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ABSTRACT EXTENSIONS OF FUNCTIONAL CALCULI

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Abstract. We describe a 'minimal' and a 'maximal' method of extending a certain basic functional calculus towards unbounded operators. The maximal method is a generalisation of (and consistent with) the approach via regularisation, favoured in the author's earlier papers, and it covers also Balakrishnan's calculus for generators of C_0-semigroups. It is shown that for many special cases the maximal and the minimal extensions coincide. However, some simple but rather startling examples are devised to show not only that the two extensions may differ, but also that each of them may yield an intuitively 'wrong' result.

1. Introduction. Although[1] the notion of a functional calculus for a linear operator A on a Banach space X is nowadays quite common, it has not yet found a definite formal realisation. Intuitively, to a function f one wants to associate a new operator — written $f(A)$ — that deserves this notation. The guiding example for, and at the same time the most trivial instance of, a functional calculus is the case of A being multiplication with a function a on some function space; then $f(A)$ should be multiplication with the function $f \circ a$. The same example in disguise is given by (one form of) the spectral theorem for a normal operator on a Hilbert space; this functional calculus has been used so successfully that it is natural to generalise it as far as possible towards other classes of operators.

It is clear from this fundamental example that in general it is not desirable to restrict to bounded operators A nor to aim exclusively at bounded operators $f(A)$: many important operators are unbounded, and if A is not bounded then $A^{1/2}$ — if reasonably defined — is not bounded too. However, bounded operators may be easier to work with for the time being, and so one usually employs a two-step procedure: first define $f(A)$ as a bounded

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[1]This article is a revised version of the author's earlier paper [11].

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operator for many functions f, then use this to define $f(A)$ as a possibly unbounded operator, for even more functions f.

It seems reasonable to require that the first step comprises a whole *algebra* of functions and that the assignment $f \mapsto f(A)$ is a homomorphism of algebras. (This is not without debate and there are instances in the literature deviating from this principle, see e.g., [5, 15, 13]. See also Section 8 below.) Put abstractly, the first step provides us with a mapping

$$\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X),$$

where X is a Banach space, $\mathcal{E}$ is a commutative algebra (frequently an algebra of functions defined at least on the spectrum of the operator A) and Φ is a homomorphism of algebras, i.e., a representation. (In concrete situations, the construction of Φ uses analytical properties of A, and $\mathcal{E}$ carries a natural topology such that Φ is continuous.)

The *extension problem* for the calculus $(\mathcal{E}, \Phi)$ is now: Is there a commutative algebra $\mathcal{F}$ containing $\mathcal{E}$ as a subalgebra and a “natural” extension of Φ to $\mathcal{F}$ yielding possibly unbounded operators on X? And if yes, how large will such an extension be, maximally?

According to the imprecise formulation of these questions, no definite general answer is to be expected. In some cases the answer to the first question is “yes” simply because in the original construction of Φ one was not clever enough, i.e., $\mathcal{E}$ was chosen unnecessarily small. For instance, let A be a bounded operator on X and let $\mathcal{E}$ be the algebra of all complex polynomials. Then one may extend the natural representation $p \mapsto p(A)$ in a straightforward manner to rational functions with poles outside $\sigma(A)$, and eventually by using Cauchy-integrals to the Dunford–Riesz calculus, see [6, Sect. VII.4]. Using more information about the operator, for example sectoriality in certain boundary points of the spectrum, gives rise to even larger (bounded) functional calculi.

Extensions of this kind have their reason in intrinsic (and often analytical) properties of the operators and the function algebras involved. In this paper, we address the *algebraic extension problem*: find a “natural” extension of Φ from $\mathcal{E}$ to a (given) larger algebra $\mathcal{F} \supseteq \mathcal{E}$ using only the algebraic properties of the representation Φ.

Following ideas of Bade [2] and McIntosh [14], deLaubenfels [7, 8] in the 1990s suggested different solutions to the extension problem. His “Construction Two” from [8] coincides in principle with what has been called *extension by regularisation* in [9][2]. By a *regularisable* element we mean $f \in \mathcal{F}$ such that there is an element $e \in \mathcal{E}$ (the *regulariser* of f) such that $ef \in \mathcal{E}$ and $\Phi(e)$ is injective. Using this regulariser we may define

$$\Phi(f) := \Phi(e)^{-1}\Phi(ef), \tag{1}$$

yielding a closed but possibly unbounded operator $\Phi(f)$. (The construction is of course independent of the regulariser.) Although this approach is elegant and simple and has been successfully used in a variety of situations (see, e.g., [10]), there are reasons to believe that this is not the final word on the matter.

[2]Regrettably, the author learned only later about deLaubenfels' paper, and hence reference to it appears neither in [9] nor in [10].

Example. Let $X = \mathrm{C}[0,1]$ with supremum norm, $\mathcal{E} = \mathrm{C}[0,1]$ as well and let $\mathcal{F}$ be the algebra of *all* complex functions on $[0,1]$. For $f \in \mathcal{F}$ define the *multiplication operator* M_f on X by

$$\mathrm{dom}(M_f) = \{x \in X \mid fx \in \mathrm{C}[0,1]\}, \quad M_f x = fx \qquad (x \in \mathrm{dom}(M_f)).$$

This is a closed operator, as is easily seen. Let $\Phi : \mathcal{E} \to \mathcal{L}(X)$ defined by $\Phi(e) = M_e$. This is the functional calculus for the operator $A = M_a$, with $a(t) = t$, $t \in [0,1]$.

Consider the function $f = \mathbf{1}_{(0,1]}$. It is easy to see that M_f is the restriction of the identity operator to the closed subspace $\mathrm{dom}(M_f) = \{x \in X \mid x(0) = 0\}$; and this operator is clearly the right candidate for the operator $f(A)$. Now let us look what the regularisation approach (1) yields. Since multiplication with a is injective on $\mathrm{C}[0,1]$, and $af \in \mathrm{C}[0,1]$, the function a is a regulariser of f. Moreover $af = a$, and

$$\Phi(f) := \Phi(a)^{-1}\Phi(af) = \Phi(a)^{-1}\Phi(a) = \mathrm{I}$$

is the full identity operator. Hence the functional calculus extended by regularisation does not see the discontinuity of the function f, and $M_f \neq \Phi(f)$.

The given example indicates that extension via regularisation is not always the method of choice. A second motivation to go beyond regularisation comes from Balakrishnan's construction of an unbounded extension of the Hille–Phillips calculus for semigroup generators [4]. It could be shown in [3] that in the case that the semigroup is actually a bounded *group*, Balakrishnan's approach is entirely covered (and extended) by the regularisation approach. Whether this is also true in the generic semigroup case, however, remains still an open problem.

It is one consequence of the present paper that the answer to this question does not matter too much. As we shall see below, the regularisation approach can be slightly generalised to what is called here the "upper extension", which is as simple and efficient and has more or less the same properties (Section 3). Balakrishnan's approach, by contrast, is an instance of what is called here the "lower extension". This method appears to have less good properties in general (Section 4), but in the special case of C_0-semigroup generators both methods coincide (Section 7.5).

Upper and lower extension. Let us return to the abstract situation, now a little less cursory. By an *abstract functional calculus* (afc) we mean a triple $(\mathcal{E}, \mathcal{F}, \Phi)$, where $\mathcal{F}$ is a commutative algebra (over $\mathbb{C}$) with unit $\mathbf{1}$, $\mathcal{E} \subseteq \mathcal{F}$ is a non-trivial subalgebra (without unit in general), and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is a representation of $\mathcal{E}$ on a complex Banach space X. We aim at a "reasonable" extension of Φ to a mapping from $\mathcal{F}$ to the set of all closed, possibly multi-valued operators. (By a multi-valued operator we simply mean a linear subspace of $X \oplus X$, identifying usual operators with their graph, see [10, App. A]. The use of multi-valued operators in the given context turns out to be quite natural and actually simplifies the proofs.)

For $f \in \mathcal{F}$ we define

$$[f]_{\mathcal{E}} := \{e \in \mathcal{E} \mid ef \in \mathcal{E}\},$$

which is an ideal of $\mathcal{E}$. From a tentative extension $\hat{\Phi}$ of Φ to $\mathcal{F}$ it is reasonable to expect the validity of the identities

$$\Phi(e)\hat{\Phi}(f)x = \Phi(ef)x = \hat{\Phi}(f)\Phi(e)x \qquad (e \in [f]_{\mathcal{E}}) \tag{2}$$

for an appropriate choice of $x \in X$. This translates into two postulates:

1) If $x \in X$ and $e \in [f]_{\mathcal{E}}$ then $(\Phi(e)x, \Phi(ef)x) \in \hat{\Phi}(f)$.
2) If $(x, y) \in \hat{\Phi}(f)$ and $e \in [f]_{\mathcal{E}}$ then $\Phi(ef)x = \Phi(e)y$.

(The right-hand equality of (2) accounts for 1), the left-hand for 2).) We hence obtain

$$\underline{\Phi}(f) \subseteq \hat{\Phi}(f) \subseteq \overline{\Phi}(f),$$

where the (multi-valued) operators $\underline{\Phi}(f)$, $\overline{\Phi}(f)$ are defined by

$$\underline{\Phi}(f) := \overline{\operatorname{span}}\big\{\big(\Phi(e)x, \Phi(ef)x\big) \,|\, e \in [f]_{\mathcal{E}},\ x \in X\big\},$$
$$\overline{\Phi}(f) := \big\{(x, y) \,|\, \Phi(ef)x = \Phi(e)y \ \forall\, e \in [f]_{\mathcal{E}}\big\},$$

and are called the *lower extension* and the *upper extension* of Φ, respectively. Since obviously $\underline{\Phi}(f) \subseteq \overline{\Phi}(f)$, both $\underline{\Phi}$ and $\overline{\Phi}$ are themselves natural candidates for the desired extension. It turns out that if f is regularisable by an element e, then

$$\Phi(e)^{-1}\Phi(ef) = \overline{\Phi}(f)$$

(see Section 3.1 below). On the other hand it is easy to see that in the above mentioned example one has $\underline{\Phi}(f) = M_f$ for all $f : [0, 1] \to \mathbb{C}$. But we shall see in Section 7.1 that there are situations where $\overline{\Phi}(f)$ is expected and $\underline{\Phi}(f)$ is too small. So is one of them "more natural" in general? Or is there a "natural" choice of operator lying in between? Although these questions have to remain open for the time being, we hope to contribute with the following investigations to their eventual resolution.

Terminology. In this paper X denotes a non-trivial complex Banach space and X' is its dual space. A (multi-valued) *operator* is simply a linear subspace of $X \oplus X$, i.e., a linear relation. The operator $A \subseteq X \oplus X$ is called *closed* if it is a closed subspace of $X \oplus X$. Associated with an operator $A \subseteq X \oplus X$ is its *inverse* $A^{-1} := \{(x, y) \,|\, (y, x) \in A\}$, its *domain* $\operatorname{dom}(A) := \{x \in X \,|\, \exists\, y \in X : (x, y) \in A\}$, its *range* $\operatorname{ran}(A) := \operatorname{dom}(A^{-1})$, and its *kernel* $\ker(A) := \{x \,|\, (x, 0) \in A\}$. The subspace

$$\operatorname{mul}(A) := \ker(A^{-1}) = \{y \,|\, (0, y) \in A\}$$

has no established name, but it is a measure of the "multi-valuedness" of A. If $\operatorname{mul}(A) = \{0\}$ then A is called *single-valued*, and A is a functional relation, i.e., a graph of a proper operator. If $\operatorname{dom}(A) = X$ then A is called *fully defined*. The set of single-valued, fully defined bounded linear operators on X is denoted by $\mathcal{L}(X)$. For multi-valued operators one can define sum, product and scalar multiples in a canonical way. Also, there is a natural definition of the *adjoint* $A' \subseteq X' \oplus X'$. For more information on multi-valued operators we refer to [10, App. A].

If X is a Banach space and a topological notion has to be understood with respect to the weak* topology on X', we shall indicate this with a "w*-" in front. For instance, we shall speak of w*-closed sets, of w*-convergent sequences and so on. As X is the dual space of X' with respect to the w*-topology, any operator B on X' has a w*-*adjoint*, which is simply $B' \cap X \oplus X$.

2. Basic theory. Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an abstract functional calculus over X. In general it may happen that for $e \in \mathcal{E}$ neither $\underline{\Phi}(e)$ nor $\overline{\Phi}(e)$ coincides with $\Phi(e)$. This is certainly an undesirable situation, as then the name "extension" has to be read *cum grano salis*. Let us examine this in more detail.

LEMMA 2.1.

a) *The following assertions are equivalent*:
   (i) $\underline{\Phi}(\mathbf{1}) = \mathrm{I}$.
   (ii) $\underline{\Phi}(e) = \Phi(e)$ *for all* $e \in \mathcal{E}$.
   (iii) $\bigcup_{e\in\mathcal{E}} \operatorname{ran}(\Phi(e))$ *is a total subset of* X.

b) *The following assertions are equivalent*:
   (i) $\overline{\Phi}(\mathbf{1}) = \mathrm{I}$.
   (ii) $\overline{\Phi}(e) = \Phi(e)$ *for all* $e \in \mathcal{E}$.
   (iii) $\bigcap_{e\in\mathcal{E}} \ker(\Phi(e)) = \{0\}$.

Proof. This is straightforward. ∎

We call the afc $(\mathcal{E}, \mathcal{F}, \Phi)$ *upper non-degenerate* if $\overline{\Phi}(\mathbf{1}) = \mathrm{I}$, and *lower non-degenerate* if $\underline{\Phi}(\mathbf{1}) = \mathrm{I}$. An afc is called *standard* if it is both lower and upper non-degenerate.

REMARK 2.2. In [9, 10] an afc is called non-degenerate, if there is $e \in \mathcal{E}$ such that $\Phi(e)$ is injective. Clearly, if an afc is non-degenerate in this sense, it is upper non-degenerate, but nothing indicates that this should also imply lower non-degeneracy.

2.1. Remedies for degeneracy. Clearly, an afc is standard if $\mathbf{1} \in \mathcal{E}$ and $\Phi(\mathbf{1}) = \mathrm{I}$. If $\mathbf{1} \notin \mathcal{E}$, then one may enlarge Φ to a unital representation of the algebra $\mathcal{E} \oplus \mathbb{C}\mathbf{1}$, thus leaving us with a standard afc. If $\mathbf{1} \in \mathcal{E}$ and $P := \Phi(\mathbf{1}) \neq \mathrm{I}$ then P is a nontrivial projection onto the space $Y := PX$, and one has an induced standard afc on Y. Hence there is no loss of generality in supposing that an afc is standard, and we shall sometimes do so.

2.2. The dual calculus. As the algebra $\mathcal{F}$ is commutative, with each afc $(\mathcal{E}, \mathcal{F}, \Phi)$ over a Banach space X there comes along an afc $(\mathcal{E}, \mathcal{F}, \Phi')$ over the dual space X', defined by

$$\Phi'(e) := \Phi(e)' \qquad (e \in \mathcal{E}).$$

Iterating this yields the calculi $\Phi'', \Phi''', \ldots$. The next technical result helps to relate the upper and the lower extension by means of the dual calculi.

THEOREM 2.3. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc and $f \in \mathcal{F}$. Then*

$$\overline{\Phi''}(f) \cap (X \oplus X) = \overline{\Phi}(f) \quad \textit{and} \quad \underline{\Phi}(f)' = \overline{\Phi'}(f).$$

In particular, the operator $\overline{\Phi'}(f)$ is w^*-closed. Furthermore

$$\overline{\Phi}(f)' = \overline{\underline{\Phi'}(f)}^{w^*} \quad \textit{and} \quad \underline{\Phi''}(f) \cap (X \oplus X) = \underline{\Phi}(f).$$

Proof. The first assertion follows from $\Phi(e)'' \cap (X \oplus X) = \Phi(e)$ for all $e \in \mathcal{E}$. Take $x', y' \in X'$. Then, identifying $(X \oplus X)' = X' \oplus X'$,

$$\begin{aligned}
(x', y') \in \overline{\Phi'}(f) \quad &\iff \quad \forall\, e \in [f]_{\mathcal{E}} :\ \Phi(ef)'x' = \Phi(e)'y' \\
&\iff \quad \forall\, e \in [f]_{\mathcal{E}},\ x \in X :\ \langle \Phi(ef)x, x' \rangle = \langle \Phi(e)x, y' \rangle \\
&\iff \quad \forall\, e \in [f]_{\mathcal{E}},\ x \in X :\ (y', -x') \perp (\Phi(e)x, \Phi(ef)x) \\
&\iff \quad (y', -x') \perp \underline{\Phi}(f).
\end{aligned}$$

This yields $\overline{\Phi'}(f) = \underline{\Phi}(f)'$. Using this with Φ' in place of Φ we obtain

$$\overline{\Phi}(f) = \overline{\Phi''}(f) \cap (X \oplus X) = \underline{\Phi'}(f)' \cap (X \oplus X)$$

and this is equal to the w*-adjoint of $\underline{\Phi'}(f)$. Hence $\overline{\Phi}(f)'$ is equal to the w*-double adjoint of $\underline{\Phi'}(f)$, which is its w*-closure. Using the established facts yields

$$\overline{\underline{\Phi''}(f)}^{w^*} = \overline{\Phi'}(f)' = \underline{\Phi}(f)'',$$

and intersecting this with $X \oplus X$ we arrive at

$$\underline{\Phi''}(f) \cap (X \oplus X) \subseteq \overline{\underline{\Phi''}(f)}^{w^*} \cap (X \oplus X) = \overline{\Phi'}(f)' \cap (X \oplus X) = \underline{\Phi}(f).$$

Finally, since the embedding $X \subseteq X''$ is isometric, $\underline{\Phi}(f) \subseteq \underline{\Phi''}(f) \cap (X \oplus X)$. ∎

Theorem 2.3 exhibits a certain duality between the notions of upper and lower extension. Therefore, some properties of the lower extension can be derived from properties of the upper extension. These will be investigated in the coming section.

3. Properties of the upper extension. In this section we suppose that $(\mathcal{E}, \mathcal{F}, \Phi)$ is an afc over a Banach space X. We shall study the upper extension

$$\overline{\Phi} : \mathcal{F} \longrightarrow \{\text{closed multi-valued operators on } X\}$$

(which, we recall, is only an extension in the proper sense if the afc is upper non-degenerate). It is certainly of interest to have criteria for the case that $\overline{\Phi}(f)$ is single-valued. Let us call a subset $E \subseteq \mathcal{E}$ an $\mathcal{E}$-*anchor* if

$$\bigcap_{e \in E} \ker(\Phi(e)) = \{0\}.$$

If $\mathcal{E}$ is understood we drop reference to it and just speak of an *anchor*. If $E \subseteq [f]_{\mathcal{E}}$ is an anchor then we call it an *anchor for* f; and we call $f \in \mathcal{F}$ *anchored* (in $\mathcal{E}$) if it has an ($\mathcal{E}$-)anchor. Obviously, the existence of an anchored element is equivalent to the upper non-degeneracy of Φ.

The next lemma shows that for an anchored element f the operator $\overline{\Phi}(f)$ is determined by any anchor for f.

Lemma 3.1. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc and let $f \in \mathcal{F}$. Then*

$$\operatorname{mul}\big(\overline{\Phi}(f)\big) = \{y \mid (0, y) \in \overline{\Phi}(f)\} = \bigcap_{e \in [f]_{\mathcal{E}}} \ker\big(\Phi(e)\big).$$

In particular, the operator $\overline{\Phi}(f)$ is single-valued if and only if f is anchored. If $E \subseteq \mathcal{E}$ is an anchor for f and $x, y \in X$, then

$$(x, y) \in \overline{\Phi}(f) \quad \iff \quad \Phi(ef)x = \Phi(e)y \quad \textit{for all } e \in E.$$

Proof. Clearly $(0, y) \in \overline{\Phi}(f)$ is equivalent to $\Phi(e)y = 0$ for all $e \in [f]_{\mathcal{E}}$. In the equivalence only the implication "$\Leftarrow$" is not trivial. So let $\tilde{e} \in [f]_{\mathcal{E}}$ be arbitrary. Then

$$\Phi(e)\Phi(\tilde{e}f)x = \Phi(e\tilde{e}f)x = \Phi(\tilde{e})\Phi(ef)x = \Phi(\tilde{e})\Phi(e)y = \Phi(e)\Phi(\tilde{e})y$$

for all $e \in E$. Since E is an anchor, $\Phi(\tilde{e}f)x = \Phi(\tilde{e})y$. It follows that $(x, y) \in \overline{\Phi}(f)$ as claimed. ■

Let us introduce the set

$$\mathcal{F}_{\mathrm{sv}} = \mathcal{F}_{\mathrm{sv}}(\overline{\Phi}) := \{f \in \mathcal{F} \mid \overline{\Phi}(f) \text{ is single-valued}\}.$$

Then we obtain the following result.

THEOREM 3.2. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc over a Banach space X. Then the following assertions hold.*

a) *If $S \in \mathcal{L}(X)$ commutes with every $\Phi(e), e \in \mathcal{E}$, then S commutes with every $\overline{\Phi}(f)$, $f \in \mathcal{F}$.*
b) *The afc is upper non-degenerate if and only if $\mathcal{F}_{\mathrm{sv}} \neq \emptyset$.*
c) *Let $f, g \in \mathcal{F}$ such that $f \in \mathcal{F}_{\mathrm{sv}}$. Then*

$$\overline{\Phi}(f) + \overline{\Phi}(g) \subseteq \overline{\Phi}(f+g), \qquad \overline{\Phi}(f)\overline{\Phi}(g) \subseteq \overline{\Phi}(fg).$$

If, in addition, $fg \in \mathcal{F}_{\mathrm{sv}}$ then

$$\mathrm{dom}(\overline{\Phi}(fg)) \cap \mathrm{dom}(\overline{\Phi}(g)) = \mathrm{dom}(\overline{\Phi}(f)\overline{\Phi}(g)).$$

d) *If $f, g \in \mathcal{F}_{\mathrm{sv}}$ then $f + g, fg \in \mathcal{F}_{\mathrm{sv}}$.*
e) *If $fg = \mathbf{1}$ then $\overline{\Phi}(f)^{-1} = \overline{\Phi}(g)$.*

Proof. a) If $(x, y) \in \overline{\Phi}(f)$ then $\Phi(e)y = \Phi(ef)x$ for all $e \in [f]_{\mathcal{E}}$. By hypothesis, this implies $\Phi(e)Sy = \Phi(ef)Sx$ for all $e \in [f]_{\mathcal{E}}$, hence $(Sx, Sy) \in \overline{\Phi}(f)$.

b) This is a straightforward consequence of Lemma 2.1.

c) Take $(x, y) \in \overline{\Phi}(f)$ and $(x, z) \in \overline{\Phi}(g)$, and fix $e \in [f+g]_{\mathcal{E}}$ and $e' \in [f]_{\mathcal{E}}$. Then clearly $ee' \in [f+g]_{\mathcal{E}} \cap [g]_{\mathcal{E}} \cap [f]_{\mathcal{E}}$. Hence

$$\begin{aligned}\Phi(e')\Phi(e(f+g))x &= \Phi(ee'(f+g))x = \Phi(ee'f)x + \Phi(ee'g)x \\ &= \Phi(ee')y + \Phi(ee')z = \Phi(e')\Phi(e)[y+z].\end{aligned}$$

As $[f]_{\mathcal{E}}$ is an anchor, we conclude that $\Phi(e(f+g))x = \Phi(e)(y+z)$, and as $e \in [f+g]_{\mathcal{E}}$ was arbitrary, this yields $(x, y+z) \in \overline{\Phi}(f+g)$.

Similarly, take $(x, y) \in \overline{\Phi}(g)$ and $(y, z) \in \overline{\Phi}(f)$. We have to show that $(x, z) \in \overline{\Phi}(fg)$. Fix $e \in [fg]_{\mathcal{E}}$ and take $e' \in [f]_{\mathcal{E}}$. Then $ee'f \in [g]_{\mathcal{E}}$ and hence

$$\Phi(e')\Phi(efg)x = \Phi(ee'fg)x = \Phi(ee'f)y = \Phi(ee')z = \Phi(e')\Phi(e)z.$$

As $[f]_{\mathcal{E}}$ is an anchor, we conclude that $\Phi(efg)x = \Phi(e)z$, and as $e \in [fg]_{\mathcal{E}}$ was arbitrary, this yields $(x, z) \in \overline{\Phi}(fg)$.

To prove the statement concerning the domains, note that the inclusion "$\supseteq$" follows from what we already know. Suppose that $(x, y) \in \overline{\Phi}(g)$ and $(x, z) \in \overline{\Phi}(fg)$. Take $e \in [f]_{\mathcal{E}}$ and $e' \in [fg]_{\mathcal{E}}$; then $ee'f \in [g]_{\mathcal{E}}$ and hence

$$\Phi(e')\Phi(ef)y = \Phi(ee'f)y = \Phi(ee'fg)x = \Phi(ee')z = \Phi(e')\Phi(e)z.$$

By hypothesis, $[fg]_{\mathcal{E}}$ is an anchor and hence $\Phi(ef)y = \Phi(e)z$. As $e \in [f]_{\mathcal{E}}$ was arbitrary, $(y, z) \in \overline{\Phi}(f)$.

d) The set $\{ee' \mid e \in [f]_{\mathcal{E}}, e' \in [g]_{\mathcal{E}}\}$ is an anchor for fg and $f + g$.

e) Suppose that $(x, y) \in \overline{\Phi}(f)$ and let $e \in [g]_{\mathcal{E}}$. Since $egf = e \in \mathcal{E}$, $eg \in [fg]_{\mathcal{E}}$. Hence $\Phi(eg)y = \Phi(egf)x = \Phi(e)x$, and as $e \in [g]_{\mathcal{E}}$ was arbitrary, we conclude that $(y, x) \in \overline{\Phi}(g)$. This gives $\overline{\Phi}(f) \subseteq \overline{\Phi}(g)^{-1}$, and by symmetry it follows that $\overline{\Phi}(g)^{-1} = \overline{\Phi}(f)$. ∎

Remark 3.3. By Theorem 3.2, the upper extension shares all formal properties with the calculi based on regularisation, cf. [10, Chapter 1].

Theorem 3.2 implies that $\mathcal{F}_{\mathrm{sv}}$ is empty or a subalgebra of $\mathcal{F}$. Let us write

$$\mathcal{F}_{\vee} = \mathcal{F}_{\vee}(\Phi) := \{f \in \mathcal{F} \mid \overline{\Phi}(f) \in \mathcal{L}(X)\}.$$

This is also either empty or a subalgebra of $\mathcal{F}_{\mathrm{sv}}$.

Corollary 3.4. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc. Then the following statements hold.*

a) *If $\overline{\Phi}(f) \in \mathcal{L}(X)$ then $\overline{\Phi}(f)$ commutes with every $\overline{\Phi}(g)$, $g \in \mathcal{F}$.*

b) *If $\overline{\Phi}(g) \in \mathcal{L}(X)$ then*

$$\overline{\Phi}(f) + \overline{\Phi}(g) = \overline{\Phi}(f + g) \qquad (f \in \mathcal{F}).$$

c) *If $f \in \mathcal{F}_{\mathrm{sv}}$ and $\overline{\Phi}(g) \in \mathcal{L}(X)$ then*

$$\overline{\Phi}(f)\overline{\Phi}(g) = \overline{\Phi}(fg).$$

d) *$\mathcal{F}_{\vee}$ is either empty or a subalgebra of $\mathcal{F}$ with $\mathbf{1}$, and in the latter case*

$$\overline{\Phi} : \mathcal{F}_{\vee} \longrightarrow \mathcal{L}(X)$$

is a unital representation.

e) *Let $f \in \mathcal{F}_{\mathrm{sv}}$ and $\overline{\Phi}(g) \in \mathcal{L}(X)$. Then*

$$\operatorname{ran}(\overline{\Phi}(g)) \subseteq \operatorname{dom}(\overline{\Phi}(f)) \qquad \Longleftrightarrow \qquad \overline{\Phi}(fg) \in \mathcal{L}(X).$$

f) *Let $f \in \mathcal{F}_{\mathrm{sv}}$ and $\Phi(g) \in \mathcal{L}(X)$. If $\overline{\Phi}(g)$ is injective, then*

$$\overline{\Phi}(f) = \overline{\Phi}(g)^{-1}\overline{\Phi}(fg).$$

Proof. a), c) and d) are immediate from Theorem 3.2.

b) Changing the roles of f and g in Theorem 3.2 c) we have $\overline{\Phi}(f) + \overline{\Phi}(g) \subseteq \overline{\Phi}(f + g)$. Likewise, $\overline{\Phi}(f + g) - \overline{\Phi}(g) \subseteq \overline{\Phi}(f)$, and combining both yields what we want.

To prove e), note that $f \in \mathcal{F}_{\mathrm{sv}}$, $g \in \mathcal{F}_{\vee}$ by Theorem 3.2 imply that $fg \in \mathcal{F}_{\mathrm{sv}}$ and hence $\operatorname{dom}(\overline{\Phi}(fg)) = \operatorname{dom}(\overline{\Phi}(f)\overline{\Phi}(g))$. But $\operatorname{dom}(\overline{\Phi}(fg)) = X$ if and only if $\overline{\Phi}(fg) \in \mathcal{L}(X)$, and $\operatorname{dom}(\overline{\Phi}(f)\overline{\Phi}(g)) = X$ if and only if $\operatorname{ran}(\overline{\Phi}(g)) \subseteq \operatorname{dom}(\overline{\Phi}(f))$.

f) follows easily from the other statements. ∎

Suppose that the afc $(\mathcal{E}, \mathcal{F}, \Phi)$ is upper non-degenerate. Then we have a new non-degenerate functional calculus

$$(\mathcal{F}_{\vee}, \mathcal{F}, \overline{\Phi}),$$

and one may wonder how the lower and upper extension of it relate to the extensions of Φ. This question shall be postponed until Section 5.4.

3.1. Regularisable elements. Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a non-degenerate afc. An element $f \in \mathcal{F}$ is called *regularisable* if there is $e \in [f]_{\mathcal{E}}$ such that $\Phi(e)$ is injective. Such an element e is then called a *regulariser* for f. Hence $e \in \mathcal{E}$ is a regulariser for f if and only if $\{e\}$ is an anchor of f. In particular, a regularisable element is anchored, i.e.,

$$\mathcal{F}_{\mathrm{re}} \subseteq \mathcal{F}_{\mathrm{sv}}$$

with $\mathcal{F}_{\mathrm{re}} = \mathcal{F}_{\mathrm{re}}(\Phi) := \{f \in \mathcal{F} \mid f \text{ is regularisable}\}$. If e is a regulariser for f then

$$\overline{\Phi}(f) = \Phi(e)^{-1}\Phi(ef)$$

using the usual operator product. This shows that on regularisable elements, the upper extension coincides with the extension introduced in [8, Construction Two], [9], and used extensively in [10].

4. Properties of the lower extension. Let us now turn to the lower extension $\underline{\Phi}$. Contrary to the upper extension, the lower one seems to behave less nice in general.

We introduce the set

$$\mathcal{F}_{\mathrm{dd}} = \mathcal{F}_{\mathrm{dd}}(\Phi) := \mathcal{F}_{\mathrm{sv}}(\Phi') = \{f \in \mathcal{F} \mid \underline{\Phi}(f) \text{ is densely defined}\}.$$

The following result is the analogue of Theorem 3.2. Recall again the notation $\mathrm{mul}(A) := \ker(A^{-1})$ for the "space of multi-valuedness" of an operator A.

THEOREM 4.1. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc over a Banach space X. Then the following assertions hold.*

a) *If $S \in \mathcal{L}(X)$ commutes with every $\Phi(e), e \in \mathcal{E}$, then S commutes with every $\underline{\Phi}(f)$, $f \in \mathcal{F}$.*
b) *The afc is lower non-degenerate if and only if $\mathcal{F}_{\mathrm{dd}} \neq \emptyset$.*
c) *Let $f, g \in \mathcal{F}$ such that $g \in \mathcal{F}_{\mathrm{dd}}$. Then*

$$\underline{\Phi}(f+g) \subseteq \overline{\underline{\Phi}(f) + \underline{\Phi}(g)}\,, \qquad \underline{\Phi}(fg) \subseteq \overline{\underline{\Phi}(f)\underline{\Phi}(g)}\,.$$

If, in addition, $fg \in \mathcal{F}_{\mathrm{dd}}$ then

$$\overline{\mathrm{mul}(\underline{\Phi}(fg)) + \mathrm{mul}(\underline{\Phi}(f))} = \mathrm{mul}(\overline{\underline{\Phi}(f)\underline{\Phi}(g)}).$$

d) *If $f, g \in \mathcal{F}_{\mathrm{dd}}$ then $f+g, fg \in \mathcal{F}_{\mathrm{dd}}$.*
e) *If $fg = \mathbf{1}$ then $\underline{\Phi}(f)^{-1} = \underline{\Phi}(g)$.*

Proof. a) and b) are straightforward.

c) Suppose that $f \in \mathcal{F}$ and $g \in \mathcal{F}_{\mathrm{dd}}$. Let $e \in [f+g]_{\mathcal{E}}$ and $e' \in [g]_{\mathcal{E}}$. Then $ee' \in [f+g]_{\mathcal{E}} \cap [f]_{\mathcal{E}} \cap [g]_{\mathcal{E}}$. For $x \in X$ clearly

$$\big(\Phi(e)\Phi(e')x, \Phi(e(f+g))\Phi(e')x\big) = \big(\Phi(ee')x, \Phi(fee')x + \Phi(gee')x\big) \in \underline{\Phi}(f) + \underline{\Phi}(g).$$

Writing $y := \Phi(e')x$, we see that $(\Phi(e)y, \Phi(e(f+g))y) \in \underline{\Phi}(f) + \underline{\Phi}(g)$ for y out of a total subset of X, hence it follows that

$$\underline{\Phi}(f+g) \subseteq \overline{\underline{\Phi}(f) + \underline{\Phi}(g)}.$$

Similarly, fix $e \in [fg]_{\mathcal{E}}$, and take $e' \in [g]_{\mathcal{E}}$. Then $ee' \in [g]_{\mathcal{E}}$ and $ee'g \in [f]_{\mathcal{E}}$, hence

$$\big(\Phi(ee')x, \Phi(ee'g)x\big) \in \underline{\Phi}(g) \quad \text{and} \quad \big(\Phi(ee'g)x, \Phi(fee'g)x\big) \in \underline{\Phi}(f)$$

for $x \in X$. Consequently,

$$\big(\Phi(e)\Phi(e')x, \Phi(fge)\Phi(e')x\big) = \big(\Phi(ee')x, \Phi(fgee')x\big) \in \underline{\Phi}(f)\underline{\Phi}(g) \subseteq \overline{\underline{\Phi}(f)\underline{\Phi}(g)}.$$

With the same argument as before we conclude that

$$\underline{\Phi}(fg) \subseteq \overline{\underline{\Phi}(f)\underline{\Phi}(g)}.$$

Finally, suppose that also $fg \in \mathcal{F}_{\mathrm{dd}}$. Then $fg \in \mathcal{F}_{\mathrm{sv}}(\Phi')$ and Theorem 3.2 (with the roles of f, g interchanged) yields that

$$\mathrm{dom}(\underline{\Phi}(fg)') \cap \mathrm{dom}(\underline{\Phi}(f)') = \mathrm{dom}(\underline{\Phi}(g)'\underline{\Phi}(f)') = \mathrm{dom}([\underline{\Phi}(f)\underline{\Phi}(g)]').$$

If we take the orthogonal in X here, we arrive at

$$\overline{\mathrm{mul}(\underline{\Phi}(fg)) + \mathrm{mul}(\underline{\Phi}(f))} = \mathrm{mul}\left(\overline{\underline{\Phi}(f)\underline{\Phi}(g)}\right).$$

as required.

d) and e) are straightforward consequences of Theorem 3.2 by dualising. ∎

Theorem 4.1 implies that $\mathcal{F}_{\mathrm{dd}}$ is a subalgebra of $\mathcal{F}$. Let us write

$$\mathcal{F}_\wedge = \mathcal{F}_\wedge(\Phi) := \{f \in \mathcal{F} \mid \underline{\Phi}(f) \in \mathcal{L}(X)\}.$$

This is then a subalgebra of $\mathcal{F}_{\mathrm{dd}}$.

Corollary 4.2. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc. Then the following statements hold.*

a) *If $\underline{\Phi}(f) \in \mathcal{L}(X)$ then $\underline{\Phi}(f)$ commutes with every $\underline{\Phi}(g)$, $g \in \mathcal{F}$.*

b) *If $\underline{\Phi}(g) \in \mathcal{L}(X)$ then*

$$\underline{\Phi}(f+g) = \underline{\Phi}(f) + \underline{\Phi}(g) \qquad (f \in \mathcal{F}).$$

c) *If $g \in \mathcal{F}_{\mathrm{dd}}$ and $\underline{\Phi}(f) \in \mathcal{L}(X)$ then*

$$\underline{\Phi}(fg) = \overline{\underline{\Phi}(f)\underline{\Phi}(g)}.$$

d) *$\mathcal{F}_\wedge$ is either empty or a subalgebra of $\mathcal{F}$ with $\mathbf{1}$, and in the latter case*

$$\underline{\Phi} : \mathcal{F}_\wedge \longrightarrow \mathcal{L}(X)$$

is a unital representation.

Proof. a) and d) are clear. For the proof of b) note that if $\underline{\Phi}(f)$ is a bounded operator, then by Theorem 4.1 c)

$$\underline{\Phi}(f+g) \subseteq \overline{\underline{\Phi}(f) + \underline{\Phi}(g)} = \underline{\Phi}(f) + \underline{\Phi}(g)$$

as $\underline{\Phi}(g)$ is closed. Replacing g by $f+g$ and f by $-f$ yields $\underline{\Phi}(g) \subseteq \underline{\Phi}(f+g) - \underline{\Phi}(f)$. Combining both leads to $\underline{\Phi}(f+g) = \underline{\Phi}(f) + \underline{\Phi}(g)$.

c) Dualising (see [10, Prop. A.4.2]) yields

$$[\underline{\Phi}(f)\underline{\Phi}(g)]' = \underline{\Phi}(g)'\underline{\Phi}(f)' = \overline{\Phi'}(g)\overline{\Phi'}(f) = \overline{\Phi'}(gf) = \underline{\Phi}(fg)'.$$

This implies the desired identity. (Here we used Corollary 3.4; note that $\overline{\Phi'}(g)$ is single-valued since it is the dual of a densely defined operator.) ∎

5. Miscellaneous results

5.1. Coincidence of lower and upper extension. From the discussion in Section 1 it appears to be important to find criteria to ensure that $\underline{\Phi}(f) = \overline{\Phi}(f)$.

PROPOSITION 5.1. *One has* $\underline{\Phi}(f) = \overline{\Phi}(f)$ *if* $\overline{\Phi}(f)$ *is single-valued and* $\operatorname{span}\{\Phi(e)x \mid e \in [f]_{\mathcal{E}},\ x \in X\}$ *is a core of* $\overline{\Phi}(f)$. *This is satisfied, for instance, in the following cases*:

1) $\overline{\Phi}(f)$ *is single-valued and* $\operatorname{dom}(\overline{\Phi}(f)) = \operatorname{ran}(\Phi(e))$ *for some* $e \in [f]_{\mathcal{E}}$.
2) $\overline{\Phi}(f)$ *is single-valued and* $\underline{\Phi}(f)$ *is fully defined.*
3) *Both* $\overline{\Phi}(f), \underline{\Phi}(f)$ *are bounded operators.*

Proof. This is straightforward from the definition and the fact that $\underline{\Phi}(f) \subseteq \overline{\Phi}(f)$ always. ■

In [10] the approach via regularisation (see above) was used to define fractional powers and the logarithm of a sectorial operator. We have seen above that this amounts to using the upper extension of the basic calculus. Now, Proposition 5.1 shows that we could have used the lower extension with the same outcome. Compare also Section 7.4 below.

One would expect that for a standard afc (see definition on page 157) the set

$$\mathcal{F}_{\mathrm{u}} = \mathcal{F}_{\mathrm{u}}(\Phi) := \{f \in \mathcal{F} \mid \underline{\Phi}(f) = \overline{\Phi}(f)\}$$

is a subalgebra of $\mathcal{F}$. However, we are unable to prove or disprove this at the moment. The best we can say is that if $fg = \mathbf{1}$ and if $f \in \mathcal{F}_{\mathrm{u}}$, then also $g \in \mathcal{F}_{\mathrm{u}}$, but this is just a trivial consequence of Theorems 3.2 and 4.1. For $f \in \mathcal{F}_{\mathrm{u}}$ we may write $\Phi(f) := \underline{\Phi}(f) = \overline{\Phi}(f)$ without danger.

PROPOSITION 5.2. *The following statements hold*:

a) *If* $f, g, fg \in \mathcal{F}_{\mathrm{u}}$ *and* $g \in \mathcal{F}_{\mathrm{dd}}$, $f \in \mathcal{F}_{\mathrm{sv}}$, *then*
$$\overline{\Phi(f)\Phi(g)} = \Phi(fg).$$

b) *If* $f, g, f+g \in \mathcal{F}_{\mathrm{u}}$ *and if at least one of* $\{f, g\}$ *is in* $\mathcal{F}_{dd}$ *and one is in* $\mathcal{F}_{\mathrm{sv}}$, *then*
$$\overline{\Phi(f) + \Phi(g)} = \Phi(f+g).$$

Proof. a) By Theorems 4.1 c) and 3.2 c)

$$\Phi(fg) = \underline{\Phi}(fg) \subseteq \overline{\underline{\Phi}(f)\underline{\Phi}(g)} = \overline{\overline{\Phi}(f)\overline{\Phi}(g)} \subseteq \overline{\overline{\Phi}(fg)} = \overline{\Phi}(fg) = \Phi(fg).$$

b) is proved similarly. ■

5.2. Approximate identities. A (*weak*) *approximate identity* is a sequence (or a net) $(e_k)_k \subseteq \mathcal{E}$ such that $\Phi(e_k)x \to x$ (weakly) for all $x \in X$. A (weak) approximate identity $(e_k)_k$ is called *bounded* if $\sup_k \|\Phi(e_k)\| < \infty$.

LEMMA 5.3. *Let* $(e_k)_k$ *be a weak approximate identity. Then*

$$M := \{f \in \mathcal{F} \mid e_k \in [f]_{\mathcal{E}}\ \forall\, k\}$$

is a subspace of $\mathcal{F}_{\mathrm{dd}} \cap \mathcal{F}_{\mathrm{sv}} \cap \mathcal{F}_{\mathrm{u}}$ *and*

$$\overline{\Phi(f) + \Phi(g)} = \Phi(f+g) \quad \textit{for all } f, g \in M.$$

Proof. If $f \in M$ then $\overline{\Phi}(f)$ is single-valued since $\{e_k \,|\, k\}$ is an anchor. If $x' \in X'$ is such that $x' \perp \Phi(e_k)x$ for every $x \in X$ and k, then by taking the weak limit, $x' \perp X$, hence $x' = 0$. Hence $\underline{\Phi}(f)$ is densely defined. Finally, if $(x, y) \in \overline{\Phi}(f)$ then $\Phi(e_k f)x = \Phi(e_k)y$, and $(\Phi(e_k)x, \Phi(e_k)y) \to (x, y)$ weakly. Since weak closure and strong closure coincide for subspaces by Mazur's theorem, $(x, y) \in \underline{\Phi}(f)$. This shows that $\underline{\Phi}(f) = \Phi(f)$. The remaining assertion follows immediately from Proposition 5.2 b) above. ■

We say that $f \in \mathcal{F}$ *admits* a (weak, bounded) approximate identity, if there is a (weak, bounded) approximate identity contained in $[f]_{\mathcal{E}}$.

Proposition 5.4. *If f admits a bounded weak approximate identity and g admits a strong approximate identity, then $f, g, f+g, fg$ admit a common weak approximate identity. In particular,*

$$f, g, f+g, fg \in \mathcal{F}_{\mathrm{dd}} \cap \mathcal{F}_{\mathrm{sv}} \cap \mathcal{F}_{\mathrm{u}} \tag{3}$$

and

$$\overline{\Phi(f) + \Phi(g)} = \Phi(f+g), \qquad \overline{\Phi(f)\Phi(g)} = \Phi(f+g). \tag{4}$$

Proof. Let $(e_j)_j$ be a bounded weak approximate identity in $[f]_{\mathcal{E}}$ and let $(\tilde{e}_k)_k$ be a strong approximate identity in $[g]_{\mathcal{E}}$. By taking the Cartesian product of the index sets, we may suppose that the nets are defined over the same directed set. Then

$$\Phi(e_k \tilde{e}_k) - \mathrm{I} = \Phi(e_k)(\Phi(\tilde{e}_k) - \mathrm{I}) + (\Phi(e_k) - \mathrm{I})$$

tends weakly to 0, by hypothesis, and $e_k \tilde{e}_k \in [f]_{\mathcal{E}} \cap [g]_{\mathcal{E}} \cap [f+g]_{\mathcal{E}} \cap [fg]_{\mathcal{E}}$. Assertion (3) as well as the first formula in (4) follows from Lemma 5.3. The second formula follows from Proposition 5.2 a). ■

Approximate identities play a fundamental role in Balakrishnan's construction, see Section 7.5.

5.3. Morphisms. We shall now address the question what happens when one enlarges the initial definition of Φ. This is best treated within the setting of morphisms.

Let $(\mathcal{E}, \mathcal{F}, \Phi)$ and $(\mathcal{E}_1, \mathcal{F}_1, \Phi_1)$ be afc[3] over a Banach space X. A *morphism*

$$\theta : (\mathcal{E}, \mathcal{F}, \Phi) \longrightarrow (\mathcal{E}_1, \mathcal{F}_1, \Phi_1)$$

consists of a homomorphism $\theta : \mathcal{F} \to \mathcal{F}_1$ such that $\theta(\mathcal{E}) \subseteq \mathcal{E}_1$ and $\Phi_1(\theta e) = \Phi(e)$ (see [10, p. 8]).

Proposition 5.5. *Let $\theta : (\mathcal{E}, \mathcal{F}, \Phi) \to (\mathcal{E}_1, \mathcal{F}_1, \Phi_1)$ be a morphism of afc over X. Then*

$$\theta[f]_{\mathcal{E}} \subseteq [\theta f]_{\mathcal{E}_1} \qquad \text{and} \qquad \underline{\Phi}(f) \subseteq \underline{\Phi_1}(\theta f) \subseteq \overline{\Phi_1}(\theta f) \subseteq \overline{\Phi}(f)$$

for every $f \in \mathcal{F}$. Moreover, if $f \in \mathcal{F}_{\mathrm{sv}}(\Phi)$ then $\overline{\Phi_1}(\theta f) = \overline{\Phi}(f)$ and if $f \in \mathcal{F}_{\mathrm{dd}}(\Phi)$ then $\underline{\Phi_1}(\theta f) = \underline{\Phi}(f)$.

Proof. Fix $e \in [f]_{\mathcal{E}}$. Then $ef \in \mathcal{E}$ and hence

$$(\theta e)(\theta f) = \theta(ef) \in \mathcal{E}_1,$$

[3]Since the plural of "calculus" is "calculi", the plural of "afc" is again "afc".

hence $\theta e \in [\theta f]_{\mathcal{E}_1}$. Suppose that $(x, y) \in \overline{\Phi_1}(\theta f)$. This means that $\Phi_1(e_1\theta f)x = \Phi_1(e_1)y$ for all $e_1 \in [\theta f]_{\mathcal{E}_1}$, whence in particular for all $e_1 \in \theta[f]_{\mathcal{E}}$. By the morphism property, this implies that $(x, y) \in \overline{\Phi}(f)$.

In the case that $\Phi(f)$ is single-valued, let $E \subseteq \mathcal{E}$ be an anchor for f. By the morphism property, $\theta E \subseteq \mathcal{E}_1$ is an anchor for θf, and so $\overline{\Phi}(f) = \overline{\Phi_1}(\theta f)$ follows from Lemma 3.1. Now note that θ induces also a morphism of the dual calculi. Thus the remaining statements follow from dualising. ∎

5.4. Enlargements. The previous proposition has many applications. Suppose for example that $\mathcal{E} \subseteq \mathcal{E}_1 \subseteq \mathcal{F}$ is another subalgebra and $\Phi_1 : \mathcal{E}_1 \to \mathcal{L}(X)$ is a representation such that $\Phi_1\big|_{\mathcal{E}} = \Phi$. Then we call the afc $(\mathcal{E}_1, \mathcal{F}, \Phi_1)$ an *enlargement* of the afc $(\mathcal{E}, \mathcal{F}, \Phi)$. For instance, $(\mathcal{F}_\vee, \mathcal{F}, \overline{\Phi})$ is such an enlargement if the original afc is upper non-degenerate, and $(\mathcal{F}_\wedge, \mathcal{F}, \underline{\Phi})$ is an enlargement if it is lower non-degenerate.

COROLLARY 5.6. *Let $(\mathcal{E}_1, \mathcal{F}, \Phi_1)$ be an enlargement of $(\mathcal{E}, \mathcal{F}, \Phi)$. Then*

$$\theta[f]_{\mathcal{E}} \subseteq [f]_{\mathcal{E}_1} \qquad \textit{and} \qquad \underline{\Phi}(f) \subseteq \underline{\Phi_1}(f) \subseteq \overline{\Phi_1}(f) \subseteq \overline{\Phi}(f)$$

for every $f \in \mathcal{F}$. Moreover,

$$\overline{\Phi_1}(f) = \overline{\Phi}(f) \quad \textit{if } f \in \mathcal{F}_{\mathrm{sv}}(\Phi), \quad \textit{and} \quad \underline{\Phi_1}(f) = \underline{\Phi}(f) \quad \textit{if } f \in \mathcal{F}_{\mathrm{dd}}(\Phi).$$

Proof. This is simply Proposition 5.5 with $\theta = \mathrm{id}$. ∎

The corollary tells us that whenever $\overline{\Phi}(f)$ is already single-valued then $\overline{\Phi_1}(f) = \underline{\Phi}(f)$. So enlarging Φ may turn certain Φ-multi-valued elements into Φ_1 single-valued ones, but leaves untouched the ones that are already Φ-single-valued. Analogous statements hold for the lower extensions.

An enlargement $(\mathcal{E}_1, \mathcal{F}, \Phi_1)$ of $(\mathcal{E}, \mathcal{F}, \Phi)$ is called *upper/lower harmless* if $\overline{\Phi_1}(f) = \overline{\Phi}(f)$, respectively $\underline{\Phi_1}(f) = \underline{\Phi}(f)$ for all $f \in \mathcal{F}$. The next lemma shows in particular that (in the case of upper non-degeneracy) the *upper enlargement*

$$(\mathcal{E}, \mathcal{F}, \Phi) \subseteq (\mathcal{F}_\vee, \mathcal{F}, \overline{\Phi})$$

is upper harmless and (given lower non-degeneracy) the *lower enlargement*

$$(\mathcal{E}, \mathcal{F}, \Phi) \subseteq (\mathcal{F}_\wedge, \mathcal{F}, \underline{\Phi})$$

is lower harmless.

LEMMA 5.7. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be an afc and let $f \in \mathcal{F}$. Then the following assertions hold.*

a) *If there is $g \in \mathcal{F}$ such that $\overline{\Phi}(g), \overline{\Phi}(fg) \in \mathcal{L}(X)$ then $(x, y) \in \underline{\Phi}(f)$ implies $\Phi(gf)x = \Phi(g)y$ for all $x, y \in X$.*
b) *If there is $g \in \mathcal{F}$ such that $\underline{\Phi}(g), \underline{\Phi}(gf) \in \mathcal{L}(X)$ then $(\underline{\Phi}(g)x, \underline{\Phi}(fg)x) \in \underline{\Phi}(f)$ for all $x \in X$.*

Proof. a) Let $e \in [g]_{\mathcal{E}}$ and $e' \in [fg]_{\mathcal{E}}$. Then $ee'g \in [f]_{\mathcal{E}}$ and hence

$$\Phi(e)\Phi(e')\Phi(gf)x = \Phi(ee'gf)x = \Phi(ee'g)y = \Phi(e)\Phi(e')\Phi(g)y.$$

Since $[g]_{\mathcal{E}}$ and $[fg]_{\mathcal{E}}$ are both anchors, $\Phi(gf)x = \Phi(g)y$ as desired.

b) Take $e \in [g]_{\mathcal{E}}$ and $e' \in [fg]_{\mathcal{E}}$ and $x \in X$. Then $ee'g \in [f]_{\mathcal{E}}$ and hence

$$\big(\underline{\Phi}(g)\Phi(e')\Phi(e)x, \underline{\Phi}(fg)\Phi(e')\Phi(e)x\big) = \big(\Phi(ge'e)x, \Phi(fge'e)x\big) \in \underline{\Phi}(f).$$

Since both $\underline{\Phi}(g)$ and $\underline{\Phi}(fg)$ are bounded operators, the set

$$\big\{\Phi(e')\Phi(e)x \,|\, e \in [g]_{\mathcal{E}},\ e' \in [fg]_{\mathcal{E}},\ x \in X\big\}$$

is total in X, and — again by boundedness of $\underline{\Phi}(g), \underline{\Phi}(fg)$ — we arrive at $(\underline{\Phi}(g)x, \underline{\Phi}(fg)x) \in \underline{\Phi}(f)$ for *all* $x \in X$ as desired. ∎

We shall study enlargements in more detail in the coming section.

6. Saturated functional calculi. Suppose we have a standard afc $(\mathcal{E}, \mathcal{F}, \Phi)$ over a Banach space X. As we have seen above, there are then two natural enlargements of this afc: the *upper enlargement* $(\mathcal{F}_\vee, \mathcal{F}, \overline{\Phi})$ and the *lower enlargement* $(\mathcal{F}_\wedge, \mathcal{F}, \underline{\Phi})$. The first is upper harmless, i.e., the upper extensions of the original and the enlarged afc coincide, whereas the lower extensions may differ, but in a consistent way (see Proposition 5.5). The second is lower harmless, i.e., stable with respect to lower extensions.

Let us call a standard afc $(\mathcal{E}, \mathcal{F}, \Phi)$ *saturated* if for every $f \in \mathcal{F}$ one has

$$\overline{\Phi}(f) \in \mathcal{L}(X) \quad \Longleftrightarrow \quad f \in \mathcal{E} \quad \Longleftrightarrow \quad \underline{\Phi}(f) \in \mathcal{L}(X).$$

Clearly, in a saturated afc the upper and the lower enlargements are not proper.

An afc is called *almost saturated* if $\mathcal{F}_\wedge = \mathcal{F}_\vee$. Here, the upper and lower enlargement coincide, and lead to a saturated afc.

If $(\mathcal{E}, \mathcal{F}, \Phi)$ fails to be saturated thanks to $\mathcal{F}_\wedge \neq \mathcal{F}_\vee$, then one may ask whether one can still find a — in some sense canonical — saturated enlargement of it. In the following we shall show that this is indeed the case.

It is a standard argument involving Zorn's lemma to produce at least one saturated enlargement. Indeed, note that if we fix an algebra $\mathcal{F}$ and a Banach space X, the set of all $(\mathcal{E}, \Phi)$ such that $(\mathcal{E}, \mathcal{F}, \Phi)$ is a standard afc on X, is naturally ordered by

$$(\mathcal{E}_1, \Phi_1) \leq (\mathcal{E}_2, \Phi_2) \quad :\Longleftrightarrow \quad \mathcal{E}_1 \subseteq \mathcal{E}_2, \quad \Phi_2\big|_{\mathcal{E}_1} = \Phi_1.$$

If $(\mathcal{E}, \Phi)$ is a maximal element with respect to this ordering, we call the resulting afc $(\mathcal{E}, \mathcal{F}, \Phi)$ a *maximal* afc.

LEMMA 6.1. *Every maximal enlargement of a standard afc is saturated.*

Proof. Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be maximal and standard. Then by maximality, the upper and the lower enlargements must both coincide with Φ itself. But this means that the afc is saturated. ∎

THEOREM 6.2. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a standard afc. Then it has a minimal saturated enlargement.*

Proof. We consider

$$\mathcal{M} := \{\mathcal{C} := (\mathcal{E}^\sim, \Phi^\sim) \,|\, \mathcal{C} \text{ is a saturated enlargement of } (\mathcal{E}, \Phi)\}.$$

By the previous remarks we know that $\mathcal{M} \neq \emptyset$. Define

$$\mathcal{E}_0 := \Big\{f \in \bigcap_{(\mathcal{E}_1, \Phi_1) \in \mathcal{M}} \mathcal{E}_1 \,\Big|\, \text{all the } \Phi_1\text{'s coincide on } f\Big\}.$$

It is obvious that $\mathcal{E}_0$ is a subalgebra of $\mathcal{F}$ that contains $\mathcal{E}$. Moreover, for $f \in \mathcal{E}_0$ we may define $\Phi_0(f) := \Phi_1(f)$, where Φ_1 is any saturated enlargement of Φ.

We claim that Φ_0 is itself saturated. Take $f \in \mathcal{F}$ such that $\overline{\Phi_0}(f) \in \mathcal{L}(X)$. Take $(\mathcal{E}_1, \Phi_1) \in \mathcal{M}$. As Φ_1 extends Φ_0, by Proposition 5.5, $\overline{\Phi_1}(f) = \overline{\Phi_0}(f) \in \mathcal{L}(X)$. By hypothesis, this implies that $f \in \mathcal{E}_1$, and hence $f \in \mathcal{E}_0$, by definition.

Similarly, suppose that $\underline{\Phi_0}(f) \in \mathcal{L}(X)$ and take $(\mathcal{E}_1, \Phi_1) \in \mathcal{M}$. As Φ_1 extends Φ_0, again by Proposition 5.5 we must have $\underline{\Phi_1}(f) = \underline{\Phi_0}(f) \in \mathcal{L}(X)$. By hypothesis, this implies that $f \in \mathcal{E}_1$, and hence $f \in \mathcal{E}_0$, by definition. ∎

7. Examples. The discussion of the following examples is cursory and concentrates on a few important aspects; a thorough investigation is still to be done.

7.1. Multiplication operator on C(K). We come back to the example from Section 1. Let K be a compact set, $X := \mathrm{C}(K)$ and $\mathcal{E} := \mathrm{C}(K)$ with $\Phi(e)x := ex$ defined for all $e \in \mathcal{E}, x \in X$. The algebra $\mathcal{F}$ is the algebra of *all* complex functions on K.

For $f \in \mathcal{F}$ define the multiplication operator M_f by

$$\mathrm{dom}(M_f) = \{x \in X \mid fx \in X\} = [f]_{\mathcal{E}}$$

with $M_f x = fx$ for $x \in \mathrm{dom}(M_f)$. Then $\underline{\Phi}(f) = M_f$. Indeed,

$$\begin{aligned}\underline{\Phi}(f) &= \overline{\mathrm{span}}\{(\Phi(e)x, \Phi(ef)x) \mid x \in X,\ e \in [f]_{\mathcal{E}}\} \\ &= \overline{\mathrm{span}}\{(e, ef) \mid e \in \mathrm{dom}(M_f)\} = M_f.\end{aligned}$$

Clearly, if f is not already continuous, then $\underline{\Phi}(f)$ cannot be densely defined.

On the other hand, define on $\mathcal{F}$ the relation $\sim$ by

$$f \sim g \quad :\Longleftrightarrow \quad \overline{\{f \neq g\}}^{\circ} = \emptyset.$$

This is an equivalence relation (by Baire's theorem) and it is easy to show that if $f \sim g$ then $\overline{\Phi}(f) = \overline{\Phi}(g)$. Also, $\overline{\Phi}(f)$ is single-valued if and only if the set of discontinuities of f is nowhere dense. Hence if f is not continuous but coincides with a continuous function apart from a nowhere dense set (as in the example of Section 1), then $\underline{\Phi}(f) \neq \overline{\Phi}(f) \in \mathcal{L}(X)$.

Specify $K := [0,1]$ and $f := \mathbf{1}_{[0,1/2]}$. Then one can show that $\overline{\Phi}(f) = \underline{\Phi}(f) = M_f$. This raises the question under which conditions on a general f one can find g such that $g \sim f$ and $\underline{\Phi}(g) = \overline{\Phi}(g)$.

7.2. Multiplication operators on M(K). The following example shows that neither extension may lead to the expected operator. Let again K be a compact set, let $X = \mathrm{M}(K)$ be the dual of $\mathrm{C}(K) =: \mathcal{E}$ and let $\Phi(e)\mu = e\mu$ be given by multiplication ($e \in \mathcal{E}, \mu \in X$). Here, neither $\underline{\Phi}$ nor $\overline{\Phi}$ gives the "just" extension. Let us look at the special case $f = \mathbf{1}_{(0,1]}$. Then one finds that

$$\begin{aligned}&\overline{\Phi}(f) = \mathrm{I} + \{(0, \lambda\delta_0) \mid \lambda \in \mathbb{C}\} \\ &\underline{\Phi}(f) = \{(\mu, \mu) \mid \mu \in \mathrm{M}[0,1],\ \mu\{0\} = 0\}.\end{aligned}$$

As "just" one could consider the multiplication operator with $\mathbf{1}_{(0,1]}$.

7.3. Multiplication operators on $\mathrm{L}^p(\Omega)$. The next example shows that in the case of the classical spectral theorem lower and upper extensions coincide.

Let (Ω, Σ, μ) be any measure space and let $1 \le p < \infty$ and $X := \mathrm{L}^p(\Omega, \mu)$. Let $\mathcal{E} := \mathrm{L}^\infty(\Omega, \mu)$ with $\Phi : \mathcal{E} \to \mathcal{L}(X)$ being ordinary multiplication. Let $\mathcal{F}$ be the algebra of equivalence classes of measurable functions on Ω modulo equality almost everywhere. If $f \in \mathcal{F}$ then $(n/(n+|f|))_{n\in\mathbb{N}}$ is an approximate identity for f. Hence $\underline{\Phi}(f) = \overline{\Phi}(f)$, and it is easy to see that this coincides with the multiplication operator M_f, with the usual domain. Note that $(1 + |f|)^{-1}$ is a regulariser for f.

(The same results hold in the case of a non-localisable measure space with $\mathcal{F}$ being the algebra of equivalence classes of locally measurable functions on Ω modulo equality locally almost everywhere; and $\mathcal{E}$ is the subalgebra of locally essentially bounded functions.)

7.4. Sectorial operators. We consider the situation of [9] or [10]. Let $A \in \mathrm{Sect}(S_\varphi)$ on a Banach space X. One considers $\mathcal{F}$ to be the germs of meromorphic functions on $\overline{S_\varphi} \setminus \{0\}$. There is a basic functional calculus $\Phi : \mathcal{E} \to \mathcal{L}(X)$ with

$$\mathcal{E} = \mathrm{H}_0^\infty(S_\varphi) \oplus \mathbb{C}(1+z)^{-1} \oplus \mathbb{C}\mathbf{1}.$$

The operator A need not be single-valued nor injective. In [10] we considered the single-valued case and defined extension via regularisers. Our discussion in Section 3.1 shows that this is consistent with the upper extension. Proposition 5.1 implies that on important functions like the fractional powers the upper extension coincides with the lower extension.

However, this fails for the logarithm if A is injective but not densely defined. To see this, take $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > \pi$ and consider the function $f(z) = (\lambda - \log z)^{-1}$. It is known [10, Chap. 4] that $\overline{\Phi}(f) = f(A) \in \mathcal{L}(X)$. However, if $e \in [f]_{\mathcal{E}}$ then $e \in \mathrm{H}_0^\infty(S_\varphi)$ and $\Phi(e)x = e(A)x \in \overline{\mathrm{dom}}(A) \neq X$ since $e(A)$ is defined by a Cauchy integral. Hence $\underline{\Phi}(e) \notin \mathcal{L}(X)$ and therefore $\underline{\Phi}(e) \neq \overline{\Phi}(e)$.

A detailed study of the relation of upper and lower extension for sectorial operators is still to be done. In particular, it would be of interest whether the upper enlargement is saturated, and how that actually looks like.

We remark that we do not require the sectorial operator to be single-valued. Our approach therefore allows, e.g. via the upper extension, to define fractional powers of multi-valued sectorial operators. This has been done in [1] and [12], and it remains to be studied how the two approaches relate.

7.5. Semigroup generators. Let $-A$ be the generator of a bounded C_0-semigroup $T = (T(s))_{s\in\mathbb{R}}$. For a bounded measure $\mu \in \mathrm{M}(\mathbb{R}_+)$ one defines

$$\Phi(\widehat{\mu})x := \int_0^\infty T(s)x\, \mu(\mathrm{d}s) \qquad (x \in X),$$

where $\widehat{\mu}(x) := \int_0^\infty e^{-zs}\, \mu(\mathrm{d}s)$ is the Laplace transform of μ. (The definition makes sense since the Laplace transform is injective.) The set $\mathcal{E} := \{\widehat{\mu} \mid \mu \in \mathrm{M}(\mathbb{R}_+)\}$ is an algebra and the mapping $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is a well-defined representation, the so-called *Hille–Phillips calculus* for A. We embed $\mathcal{E}$ into the algebra $\mathcal{F}$ of all functions on $\mathbb{C}_+ = \{z \mid \operatorname{Re} z > 0\}$ and obtain a standard afc $(\mathcal{E}, \mathcal{F}, \Phi)$. Balakrishnan [4] has constructed an extension of Φ to a subalgebra $\mathcal{M}$ of $\mathcal{F}$, and we shall show that our approach covers and extends this.

Every function $\varphi \in \mathcal{F}$ defines an unbounded *Laplace–Stieltjes* multiplier M_φ on $\mathrm{L}^1(\mathbb{R}_+)$ by setting

$$(f, g) \in M_\varphi \quad :\Longleftrightarrow \quad \varphi\widehat{f} = \widehat{g}$$

for $f, g \in \mathrm{L}^1(\mathbb{R}_+)$. The domain $\mathrm{dom}(M_\varphi)$ is a convolution ideal of $\mathrm{L}^1(\mathbb{R}_+)$. We set $\mathcal{M} := \{\varphi \,|\, \mathrm{dom}(M_\varphi) \text{ is dense in } \mathrm{L}^1(\mathbb{R}_+)\}$. For $\varphi \in \mathcal{M}$ one can find an *approximate identity* $(e_k)_k \subseteq \mathrm{dom}(M_\varphi)$, i.e., one has $\lim_k e_k * f \to f$ for all $f \in \mathrm{L}^1(\mathbb{R}_+)$. By the density of $\mathrm{dom}(A)$, one concludes that also $\Phi(\widehat{e_k}) \to \mathrm{I}$ strongly on X.

Suppose that $\varphi \in \mathcal{M}$. Balakrishnan defines

$$C_0(\Phi(\widehat{f})x) := \Phi(\varphi\widehat{f})x \quad \text{for} \quad x \in X,\ f \in \mathrm{dom}(M_\varphi),$$

and using a approximating identity within $\mathrm{dom}(M_\varphi)$ he shows that C_0 is in fact a well-defined closable linear operator

$$C_0 : \mathrm{span}\{\Phi(f)x \,|\, f \in \mathrm{dom}(M_\varphi),\ x \in X\} \longrightarrow X.$$

He then defines $\varphi(A)$ as the closure $\overline{C_0}$ of C_0. We claim that $\varphi(A) = \underline{\Phi}(\varphi) = \overline{\Phi}(\varphi)$. Indeed, note that $\mathrm{dom}(M_\varphi) \subseteq [\varphi]_{\mathcal{E}}$. This yields the inclusion $\varphi(A) \subseteq \underline{\Phi}(\varphi)$, and the converse inclusion is easily proved by using an approximate identity in $\mathrm{L}^1(\mathbb{R}_+)$ and observing that $\mu * f \in \mathrm{L}^1(\mathbb{R}_+)$ for all $\mu \in \mathrm{M}(\mathbb{R}_+)$. The presence of an approximate identity for each $\varphi \in \mathcal{M}$ in the sense of Section 5.2 then implies that $\overline{\Phi}(\varphi) = \underline{\Phi}(\varphi)$, by Proposition 5.4.

8. Conclusion and open questions. Let us review what we have achieved so far. Given a standard abstract functional calculus $(\mathcal{E}, \mathcal{F}, \Phi)$ we have defined two extensions $\underline{\Phi}, \overline{\Phi}$ of Φ towards all of $\mathcal{F}$. The upper extension $\overline{\Phi}$ generalises the extension via regularisation favoured in the author's earlier papers. It shows the very same (nice) properties and behaves even better with respect to inverses. However, the example in Section 1 shows that it sometimes does not yield the expected result. Other examples (Section 7) show that also the lower extension cannot be favoured in general. So what is then the "right" extension? Here are some open problems or questions that may be important.

1) Can one prove that $\{f \in \mathcal{F} \,|\, \underline{\Phi}(f) = \overline{\Phi}(f)\}$ is an algebra?
2) The pathological examples live on non-reflexive spaces. What does reflexivity help?
3) Even on non-reflexive spaces there is no example yet known for $\underline{\Phi}(f)$ bounded but $\overline{\Phi}(f)$ not bounded. Is that possible?
4) The lower extension $\underline{\Phi}$ appears to have worse computational properties compared to the upper extension. Is this really so? Namely, in the introductory example, where $\underline{\Phi}$ would be to favour to the upper extension $\underline{\Phi}$, it has good properties.
5) How do $\underline{\Phi}$ and $\overline{\Phi}$ relate for sectorial operators? How do they relate to constructions in the literature?
6) Is the saturation of an afc really important?

Final Remark. The construction of the upper and lower extension are based only on the multiplicative structure of $\mathcal{F}$ and can be done even in the following more general situation: $\mathcal{F}$ is a commutative monoid, $\mathcal{E} \subseteq \mathcal{F}$ is a subsemigroup, and $\Phi : \mathcal{E} \to \mathcal{L}(X)$ is multiplicative. All results, except for those explicitly referring to the additive structure, remain true in this more general context.

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