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AN EXPLICIT BOUND FOR THE POINCARÉ CONSTANT ON A LIPSCHITZ DOMAIN

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Abstract. We construct an explicit bound for the constant in the Neumann-type Poincaré inequality for a Lipschitz domain. The bound depends on only four parameters for the geometry of the domain.

1. Introduction. If Ω is a bounded domain in $\mathbb{R}^n$ such that the Sobolev space $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, then a Neumann-type Poincaré inequality is valid. That is, there exists a constant C_Ω, to be chosen minimal, such that

$$\|u - \overline{u}_\Omega\|_{L_2(\Omega)} \leq C_\Omega \|\nabla u\|_{L_2(\Omega)} \text{ for all } u \in H^1(\Omega).$$

Weak regularity conditions on the boundary of Ω imply this compact embedding, for example a continuous boundary suffices (see [EE], Theorem V.4.17). The compactness of the embedding gives the existence of the constant C_Ω, the Poincaré constant for Ω, but it gives no estimate on its value. For a Lipschitz domain four parameters are needed to quantify it: the dimension, the diameter, the Lipschitz constant for the boundary and one more (see Definition 1.1). Boulkhemair and Chakib ([BC], Theorem 1) showed that given a set of four parameters there exists a $C > 0$ such that $C_\Omega \leq C$, for every Lipschitz domain Ω with these parameters. The proof in [BC] is also based on a compactness argument, and does not allow to construct an explicit bound. We will present an explicit bound for C_Ω in terms of the four parameters.

In general no explicit expression for C_Ω is known. In certain specific cases there are bounds available. If Ω is convex with diameter D, then $C_\Omega \leq \frac{D}{\pi}$ (see [Beb], [PW]).

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If $Q(0,2a) \subset \Omega \subset B_0(b)$ and Ω is star-shaped with respect to the origin, then $C_\Omega \leq 2^{n+6} b (\frac{b}{a})^{(n-1)/2}$ by [SS], Theorem 5, where $Q(0,2a)$ is the cube with centre 0 and side length $2a$. On a general Lipschitz domain Alessandrini, Morassi and Rosset [AMR] gave a method which allows the reader, after much more work, to construct an explicit bound for C_Ω. Compared with [AMR] our method is different and our bound for C_Ω is sharper for large Lipschitz constant L or large dimension.

Throughout this paper we will take a *domain* $\Omega \subset \mathbb{R}^n$ to be a non-empty, open, bounded, connected set.

For any integrable function u defined on some bounded measurable set A with positive measure, we write $\overline{u}_A$ for the *mean of u over A*; that is,

$$\overline{u}_A := \frac{1}{|A|} \int_A u,$$

where $|A|$ denotes the Lebesgue measure of A.

Definition 1.1. Let $L, w > 0$. A domain $\Omega \subset \mathbb{R}^n$ is said to be *Lipschitz with parameters L, w* if for every $z_0 \in \partial\Omega$ there exist an isometry Φ, an open neighbourhood U of z_0 and a Lipschitz-continuous function $f \colon (-w,w)^{n-1} \to \mathbb{R}$ with Lipschitz constant L such that $\Phi(z_0) = 0$, $f(0) = 0$ and $\Phi(U \cap \Omega) = \big\{(x,y) \in (-w,w)^{n-1} \times \mathbb{R} : -2Lw < y < f(x)\big\}$, and $\Phi(U \cap \partial\Omega) = \{(x, f(x)) : x \in (-w,w)^{n-1}\}$.

Obviously for every Lipschitz domain there are $L, w > 0$ such that it is Lipschitz with parameters L, w. We may now state the main theorem.

Theorem 1.2 (Poincaré inequality for Lipschitz domains). *If $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with parameters L, w and diameter D, then*

$$\|u - \overline{u}_\Omega\|_{L_2(\Omega)} \leq C \|\nabla u\|_{L_2(\Omega)}, \tag{1}$$

for all $u \in H^1(\Omega)$, where

$$C = 2^{(n+3)/2} \Big(\frac{D}{\delta}\Big)^{n/2} \Big\lceil \frac{D\sqrt{n}}{\delta} \Big\rceil^{n/2} \Big(\delta \Big\lceil \frac{D}{\delta} \Big\rceil^n + 2Lw\Big)$$

and

$$\delta = \frac{1}{8\sqrt{n}} \, \frac{w \min(2L, 1)}{3 + 2\sqrt{1 + L^2}} \, .$$

Remark 1.3. Note that C depends only on the four parameters w, L, n and D. Also note that C has the right scaling behaviour.

The proof of Theorem 1.2 uses Jamet's notion of tubes [Jam] (see also Section 2). If $\Omega_0 \subset \Omega$ is an open set such that Ω is the union of N open subsets, each of which is linked to Ω_0 by a tube of length at most K, then

$$\|u\|_{L_2(\Omega)} \leq \sqrt{N} \, \|u\|_{L_2(\Omega_0)} + K\sqrt{N} \, \|\nabla u\|_{L_2(\Omega)} \tag{2}$$

for all $u \in H^1(\Omega)$. Using another idea of Jamet one can write $u - \overline{u}_{\Omega_0} = u_1 + E(u_2|_{\Omega_0})$, where $u_1 \in H^1(\Omega)$ vanishes on Ω_0, the function $u_2 \in H^1(\Omega)$ has mean zero on Ω_0 and E is an extension operator which leaves constants invariant, from $H^1(\Omega_0)$ to the H^1-functions on a big ball which contains Ω. Therefore (2) applied to u_1 gives

$$\|u - \overline{u}_\Omega\|_{L_2(\Omega)} \leq \|u - \overline{u}_{\Omega_0}\|_{L_2(\Omega)} \leq K\sqrt{N} \, \|\nabla u_1\|_{L_2(\Omega)} + \|E(u_2|_{\Omega_0})\|_{L_2(\Omega)}$$

and the result almost follows.

In Section 2 we introduce the notion of tubes and prove (2). In Section 3 we consider the extension operator E, with suitable norm estimates. Next, in Section 4 we show the existence of the open set Ω_0 and give estimates for K and N in terms of the four parameters for the Lipschitz domain. Finally we complete the proof of the theorem.

2. Tubes. Given a bounded open subset $G_0 \subset \mathbb{R}^n$ and a piecewise C^1, arc-length parameterised curve $c\colon [0,\ell] \to \mathbb{R}^n$ with $c(0) = 0$, define for all $\lambda \in [0,\ell]$ the transport operator $T_\lambda\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T_\lambda x = x + c(\lambda)$. Define also

$$G_\lambda := T_\lambda G_0 \qquad \text{and} \qquad G_\ell^* := \bigcup_{0\le\lambda\le\ell} G_\lambda.$$

The following lemma is a generalisation of Lemma 1.2 in [Jam].

Lemma 2.1. *If $u \in H^1(G_\ell^*)$, then*

$$\|u\|_{L_2(G_\ell)} \le \|u\|_{L_2(G_0)} + \ell\|\nabla u\|_{L_2(G_\ell^*)}.$$

Proof. It suffices to prove the lemma for C^1-functions, since the subspace $C^1(G_\ell^*) \cap H^1(G_\ell^*)$ is dense in $H^1(G_\ell^*)$ (see [EE], Theorem V.3.2). Let $u \in C^1(G_\ell^*) \cap H^1(G_\ell^*)$. Then

$$u(x + c(\ell)) = u(x) + \int_0^\ell \big\langle (\nabla u)(x + c(t)), c'(t)\big\rangle\, dt$$

for all $x \in G_0$. Therefore

$$\|u\|_{L_2(G_\ell)} \le \|u\|_{L_2(G_0)} + \bigg(\int_{G_0} \bigg|\int_0^\ell \big\langle (\nabla u)(x + c(t)), c'(t)\big\rangle\, dt\bigg|^2\, dx\bigg)^{1/2}.$$

Now we apply the Cauchy–Schwarz inequality twice to the above integral to obtain

$$\begin{aligned}\int_{G_0} \bigg|\int_0^\ell \big\langle (\nabla u)(x + c(t)), c'(t)\big\rangle\, dt\bigg|^2\, dx &\le \ell \int_{G_0}\int_0^\ell \big|\big\langle (\nabla u)(x + c(t)), c'(t)\big\rangle\big|^2\, dt\, dx\\ &\le \ell \int_0^\ell \int_{G_0} \big|(\nabla u)(x + c(t))\big|^2\, dx\, dt \le \ell^2 \int_{G_\ell^*} |\nabla u|^2.\end{aligned}$$

The penultimate step uses that c is arc-length parameterised. ■

Definition 2.2. Given a domain Ω and a non-empty open subset $\Omega_0 \subset \Omega$, we say that an open $G \subset \Omega$ is *linked to Ω_0 in Ω by a tube of length ℓ* if there exists an open $G_0 \subset \Omega_0$ and a piecewise C^1, arc-length parameterised curve $c\colon [0,\ell] \to \mathbb{R}^n$ with $c(0) = 0$ such that $G_0 + c(\ell) = G$ and $G_0 + c(t) \subset \Omega$ for all $t \in [0,\ell]$.

Two balls $B_\delta(x)$ and $B_\delta(y)$ are said to be *linked by a tube in Ω* if there exists a piecewise C^1, arc-length parameterised curve $c\colon [0,\ell] \to \Omega$ with $c(0) = x$, $c(\ell) = y$ and $B_\delta(c(t)) \subset \Omega$ for all $t \in [0,\ell]$.

Lemma 2.3. *Let Ω be a domain and $\Omega_0 \subset \Omega$ a non-empty open subset. Suppose Ω is the union of N open subsets, each of which is linked to Ω_0 in Ω by a tube. Then*

$$\|u\|_{L_2(\Omega)} \le \sqrt{N}\,\|u\|_{L_2(\Omega_0)} + K\sqrt{N}\,\|\nabla u\|_{L_2(\Omega)}$$

for all $u \in H^1(\Omega)$, where K is the maximum length of any such tube.

Proof. The proof is similar to the proof of Theorem 1.2 in [Jam]. Denote the N subsets by G_i, with $1 \le i \le N$. It follows from Lemma 2.1 that $\|u\|_{L_2(G_i)} \le A + B$ for all the G_i, with

$$A = \|u\|_{L_2(\Omega_0)} \quad \text{and} \quad B = K \|\nabla u\|_{L_2(\Omega)}.$$

Hence

$$\|u\|_{L_2(\Omega)} \le \Big(\sum_{i=1}^{N} \|u\|_{L_2(G_i)}^2 \Big)^{1/2} \le \sqrt{N}\,(A+B)$$

as required. ∎

In Section 4 we will show that for any Lipschitz domain the values of K and N in Lemma 2.3 can be estimated by the four parameters that quantify the Lipschitz domain.

3. Extension operator.

For the proof of Theorem 1.2 we require a continuous extension operator from a small ball to a larger one. We also need it to preserve the constant functions and we deduce some norm estimates.

LEMMA 3.1. *Suppose* $0 < \delta < D$ *and* $n \ge 2$. *Let* Ω_0 *and* Ω_1 *be concentric balls in* $\mathbb{R}^n$ *of radii* δ *and* D *respectively. Then there exists a continuous linear extension operator* $E \colon L_2(\Omega_0) \to L_2(\Omega_1)$ *which preserves constant functions such that* $EH^1(\Omega_0) \subset H^1(\Omega_1)$,

$$\|E\|_{L_2(\Omega_0)\to L_2(\Omega_1)} \le 2^{(n+2)/2} \Big(\frac{D}{\delta}\Big)^{n/2} \tag{3}$$

and

$$\|\nabla E u\|_{L_2(\Omega_1)} \le 2^{(n+2)/2} \Big(\frac{D}{\delta}\Big)^{n/2} \|\nabla u\|_{L_2(\Omega_0)} \tag{4}$$

for all $u \in H^1(\Omega_0)$.

Proof. We may assume that the centre of the balls is the origin and by scaling we may assume that $\delta = 1$. Define $\tau_1 : \mathbb{R} \to \mathbb{R}$ by

$$\tau_1(r) = \begin{cases} r - 1 & \text{if } D < \frac{3}{2}, \\ r - 1 & \text{if } D \ge \frac{3}{2} \text{ and } r < \frac{5}{4}, \\ \dfrac{1}{4} + \dfrac{1}{4}\,\dfrac{r - 5/4}{D - 5/4} & \text{if } D \ge \frac{3}{2} \text{ and } r \ge \frac{5}{4}. \end{cases}$$

Then τ_1 is strictly increasing, bijective, Lipschitz, $\tau_1(1) = 0$, $\tau_1(D) \le \frac{1}{2}$ and $\frac{1}{4D} \le \tau_1'(r) \le 1$ for all $r \in \mathbb{R} \setminus \{\frac{5}{4}\}$. Let $\chi \in C_c^\infty(-\frac{1}{8}, \frac{1}{8})$ be such that $\chi \ge 0$, $\int \chi = 1$ and $\chi(-t) = \chi(t)$ for all $t \in \mathbb{R}$. Define $\tau \colon \mathbb{R} \to \mathbb{R}$ by $\tau = \chi * \tau_1$. Then $\tau \in C^\infty(\mathbb{R})$, τ is strictly increasing, bijective, $\tau(r) = r - 1$ for all $r < \frac{9}{8}$ and $\frac{1}{4D} \le \tau'(r) \le 1$ for all $r \in \mathbb{R}$. Hence τ is invertible and $|(\tau^{-1})'(t)| \le 4D$ for all $t \in \mathbb{R}$. Set $p = \tau(D)$. Then $0 < p \le \frac{1}{2}$.

Define $E \colon L_2(\Omega_0) \to L_2(\Omega_1)$ by

$$(Eu)(x) = \begin{cases} u(x) & \text{if } |x| < 1, \\ u\Big(\dfrac{1 - \tau(|x|)}{|x|}\, x\Big) & \text{if } 1 < |x| < D. \end{cases}$$

Then $EH^1(\Omega_0) \subset H^1(\Omega_1)$ by an argument similar to the proof of Lemma 3.4 in [Giu]. Moreover, E preserves constant functions. Let S denote the unit sphere in $\mathbb{R}^n$ and use spherical coordinates. Let $u \in L_2(\Omega_0)$. Then

$$\begin{aligned}\int_{\Omega_1\setminus\Omega_0} |Eu|^2 &= \int_1^D \int_S \Big|u\big((1-\tau(r))\omega\big)\Big|^2 r^{n-1}\,d\omega\,dr\\ &\le D^{n-1}\int_1^D \int_S \Big|u\big((1-\tau(r))\omega\big)\Big|^2 \,d\omega\,dr\\ &\le 4D^n \int_{1-p}^1 \int_S |u(r\omega)|^2\,d\omega\,dr \le 2^{n+1}D^n \int_{\Omega_0} |u|^2\end{aligned}$$

and (3) follows.

Similarly, with $\frac{\partial}{\partial\omega}$ denoting schematically the contribution of the spherical derivatives one obtains for all $u \in H^1(\Omega_0)$ that

$$\begin{aligned}\int_{\Omega_1\setminus\Omega_0} |\nabla Eu|^2 &= \int_1^D \int_S \bigg(\Big|\frac{\partial}{\partial r}u\big((1-\tau(r))\omega\big)\Big|^2 + \frac{1}{r^2}\Big|\frac{\partial}{\partial\omega}u\big((1-\tau(r))\omega\big)\Big|^2\bigg) r^{n-1}\,d\omega\,dr\\ &\le D^{n-1}\cdot 4D\cdot 2^{n-1}\int_{1-p}^1 \int_S \Big|\frac{\partial}{\partial r}u(r\omega)\Big|^2 r^{n-1}\,d\omega\,dr\\ &\quad + D^{n-3}\cdot 4D\cdot 2^{n-3}\cdot 2^2\int_{1-p}^1 \int_S \Big|\frac{\partial}{\partial\omega}u(r\omega)\Big|^2 r^{n-1}\,d\omega\,dr\\ &\le 2^{n+1}D^n\int_{\Omega_0}|\nabla u|^2.\end{aligned}$$

Now the lemma follows. ■

4. Lipschitz domains. The lemmas in Section 2 are valid for any domain. In this section we shall estimate the values of N and K in Lemma 2.3 for Lipschitz domains, expressed in the four parameters w, L, n and D. We first show that for sufficiently small $\zeta > 0$, any two balls with radius ζ in Ω are linked by a tube in Ω. Clearly, any set in Ω with diameter at most ζ and which is close to the boundary of Ω can be linked to a ball away from the boundary in Ω with radius ζ by a simple translation. Obviously Ω can be covered with a finite number of balls in $\mathbb{R}^n$ with radius ζ, which gives an estimate on the number N. This gives, however, no estimate on the length of the required tubes.

Reducing ζ and using a grid, we can force the tubes to follow gridpoints. But such a tube can be shortened such that it passes through each gridpoint at most once. Then it is easy to estimate the length K.

To show that for sufficiently small ζ, any two balls with radius ζ in Ω are linked by a tube in Ω, we present a method to turn any piecewise linear curve between their centres into one that has always distance at least ζ to the boundary of Ω.

PROPOSITION 4.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with parameters L, w. Define*

$$\zeta := \frac{w\min(2L,1)}{3+2\sqrt{1+L^2}}\,.$$

Let $x, y \in \Omega$ be such that $B_\zeta(x), B_\zeta(y) \subset \Omega$. Then $B_\zeta(x)$ and $B_\zeta(y)$ are linked by a tube in Ω.

Proof. Let $x, y \in \Omega$ and suppose that $d(x, \partial\Omega) \geq \zeta$ and $d(y, \partial\Omega) \geq \zeta$. Since Ω is open and connected there exist $\ell > \zeta$ and a piecewise linear curve $\gamma \colon [0, \ell] \to \Omega$, parameterised by arc-length with $\gamma(0) = x$ and $\gamma(\ell) = y$. Extend γ to $[0, \ell + 2\zeta]$ by setting $\gamma(t) = y$ for all $t \in [\ell, \ell + 2\zeta]$.

We now turn the curve γ into a 'good' curve (i.e. one that stays at least a distance ζ away from $\partial\Omega$) by induction. Note that this 'good' curve is no longer parameterised by arc-length. For all $k \in \mathbb{N}_0$ let $P(k)$ be the hypothesis "there exists a piecewise linear curve $\tilde{\gamma} \colon [0, \ell + 2\zeta] \to \Omega$ such that

$$\begin{aligned} \tilde{\gamma}(0) &= x, && && (5)\\ \tilde{\gamma}(\ell + 2\zeta) &= y, && && (6)\\ d(\tilde{\gamma}(t), \partial\Omega) &\geq \zeta && \text{for all } t \in [0, k\zeta], && (7)\\ |\tilde{\gamma}(t) - \gamma(t)| &\leq \zeta\sqrt{1 + L^2} && \text{for all } t \in [k\zeta, (k+1)\zeta] \text{ and} && (8)\\ \tilde{\gamma}(t) &= \gamma(t) && \text{for all } t \in [(k+1)\zeta, \ell + 2\zeta]." && (9) \end{aligned}$$

The case $k = 0$ is trivial. For the inductive step, let $k \in \mathbb{N}$ and suppose $P(k-1)$ is valid. So there exists a piecewise linear curve $\tilde{\gamma} \colon [0, \ell + 2\zeta] \to \Omega$ such that

$$\begin{aligned} \tilde{\gamma}(0) &= x, \\ \tilde{\gamma}(\ell + 2\zeta) &= y, \\ d(\tilde{\gamma}(t), \partial\Omega) &\geq \zeta && \text{for all } t \in [0, (k-1)\zeta], \\ |\tilde{\gamma}(t) - \gamma(t)| &\leq \zeta\sqrt{1 + L^2} && \text{for all } t \in [(k-1)\zeta, k\zeta], \text{ and} \\ \tilde{\gamma}(t) &= \gamma(t) && \text{for all } t \in [k\zeta, \ell + 2\zeta]. \end{aligned}$$

We now consider two cases.

Case 1. Suppose $d(\gamma(k\zeta), \partial\Omega) \geq \zeta(2 + \sqrt{1 + L^2})$. Then

$$\begin{aligned} d(\tilde{\gamma}(t), \partial\Omega) &\geq d(\gamma(k\zeta), \partial\Omega) - |\tilde{\gamma}(t) - \gamma(t)| - |\gamma(t) - \gamma(k\zeta)| \\ &\geq \zeta(2 + \sqrt{1 + L^2}) - \zeta\sqrt{1 + L^2} - \zeta = \zeta \end{aligned}$$

for all $t \in [(k-1)\zeta, k\zeta]$ and so we may take our new curve $\hat{\gamma}$ to be the same as $\tilde{\gamma}$.

Case 2. Suppose $d(\gamma(k\zeta), \partial\Omega) < \zeta(2 + \sqrt{1 + L^2})$. Then there exists a $z_0 \in \partial\Omega$ such that $|\gamma(k\zeta) - z_0| < \zeta(2 + \sqrt{1 + L^2})$. Let Φ, U, f be as in Definition 1.1. Without loss of generality we may assume that Φ is the identity map. If $t \in [(k-1)\zeta, k\zeta]$, then

$$\begin{aligned} |\tilde{\gamma}(t)| &\leq |\tilde{\gamma}(t) - \gamma(t)| + |\gamma(t) - \gamma(k\zeta)| + |\gamma(k\zeta)| \\ &< \zeta\sqrt{1 + L^2} + \zeta + \zeta(2 + \sqrt{1 + L^2}) = w \min(2L, 1). \end{aligned}$$

This implies that $\tilde{\gamma}(t) \in U$ for all $t \in [(k-1)\zeta, k\zeta]$. To see that $\tilde{\gamma}(t) \in U$ for all $t \in [k\zeta, (k+1)\zeta]$, observe that

$$|\tilde{\gamma}(t)| = |\gamma(t)| \leq |\gamma(k\zeta)| + |\gamma(k\zeta) - \gamma(t)| \leq \zeta(3 + \sqrt{1 + L^2}) < w \min(2L, 1).$$

Hence $\tilde{\gamma}(t) \in U \cap \Omega$ for all $t \in [(k-1)\zeta, (k+1)\zeta]$.

Define the points

$$p := \tilde{\gamma}((k-1)\zeta) - \Big(\langle \tilde{\gamma}((k-1)\zeta), \mathbf{e}_n\rangle + \frac{3}{2}Lw\Big)\mathbf{e}_n,$$
$$q := \gamma(k\zeta) - \Big(\langle \gamma(k\zeta), \mathbf{e}_n\rangle + \frac{3}{2}Lw\Big)\mathbf{e}_n,$$

where $\mathbf{e}_n$ is the n-th basis vector. Obviously $p, q \in U \cap \Omega$ (see Figure 1). If $t \in [k\zeta, (k+1)\zeta]$, then

$$\big|\gamma(t) - ((k-1)\zeta - t)\sqrt{1+L^2}\,\mathbf{e}_n\big| \le |\gamma(k\zeta)| + |\gamma(t) - \gamma(k\zeta)| + \zeta\sqrt{1+L^2}$$
$$< \zeta(2 + \sqrt{1+L^2}) + \zeta + \zeta\sqrt{1+L^2} = w\min(2L, 1).$$

Therefore we can define the piecewise linear curve $\hat{\gamma}\colon [0, \ell + 2\zeta] \to \Omega$ by

$$\hat{\gamma}(t) := \begin{cases} \tilde{\gamma}(t) & \text{if } t \in [0, (k-1)\zeta] \\ \frac{3((k-2/3)\zeta - t)}{\zeta}\,\tilde{\gamma}((k-1)\zeta) + \frac{3(t-(k-1)\zeta)}{\zeta}\,p & \text{if } t \in ((k-1)\zeta, (k-\frac{2}{3})\zeta] \\ \frac{3((k-1/3)\zeta - t)}{\zeta}\,p + \frac{3(t-(k-2/3)\zeta)}{\zeta}\,q & \text{if } t \in ((k-\frac{2}{3})\zeta, (k-\frac{1}{3})\zeta] \\ \frac{3(k\zeta - t)}{\zeta}\,q + \frac{3(t-(k-1/3)\zeta)}{\zeta}\big(\gamma(k\zeta) - \zeta\sqrt{1+L^2}\mathbf{e}_n\big) & \text{if } t \in ((k-\frac{1}{3})\zeta, k\zeta] \\ \gamma(t) - \big((k+1)\zeta - t\big)\sqrt{1+L^2}\mathbf{e}_n & \text{if } t \in (k\zeta, (k+1)\zeta] \\ \gamma(t) & \text{if } t \in ((k+1)\zeta, \ell + 2\zeta]. \end{cases}$$

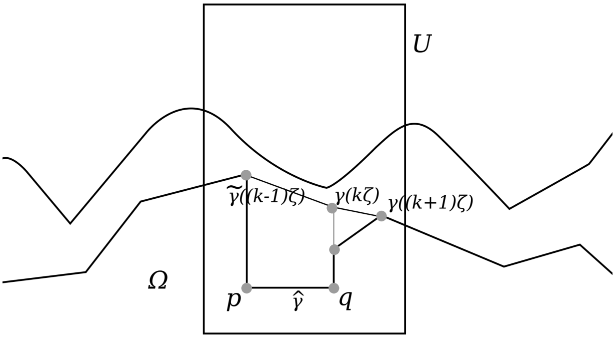

Fig. 1. The construction of $\hat{\gamma}$

It is obvious that $\hat{\gamma}$ satisfies conditions (5), (6) and (9) in $P(k)$. For condition (7) note that if $t \le (k-1)\zeta$, then $d(\hat{\gamma}(t), \partial\Omega) \ge \zeta$ by the inductive hypothesis. If $t \in [(k-1)\zeta, k\zeta]$, then $d\big(\hat{\gamma}(t), \partial\Omega\big) \ge \min\big(d(\hat{\gamma}((k-1)\zeta), \partial\Omega), d(\hat{\gamma}(k\zeta), \partial\Omega)\big)$. Clearly $d\big(\hat{\gamma}((k-1)\zeta), \partial\Omega\big) = d\big(\tilde{\gamma}((k-1)\zeta), \partial\Omega\big) \ge \zeta$. For $\hat{\gamma}(k\zeta)$, observe that $\tilde{\gamma}(k\zeta) \in \Omega$ and f has Lipschitz constant L. Therefore

$$d\big(\hat{\gamma}(k\zeta), \partial\Omega\big) \ge \frac{\zeta\sqrt{1+L^2}}{\sqrt{1+L^2}} = \zeta.$$

For condition (8) note that for all $t \in [k\zeta, (k+1)\zeta]$ we have

$$|\hat{\gamma}(t) - \gamma(t)| \le ((k+1)\zeta - t)\sqrt{1+L^2} \le \zeta\sqrt{1+L^2}\,.$$

So indeed $P(k)$ is satisfied. This completes the induction.

Finally, there exists a $k \in \mathbb{N}$ with $\ell \leq k\zeta < (k+1)\zeta \leq \ell+2\zeta$. If $\hat{\gamma}$ is as in $P(k)$ in Case 2, then $\hat{\gamma}(k\zeta) = \gamma(k\zeta) - \zeta\sqrt{1+L^2}\,\mathbf{e}_n = y - \zeta\sqrt{1+L^2}\,\mathbf{e}_n$. Define then $\check{\gamma}\colon [0, \ell+2\zeta] \to \Omega$ by

$$\check{\gamma}(t) := \begin{cases} \hat{\gamma}(t) & \text{if } t \in [0, k\zeta] \\ y - ((k+1)\zeta - t)\sqrt{1+L^2}\,\mathbf{e}_n & \text{if } t \in (k\zeta, (k+1)\zeta] \\ y & \text{if } t \in ((k+1)\zeta, \ell+2\zeta]. \end{cases}$$

Then $\check{\gamma}$ is a curve from x to y in Ω which is always at least ζ from $\partial\Omega$. Alternatively, if $\hat{\gamma}$ is as in $P(k)$ in Case 1, then $\hat{\gamma}$ is a curve from x to y in Ω which is always at least ζ from $\partial\Omega$.

Hence $B_\zeta(x)$ and $B_\zeta(y)$ are connected by a tube in Ω. ■

It is clear that we may shrink ζ.

Corollary 4.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with parameters L, w. Then for all*

$$\zeta < \frac{w \min(2L, 1)}{3 + 2\sqrt{1+L^2}}$$

and $x, y \in \Omega$ with $B_\zeta(x), B_\zeta(y) \subset \Omega$, the balls $B_\zeta(x)$ and $B_\zeta(y)$ are linked by a tube in Ω.

For all $r > 0$ define

$$\Omega_r = \{x \in \Omega : d(x, \partial\Omega) \geq r\}.$$

Corollary 4.2 can be reformulated.

Corollary 4.3. *If Ω is a Lipschitz domain with parameters L, w and*

$$r < \frac{w \min(2L, 1)}{3 + 2\sqrt{1+L^2}},$$

then Ω_r is connected.

We next control the length of tubes connecting small balls in Ω. To do this we force their curves to follow the edges of a cubic lattice.

Definition 4.4. For all $c, d \in \mathbb{Z}^n$ define the adjacency relation $\sim$ by $c \sim d$ if there exists an $i \in \{1, \ldots, n\}$ such that $c_j = d_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$ and $|c_i - d_i| = 1$. Let Ω be a Lipschitz domain with parameters L, w. Define

$$\delta := \frac{1}{8\sqrt{n}}\,\zeta, \tag{10}$$

where ζ is as in Proposition 4.1. Set

$$A = \{c \in \mathbb{Z}^n : \delta c \in \Omega_\zeta\} \quad \text{and} \quad B = \{c \in \mathbb{Z}^n : \delta c \in \Omega_{\zeta/2}\}.$$

The next lemma allows us to approximate arbitrary curves by ones that follow the edges of the cubic lattice $\delta\mathbb{Z}^n \cap \Omega_{\zeta/2}$.

Lemma 4.5. *For all $c, d \in A$ there exist $J \in \mathbb{N}_0$ and $m^{(0)}, \ldots, m^{(J)} \in B$ such that $m^{(0)} = c$, $m^{(J)} = d$ and $m^{(k-1)} \sim m^{(k)}$ for all $k \in \{1, \ldots, J\}$.*

Proof. By Proposition 4.1, there exists a piecewise C^1-curve $\gamma\colon [0,1] \to \Omega_\zeta$ such that $\gamma(0) = \delta c$ and $\gamma(1) = \delta d$. Let W be the set of all $t \in [0,1]$ such that there exist $p \in B$, $J \in \mathbb{N}_0$ and $m^{(0)}, \ldots, m^{(J)} \in B$ such that

$$\left.\begin{aligned} &|\delta p - \gamma(t)| < \sqrt{n}\,\delta, \\ &m^{(0)} = c,\ m^{(J)} = p, \\ &m^{(k-1)} \sim m^{(k)} \text{ for all } k \in \{1,\ldots,J\}. \end{aligned}\right\} \tag{11}$$

Then it is clear that $0 \in W$ and W is open. Let $s = \sup W$. Then $s > 0$. We shall show that $s \in W$. This then implies that $s = 1$ and the lemma follows.

By the continuity of γ there exists a $t \in W$ such that $|\gamma(t) - \gamma(s)| < \delta$. By definition of W there exist $p \in B$, $J \in \mathbb{N}_0$ and $m^{(0)}, \ldots, m^{(J)} \in B$ such that (11) is valid. There exists a $q \in \mathbb{Z}^n$ such that

$$|\delta q - \gamma(s)| < \sqrt{n}\,\delta.$$

Then $|\delta p - \delta q| < (1 + 2\sqrt{n})\,\delta$ and so $|p - q| < 1 + 2\sqrt{n}$. There are $M \in \mathbb{N}_0$ and $r^{(0)}, \ldots, r^{(M)} \in \mathbb{Z}^n$ such that $r^{(0)} = p$, $r^{(M)} = q$, $r^{(k-1)} \sim r^{(k)}$ and $|r^{(k)} - p| < 1 + 2\sqrt{n}$ for all $k \in \{1, \ldots, M\}$. Then $|\delta r^{(k)} - \gamma(t)| < (1 + 3\sqrt{n})\,\delta$ for all $k \in \{1, \ldots, M\}$. Hence

$$d(\delta r^{(k)}, \partial\Omega) \geq d(\gamma(t), \partial\Omega) - |\delta r^{(k)} - \gamma(t)| \geq \zeta - (1 + 3\sqrt{n})\,\delta \geq \frac{1}{2}\zeta$$

and so $r^{(k)} \in B$ for all $k \in \{1, \ldots, M\}$. Therefore $m^{(0)}, \ldots, m^{(J)}, r^{(1)}, \ldots, r^{(M)}$ is a sequence of the sort we are looking for. This proves the lemma. ∎

We now use the cubic lattice structure to estimate N and K.

LEMMA 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with parameters L, w and diameter D and define δ as in* (10). *Then there exists a ball $\Omega_0 \subset \Omega$ of radius δ such that Ω is the union of N open sets linked to Ω_0 in Ω by tubes of length at most K, where*

$$N = \left\lceil \frac{D\sqrt{n}}{\delta} \right\rceil^n$$

and

$$K = \delta \left\lceil \frac{D}{\delta} \right\rceil^n + \sqrt{n}\,\delta + 2Lw.$$

Proof. We use the notation of Definition 4.4. Clearly $A \neq \emptyset$ by Definition 1.1. Choose $d \in A$ and take $\Omega_0 = B_\delta(d)$. A ball of radius δ contains an open cube of side-length $\frac{2\delta}{\sqrt{n}}$. Let Q be a cube of side-length D such that $\Omega \subset Q$. It is clear that Q can be covered with $\lceil \frac{D\sqrt{n}}{\delta} \rceil^n$ closed cubes of side-length $\frac{\delta}{\sqrt{n}}$, hence Ω can be covered with $\lceil \frac{D\sqrt{n}}{\delta} \rceil^n$ open cubes of side-length $\frac{2\delta}{\sqrt{n}}$. Moreover, we may assume that all of these cubes have parallel axes.

Let Q' be one of these open cubes of side-length $\frac{2\delta}{\sqrt{n}}$ and suppose that $Q' \cap \Omega \neq \emptyset$. We shall show that $Q' \cap \Omega$ is linked to Ω_0 in Ω by a tube of length at most K.

Let x be the centre of the cube Q'. We distinguish two cases.

Case 1. Suppose $d(x, \partial\Omega) < \zeta + 2\sqrt{n}\,\delta$. Then there exists a $z_0 \in \partial\Omega$ such that $|x - z_0| < \zeta + 2\sqrt{n}\delta$. Let Φ, U, f be as in Definition 1.1. Without loss of generality we may again assume that Φ is the identity map. Then $z_0 = 0$. Write $x = (x', x_n)$. Then $|x| < (1 + \frac{1}{4})\zeta \leq$

$\frac{1}{4}w\min(2L,1)$. Moreover, if $y \in Q'$, then $|x-y| < \delta \le \frac{1}{40}w\min(2L,1)$. So $Q' \cap \Omega \subset U \cap \Omega$. Let $\ell = x_n + \frac{3}{2}Lw$. Note that $\ell \le 2Lw$. Consider the curve $c\colon [0,\ell] \to \mathbb{R}^n$ defined by $c(t) = (0,\ldots,0,-t)$. Then $c(t) + (Q' \cap \Omega) \subset \Omega$ for all $t \in [0,\ell]$. Take $p = (x', -\frac{3}{2}Lw)$. We then have

$$c(\ell) + (Q' \cap \Omega) \subset B_\delta(p) \subset \Omega_{\zeta+\sqrt{n}\,\delta}.$$

This last inclusion is because $\frac{Lw}{2} - \delta \ge \zeta + \sqrt{n}\delta$, which follows from the fact that $\zeta \le \frac{2Lw}{5}$.

Case 2. Suppose $d(x,\partial\Omega) \ge \zeta + 2\sqrt{n}\delta$. Then $x \in \Omega$ since $Q' \cap \Omega \ne \emptyset$. So $x \in \Omega_{\zeta+2\sqrt{n}\delta}$. Then

$$(Q' \cap \Omega) \subset B_\delta(p) \subset \Omega_{\zeta+\sqrt{n}\delta},$$

where now $p = x$.

In both cases there exists a $c \in A$ such that $|\delta c - p| < \sqrt{n}\delta$. Thus the straight line segment from p to δc is in Ω_ζ and has length at most $\sqrt{n}\delta$. It follows from Lemma 4.5 that there exists a curve from c to d along the edges of the cubic lattice $(B,\sim)$. There are at most $\lceil \frac{D}{\delta} \rceil^n$ points in B, so the shortest curve from c to d has length at most $\delta \lceil \frac{D}{\delta} \rceil^n$. Combining the two or three curves, one deduces that $Q' \cap \Omega$ is linked to Ω_0 in Ω by a tube of length at most

$$K = \delta \left\lceil \frac{D}{\delta} \right\rceil^n + \sqrt{n}\,\delta + 2Lw,$$

and the proof of the lemma is complete. ∎

In two dimensions it is also possible to give a different estimate for K, based on Pappus' theorem.

Now we have all that we need for the proof of our theorem. The proof is a modification of the proof of Theorem 1.1 in [Jam].

Proof of Theorem 1.2. Let N, K and δ be as in Lemma 4.6. Let Ω_0 and Ω_1 be concentric balls of radius δ and D respectively such that $\Omega_0 \subset \Omega \subset \Omega_1$. Let $E\colon L_2(\Omega_0) \to L_2(\Omega_1)$ be the extension operator as in Lemma 3.1. We denote the restriction of a function f to a set V by $r_V f$. Let $u \in H^1(\Omega)$. Since E leaves invariant the constant functions, we can decompose

$$u - \overline{u}_{\Omega_0} = (u - r_\Omega E r_{\Omega_0} u) + r_\Omega E r_{\Omega_0}(u - \overline{u}_{\Omega_0}).$$

Because $\overline{u}_\Omega$ is the best constant approximation to u in $L_2(\Omega)$, one deduces that

$$\|u - \overline{u}_\Omega\|_{L_2(\Omega)} \le \|u - \overline{u}_{\Omega_0}\|_{L_2(\Omega)} \le \|u - r_\Omega E r_{\Omega_0} u\|_{L_2(\Omega)} + \|E r_{\Omega_0}(u - \overline{u}_{\Omega_0})\|_{L_2(\Omega_1)}. \quad (12)$$

We estimate the two terms separately. First, $u - r_\Omega E r_{\Omega_0} u$ vanishes on Ω_0. Therefore by Lemma 2.3

$$\begin{aligned}\|u - r_\Omega E r_{\Omega_0} u\|_{L_2(\Omega)} &\le K\sqrt{N}\,\|\nabla(u - r_\Omega E r_{\Omega_0} u)\|_{L_2(\Omega)} \\ &\le K\sqrt{N}\,\big(\|\nabla u\|_{L_2(\Omega)} + \|\nabla E r_{\Omega_0} u\|_{L_2(\Omega_1)}\big).\end{aligned}$$

Then (4) gives

$$\|\nabla E r_{\Omega_0} u\|_{L_2(\Omega_1)} \le 2^{(n+2)/2}\Big(\frac{D}{\delta}\Big)^{n/2} \|\nabla r_{\Omega_0} u\|_{L_2(\Omega_0)} \le 2^{(n+2)/2}\Big(\frac{D}{\delta}\Big)^{n/2} \|\nabla u\|_{L_2(\Omega)}.$$

Next consider the second term in the right-hand side of (12). By (3) and the Poincaré inequality on the ball Ω_0 (see [Beb]) one has

$$\begin{aligned}\|Er_{\Omega_0}(u-\overline{u}_{\Omega_0})\|_{L_2(\Omega_1)} &\le 2^{(n+2)/2}\Big(\frac{D}{\delta}\Big)^{n/2}\|u-\overline{u}_{\Omega_0}\|_{L_2(\Omega_0)}\\ &\le 2^{(n+2)/2}\Big(\frac{D}{\delta}\Big)^{n/2}\frac{2\delta}{\pi}\|\nabla u\|_{L_2(\Omega_0)}\\ &\le 2^{(n+2)/2}\delta\Big(\frac{D}{\delta}\Big)^{n/2}\|\nabla u\|_{L_2(\Omega)}.\end{aligned}$$

So

$$C_\Omega \le 2^{(n+2)/2}\,\delta\,\Big(\frac{D}{\delta}\Big)^{n/2} + K\sqrt{N}\,\Big(1+2^{(n+2)/2}\Big(\frac{D}{\delta}\Big)^{n/2}\Big).$$

Since $\delta \le \frac{D}{8\sqrt{n}}$ the estimate (1) follows. ■

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