly varying sequences (which will be assumed eventually monotone); for some direct proofs see [5].

Proposition 5.6. *Let $1 < r \le \infty$ and let $\{\beta_n\}$ be a Kaluza sequence with $\beta_n \asymp b(n)/n^{1-1/r}$ for some slowly varying eventually monotone sequence $\{b(n)\}$, such that the corresponding $\{|\alpha_n|\}$ is eventually non-increasing. Let T be a Dunford–Schwartz operator on $L_1(S,\Sigma,\mu)$ of a probability space. Then for every $p > r/(r-1)$ there exists $C = C_p > 0$, such that*

$$\Big\|\sup_{n\ge1}|M_n(T)h|\Big\|_p \le C_p\|h\|_p \quad \forall h \in L_p(\mu).$$

Proof. Since $\{\beta_n\}$ is a bounded Kaluza sequence, it is non-increasing. By the Lemma in [24, p. 280], the assumed estimate for β_n yields $\sum_{k=0}^n \beta_k \asymp L(n)n^{1/r}$, for some slowly varying sequence $\{L(n)\}$ (which depends on r). By inequality (8) we have $\sum_{m\ge n}|\alpha_m| = O\big(1/(L(n)n^{1/r})\big)$. Using the monotonicity of $|\alpha_n|$ we also have (eventually) $|\alpha_n| = O\big(1/(L(n)n^{1+1/r})\big)$. Then for $h \in L_p$

$$\begin{aligned}|M_n(T)h| &\le \sum_{m=n+1}^{3n}\Big(\sum_{k=0}^{[n/2]}\beta_k|\alpha_{m-k}|\Big)\mathbf{T}^m|h| + \sum_{m=n+1}^{3n}\Big(\sum_{k=[n/2]+1}^{n}\beta_k|\alpha_{m-k}|\Big)\mathbf{T}^m|h|\\ &\le \frac{C'}{L(n/2)(n/2)^{1+1/r}}\Big(\sum_{k=0}^{[n/2]}\beta_k\Big)S_{3n} + \beta_{[n/2]}\sum_{m=n+1}^{3n}\Big(\sum_{k=m-n}^{m-[n/2]-1}|\alpha_k|\Big)\mathbf{T}^m|h|\\ &\le C\,\frac{S_{3n}}{n} + \frac{CL(n)}{n^{1-1/r}}\sum_{m=n+1}^{3n}\frac{\mathbf{T}^m|h|}{L(m-n)(m-n)^{1/r}}\,.\end{aligned}$$

By the Dunford–Schwartz ergodic theorem, the first term satisfies a strong p-p maximal inequality. Hence one has to deal with the second term. Our approach is similar to Déniel's [17].

Since $p > r/(r-1)$, one can find p', with $p > p' > r/(r-1)$. Then $h \in L_{p'}$ (since μ is a probability), and $(\mathbf{T}|h|)^{p'} \le \mathbf{T}(|h|^{p'})$ (see [36, p. 65]). If $q' = p'/(p'-1)$, then $q'/r < 1$, and by Hölder's inequality

$$\begin{aligned}\sum_{m=n+1}^{3n}\frac{\mathbf{T}^m|h|}{L(m-n)(m-n)^{1/r}} &\le \Big(\sum_{m=n+1}^{3n}\mathbf{T}^m(|h|^{p'})\Big)^{1/p'}\\ &\quad\times\Big(\sum_{m=n+1}^{3n}\frac{1}{(L(m-n))^{q'}(m-n)^{q'/r}}\Big)^{1/q'} \le \frac{Cn^{1/q'-1/r}}{L(n)}\Big(\sum_{m=n+1}^{3n}\mathbf{T}^m(|h|^{p'})\Big)^{1/p'}.\end{aligned}$$

Since $p/p' > 1$, we apply the Dunford–Schwartz theorem for $\mathbf{T}$ to the function $|h|^{p'} \in L_{p/p'}(S,\mu)$ and obtain

$$\begin{aligned}&\Big\|\sup_{n\ge1}\frac{L(n)}{n^{1-1/r}}\sum_{m=n+1}^{3n}\frac{\mathbf{T}^m|h|}{L(m-n)(m-n)^{1/r}}\Big\|_p\\ &\qquad\le \Big\|\Big(\sup_{n\ge1}\frac{\sum_{m=1}^{3n}\mathbf{T}^m(|h|^{p'})}{n}\Big)^{1/p'}\Big\|_p \le C\|h\|_p. \ \blacksquare\end{aligned}$$

Remark. In fact, $B(x) = b(x)x^{1/r-1}$ is regularly varying, and by Karamata's theorem [24, p. 281, Theorem 1(b)] we obtain $\lim_n \frac{b(n)}{L(n)} = \frac{1}{r}$. In the proof we obtained $\sum_{m\geq n} |\alpha_m| = O\big(1/(L(n)n^{1/r})\big)$; this is related to relations obtained in [26].

Proposition 5.7. *Let $r > 1$ and $\{\beta_n\}$ a Kaluza sequence as in Proposition 5.6. Let $p > r/(r-1)$ and let T be a positive contraction of $L_p(S,\Sigma,\mu)$. Then there exists $C > 0$ such that*

$$\Big\| \sup_{n\geq 1} |M_n(T)h| \Big\|_p \leq C\|h\|_p \quad \forall h \in L_p(\mu).$$

Proof. By the computation in the previous proof, we have

$$|M_n(T)h| \leq C\,\frac{S_{3n}}{n} + \frac{CL(n)}{n^{1-1/r}} \sum_{m=n+1}^{3n} \frac{T^m|h|}{L(m-n)(m-n)^{1/r}}\,.$$

By Akcoglu's theorem we have a maximal inequality for the first term, so it is enough to prove that

$$\Big\| \sup_n \frac{L(n)}{n^{1-1/r}} \sum_{m=n+1}^{3n} \frac{T^m|h|}{L(m-n)(m-n)^{1/r}} \Big\|_p \leq C\|h\|_p\,. \tag{25}$$

Step 1. Proof for T an invertible isometry. We apply Lamperti's extension of a result of Banach [37]; since our T is positive and invertible, Lamperti's representation yields that there exists a positive function $u \in L_p$ and a linear positive lattice isomorphism U on L_∞ such that $Tf = u\cdot Uf$ (for $p=2$ see [31, footnote 3]). Since a lattice isomorphism preserves disjointness, we have $U(|f|^t) = |Uf|^t$ for any positive t. We proceed as in the proof of the previous proposition. Let $p' \in (\frac{r}{r-1}, p)$. We define for $g \in L_\infty$ the operator $Sg = u^{p'}\cdot Ug$, which is clearly linear and positive, and obtain:

$$\begin{aligned}\int |Sg|^{p/p'}\,d\mu &= \int u^p \cdot |Ug|^{p/p'}\,d\mu = \int u^p\cdot \big(U(|g|^{1/p'})\big)^p\,d\mu \\ &= \int \big(T(|g|^{1/p'})\big)^p\,d\mu = \int (|g|^{1/p'})^p\,d\mu = \int |g|^{p/p'}\,d\mu.\end{aligned}$$

Since $g \geq 0$ is in $L_{p/p'}$ if and only if $g^{1/p'} \in L_p$, we see that S extends to a positive isometry of $L_{p/p'}(\mu)$. Now if $f \in L_p$ then $|f|^{p'}$ is in $L_{p/p'}(\mu)$, and $|Tf|^{p'} \leq (T|f|)^{p'} = S(|f|^{p'})$. Induction yields $|T^m f|^{p'} \leq (T^m|f|)^{p'} = S^m(|f|^{p'})$ for $m \geq 1$. We now prove (25): By Hölder's inequality

$$\begin{aligned}&\sum_{m=n+1}^{3n} \frac{T^m|h|}{L(m-n)(m-n)^{1/r}} \\ &\qquad \leq \Big(\sum_{m=n+1}^{3n} (T^m|h|)^{p'}\Big)^{1/p'} \Big(\sum_{m=n+1}^{3n} \frac{1}{(L(m-n))^{q'}(m-n)^{q'/r}}\Big)^{1/q'} \\ &\qquad \leq \frac{Cn^{1/q'-1/r}}{L(n)} \Big(\sum_{m=n+1}^{3n} S^m(|h|^{p'})\Big)^{1/p'}.\end{aligned}$$

Since $p/p' > 1$, we apply Akcoglu's theorem for S to $|h|^{p'} \in L_{p/p'}(S,\mu)$ and obtain

$$\Big\| \sup_{n\ge1} \frac{L(n)}{n^{1-1/r}} \sum_{m=n+1}^{3n} \frac{T^m|h|}{L(m-n)(m-n)^{1/r}} \Big\|_p \le \Big\| \Big(\sup_{n\ge1} \frac{\sum_{m=1}^{3n} S^m(|h|^{p'})}{n}\Big)^{1/p'} \Big\|_p \le C\|h\|_p\,.$$

Step 2. Proof for T a positive contraction. Recall that we (may) assume that (S,Σ,μ) is a probability space. By the Akcoglu–Sucheston dilation theorem [3] (see [2] for a constructive proof for separable L_p spaces), there is a positive invertible isometry $\hat{T}$ on a probability space $(\hat{S},\hat{\Sigma},\hat{\mu})$ and a positive isometric isomorphism J of $L_p(S,\Sigma,\mu)$ into $L_p(\hat{S},\hat{S},\hat{\mu})$ such that $T^n = J^{-1}E\hat{T}^nJ$, with $E : L_p(\hat{\mu}) \longrightarrow JL_p(\mu) \subset L_p(\hat{\mu})$ (see remarks at the end of [3]) a positive norm-1 projection. Then for every n, by positivity of the operators,

$$|M_n(T)h| = |J^{-1}EM_n(\hat{T})Jh| \le J^{-1}E|M_n(\hat{T})Jh| \le J^{-1}E\sup_{k\ge1}|M_k(\hat{T})Jh|,$$

which yields, by using Step 1,

$$\Big\|\sup_{n\ge1}|M_n(T)h|\Big\|_p \le \Big\|\sup_{k\ge1}|M_k(\hat{T})Jh|\Big\|_p \le C\|Jh\|_p = C\|h\|_p\,. \ \blacksquare$$

THEOREM 5.8. *Let $1 < r \le \infty$ and $\{\beta_n\}$ a Kaluza sequence as in Proposition 5.6. Fix $p > r/(r-1)$ and let T be a contraction of $L_p(S,\Sigma,\mu)$, which is either positive, or the restriction of a Dunford–Schwartz operator. Let $f \in L_p(\mu)$ such that $\sum_{n\ge1}\beta_nT^nf$ converges in $L_p(\mu)$. Then the series converges μ-a.e.*

Proof. It follows from Lemma 4.1 and either Proposition 5.6 or Proposition 5.7, that for every $h \in L_p(\mu)$, $\sup_{n\ge1}|H_n(T)G(T)h| < \infty$ μ-a.e. Hence, as in the proof of Theorem 5.3, using Banach's principle, it is enough to show that $H_n(T)G(T)h$ converges μ-a.e. for every $h = u - Tu$. With $v = G(T)u$, we have

$$H_n(T)v = \beta_n(v - T^{n+1}v) + \sum_{k=0}^{n}(\beta_k - \beta_{k+1})(v - T^{k+1}v).$$

The series is absolutely convergent since $\{\beta_n\}$ is decreasing to 0. Moreover, $\beta_nT^{n+1}v \to 0$ μ-a.e. since $p > r/(r-1)$ implies

$$\sum_{n\ge1}\beta_n^p\|T^{n+1}\|_p^p \le \sum_{n\ge1}\frac{C(b(n))^p}{n^{p(1-1/r)}} < \infty. \ \blacksquare$$

EXAMPLE 5 (Fractional coboundaries). Let $r > 1$, and let T be a power-bounded operator on a Banach space X. Derriennic and Lin [18] defined the operator $(I-T)^{1/r}$ by a power series with the coefficients of $(1-t)^{1/r} = \sum_{n=0}^{\infty}\alpha_nt^n$ (where $\alpha_0 = 1$, $\alpha_1 = -1/r$ and $\alpha_n = \frac{-1}{r\cdot n!}\prod_{k=1}^{n-1}(k-\frac{1}{r})$ for $n \ge 2$). It is proved in [18] that when T is mean ergodic, $x \in (I-T)^{1/r}X$ if and only if the series $\sum_{n=1}^{\infty}T^nx/n^{(r-1)/r}$ converges in norm. Let $(1-t)^{-1/r} = \sum_{n=0}^{\infty}\beta_nt^n$. Then $\{\beta_n\}$ (also computed in [18]) is a Kaluza sequence (with $G(z) = (1-z)^{1/r}$), and the assumptions of Theorem 5.8 are satisfied, so for $p > r/(r-1)$ the theorem yields that if T is a Dunford–Schwartz operator on (S,Σ,μ) and $f \in (I-T)^{1/r}L_p(\mu)$, then $\sum_{n=0}^{\infty}\beta_nT^nf$ converges a.e., and the precise estimate of β_n in [44, vol. I, p. 77] yields that also the series $\sum_{n=1}^{\infty}T^nf/n^{(r-1)/r}$ converges a.e. This improves

Theorem 3.6(ii) of [18]. When T is just a positive contraction of L_p (with $p > r/(r-1)$ fixed), we obtain for $f \in (I-T)^{1/r}L_p$ the a.e. convergence of $\sum_{n=1}^{\infty} T^n f/n^{(r-1)/r}$, improving (considerably) part of [18, Theorem 3.12].

Remark. For $p = r/(r-1)$ the theorem is false; we give a counter-example for $r=2$ with $p=2$. By [18, Proposition 3.8(ii)] there exists T on $L_2([0,1],dt)$ induced by a measure preserving transformation, and $f \in (I-T)^{1/2}L_2$ such that $(1/\sqrt{n})\sum_{k=1}^{n} T^k f$ is a.e. non-convergent, hence $\sum_{k=1}^{n} T^k f/\sqrt{k}$ is a.e. non-convergent. Let $a_1^{(1/2)} = 1/2$ and $a_k^{(1/2)} = (1/2k!)\Pi_{j=1}^{k-1}(j-1/2)$ for $k \geq 2$ be the coefficients in $(1-t)^{1/2} = 1 - \sum_{k=1}^{\infty} a_k^{(1/2)} t^k$ [18], and put $\beta_k = 2(k+1)a_{k+1}^{(1/2)}$. Since $f \in (I-T)^{1/2}L_2$, the series $\sum_{k=0}^{\infty} \beta_k T^k f$ converges in L_2-norm [18], but not a.e. by comparison with $\sum_{k=1}^{\infty} T^k f/\sqrt{k}$ (because $\beta_k = (k^{-1/2}/\Gamma(1/2))[1+O(1/k)]$ [44, vol. I, p. 77]). By the construction, $\alpha_k = -a_k^{(1/2)}$, and $|\alpha_k|$ decreases. The sequence $\{\beta_k\}$ is a Kaluza sequence (so it is decreasing), which satisfies the assumptions of Proposition 5.1 and $\beta_n = O(1/n^{1/2}) \to 0$, but $n\beta_n \to \infty$. For additional information see [14].

This example shows in particular that in Theorem 5.3 the assumption $\beta_n = O(1/n)$ cannot be weakened to $\beta_n \to 0$.

Corollary 5.9. *Let $0 < \gamma < 1$ and $p > 1/\gamma$. Let T be a positive invertible isometry of $L_p(S,\Sigma,\mu)$. Let $f \in L_p(S,\mu)$ such that, with $\overline{p} = \min(p,2)$,*

$$\sum_{n\geq 1} \frac{\|f + \ldots + T^{n-1}f\|_p^{\overline{p}}}{n^{1+\gamma\overline{p}}} < \infty. \tag{26}$$

Then $\sum_{n\geq 1} \frac{T^n f}{n^\gamma}$ converges in norm and μ-a.e.

Proof. The norm convergence is proved in [15, Theorem 4.7]. If we take $\gamma = 1 - 1/r$, the norm convergence implies that $f \in (I-T)^{1/r}L_p$ [18], and the result follows from Example 5. ∎

Theorem 5.10. *Let $0 < \gamma < 1$. For θ a measure preserving transformation of a Lebesgue probability space (S,Σ,μ) with $\theta(S) = S$, put $Tf = f \circ \theta$. Let $p > 1/\gamma$. If $f \in L_p(S,\mu)$ satisfies* (26)*, then $\sum_{n\geq 1} T^n f/n^\gamma$ converges in norm and μ-a.e.*

Proof. The operator T is an isometry of L_p. If θ is invertible, the result follows from Corollary 5.9 (even without the assumption of a Lebesgue space).

When θ is not invertible, we use the construction of its *natural extension* (e.g. [12, p. 240]): there exists an *invertible* probability preserving $\hat{\theta}$ on $(\hat{S},\hat{\Sigma},\hat{\mu})$ with a factor map $\pi : \hat{S} \longrightarrow S$ such that $\hat{\mu}(\pi^{-1}A) = \mu(A)$ for $A \in \Sigma$ and $\pi \circ \hat{\theta} = \theta \circ \pi$, which yields $\pi \circ \hat{\theta}^n = \theta^n \circ \pi$ for $n \geq 0$. We put $\hat{T}\hat{g} = \hat{g} \circ \hat{\theta}$ for $\hat{g} \in L_1(\hat{S},\hat{\mu})$. By invertibility of $\hat{\theta}$ we have that $\hat{T}$ is an invertible positive isometry of $L_p(\hat{S},\hat{\mu})$. For $f \in L_p(S,\mu)$ we define $\tilde{f} = f \circ \pi$; then

$$\widetilde{T^k f} = \widetilde{f \circ \theta^k} = (f \circ \theta^k) \circ \pi = (f \circ \pi) \circ \hat{\theta}^k = \tilde{f} \circ \hat{\theta}^k = \hat{T}^k \tilde{f}.$$

the identity $\widetilde{1_A} = 1_{\pi^{-1}A}$ yields that $\|\tilde{f}\|_p = \|f\|_p$, so $\|\sum_{k=0}^{n-1} T^k f\|_p = \|\widetilde{\sum_{k=0}^{n-1} T^k f}\|_p = \|\sum_{k=0}^{n-1} \hat{T}^k \tilde{f}\|_p$. Hence if $f \in L_p(S,\mu)$ satisfies (26), so does $\tilde{f}$ with respect to $\hat{T}$. By [15, Theorem 4.7] the series $\sum_{n\geq 1} \frac{\hat{T}^n \tilde{f}}{n^\gamma}$ converges in L_p-norm, which yields (is equivalent to) convergence in L_p-norm of $\sum_{n\geq 1}\frac{T^nf}{n^\gamma}$. Since $p>1/\gamma$, we obtain a.e. convergence of $\sum_{n\geq 1}\frac{T^nf}{n^\gamma}$ by Example 5 (with $r=1/(1-\gamma)$). ■

Proposition 5.11. *Let $1<r\leq\infty$ and $\beta\in\mathbb{R}$. Fix $p>r/(r-1)$ and let T be a contraction of $L_p(S,\Sigma,\mu)$, which is either positive, or the restriction of a Dunford–Schwartz operator. Let $f\in L_p(\mu)$ such that $\sum_{n=0}^\infty\frac{(\log(n+1))^\beta}{(n+1)^{1-1/r}}T^nf$ converges in $L_p(\mu)$-norm. Then the series converges μ-a.e.*

Proof. Step 1. We prove that *for every $\beta\in\mathbb{R}$ and every $\alpha\in(0,1]$, there exists $\delta>0$ such that the function $\frac{(\log(1+t^\delta))^\beta}{t^\alpha}$ is completely monotone for $t>0$.*

For $\beta=0$ the result follows since $t^{-\alpha}$ is completely monotone by checking the definition. Assume $\beta<0$. By the above t^β is completely monotone, and since $\log(1+t)$ has a completely monotone derivative, the composition $(\log(1+t))^\beta$ is completely monotone [40, Theorem 2]. Hence the product $(\log(1+t))^\beta/t^\alpha$ is completely monotone.

Assume now $\beta>0$. Since $\log(1+t)/t$ has a positive inverse Laplace transform [41, p. 269], it is completely monotone by Bernstein's theorem [42] (for another proof see [40, Corollary to Theorem 5]). Hence $(\frac{\log(1+t)}{t})^{[\beta]+1}$ is also completely monotone. As noted in the previous case, $(\log(1+t))^{\beta-[\beta]-1}$ is completely monotone, so the product $\frac{(\log(1+t))^\beta}{t^{[\beta]+1}}$ is completely monotone. To conclude, we note that $t^{\alpha/([\beta]+1])}$ is positive with completely monotone derivative, hence the composed function $\frac{(\log(1+t^\delta))^\beta}{t^\alpha}$, with $\delta=\alpha/([\beta]+1])$, is completely monotone for $t>0$.

Step 2. Let $\sum_{n\geq 1}x_n$ be a convergent series in a Banach space X, and let $\{u_n\}_{n\geq 1}$ be a convergent monotone sequence. Then the series $\sum_{n\geq 1}u_nx_n$ converges in X. This assertion may be proved using Abel's summation by parts.

Step 3. We now prove the proposition. Let $1<r\leq\infty$ and $\beta\in\mathbb{R}$. Let $p>r/(r-1)$ and T be a contraction as in the proposition, and assume that $\sum_{n\geq 0}\frac{(\log(n+2)^\beta)T^nf}{(n+1)^{1-1/r}}$ converges in L_p. Put $\alpha=1-\frac{1}{r}$, so $\alpha\in(0,1]$.

Let $\delta>0$ be given by Step 1, such that $\frac{(\log(1+t^\delta))^\beta}{t^\alpha}$ is completely monotone. Then the sequence $\beta_n=(\log 2)^{-\beta}\frac{(\log(1+(n+1)^\delta))^\beta}{(n+1)^\alpha}$ is a completely monotone Kaluza sequence [42, p. 158], so it satisfies the conditions of Proposition 4.6.

Now, note that for $n\geq 1$

$$\big(\log(1+(n+1)^\delta)\big)^\beta=\Big(\delta\log(n+1)+\log\Big(1+\frac{1}{(n+1)^\delta}\Big)\Big)^\beta$$
$$=\delta^\beta(\log(n+1))^\beta\cdot\Big(1+\frac{\log(1+1/(n+1)^\delta)}{\delta\log(n+1)}\Big)^\beta.$$

Put $u_n=(1+\frac{\log(1+1/(n+1)^\delta)}{\delta\log(n+1)})^\beta$. Since $\{u_n\}_{n\geq 1}$ is monotone converging to 1, it follows from the assumption on f and Step 2 that the series $\sum_{n\geq 1}\beta_nT^nf$ converges in L_p, hence, by Theorem 5.8, it converges a.e.

To finish the proof we use again Step 2 (in the space $\mathbb{R}$ or $\mathbb{C}$, for pointwise convergence), noting that $u_n>0$ and

$$(\log(n+1))^\beta=\frac{1}{\delta^\beta}u_n^{-1}(\log(1+(n+1)^\delta))^\beta.\ ■$$

Remark. Let T be a (positive) Dunford–Schwartz operator of a probability space. Proposition 5.11 yields, for $r = \infty$, that if $f \in L_p$, $p > 1$, and the series

$$\sum_{n=0}^{\infty} \frac{(\log(n+1))^\beta}{n+1} T^n f$$

converges in L_p-norm, then we have a.e. convergence. For $\beta \le 0$, it is shown in Step 1 of the previous proof that $\{(\log(n+2))^\beta/(n+1)\}$ is completely monotone, hence the L_1-norm convergence of the series $\sum_{n=0}^{\infty} \frac{(\log(n+2))^\beta}{n+1} T^n f$ implies a.e. convergence by Corollary 5.4 (but this cannot be deduced from Proposition 5.11); this is no longer true for $\beta > 0$. Applying Step 3 of the previous proof (in reverse direction) to $\beta_n = (\log 2)^{-\beta} \frac{(\log(1+(n+1)^\delta))^\beta}{(n+1)}$ and using Example 6 below, we obtain that Theorem 5.3 for L_1 may fail under the weaker assumption of $\beta_n \asymp b(n)/n$ for some slowly varying sequence $\{b(n)\}$, even for Hausdorff moment sequences.

Example 6 (An operator T induced by a probability preserving transformation, such that for $\beta > 0$ there is $f \in L_1$ with $\sum_{n=0}^{\infty} \frac{(\log(n+2))^\beta}{n+1} T^n f$ norm-convergent but a.e. divergent). Let ν be a probability on $\mathbb{R}$ and put $S = \mathbb{R}^{\mathbb{N}}$ with μ the infinite product measure, and define T to be induced by the shift, so for g which depends on the first coordinate $\{T^n g\}$ are independent identically distributed (iid). Fix $\beta > 0$ and let $g \ge 0$ depend on the first coordinate with $g \in L_1(\mu)$ and $\int g(\log^+ g)^\beta \, d\mu = \infty$.

Claim. $\limsup \frac{(\log(n+1))^\beta}{n} T^n g \ge \frac{1}{2}$ a.e.

Put $\varphi(t) = t/(\log t)^\beta$ for $t > \mathrm{e}$. Then φ is increasing, and $\varphi(s(\log s)^\beta)/s \to 1$ as $s \to \infty$. Hence for large n we obtain

$$\begin{aligned}\{x \in S : g(x)(\log g(x))^\beta > n\} \\ = \{x \in S : \varphi\big(g(x)(\log g(x))^\beta\big) > \varphi(n)\} \subset \{x \in S : 2g(x) > n/(\log n)^\beta\}.\end{aligned}$$

Since $T^n g$ has the distribution of g, the assumptions on g yield

$$\sum_{n \ge N} \mu\Big\{x : \frac{(\log n)^\beta}{n} T^n g(x) > \frac{1}{2}\Big\} \ge \sum_{n \ge N} \mu\big\{x : g(x)(\log g(x))^\beta > n\big\} = \infty.$$

Since $\{T^n g\}$ are independent, $\limsup \frac{(\log n)^\beta}{n} T^n g \ge \frac{1}{2}$ a.e. by the Borel–Cantelli lemma [23, Lemma 2, p. 201], which yields the claim.

Now let $f = (I - T)g$. Then

$$\begin{aligned}&\sum_{n=0}^{N} \frac{(\log(n+2))^\beta}{n+1} T^n (I-T) g \\ &\quad = (\log 2)^\beta g + \sum_{n=1}^{N} \Big(\frac{(\log(n+2))^\beta}{n+1} - \frac{(\log(n+1))^\beta}{n}\Big) T^n g - \frac{(\log(N+2))^\beta}{N+1} T^{N+1} g.\end{aligned}$$

The last term converges to zero in L_1-norm, and the sum converges in norm and absolutely a.e. Thus $\sum_{n=0}^{\infty} \frac{(\log(n+2))^\beta}{n+1} T^n f$ is norm-convergent, but by the claim it is a.e. divergent.

6. Problems. In this section we list some problems connected with the results of the previous sections. We assume that $\{\beta_n\}$ is a positive bounded sequence with $\sum_{n\geq 0}\beta_n = \infty$, satisfying the assumptions of Proposition 2.3; of particular interest are the cases of Kaluza or Hausdorff moment sequences with $\beta_n \to 0$.

PROBLEM 6.1. *Is the closed operator $-H(T) = -\sum_{n=0}^{\infty}\beta_n T^n$ an infinitesimal generator of a semi-group defined on $\overline{(I-T)X}$?*

It was shown in [9] and in [28] that for the one-sided ergodic Hilbert transform, $-\sum_{n=1}^{\infty}\frac{T^n}{n}$ is the infinitesimal generator, in $\overline{(I-T)X}$, of the semi-group $\{(I-T)^r\}_{r\geq 0}$. In Proposition 2.8 it was shown that for $\{\beta_n\}$ with $\sum_n \beta_n = \infty$, satisfying the hypotheses of Proposition 2.3, the operator $H(T) = \sum_n \beta_n T^n$ is a closed operator which is densely defined in $\overline{(I-T)X}$, so our problem is a natural one. If (or when) the answer is positive, the problem is then to provide some description of the generated semi-group.

PROBLEM 6.2. *Is there a Kaluza sequence $\{\beta_n\}$ with divergent sum and $\beta_n \to 0$ which has a rate $r_n \to 0$ such that for every power-bounded T on a reflexive Banach space X,*

$$\sum_{n=0}^{\infty}\beta_n T^n x \ \textit{converges} \quad \Longleftrightarrow \quad \Big\|\frac{1}{n}\sum_{k=0}^{n-1}T^k x\Big\| = O(r_n). \tag{27}$$

In view of Proposition 2.6 and the remark following it, for $\{\beta_n\}$ as in the problem and T power-bounded in X reflexive, (27) can be reformulated as

$$x \in G(T)X \quad \text{if and only if} \quad \Big\|\frac{1}{n}\sum_{k=0}^{n-1}T^k x\Big\| = O(r_n). \tag{28}$$

In this reformulation, Browder's theorem [6] shows that for the constant sequence $\beta_n \equiv 1$, for which $G(T) = I-T$, (28) holds with the rate $\{\frac{1}{n}\}$. However in this example we do not have $\beta_n \to 0$ (so Proposition 2.6 does not apply), and (27), which is no longer equivalent to (28), does not hold (since $\sum_{n=0}^{\infty}T^n x$ need not converge for $x \in (I-T)X$ when T is not mixing).

Theorem 2.10 (see also [27, Section 5]) yields that when $\{\beta_n\}$ is Kaluza with divergent sum and $\beta_n \to 0$, whenever $\sum_{n=0}^{\infty}\beta_n T^n x$ converges we have

$$\Big\|\frac{1}{n}\sum_{k=0}^{n-1}T^k x\Big\| = o\Big(1/\sum_{k=0}^{n-1}\beta_k\Big). \tag{29}$$

Theorem 2.14 shows that when 1 is not isolated in $\sigma(T)$, the rate in (29) is optimal. The examples in [18, pp. 121, 127] show, when $\beta_n = 1/\sqrt{n+1}$, that for unitary or symmetric contractions in a Hilbert space the rate (29) does not imply $x \in G(T)X$. We therefore formulate a (closely related) problem as follows: *Is there a Kaluza sequence* (*as in the problem*) *such that for every power-bounded T on a reflexive space X, the rate* (29) *implies convergence of $\sum_{n=0}^{\infty}\beta_n T^n x$ (i.e. $x \in G(T)X$) ?*

Another related problem is whether the optimality of Theorem 2.14 is still true when $X = \overline{(I-T)X}$ and 1 is isolated in $\sigma(T)$ (these assumptions exclude uniform ergodicity). Combining the second remark to Theorem 2.14 with the decomposition of X obtained from the Lorch–Dunford theory of spectral sets [22, Theorem VII.3.20], the problem is reduced to the question whether Theorem 2.14 is true when $X = \overline{(I-T)X}$ and $\sigma(T) = \{1\}$.

PROBLEM 6.3. *Let $\{\beta_n\}$ be a Kaluza sequence with $B_n = \sum_{k=0}^{n} \beta_k \to \infty$ and $\beta_n \to 0$. Are the following three conditions equivalent for every normal contraction T on $\mathcal{H}$ and $x \in \mathcal{H}$:*

(i) $\sum_{n=0}^{\infty} \beta_n T^n x$ *converges.*
(ii) $\sum_{n=0}^{\infty} \beta_n B_n \langle T^n x, x\rangle$ *converges.*
(iii) $\sum_{n=1}^{\infty} \frac{\beta_n B_n}{n^2} \left\|\sum_{k=1}^{n} T^k x\right\|^2 < \infty$.

The equivalence of the three conditions for the one-sided EHT (when T is normal) was proved in [10].

It was shown in Proposition 4.3 that (iii) implies (i), but for the converse an additional condition on $\{\beta_n\}$ was used (see Corollary 4.7). Is this additional condition really needed?

In Theorem 4.9 it is shown that (i) implies (ii) when $\{\beta_n B_n\}$ decreases to zero. Is this implication true in general? For the converse implication some restrictions are needed; when $\{\beta_n\}$ are the coefficients of $(1-t)^{-1/2}$ (see Example 5), we have $\beta_n \asymp 1/\sqrt{n+1}$ and $\beta_n B_n \sim C$, but an example at the end of [19] has a unitary T on L_2 with a function $f \in L_2$ such that $\sum_{n=0}^{\infty} |\langle T^n f, f\rangle| < \infty$, but $f \notin (I-T)^{1/2} L_2 = G(T) L_2$. Does (ii) imply (i) when $\{\beta_n B_n\}$ decreases to zero?

Since the above sequence $\beta_n \asymp 1/\sqrt{n+1}$ satisfies the assumptions of Corollary 4.7, the above example of [19] shows that without additional assumptions (ii) does not imply (iii). Does (ii) imply (iii) when $\{\beta_n B_n\}$ decreases to zero? If (ii) does not imply (i), it cannot imply (iii), by Proposition 4.3.

For $\{\beta_n\}$ with $\beta_n = O(1/n)$ and $\{\beta_n B_n\}$ decreasing, if we knew that (ii) implies (i) for normal contractions, we would deduce that *for any contraction T on $\mathcal{H}$ conditions* (i) *and* (ii) *are equivalent.* If (ii) holds for T, then it holds also for the unitary dilation U, so U satisfies (i), hence so does T. Finally, if T satisfies (i) then (ii) holds by Theorem 4.9, since $\beta_n = O(1/n)$.

The use of Theorem 4.9 leads to the question for which Kaluza sequences $G(T)\mathcal{H} = G(T^*)\mathcal{H}$ (i.e. (18) holds) for every contraction T. It is true for $\beta_n = O(1/n)$ (Proposition 4.12), and false for $\beta_n \asymp 1/(n+1)^{1-\alpha}$ when $\alpha \geq 1/2$ ([18, p. 125]). Is it true when $\beta_n = O(1/n^{1-\alpha})$ with $\alpha < 1/2$?

PROBLEM 6.4. *Let T be a positive Dunford–Schwartz operator in a probability space with $T\mathbf{1} = \mathbf{1}$, and let $\{\beta_n\}$ be a Kaluza sequence, with $\beta_n = O(\frac{1}{n})$ and the corresponding $\{|\alpha_n|\}$ eventually non-increasing. If for $f \in L_1$ the series $\sum_{k=1}^{\infty} \beta_k T^k f$ converges a.e., does it converge in norm?*

In Theorem 5.3 it is proved that if $\sum_{k=1}^{\infty} \beta_k T^k f$ converges in norm, then it converges a.e. Thus the problem is whether the converse implication is also true, i.e., are the a.e. and norm convergence equivalent? Of course, we need to know that the pointwise limit is in L_1, otherwise norm convergence cannot hold. In fact, for norm convergence f must be in $\overline{(I-T)L_1}$, by Lemma 2.4, so also the limit must be in $\overline{(I-T)L_1}$.

The answer is not yet known even for the one-sided ergodic Hilbert transform. Note that without the assumption that $T\mathbf{1} = \mathbf{1}$, it is possible for the one-sided EHT to converge a.e. for an L_1 function to a non-integrable limit [10, Example 2]. In that example there is $f \in L_\infty$ for which the a.e. limit is in L_1 (in fact in L_p for $1 \le p < 2$), but not in L_2. It will be of interest to know the answer to the stated problem even under the additional assumption that $f \in L_2$, and whether there is L_2 convergence when the a.e. limit is in L_2.

By Proposition 2.3, if $g \in \overline{(I-T)L_p}$, then $H_n(T)G(T)g \to g$ in L_p-norm, and by Theorem 5.3 $\sum_{k=0}^\infty \beta_k T^k G(T)g$ converges a.e. to g. Now, let $\sum_{k=0}^\infty \beta_k T^k f$ converge a.e. to $g \in \overline{(I-T)L_p}$; then by looking at $f - G(T)g$, the problem is reduced to the question whether a.e. convergence to zero of $H_n(T)f$ for $f \in L_p$ implies L_p-norm convergence.

We note that a positive answer for the one-sided EHT, for a class of moving averages, is given in [8, Proposition 3.1.1]. For this class, similar arguments can be used to obtain a positive answer to the stated problem.

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