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ON TIME REGULARITY PROPERTIES OF BOUNDED LINEAR OPERATORS

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Abstract. We offer a survey on the decay rates of subsequent differences of operator powers.

1. Introduction. One of the basic properties of bounded linear operators acting on Banach spaces is the asymptotic behaviour of their iterates. One can study the stability properties in several different topologies; e.g. in the norm one, in the strong topology or in the weak one. In this survey, we shall focus on the relevant results in the operator algebra; that is, the norm convergence of the operator differences $T^{n+1} - T^n$.

Let X stand for a complex Banach space and T denote a bounded, linear operator acting on X. The notation $\sigma(T)$ is used for the spectrum of T, as usual. It is a simple matter to give a spectral characterization of the stability $T^n \to 0$ (in norm) when $n \to \infty$. Namely, this convergence holds if and only if $\sigma(T) \subseteq \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc of the complex plane. However, it is more challenging to provide a useful description of the norm stability $T^n(T - I) \to 0$ $(n \to \infty)$. We recall that the operator sequence $T^n(T - I)$ was essentially studied by J. Esterle, who proved that if T is power bounded, i.e. $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$, and $\sigma(T) = \{1\}$ then this sequence is always stable in operator norm ([13]). Motivated by the $0 - 2$ laws in operator ergodic theory, some years later Y. Katznelson and L. Tzafriri ([19]) provided the complete characterization for power bounded operators, namely $T^n(T - I) \to 0$ if and only if $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$. One might see that the stability condition here readily implies the given spectral condition. Actually, the difficult part of their result is the converse direction. Quite soon the result became a fundamental theorem in operator theory, because of its close connection with the classical result of Gelfand on invertible operators (see the survey [46]), and the celebrated Arendt–Batty–Lyubich–Vũ stability theorem ([2], [28]) (see also [3], [7]).

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The Esterle–Katznelson–Tzafriri theorem has an extensive literature and comprehensive studies were made in several directions concerning it, see e.g. [14], [15], [21], [37], [43], [44], [45] and the references therein. The aim of this survey is to present the recent results on the rate of the convergence in the classical Esterle–Katznelson–Tzafriri theorem. In advance, we can say that for a given operator it can be a difficult problem to determine the exact decay rates. The first essential and general result here is related to J. Esterle, who pointed out that the decay rates cannot be arbitrarily fast (except for trivial cases). For any bounded, linear operator $T \neq I$ such that $\sigma(T) = \{1\}$, he found that $\liminf_{n\to\infty} n\|T^n(T-I)\| > \text{const} > 0$ any Banach space [13]. The exact value of the constant appeared in his theorem is $1/e$ which was determined later by N. Kalton et al. [17], O. Nevanlinna et al. [29] and N. Dungey [10]. Dropping the minimal spectral condition, now let us choose any bounded operator T acting on a Banach space X. We know that either $\limsup_n n\|T^n(T-I)\| \geq 1/e$ or X can be decomposed into the direct sum of two closed T-invariant subspaces X_0 and X_1 such that T is the identity on X_1 and the restriction of T onto X_0 has spectral radius strictly less than 1 (see [17, Theorem 3.1], [35, Theorem 2.1]). This means that we can get faster decay rates than $O(n^{-1})$ only in very special situations.

Concerning the time regularity property, one can simply construct operators having various decay rates if one allows the spectrum to be large. Throughout the paper we shall use the notation $\Theta(a_n) = b_n$ for positive sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ which means that there does exist a positive constant C such that $C^{-1}a_n \leq b_n \leq Ca_n$ for all n. For instance, let us consider a simple geometric construction.

EXAMPLE 1.1. Fix an $0 < \varepsilon < 1$. Let Ω_ε denote the inner domain of the Jordan curve

$$w_\varepsilon \colon t \mapsto |t^3/3 - t|^{1+\varepsilon} + i(t^3/3 - t)(1-t^2), \quad t \in [-1,1].$$

We shall denote the Bergman space on Ω_ε by $L^2(\Omega_\varepsilon)$. If M_z is the multiplication operator $M_z \colon f(z) \mapsto zf(z)$, $f \in L^2(\Omega_\varepsilon)$, the norm of the operators $M_z^n - M_z^{n+1}$ can be readily estimated. In fact, one gets

$$\|M_z^n - M_z^{n+1}\| = \|M_{z^n - z^{n+1}}\| = \sup_{z\in\Omega_\varepsilon} |z^n - z^{n+1}| = \Theta(n^{-1/(1+\varepsilon)}).$$

Note that the spectrum of M_z is large $\sigma(M_z) = \overline{\Omega}_\varepsilon$.

On the other hand, to get more light on Esterle's result and the decay rates under the single-point assumption $\sigma(T) = \{1\}$ we shall need much more refined constructions.

2. Ritt operators. Roughly speaking, Ritt operators are power bounded operators with extremal decay rates. However, the definition of the Ritt property looks quite different at first. It was actually described by means of a resolvent estimate, instead of the decay rates. Precisely, we say that an operator T is *Ritt* if it satisfies the resolvent condition

$$\|(T - \lambda I)^{-1}\| \leq \frac{\text{const}}{|\lambda - 1|} \quad \text{for all } |\lambda| > 1.$$

As was observed by Yu. Lyubich [25] and independently by B. Nagy and J. Zemánek [31], the above resolvent condition can be extended to a sectorial estimate; i.e. there exists a $\delta > 0$ such that the set

$$\{|\lambda - 1|\, \|(T - \lambda I)^{-1}\| \colon \lambda \in K_\delta\}$$

is bounded, where $K_\delta = \{1 + re^{i\theta} : r > 0,\ |\theta| < \frac{\pi}{2} + \delta\}$. They proved that T is Ritt if and only if

$$\sup_n \|T^n\| < \infty \quad \text{and} \quad \sup_n n\|T^{n+1} - T^n\| < \infty.$$

Note that the spectrum of any Ritt operator is lying in a proper sector with a vertex at 1. Next we collect a few non-trivial examples of Ritt operators.

EXAMPLE 2.1. For $\alpha > 0$, let us consider the fractional integral operator acting on $L^p[0,1]$

$$(V^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\, ds, \qquad 0 \le x \le 1.$$

In fact, $I - V^\alpha$ $(0 < \alpha < 1)$ is a Ritt operator on every $L^p[0,1]$ space, $1 \le p \le \infty$, see [26]. This example by Yu. Lyubich tells us that it may happen that a Ritt operator is the sum of the identity and a quasinilpotent operator. For further examples which exploit the method of fractional powers the reader might see the papers [12] and [11].

EXAMPLE 2.2. Choose a conformal map f from $\mathbb{D}$ into a sector of the right half complex plane with vertex at 0. If one assumes that the modulus of f is relatively small which precisely means that the range of $f \subseteq (\mathbb{D} \cap (1+\mathbb{D}))$ and $\lim_{z\to 1} f(z) = 0$ then the operator $I - f(I - 2V)$ is Ritt on $L^2[0,1]$ (see [23, Theorem 2.1]). Below we shall make it precise how to define the operator $f(I-2V)$. Here, the geometric condition on the range of f can be readily stated with the particular choice of $f(z) = \left(\frac{1-z}{2}\right)^\alpha$ $(z \in \mathbb{D},\ 0 < \alpha < 1)$. Thus we instantly deduce that $I - V^\alpha$ is Ritt on $L^2[0,1]$ ([23, Corollary 2.2]).

EXAMPLE 2.3. Markov operators induced by random walks on graphs and groups have the Ritt property as well (e.g. [9] and [8]).

Motivated by Example 2.1, N. Dungey proved that any Ritt operator can be described by means of a fractional power approach. We say that an operator T is *Kreiss* if it satisfies the resolvent condition

$$\|(T - \lambda I)^{-1}\| \le \frac{\text{const}}{|\lambda| - 1} \qquad \text{for all } |\lambda| > 1.$$

Now the characterization reads as follows.

THEOREM 2.4 ([12, Theorem 1.3]). *Let T be a bounded, linear operator in a Banach space. Then the following statements are equivalent.*

(i) *T is Ritt.*
(ii) *There exist a Kreiss operator S and an $\alpha \in (0,1)$ such that $T = I - (I-S)^\alpha$.*
(iii) *There exist a power bounded operator S and an $\alpha \in (0,1)$ such that $T = I - (I-S)^\alpha$.*

On the other hand, Ritt operators can be treated as the discrete analogues of bounded analytic operator semigroups. Essentially O. Nevanlinna proved that T is Ritt if and only if the semigroup $(e^{-t(I-T)})_{t\ge 0}$ is a bounded, analytic one and $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ (see [32, Theorem 4.5.4], [33, Theorem 2.1]). Furthermore, it is easy to check that if $(T_t)_{t\ge 0}$ is a bounded, analytic C_0-semigroup then T_t is (uniformly) Ritt for every $t \ge 0$ and by P. Vitse the converse statement holds as well [42, Section 3]. For further analogues, for instance, to the maximal regularity property, interpolation property and the functional calculus of analytic semigroups, the reader may consult [4], [5], [18], [34], [20], [42] and the references therein.

Nevertheless, we note that the previous boundedness assumptions of the Ritt property,

$$\sup_n \|T^n\| < \infty \quad \text{and} \quad \sup_n n\|T^{n+1} - T^n\| < \infty,$$

are independent. In general the time regularity property does not imply that the operator T is power bounded (for an example in $L^1(\mathbb{R})$, see [17, Theorem 3.3]). However, we do not know whether the conditions $\sigma(T) = \{1\}$ and $\|T^{n+1} - T^n\| = O(1/n)$ $(n \to \infty)$ might imply $\sup_{n\geq 0} \|T^n\| < \infty$. (This question originates from [31, p. 147].) Regarding the assumptions of the Esterle–Katznelson–Tzafriri theorem, note that whenever the convergence $T^{n+1} - T^n \to 0$ $(n \to \infty)$ holds, the convergence $T^n/n \to 0$ follows as well. In [1], G. Allan's question was the converse of whether the latter condition with the spectral one $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ implies the stability of the subsequent differences $T^{n+1} - T^n$. This would mean that the power boundedness assumption could be relaxed quite significantly. Y. Tomilov and J. Zemánek provided a negative answer to Allan's question building upon an operator matrix construction [40]. In fact, the operator

$$\mathcal{V} = \begin{pmatrix} I - V & V \\ 0 & I - V \end{pmatrix}$$

gives a counterexample on $L^1[0,1] \oplus L^1[0,1]$, even on the Hilbert space $L^2[0,1] \oplus L^2[0,1]$ (see [22]); that is, $\mathcal{V}^n/n \to 0$, but $\|\mathcal{V}^n - \mathcal{V}^{n+1}\| \not\longrightarrow 0$ $(n \to \infty)$. For further discussions on different types of boundedness and their consequences on the iterates and ergodic properties of linear operators, the reader can see [39].

3. Functions of the Volterra operator. In general, the calculation of the exact decay rates of the differences $T^n - T^{n+1}$ is a subtle and complicated task. In the case of the operator $I - V$, these estimates were settled by A. Montes-Rodríguez, J. Sánchez-Álvarez and J. Zemánek, in their paper [30]. Relying on asymptotic formulas for the Laguerre polynomials and the Riesz–Thorin interpolation theorem, they were able to prove the following fundamental theorem.

THEOREM 3.1 ([30, Theorem 2.5]). *Let V stand for the Volterra operator on $L^p[0,1]$ $(1 \leq p \leq \infty)$. Then there are positive constants c, C (depending on p) such that*

$$cn^{-1/2+|1/4-1/(2p)|} \leq \|(I-V)^{n+1} - (I-V)^n\|_p \leq Cn^{-1/2+|1/4-1/(2p)|}.$$

The Montes–Sánchez–Zemánek result provided the very first examples for operators with small spectrum where it became possible to determine the sharp decay rates. We remark that exploiting their approximation method, one can even prove a more general version of the above theorem. Actually, for any $\alpha > 0$,

$$cn^{-\alpha/2+|1/4-1/(2p)|} \leq \|(I-V)^n V^\alpha\|_p \leq Cn^{-\alpha/2+|1/4-1/(2p)|}$$

(the positive constants c, C are only depending on p and α) [23, Theorems 3.3 and 3.5]. The special case $\alpha = 0$ appeared in [30, Theorem 2.2]. Despite of the accurate and long calculations, these results do not answer the question if there exists a quasinilpotent perturbation T of the identity such that $\|T^{n+1} - T^n\| = \Theta(n^{-\alpha})$ for some $1 < \alpha < 1/2$. In [27] Yu. Lyubich investigated the subsequent differences between the powers of analytic functions of the Volterra operator. Of course, for any function f which is analytic around 0, we can define the operator $f(V)$ by means of the Riesz–Dunford functional calculus. Lyubich remarks that the operators $f(V)$ and $I + f'(0)V$ are similar in $L^2[0,1]$ and the last one is power bounded if and only if $f'(0)$ is a negative real [41, Theorem 1]. On the other hand, the iterates of $I - rV$ $(r > 0)$ and $I - V$, and their differences as well, possess the same asymptotic properties [30, Section 5]. Hence we have the following theorem.

THEOREM 3.2. *Let $1 \not\equiv f$ be an analytic function around 0 such that $f(0) = 1$ and $f(V)$ is power bounded on $L^2[0,1]$. Then*

$$\|f(V)^n - f(V)^{n+1}\|_2 = \Theta(n^{-1/2}).$$

Now let us choose a function f which is analytic on $\mathbb{D}$ and assume that f has a holomorphic extension to a neighbourhood of 1. The above theorem simply implies that the decay rate of the subsequent differences of the powers $(I - f(I - 2V))^n$, as long as they are uniformly bounded in norm, is $n^{-1/2}$ (up to constant factors). However, if we require a less strict boundary condition on f; i.e. the range of f has only $C^{1,\varepsilon}$ smooth boundary we can get faster decays. To make this idea clear, we collect a few facts about the shift operator and its compressions on the Hardy space $H^2(\mathbb{D})$.

Let S denote the unilateral shift operator on $H^2(\mathbb{D})$ defined by $(Sh)(z) = zh(z)$. The function ψ stands for the singular inner function

$$\psi(z) := \exp\Big(-\frac{1+z}{1-z}\Big), \quad z \in \mathbb{D}.$$

Let T be the compression of S into the subspace $P_\psi H^2(\mathbb{D})$; that is,

$$T = P_\psi S P_\psi,$$

where P_ψ denotes the orthogonal projection from $H^2(\mathbb{D})$ onto $H^2(\mathbb{D}) \ominus \psi H^2(\mathbb{D})$. A classical result by Sarason shows us that $(I+V)^{-1}$ is unitarily equivalent to the operator $(T+I)/2$ [35, Theorem 1]. Moreover, from T. V. Pedersen's similarity relation [1] we have $M^{-1}(I+V)^{-1}M = I - V$, where M is the multiplication operator $(Mf)(x) = e^{-x}f(x)$ on $L^2[0,1]$. Hence the above chain of reasoning tells us that the operator $I - 2V$ is actually similar to the contraction T. Now let $J\colon L^2[0,1] \to H^2 \ominus \psi H^2$ denote the bounded linear bijection for which

$$J^{-1}TJ = I - 2V.$$

This similarity enables us to define an H^∞-calculus for $I - 2V$; i.e.

$$f(I - 2V) := J^{-1}f(T)J$$

for every $f \in H^\infty(\mathbb{D})$. Furthermore, we recall that the algebra $\{f(T) : f \in H^\infty(\mathbb{D})\}$ and the factor algebra $H^\infty(\mathbb{D})/\psi H^\infty(\mathbb{D})$ are isometrically isomorphic which means $\|f(T)\| = \|f + \psi H^\infty(\mathbb{D})\|_\infty$ for all $f \in H^\infty(\mathbb{D})$ (see [36]). Hence it is enough to determine the time regularity of the differences $f^n - f^{n+1}$ in $H^\infty(\mathbb{D})/\psi H^\infty(\mathbb{D})$ to get that of the operators $f(T)^n - f(T)^{n+1}$. We remark that the last question requires function theory and it has nothing to do with operator theory.

Roughly speaking, we can say that the regularity properties (in the sense of function theory) and smoothness of the boundary of holomorphic functions at 1 are crucial to describe the time regularity of operators which have the form $I - f(I - 2V)$. Now let us choose a conformal map f_α that maps $\mathbb{D}$ onto the domain Ω_ε which has the $C^{1,\varepsilon}$ boundary $w_\varepsilon\colon t \mapsto |t^3/3 - t|^{1+\varepsilon} + i(t^3/3 - t)(1 - t^2)$, $t \in [-1, 1]$, for $\varepsilon = (1 - \alpha)/\alpha$ and $1/2 < \alpha < 1$ (as we used it in the Introduction). With an appropriate modification of f_α, one can even guarantee that the condition

$$0 < \lim_{\theta\to 0} \frac{\left|\operatorname{Im} f_\alpha(e^{i\theta})\right|^{1/\alpha}}{\operatorname{Re} f_\alpha(e^{i\theta})} < \infty \tag{1}$$

holds as well. This enables us to calculate the rate of the convergence in the factor algebra $H^\infty(\mathbb{D})/\psi H^\infty(\mathbb{D})$ [24, Theorem 2.5]

$$\|f_\alpha^n(1 - f_\alpha) + \psi H^\infty(\mathbb{D})\|_\infty = \Theta(n^{-\alpha}).$$

To summarize the above observations, we have

Theorem 3.3 ([24, Theorem 2.6]). *For any $1/2 < \alpha < 1$, there exists an $f_\alpha \in H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that $f_\alpha(1) = 0$ and*

$$\|(I - f_\alpha(I - 2V))^{n+1} - (I - f_\alpha(I - 2V))^n\|_2 = \Theta(n^{-\alpha}).$$

If one chooses f_α with small supremum norm, von Neumann's inequality implies that the contraction $I - f_\alpha(I - 2V)$ is indeed a (compact) quasinilpotent perturbation of the identity. This construction offers further examples with single-point spectrum and various time regularities.

Example 3.4. For any $1/2 < \alpha < 1$, define the operator matrix

$$\mathcal{T} = \begin{pmatrix} I - f_\alpha(I - 2V) & f_\alpha(I - 2V) \\ 0 & I - f_\alpha(I - 2V) \end{pmatrix}$$

on $L^2[0, 1] \oplus L^2[0, 1]$ (following the matrix construction of Y. Tomilov and J. Zemánek [40]). Then a straightforward calculation gives $\|\mathcal{T}^{n+1} - \mathcal{T}^n\| = O(n^{-2\alpha+1})$. Of course, $\sigma(\mathcal{T}) = \{1\}$ holds as well but $\|\mathcal{T}^n\| \geq \text{const} \cdot n^{1-\alpha}$; i.e. $\mathcal{T}$ is not power bounded anymore.

Unfortunately, we do not know how to construct contractions or power bounded operators on Hilbert space with single-point spectrum $\{1\}$ and time regularity slower than $n^{-1/2}$. We are not aware of such examples in Banach spaces either.

Question. Does there exist a Hilbert space contraction (or power bounded operator) T such that $\sigma(T) = \{1\}$ and

$$\|T^{n+1} - T^n\| = \Theta(n^{-\alpha}),$$

where $0 < \alpha < 1/2$?

4. Subsequent differences with decay rates $O(n^{-\alpha})$.

We recall that a classical lemma found by S. Foguel and B. Weiss [16] says that if T is a contraction in a Banach space, then the convex sum $T_\beta := (1 - \beta)I + \beta T$ $(0 < \beta < 1)$ always satisfies the estimate

$$\|T_\beta^{n+1} - T_\beta^n\| = O(n^{-1/2}).$$

Surprisingly, N. Dungey observed that the converse statement holds as well.

Theorem 4.1 ([10, Theorem 1.2]). *Let T be a bounded, linear operator on a Banach space. The following conditions are equivalent.*

(i) *The operator T is power bounded and $\sup_{n\in\mathbb{N}} n^{-1/2}\|T^n(T - I)\| < \infty$.*
(ii) *There exist $\beta \in (0, 1)$ and a power bounded operator S such that $T = \beta I + (1 - \beta)S$.*

Moreover, if these conditions hold, there is a $\beta \in (0,1)$ *such that*

$$\sigma(T) \subseteq \{z \in \mathbb{C} : |z-(1-\beta)| \le \beta\} \subseteq \mathbb{D} \cup \{1\}.$$

Condition (ii) of the theorem was first essentially studied in a paper by O. Nevanlinna (see [34, Theorem 8]). His result characterizes the operators S which are power bounded and can be written in the form $S = (1+\varepsilon)T - \varepsilon I$ for some $\varepsilon > 0$. Relying on Theorem 4.1, we note that the answer 'Yes' to our Question would immediately provide an example of a quasinilpotent operator Q in Hilbert space such that $I+Q$ is power bounded, but for any $\varepsilon > 0$ the perturbed operator $I+(1+\varepsilon)Q$ is not. (For a similar question on 'stability' of power boundedness on segments, see [48, Question 5].)

In fact, condition (i) holds for a large class of subordinated discrete semigroups of operators. Let F be a discrete probability measure on $\mathbb{Z}_+ = \{0,1,2,\dots\}$ and let T be a power bounded operator acting in a Banach space X. Then we say that the operator

$$\Psi(F;T) := \sum_{k\in\mathbb{Z}_+} F(k)T^k$$

is subordinated to T via the probability F. Dungey pointed out the same regularity properties of the subordinated operator $\Psi(F;T)$ and that of the function $\sum_{k\in\mathbb{Z}_+} F(k)z^k$ ($z \in \overline{\mathbb{D}}$) in the Wiener algebra. He proved that

$$\|\Psi(F;T)^{n+1} - \Psi(F;T)^n\| = O(n^{-1/2})$$

whenever F is an aperiodic probability measure on $\mathbb{Z}_+$ [10, Theorem 2.4]. Note that the Bernoulli probability with parameter β gives the direction (ii) $\Longrightarrow$ (i) in Theorem 4.1. For instance, if F is a zeta probability with parameter $0 < \beta < 1$,

$$F(0) = 0 \quad \text{and} \quad F(k) = \zeta(1+\beta)^{-1}k^{-1-\beta},$$

or it is given by generalized binomial coefficients, $0 < \beta < 1$,

$$F(0) = 0 \quad \text{and} \quad F(k) = (-1)^k \binom{\beta}{k},$$

the operator $\Psi(F;T)$ is Ritt [10, Theorem 1.1 and Theorem 1.2]. For further interesting examples, we refer the reader to [10, Sections 4 and 5].

Focusing on more general decay rates, let us consider the region

$$\Omega_{\alpha,a} := \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } |\operatorname{Im} z| < a(\operatorname{Re} z)^\alpha\},$$

if $\alpha > 0$ and $a > 0$. The reader might compare the definition with property (1) in Section 3. For a bounded linear operator T and for some $\alpha \in (1/2, 1]$, one can give a characterization of the time regularity in spirit that of the analytic operators. Namely, we have

THEOREM 4.2 ([10, Theorem 1.5 and Theorem 5.2]). *The following statements are equivalent.*

(i) *The operator* T *is power bounded and* $\sup_{n\in\mathbb{N}} n^{-\alpha}\|T^n(T-I)\| < \infty$.
(ii) $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ *and* $\sup_{t\ge 0}\big(\|e^{-t(I-T)}\| + t^\alpha\|(I-T)e^{-t(I-T)}\|\big) < \infty$.
(iii) $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ *and the semigroup* $e^{-z(I-T)}$ *is uniformly bounded on the domain* $\Omega_{\alpha,a}$ *for some* $a > 0$.

Here, the equivalence in the important special case $\alpha = 1$ is due to O. Nevanlinna; that is, T is Ritt if and only if the semigroup $e^{-t(I-T)}$ is bounded analytic and $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$. Unfortunately, the aforementioned theorem does not work for $\alpha \geq 1/2$.

Quite often stability results or regularities of operator powers follow from the regularity properties of the resolvent function near its singularities. For instance, we can observe this phenomenon around the definition of Ritt (or analytic) operators and its equivalent reformulation. Or a simple application of the Cauchy integral formula tells us that any Kreiss operator T satisfies the growth condition $\|T^n\| = O(n)$. Concerning the decay rates in the Esterle–Katznelson–Tzafriri theorem, a more sophisticated version of this approach was studied in detail for C_0-semigroups by Y. Tomilov and A. Borichev [6]. They proved that the polynomial resolvent growth implies rather strong stability results and studied the optimality of these results as well. Quite recently D. Seifert exploited their method to prove the discrete analogues for single operators in Banach spaces. Let $R(\lambda, T)$ stand for the resolvent operator $(T - \lambda I)^{-1}$, as usual.

THEOREM 4.3 ([38, Corollary 3.1]). *Let T be a power bounded operator such that $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$. Suppose that, for some $\alpha \geq 1$, $\|R(e^{i\theta}, T)\| = \Theta(|\theta|^\alpha)$ as $\theta \to 0$. Then there exist constants $c, C > 0$ such that*

$$\frac{c}{n^{1/\alpha}} \leq \|T^n(T - I)\| \leq C\Big(\frac{\log n}{n}\Big)^{1/\alpha}.$$

The above theorem is a special case of a general result which links the so-called dominating function for the resolvent of T to the decay rates of the differences $T^{n+1} - T^n$ (see [36, Theorem 2.11]). Theorem 4.3 is optimal in general Banach spaces which means that the logarithmic factor cannot be dropped. However, in Hilbert spaces the theorem can be significantly improved.

THEOREM 4.4 ([38, Theorem 3.10 and Remarks]). *Let T be a Hilbert space operator which is power bounded and $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$. Furthermore, let $\alpha \geq 1$. The following are equivalent.*

(i) $\|R(e^{i\theta}, T)\| = O(|\theta|^\alpha)$ *as* $\theta \to 0$.
(ii) $\|T^n(T - I)\| = O(n^{-1/\alpha})$.
(iii) $nT^n(T - I)^\alpha \to 0$ *in the strong operator topology.*
(iv) $\|T^n(T - I)x\| = o(n^{1/\alpha})$ *for every vector* x.

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References

[1] G. R. Allan, *Power-bounded elements and radical Banach algebras*, in: Linear Operators, Banach Center Publ. 38, Polish Acad. Sci. Inst. Math., Warsaw 1997, 9–16.

[2] W. Arendt, C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. 306 (1988), 837–852.

[3] C. J. K. Batty, *Asymptotic behaviour of semigroups of operators*, in: Functional Analysis and Operator Theory, Banach Center Publ. 30, Polish Acad. Sci. Inst. Math., Warsaw 1994, 35–52.

[4] S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), 157–176.

[5] S. Blunck, *Analyticity and discrete maximal regularity on L_p-spaces*, J. Funct. Anal. 183 (2001), 211–230.

[6] A. A. Borichev, Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. 347 (2010), 455–478.

[7] R. Chill, Y. Tomilov, *Stability of operator semigroups: ideas and results*, in: Perspectives in Operator Theory, Banach Center Publ. 75, Polish Acad. Sci. Inst. Math., Warsaw 2007, 71–109.

[8] N. Dungey, *Properties of centered random walks on locally compact groups and Lie groups*, Rev. Mat. Iberoam. 23 (2007), 587–634.

[9] N. Dungey, *Properties of random walks on discrete groups: time regularity and off-diagonal estimates*, Bull. Sci. Math. 132 (2008), 359–381.

[10] N. Dungey, *On time regularity and related conditions for power-bounded operators*, Proc. Lond. Math. Soc. (3) 97 (2008), 97–116.

[11] N. Dungey, *On an integral of fractional power operators*, Colloq. Math. 117 (2009), 157–164.

[12] N. Dungey, *Subordinated discrete semigroups of operators*, Trans. Amer. Math. Soc. 363 (2011), 1721–1741.

[13] J. Esterle, *Quasimultipliers, representation of H^∞, and the closed ideal problem for commutative Banach algebras*, in: Radical Banach Algebras and Automatic Continuity, Lecture Notes in Math. 975, Springer, Berlin 1983, 66–162.

[14] J. Esterle, E. Strouse, F. Zouakia, *Theorems of Katznelson–Tzafriri type for contractions*, J. Funct. Anal. 94 (1990), 273–287.

[15] J. Esterle, E. Strouse, F. Zouakia, *Stabilité asymptotique de certains semi-groupes d'opérateurs et idéaux primaires de $L^1(\mathbb{R}^+)$*, J. Operator Theory 28 (1992), 203–227.

[16] S. R. Foguel, B. Weiss, *On convex power series of a conservative Markov operator*, Proc. Amer. Math. Soc. 38 (1973), 325–330.

[17] N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz, Y. Tomilov, *Power-bounded operators and related norm estimates*, J. London Math. Soc. (2) 70 (2004), 463–478.

[18] N. Kalton, P. Portal, *Remarks on ℓ_1 and ℓ_∞ maximal regularity for power bounded operators*, J. Aust. Math. Soc. 84 (2008), 345–365.

[19] Y. Katznelson, L. Tzafriri, *On power bounded operators*, J. Funct. Anal. 68 (1986), 313–328.

[20] C. Le Merdy, *H^∞ functional calculus and square function estimates for Ritt operators*, Rev. Mat. Iberoam. 30 (2014), 1149–1190.

[21] Z. Léka, *A Katznelson–Tzafriri type theorem in Hilbert spaces*, Proc. Amer. Math. Soc. 137 (2009), 3763–3768.

[22] Z. Léka, *A note on the powers of Cesàro bounded operators*, Czechoslovak Math. J. 60 (2010), 1091–1100.

[23] Z. Léka, *On orbits of functions of the Volterra operator*, Complex Anal. Oper. Theory 7 (2013), 1321–1335.

[24] Z. Léka, *Time regularity and functions of the Volterra operator*, Studia Math. 220 (2014), 1–14.

[25] Yu. Lyubich, *Spectral localization, power boundedness and invariant subspaces under Ritt's type condition*, Studia Math. 134 (1999), 153–167.

[26] Yu. Lyubich, *The single-point spectrum operators satisfying Ritt's resolvent condition*, Studia Math. 145 (2001), 135–142.

[27] Yu. Lyubich, *The power boundedness and resolvent condition for functions of the classical Volterra operator*, Studia Math. 196 (2010), 41–63.

[28] Yu. Lyubich, Q. P. Vũ, *Asymptotic stability of linear differential equations on Banach spaces*, Studia Math. 88 (1988), 37–42.

[29] J. Malinen, O. Nevanlinna, V. Turunen, Z. Yuan, *A lower bound for the differences of powers of linear operators*, Acta Math. Sin. (Engl. Ser.) 23 (2007), 745–748.

[30] A. Montes-Rodríguez, J. Sánchez-Álvarez, J. Zemánek, *Uniform Abel–Kreiss boundedness and the extremal behaviour of the Volterra operator*, Proc. London Math. Soc. (3) 91 (2005), 761–788.

[31] B. Nagy, J. Zemánek, *A resolvent condition implying power boundedness*, Studia Math. 134 (1999), 143–151.

[32] O. Nevanlinna, *Convergence of Iterations for Linear Equations*, Lectures Math. ETH Zürich, Birkhäuser, Basel 1993.

[33] O. Nevanlinna, *On the growth of the resolvent operators for power bounded operators*, in: Linear Operators, Banach Center Publ. 38, Polish Acad. Sci. Inst. Math., Warsaw 1997, 247–264.

[34] O. Nevanlinna, *Resolvent conditions and powers of operators*, Studia Math. 145 (2001), 113–134.

[35] D. Sarason, *A remark on the Volterra operator*, J. Math. Anal. Appl. 12 (1965), 244–246.

[36] D. Sarason, *Generalized interpolation in* H^∞, Trans. Amer. Math. Soc. 127 (1967), 179–203.

[37] D. Seifert, *Some improvements of the Katznelson–Tzafriri theorem on Hilbert spaces*, Proc. Amer. Math. Soc. 143 (2015), 3827–3838.

[38] D. Seifert, *Rates of decay in the classical Katznelson–Tzafriri theorem*, J. Anal. Math. 130 (2016), 329–354.

[39] L. Suciu, J. Zemánek, *Growth conditions on Cesàro means of higher order*, Acta Sci. Math. (Szeged) 79 (2013), 545–581.

[40] Y. Tomilov, J. Zemánek, *A new way of constructing examples in operator ergodic theory*, Math. Proc. Cambridge Philos. Soc. 137 (2004), 209–225.

[41] D. Tsedenbayar, *On the power boundedness of certain Volterra operator pencils*, Studia Math. 156 (2003), 59–66.

[42] P. Vitse, *A Besov class functional calculus for bounded holomorphic semigroups*, J. Funct. Anal. 228 (2005), 245–269.

[43] Q. P. Vũ, *A short proof of the Y. Katznelson's and L. Tzafriri's theorem*, Proc. Amer. Math. Soc. 115 (1992), 1023–1024.

[44] Q. P. Vũ, *Theorems of Katznelson–Tzafriri type for semigroups of operators*, J. Funct. Anal. 103 (1992), 74–84.

[45] M. Zarrabi, *Some results of Katznelson–Tzafriri type*, J. Math. Anal. Appl. 397 (2013), 109–118.

[46] J. Zemánek, *On the Gelfand–Hille theorems*, in: Functional Analysis and Operator Theory, Banach Center Publ. 30, Polish Acad. Sci. Inst. Math., Warsaw 1994, 369–385.

[47] J. Zemánek, *Powers of operators*, in: Perspectives in Operator Theory, Banach Center Publ. 75, Polish Acad. Sci. Inst. Math., Warsaw 2007, 327–328.

[48] J. Zemánek, *Questions on Volterra operators*, in: Hilbert spaces of Analytic Functions, CRM Proc. Lecture Notes 51, Amer. Math. Soc., Providence, RI 2010, 149–152.