

THE COHOMOLOGICAL EQUATIONS IN NONSMOOTH CATEGORIES

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Abstract. The functional equation $\varphi(Fx) - \varphi(x) = \gamma(x)$ is considered in topological, measurable and related categories from the point of view of functional analysis and general theory of dynamical systems. The material is presented in the form of a self-contained survey.

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1. Introduction. This is an extended version of the author’s talk at the conference *Operator Theory and Applications: Perspectives and Challenges* (18–28 March, 2010, Jurata, Poland) supported by the project TODEQ. I am grateful to Professor Zemánek for the suggestion to write this paper.

The cohomological equations (c.e.) and some of their generalizations naturally appear and play an important role in many areas of mathematics: group representations [34], [55],

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dynamical systems and ergodic theory [3], [21], [30], [31], [46], [60], stochastic processes [16], singularity theory [4], local classification of smooth mappings [8], summation of divergent series [53], etc. We consider the c.e. as a subject of functional analysis. This approach originates from Browder's paper [14].

By definition, the *cohomological equation* is a linear functional equation

$$\varphi(Fx) - \varphi(x) = \gamma(x), \quad x \in X, \quad (1.1)$$

where F is a self-mapping of a nonempty set X , γ is a given real- or complex-valued function on X , and φ is an unknown function. Note that the complex case in (1.1) immediately reduces to the real one by separating the real and imaginary parts. For this reason we assume that all functions under consideration are real-valued, unless otherwise stated. The term *cohomological* is borrowed from a topological language where the *coboundary* means a function γ of the form (1.1), cf. e.g. [30]. (In [3] and in some subsequent papers the term “homological” is used.)

In physical language the set X is the *space of states* of a *dynamical system* (X, F) with the discrete *time variable* $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and with the *evolutionary operator* F . By the standard definition of the latter, if $x \in X$ is the state of the system at a moment n then Fx is the state at the moment $n + 1$. Thus, if x is a state at the moment $n = 0$ (the *initial state*) then the corresponding *trajectory* or *orbit* is

$$O_F(x) = (F^n x)_{n \geq 0}, \quad x \in X.$$

In the c.e. (1.1) the functions γ and φ relate respectively to the *input* and *output* of the system (X, F) . In these terms to solve the equation (1.1) means to find out the response of the system to a given sequence of input signals. This problem may be especially nontrivial if a solution has to belong to a prescribed functional space.

Since the equation (1.1) is linear, it is natural to consider it in a subspace E of the space $\Phi(X)$ of all scalar functions on X . More generally, φ should be found in E , while γ is given in another subspace. In any case the subspaces under consideration have to be invariant for the *Koopman operator* $T_F \varphi = \varphi \circ F$. Accordingly, the c.e. (1.1) can be rewritten as

$$T_F \varphi - \varphi = \gamma. \quad (1.2)$$

In any subspace E the simplest c.e. is the *homogeneous* one, i.e. such that $\gamma = 0$. It is always solvable, its *trivial solution* is $\varphi = 0$. If the function $\mathbf{1}(x) \equiv 1$ belongs to E then all constants are solutions. The general solution consists of the *invariant functions* $\psi \in E$, i.e. such that $T_F \psi = \psi$ or, equivalently, $\psi \in \ker(T_F - I)$ where I is the identity operator. Like any linear equation (algebraic, differential etc.), if the nonhomogeneous c.e. (1.1) has a solution $\varphi_0 \in E$ then its general solution in E is $\varphi_0 + \psi$ where ψ is the general solution to the corresponding homogeneous equation.

For a dynamical system (X, F) a subset $M \subset X$ is called *invariant* if $FM \subset M$. In this case the *subsystem* $(M, F|_M)$ is well defined. If the complement $X \setminus M$ is also invariant then M is called *completely invariant*. For this property it is necessary and sufficient that the indicator function of M is invariant.

The underlying subspace E is usually determined by a structure on X such as topology, measure, etc. We consider the following “nonsmooth” situations:

a) X is an arbitrary set, $E = \Phi(X)$ (Sections 3, 4) or $E = B(X)$, the space of bounded functions (Section 5);

b) X is a topological space, $E = C(X)$, the space of continuous functions, or $E = CB(X) = C(X) \cap B(X)$ (Sections 6, 8, 11, 12);

c) $X = G$, where G is a commutative topological group, $E = C(G)$ or $E = AP(G)$, the space of almost periodic functions (Sections 7, 8, 11, 12);

d) X is a set with a measure μ , $E = M(X, \mu)$, the space of all measurable functions or $E = L_p(X, \mu)$, $1 \leq p \leq \infty$ (Sections 5, 9, 12, 13).

In each of these cases the mapping F should be a morphism of the corresponding category. Thus, in the case a) F can be any self-mapping of X , while in the case b) it should be continuous, and then the dynamical system (X, F) is called *topological*. This system is called *compact* or *discrete* if such is the space X . Obviously, a) can be considered as b) with discrete topology on X .

A topological dynamical system (X, F) (as well as F itself) is called *minimal* if all orbits are dense. A weaker property is the *topological transitivity* meaning that there exists a dense orbit. For the minimality it is necessary and sufficient that any closed invariant subset is either \emptyset or X . If (X, F) is topological transitive then all continuous invariant functions are constants.

In the case c) we only consider $F = \tau_g$ that is the shift by an element $g \in G$: $\tau_g x = gx$, $x \in G$.

In the case d) F has to be *measurable*, i.e. such that for every measurable set $M \subset X$ the preimage $F^{-1}M$ is measurable. The system (X, F) is also called *measurable* in this case. We assume that $M = X$ is measurable and $0 < \mu(X) \leq \infty$. If $\mu(X) < \infty$, i.e. the measure μ is *finite*, then one can normalize it in order to get $\mu(X) = 1$. As a rule, we will deal with this case.

Later on the abbreviation “a.e.” means “almost everywhere” with respect to a given measure μ . We assume that if a set M is measurable and $\mu(M) = 0$ then all $N \subset M$ are measurable and then $\mu(N) = 0$ automatically.

In the measurable situation it makes sense to consider the *solutions a.e.* to the equation (1.1). Every such solution $\varphi(x)$ has to satisfy (1.1) on an invariant measurable subset $Y_\varphi \subset X$ such that $\mu(X \setminus Y_\varphi) = 0$. As a rule, we mean this case when speaking about solutions in $M(X, \mu)$ or $L_p(X, \mu)$.

The measure μ is called *invariant* if $\mu(F^{-1}M) = \mu(M)$ for every measurable set $M \subset X$. If μ is finite then it is invariant if and only if

$$\int_X \phi(Fx) d\mu = \int_X \phi(x) d\mu, \quad \phi \in L_1(X, \mu). \quad (1.3)$$

The Krylov–Bogolyubov theorem [40] states that there exists an invariant regular Borel measure for any compact dynamical system (X, F) . In this case μ is finite and its invariance is equivalent to (1.3) with $\phi \in C(X)$. There are several proofs of this theorem [31], [38], [40], [52]. In Section 2 we give another proof based on a “cohomological” lemma.

On any locally compact topological group there exists a regular Borel measure invariant for all left (for definiteness) shifts, the *Haar measure* [22], [23]. This measure is unique up to a multiplicative constant. This is finite if and only if the group is compact.

In this case we assume that the Haar measure is normalized, thus unique. We denote it by ν throughout the paper.

A dynamical system (X, F) with an invariant measure μ is called *ergodic* if for every completely invariant measurable set M either $\mu(M) = 0$ or $\mu(X \setminus M) = 0$. Also, F and μ are called *ergodic* in this case. An invariant measure μ is ergodic if and only if all measurable invariant functions are constants a.e. If, in addition, such a measure is unique up to a constant factor then (X, F) is called *uniquely ergodic*.

An obvious but very important consequence of (1.3) is the equality

$$\int_X \gamma d\mu = 0 \quad (1.4)$$

for every $\gamma \in L_1(X, \mu)$ such that the c.e. (1.1) is solvable in $L_1(X, \mu)$. We call (1.4) a *trivial necessary condition* (the *TNC*, for brevity). In particular, if X is a compact topological space then for the solvability of (1.1) in $C(X)$ the TNC has to be valid for all invariant regular Borel measures μ .

Remarkably, on any set X with a finite invariant measure μ if $\gamma \in L_1(X, \mu)$ is such that (1.1) has a measurable (maybe, nonintegrable!) solution then the TNC (1.4) is fulfilled. This is Theorem 1 from Anosov's paper [3]. We reproduce it in Section 9. The proof is based on the Birkhoff–Khinchin Individual Ergodic Theorem (IET) and the Poincaré Recurrence Theorem (RT). For the latter we give a “cohomological” proof.

The simplest nonhomogeneous c.e. is the *Abel equation* (A.e.)

$$\varphi(Fx) - \varphi(x) = 1, \quad x \in X, \quad (1.5)$$

coming back to [1], pp. 36–39. Since (1.5) implies $\varphi(F^n x) = \varphi(x) + n$ for all $n \geq 1$, the A.e. has no bounded solutions. A fortiori, there are no continuous solutions if X is a compact topological space. In contrast, on a noncompact X the A.e. may be solvable in $C(X)$. For example, we have $\varphi(x) = x$ if $Fx = x + 1$ in $X = \mathbb{N}$ or $X = \mathbb{R}$. The A.e. on \mathbb{R} is well studied [8], [43], [62]. The relations of A.e. to other c.e. in $C(X)$ on noncompact X were investigated in [5], [7]. Note that if μ is a finite invariant measure then (1.4) shows that the A.e. has no solutions in $L_1(X, \mu)$.

A very interesting and deep example is the c.e. generated on the unit circle \mathbb{T} by a rotation through the angle $2\pi\alpha$, $0 \leq \alpha < 1$, with α irrational. This equation is equivalent to

$$f(x + \alpha) - f(x) = h(x), \quad x \in \mathbb{R}, \quad (1.6)$$

where the functions f and h are 1-periodic. The latter was mentioned by Hilbert in context of his fifth problem (on analyticity of Euclidean topological groups) as a functional equation which may be unsolvable in analytic functions even for an analytic h satisfying the TNC

$$\int_0^1 h(x) dx = 0. \quad (1.7)$$

The equation (1.6) can be immediately solved in formal trigonometric series. Namely, if the Fourier decomposition of h is

$$h(x) \sim \sum_{n=-\infty}^{\infty} h_n e^{2\pi i n x}$$

then (1.7) means that $h_0 = 0$, and then for α irrational the formal solution is

$$\sum_{n \neq 0} \frac{h_n}{e^{2\pi i n \alpha} - 1} e^{2\pi i n x} \quad (1.8)$$

up to an additive constant. Wintner [67] noted that for an analytic h the series (1.8) determines an analytic solution if the number α is approximated by rationals slowly enough, while in the opposite case a phenomenon of “small denominators” appears: there are no analytic solutions if h is not a trigonometric polynomial and the approximation of α by rationals is fast enough. For a further development of this approach see [30], [56] and the references therein.

Anosov [3] mentioned that the flows on the torus \mathbb{T}^2 constructed by von Neumann [65], Kolmogorov [36], and Krygin [39] yield some continuous 1-periodic functions h satisfying (1.7) such that there are no measurable solutions to (1.6). Gordon [19] constructed such a “bad” h directly. In [49] the existence of a “bad” h was proven via the Closed Graph Theorem. This shows that the set of “bad” h ’s is of second Baire’s category in the space of continuous 1-periodic functions satisfying (1.7). In [6] the method of [49] was extended to the topologically transitive shifts of compact commutative groups and then to all uniformly stable dynamical systems on compact metric spaces, see Section 12. These dynamical systems closely relate to the almost periodic operator semigroups [50].

In [3] Anosov elaborated a subtle analytic technique to construct for any irrational α a continuous 1-periodic function h such that there exists a measurable but nonintegrable solution to (1.6). Using another method Kornfeld [37] extended this result to a class of compact dynamical systems (X, F) . It would be interesting to put it into the frameworks of functional analysis.

Returning to the operator form (1.2) of the c.e. (1.1) note that its solvability in a T_F -invariant space E means $\gamma \in \text{Im}(T_F - I)$. Moreover, if the operator $T_F - I$ is invertible then $\varphi = (T_F - I)^{-1}\gamma$ is a (unique) solution. However, this is not a case if $\mathbf{1} \in E$ since $\mathbf{1} \in \ker(T_F - I)$. Nevertheless, it is very useful to consider the formal solution

$$-(\gamma + T_F \gamma + T_F^2 \gamma + \dots) \quad (1.9)$$

arising from the formal expansion

$$I + T_F + T_F^2 + \dots,$$

of the operator $(I - T_F)^{-1}$. The series inside the parentheses in (1.9) is actually

$$\gamma(x) + \gamma(Fx) + \dots + \gamma(F^n x) + \dots, \quad x \in X. \quad (1.10)$$

We call it the *resolving series* for the c.e. (1.1). The behavior of its partial sums

$$s_n(x) = \sum_{k=0}^n \gamma(F^k x), \quad n \geq 0, \quad (1.11)$$

plays a crucial role in the problem of solvability of the c.e. (1.1).

Let us emphasize that the resolving series may diverge even for functions γ such that the c.e. has a very good solution. For example, let in (1.6) with irrational α the function

h be a nonzero trigonometrical polynomial

$$h(x) = \sum_{|l| \leq m} h_l e^{2\pi i l x}$$

with $h_0 = 0$. Then the trigonometrical polynomial

$$f(x) = \sum_{|l| \leq m, l \neq 0} \frac{h_l}{e^{2\pi i l \alpha} - 1} e^{2\pi i l x}$$

is a solution. On the other hand, we have

$$s_n(x) = \sum_{k=0}^n h(x + k\alpha) = \sum_{|l| \leq m, l \neq 0} h_l \frac{e^{2\pi i (n+1)l\alpha} - 1}{e^{2\pi i l \alpha} - 1} e^{2\pi i l x}.$$

It is easy to see that the sequence $(s_n(x))$ diverges at every point x .

If the resolving series (1.10) converges or, at least, is summable (say, by Cesàro method) for all $x \in X$ then the corresponding sum is a solution to the c.e. (1.1). Also, the convergence or summability in a norm yields a solution if the operator T_F is bounded. (For example, $\|T_F\| \leq 1$ in $B(X)$ and in each $L_p(X, \mu)$ with an invariant measure μ .) In this context a powerful tool is the Mean Ergodic Theorem (MET), see [45] and the references therein. Note that historically first version of the MET is due to von Neumann [64].

In [53] the IET was applied to solve a.e. the c.e. (1.1) with $\gamma \in L_1(X, \mu)$ under the TNC (1.4) where μ is a finite ergodic invariant measure. This result was obtained in frameworks of an axiomatic theory of summation of divergent series [52], [53]. The latter is based on a modern form of the summation axioms introduced by Hardy ([26], Section 1.3) and Kolmogorov [35]. We partially reproduce our theory in Section 13. As an application we consider the c.e.

$$f(qx) - f(x) = h(x), \quad x \in \mathbb{R}, \quad (1.12)$$

in 2π -periodic functions. This situation is quite different from that of (1.6). In [53] it was proven that *if $q \in \mathbb{N}$, $q \geq 2$, and h is a trigonometric polynomial such that the ratios of its frequencies are not powers of q then all solutions to (1.12) are nonmeasurable.* Our techniques is a development of that which Zygmund applied to the case $q = 2$, $h(x) = \cos x$, see [68], Chapter 5, Problem 26.

For $q = 3$ and $h(x) = \sin x$ Kolmogorov [35] formulated (without proof) the following conditional statement: *if the trigonometric series*

$$\sin x + \sin 3x + \dots + \sin 3^n x + \dots, \quad x \in \mathbb{R}, \quad (1.13)$$

is summable then one can efficiently construct a nonmeasurable function. He evidently meant that this function can be produced by a summation under his axioms. Our Theorem 13.5 implies the existence of such a summation. Note that (1.13) is the resolving series for the c.e. (1.12) with Kolmogorov's data.

According to the Hardy–Kolmogorov axioms the summations must be linear. Surprisingly, some nonlinear procedures can also be applied to solve the c.e. (1.1) in spite of its linearity, see Sections 5–8. The most important example is the formula

$$\varphi(x) = \sup_n (-s_n(x)), \quad x \in X, \quad (1.14)$$

for a bounded continuous solution in the case of minimal (X, F) and $\gamma \in CB(X)$ such that the sums $s_n(x)$ are uniformly bounded. Under these conditions for compact metric X and invertible F the existence of a continuous solution was established by Gottschalk and Hedlund ([21], Theorem 14.11). Browder [14] extended this fundamental result to any Hausdorff X and any continuous $F : X \rightarrow X$. Formula (1.14) appeared in the proof given by Lin and Sine [45]. In our version of the latter (Theorem 6.2) X is an arbitrary topological space. (See [54] for an alternative proof.) It is interesting that the nonlinearity of (1.14) can be removed by subtracting a linear functional over (1.14), see Theorem 6.11. In this sense the nonlinearity is 1-dimensional.

For the shift τ_g in a compact commutative group G the minimality condition mentioned above can be reformulated in terms of characters. This yields a new criterion of the solvability of the corresponding c.e. in $C(G)$, see Section 7. By the Bohr compactification this result extends to the almost periodic solutions on any commutative topological group G .

In an almost periodic context the minimality is redundant (Section 8). A general result of such a kind is Theorem 8.1 proved by Schauder's Fixed Point Principle. Its main application is that the precompactness of the set $\{s_n\}$ is a criterion of solvability of the c.e. (1.1) in $C(X)$ for the uniformly stable system (X, F) on a compact metric space X (Theorem 8.3). This system can be not minimal, and then the boundedness of $\{s_n\}$ is not sufficient for the solvability in $C(X)$ (Example 8.8). Note that there is no minimality if X is a convex compact in a Banach space.

In accordance with a general terminology, the c.e. (1.1) in a linear topological space E is called *normally solvable* if $\text{Im}(T_F - I)$ is closed. If the space E is complete metrizable then the absence of the normal solvability implies that $\text{Im}(T_F - I)$ is a subset of the first Baire category in its closure. In this sense the unsolvability of the c.e. (1.1) is typical if this equation is not normally solvable.

For $E = C(X)$ with compact X the normal solvability means the solvability in this space as long as the TNC (1.4) is valid for all invariant regular Borel measures. It turns out this case is very rare: *if (X, F) is a compact dynamical system then the c.e. (1.1) is normally solvable in $C(X)$ if and only if F is preperiodic, i.e. $F^{l+p} = F^l$ with a preperiod $l \geq 0$ and a period $p \geq 1$ [6].* This result is represented in Section 11. The proof is based on the Dunford–Lin Uniform Ergodic Theorem (UET) [15], [44].

On the locally compact spaces the normal solvability was investigated in [7].

2. Existence of invariant measures. In general, a dynamical system has no “good” invariant measures.

EXAMPLE 2.1. Let $X = \mathbb{N}$ and let $Fn = n + 1$, $n \in \mathbb{N}$. If a measure μ is invariant and all singletons are measurable then $\mu(\{n\}) = \mu(\{n - 1\})$ for $n \geq 1$, but $\mu(\{0\}) = \mu(\emptyset) = 0$. Hence, $\mu(\{n\}) = 0$ for all n , thus $\mu(X) = 0$.

The following fundamental theorem is due to Krylov and Bogolyubov [40].

THEOREM 2.2. *For every compact dynamical system (X, F) there exists an invariant regular Borel measure μ such that $\mu(X) = 1$.*

The proof below is based on the following “cohomological” lemma.

LEMMA 2.3. *Let X be a set, and let F be its self-mapping. In the space $B(X)$ the distance from the function $\mathbf{1}$ to the subspace generated by the coboundaries is equal to 1.*

This is a quantitative version of the unsolvability of the Abel equation in $B(X)$.

Proof. The distance in question is the infimum over $\varphi \in B(X)$ of the functional

$$\delta[\varphi] = \sup_x |1 - (\varphi(Fx) - \varphi(x))|.$$

Since $\delta[0] = 1$, it suffices to prove that $\delta[\varphi] \geq 1$.

By definition,

$$\delta[\varphi] \geq |1 - (\varphi(Fx) - \varphi(x))|$$

for all $\varphi \in B(X)$ and all $x \in X$. Changing x to $F^k x$, $k \geq 0$, we obtain

$$\delta[\varphi] \geq |1 - (\varphi(F^{k+1}x) - \varphi(F^k x))|,$$

whence

$$n\delta[\varphi] \geq \sum_{k=0}^{n-1} |1 - (\varphi(F^{k+1}x) - \varphi(F^k x))| \geq |n - (\varphi(F^n x) - \varphi(x))|, \quad n \geq 1,$$

by the triangle inequality. Therefore,

$$\delta[\varphi] \geq \left| 1 - \frac{\varphi(F^n x) - \varphi(x)}{n} \right|.$$

Passing to the limit as $n \rightarrow \infty$ we get $\delta[\varphi] \geq 1$ since $\varphi \in B(X)$. ■

Proof of Theorem 2.2. In the space $C(X)$ we consider the linear continuous functional f such that $f[\mathbf{1}] = 1$, $\|f\| = 1$ and

$$f[\varphi \circ F - \varphi] = 0, \quad \varphi \in C(X). \quad (2.1)$$

This functional exists by Lemma 2.3 and the Hahn–Banach theorem. It easy to see that $f[\varphi] \geq 0$ for $\varphi \geq 0$. Indeed, let $0 \leq \varphi(x) \leq 1$ for all $x \in X$. Then $\|\mathbf{1} - \varphi\| \leq 1$, hence, $1 - f[\varphi] = f[\mathbf{1} - \varphi] \leq 1$.

Now a required measure μ comes from the Riesz representation

$$f[\varphi] = \int_X \varphi d\mu.$$

The measure μ is invariant since (2.1) is equivalent to (1.3). In addition, $\mu(X) = f(\mathbf{1}) = 1$. ■

REMARK 2.4. In $B(X)$ Lemma 2.3 yields a nonnegative invariant additive function $\mu(M)$ on the set of all subsets $M \subset X$, $\mu(X) = 1$.

For a compact group G and a shift $\tau_g x = gx$ ($x, g \in G$) Theorem 2.2 yields an invariant regular Borel measure μ_g , $\mu_g(G) = 1$.

PROPOSITION 2.5. *Let G be a compact commutative group, and let $g \in G$ be such that the subsemigroup $[g] = \{g^n : n \geq 0\}$ is dense. Then the measure μ_g coincides with the Haar measure ν on G .*

Proof. We need to prove that the measure μ_g is invariant for all shifts. For a function $\varphi \in C(G)$ let us consider the convolution

$$\psi(h) = \int_G \varphi(xh) d\mu_g(x), \quad h \in G.$$

By commutativity of G and τ_g -invariance of μ_g we have

$$\psi(gh) = \int_G \varphi(xgh) d\mu_g(x) = \int_G \varphi(gxh) d\mu_g(x) = \int_G \varphi(xh) d\mu_g(x) = \psi(h), \quad (2.2)$$

whence $\psi(yh) = \psi(h)$ for all $y \in [g]$. The function ψ is continuous since φ is uniformly continuous and the measure μ_g is finite. Since $[g]$ is dense, we obtain $\psi(yh) = \psi(h)$ for all $y \in G$. In particular, $\psi(y) = \psi(e)$ where e is the unit of the group G . Thus,

$$\int_G \varphi(xy) d\mu_g(x) = \int_G \varphi(x) d\mu_g(x), \quad y \in G. \quad \blacksquare$$

We use this fact at the end of Section 10.

3. Discrete systems. Here we consider the c.e. (1.1) in the space $\Phi(X)$ of all functions on an arbitrary nonempty set X . Although this situation is elementary, it is a prototype for more complicated cases and also a source of some useful general information.

It follows from (1.1) that

$$\varphi(F^{k+1}x) - \varphi(F^kx) = \gamma(F^kx), \quad k \geq 0.$$

By summation over $k \in [0, n]$ we get

$$\varphi(F^{n+1}x) = \varphi(x) + \sum_{k=0}^n \gamma(F^kx) = \varphi(x) + s_n(x), \quad n \geq 0. \quad (3.1)$$

Formula (3.1) shows that the restriction of the solution φ to the orbit $O_F(x)$ is determined by the initial value $\varphi(x)$. However, the asymptotic behavior of $\varphi(F^n x)$ as $n \rightarrow \infty$ reduces to that of $s_n(x)$.

Assume that the resolving series (1.10) converges for all $x \in X$ to a function $s(x)$. Then the function $\varphi_0(x) = -s(x)$ is a solution to (1.1) such that

$$\lim_{n \rightarrow \infty} \varphi_0(F^n x) = 0$$

since

$$\varphi_0(F^n x) = -s(F^n x) = -\sum_{k=n}^{\infty} \gamma(F^k x).$$

Furthermore, for any solution φ there exists

$$\hat{\varphi}(x) = \lim_{n \rightarrow \infty} \varphi(F^n x) = \varphi(x) + s(x) \quad (3.2)$$

by (3.1). The function $\hat{\varphi}(x)$ is invariant, and

$$\varphi(x) = -s(x) + \hat{\varphi}(x) = \varphi_0(x) + \hat{\varphi}(x).$$

This is an explicit representation of any solution φ as the sum of the particular solution $\varphi_0 = -s$ and the solution $\hat{\varphi}$ to the corresponding homogeneous equation.

Those γ 's for which the resolving series converges constitute a T_F -invariant subspace of the space $\Phi(X)$.

The following theorem is a general solvability criterion for the c.e. (1.1).

THEOREM 3.1. *The c.e. (1.1) is solvable if and only if for every integer $p \geq 1$ and every periodic point x of the period p the sum $s_{p-1}(x)$ vanishes, i.e.*

$$\sum_{k=0}^{p-1} \gamma(F^k x) = 0. \quad (3.3)$$

Proof. The necessity of (3.3) follows from (3.1) immediately. To prove the sufficiency we introduce an equivalence relation on X letting $x_1 \sim x_2$ if there are some m, n such that $F^m x_1 = F^n x_2$. Since the equivalence classes are invariant, it suffices to solve (1.1) on each class separately. Thus, one can assume that all $x \in X$ are equivalent to an x_0 . We show that for an arbitrary “initial value” $\varphi(x_0)$ there is a unique solution $\varphi(x)$ to the c.e. (1.1). To this end we denote by l_x the minimal n such that $x = F^n x_0$. Let us consider two cases.

1) $x \in O_F(x_0)$. Assume that the orbit $O_F(x_0)$ is infinite. Then the points $F^k x_0$ ($k = 0, 1, 2, \dots$) are pairwise distinct. In particular, $Fx = F^{l_x+1} x_0$ is different from all $F^k x_0$ with $k \leq l_x$, hence $l_{Fx} = l_x + 1$. Letting

$$\varphi(x) = \varphi(x_0) + \sum_{k=0}^{l_x-1} \gamma(F^k x_0), \quad x \in O_F(x_0), \quad (3.4)$$

we get (1.1) since

$$\varphi(Fx) = \varphi(x_0) + \sum_{k=0}^{l_x} \gamma(F^k x_0) = \varphi(x) + \gamma(x). \quad (3.5)$$

Now assume that $O_F(x_0)$ is finite. Then the point x_0 is preperiodic of a preperiod l and a period p , i.e. $F^{l+p} x_0 = F^l x_0$. Let these values be minimal. Then the orbit reduces to

$$\tilde{O}_F(x_0) = \{x_0, \dots, F^l x_0, F^{l+1} x_0, \dots, F^{l+p-1} x_0\} \quad (3.6)$$

where all points are pairwise distinct. Obviously, we have $l_{Fx} = l_x + 1$ for all $x \in \tilde{O}_F(x_0)$, except for the last member in (3.6). With this exception we define $\varphi(x)$ by (3.4) and get (1.1) as before. In the exceptional case we have $l_{Fx} = l$, while $l_x = l + p - 1$. Accordingly, if we let

$$\varphi(x) = \varphi(x_0) + \sum_{k=0}^{l+p-2} \gamma(F^k x_0), \quad x = F^{l+p-1} x_0, \quad (3.7)$$

and use

$$\varphi(Fx) = \varphi(x_0) + \sum_{k=0}^{l-1} \gamma(F^k x_0) \quad (3.8)$$

then

$$\varphi(x) - \varphi(Fx) = \sum_{k=l}^{l+p-2} \gamma(F^k x_0) = \sum_{k=0}^{p-2} \gamma(F^k x_1) \quad (3.9)$$

with $x_1 = F^l x_0$. This is a periodic point of period p . By (3.3) the last sum in (3.9) is equal to $-\gamma(F^{p-1} x_1) = -\gamma(F^{l+p-1} x_0) = -\gamma(x)$, so (3.9) reduces to (1.1).

2) $x \notin O_F(x_0)$. Let m_x be the minimal $n \geq 1$ such that $F^n x$ belongs to $O_F(x_0)$. Such n 's exist since $x \sim x_0$. Obviously, $m_{Fx} = m_x - 1$ if $m_x > 1$. Since $\varphi(F^{m_x}x)$ is already determined in case 1), the definition

$$\varphi(x) = \varphi(F^{m_x}x) - \sum_{k=0}^{m_x-1} \gamma(F^k x), \quad x \notin O_F(x_0), \quad (3.10)$$

is correct. Accordingly,

$$\varphi(Fx) = \varphi(F^{m_x}x) - \sum_{k=1}^{m_x-1} \gamma(F^k x) = \varphi(x) + \gamma(x) \quad (3.11)$$

if $m_x > 1$. However, if $m_x = 1$ then yet (3.10) is equivalent to (1.1).

Since all constructions above follow from (3.1), the solution is unique. ■

REMARK 3.2. In the case of invertible F the equivalence class of $x_0 \in X$ is the two-sided orbit $(F^k x_0)_{k=-\infty}^{\infty}$. Hence, for $x \notin O_F(x_0)$ we have $F^m x = x_0$ with a uniquely determined $m \geq 1$. Thus, $m_x = m$ in this case.

REMARK 3.3. It suffices to have (3.3) for every periodic point x and for *one* of its periods, say, p . Indeed, p is a multiple of the smallest period p_0 , say $p = dp_0$. Accordingly, $s_{p-1}(x) = ds_{p_0-1}(x)$, whence $s_{p_0-1}(x) = 0$, and finally, $s_{q-1}(x) = 0$ for every period q .

The orbit of a periodic point x is a *cycle* Z_x . The condition (3.3) can be rewritten as

$$\int_X \gamma d\mu^{(x)} = 0 \quad (3.12)$$

where $\mu^{(x)}$ is the invariant measure concentrated and uniformly distributed on Z_x . Thus, (3.3) is just the TNC (1.4) corresponding to $\mu = \mu^{(x)}$. Theorem 3.1 states that for any discrete dynamical system the conjunction of all these TNC's is sufficient for the solvability of the c.e. (1.1). In topological terms, *a function γ is a coboundary if and only if its integrals of over all cycles vanish*. A remarkable case of sufficiency of these TNC's for the solvability of (1.1) in a smooth category is Livšic's theorem [46] concerning Hölder's solutions.

In the completely formalized proof of Theorem 3.1 one has to consider a subset of X which intersects every equivalence class by a singleton. This subset is called a *transversal*. Its existence follows from the axiom of choice. *Given a transversal X_0 , the restriction $\varphi \mapsto \varphi|X_0$ bijectively maps the set of solutions onto $\Phi(X_0)$* . The inverse mapping is described by the formulas (3.4), (3.7) and (3.10), where x_0 runs over X_0 . For example, *every solution to the homogeneous c.e. is constant on each equivalence class, and, conversely, every such function is a solution*.

The following particular case of Theorem 3.1 merits to be mentioned.

COROLLARY 3.4. *If the mapping F has no periodic points (thus, X is infinite) then with any γ the c.e. (1.1) is solvable.*

For example, any irrational rotation of the unit circle \mathbb{T} has no periodic points. Therefore, *the equation (1.6) with irrational α is solvable in 1-periodic functions for any*

1-periodic h . A more general example is the c.e.

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G,$$

where G is a group and $g \in G$ is of infinite order.

COROLLARY 3.5. *With $\gamma > 0$ (in particular, in the case of Abel equation) the c.e. (1.1) is solvable if and only if the mapping F has no periodic points.*

Proof. The condition (3.3) is not fulfilled if all $\gamma(F^k x) > 0$. ■

The set Π of periodic points of any mapping F is invariant but, in general, it is not completely invariant. Indeed, the union of the preimages $F^{-l}\Pi$, $l \geq 0$, is the set of preperiodic points. The latter is already completely invariant. By Corollary 3.4 we have

COROLLARY 3.6. *On the complement of the set of preperiodic points the c.e. (1.1) is solvable with any γ .*

EXAMPLE 3.7. Let q be an integer, $q \geq 2$. The preperiodic points of the mapping $x \mapsto qx \pmod{2\pi}$, $0 \leq x < 2\pi$, are

$$x_{n,l,p} = \frac{2\pi n}{q^l(q^p - 1)} \pmod{2\pi},$$

where $n, l, p \in \mathbb{N}$, $p \geq 1$. On the complement of this countable set the c.e. (1.12) is solvable in 2π -periodic functions for any 2π -periodic h .

4. Cesàro summation. The reference to the axiom of choice in the proof of Theorem 3.1 makes it nonconstructive. However, under some natural conditions a solution to the c.e. (1.1) can be found by “analytical” tools, say by a summation of the resolving series. In this section we consider the Cesàro summation. By the standard definition, the *Cesàro sum* of the series (1.10) is the limit $\sigma(x)$ as $N \rightarrow \infty$ of the arithmetic means

$$\sigma_N(x) = \frac{1}{N+1} \sum_{n=0}^N s_n(x) = \sum_{k=0}^N \left(1 - \frac{k}{N+1}\right) \gamma(F^k x). \quad (4.1)$$

Denote by C_γ the set of those $x \in X$ for which $\sigma(x)$ exists.

PROPOSITION 4.1. *The set C_γ is completely invariant, and the function $-\sigma(x)$ satisfies the c.e. (1.1) for $x \in C_\gamma$.*

Proof. Using the recurrence relation

$$s_n(Fx) = s_{n+1}(x) - \gamma(x) \quad (4.2)$$

we get

$$\sigma_{N-1}(Fx) = \frac{N+1}{N} (\sigma_N(x) - \gamma(x)), \quad N \geq 1. \quad (4.3)$$

It remains to pass to the limit as $N \rightarrow \infty$. ■

Now we introduce the set E_φ of those $x \in X$ for which the limit $\tau(x)$ of the arithmetic means

$$\tau_N(x) = \frac{1}{N+1} \sum_{n=0}^N \varphi(F^n x) \quad (4.4)$$

exists as $N \rightarrow \infty$. Here φ can be any function on X .

PROPOSITION 4.2. *The set E_φ is completely invariant, and the function $\tau(x)$ is invariant. If φ is a solution to the c.e. (1.1) then $E_\varphi = C_\gamma$, and on this set*

$$\sigma(x) = \tau(x) - \varphi(x). \quad (4.5)$$

Thus, the set E_φ is independent of the solution φ to (1.1) with a fixed γ . Formula (4.5) represents the solution $\varphi(x)$ to the c.e. (1.1) restricted to E_φ as the sum of the solution $-\sigma(x)$ and the solution $\tau(x)$ to the corresponding homogeneous equation.

Proof. The first statement follows from the recurrence relation

$$\tau_{N-1}(Fx) = \frac{N+1}{N} \tau_N(x) - \frac{1}{N} \varphi(x), \quad N \geq 1. \quad (4.6)$$

The second one follows from the formula

$$\sigma_N(x) = \frac{N+2}{N+1} (\tau_{N+1}(x) - \varphi(x)) \quad (4.7)$$

which, in turn, immediately follows from (4.1), (4.4) and (3.1). ■

As an application we consider a preperiodic system (X, F) . The following theorem was formulated without proof in [6].

THEOREM 4.3. *Let $F^{l+p} = F^l$, $l \geq 0$, $p \geq 1$. Then the c.e. (1.1) is solvable if and only if*

$$\sum_{k=0}^{p-1} \gamma(F^{k+l}x) = 0, \quad x \in X. \quad (4.8)$$

Under this condition a solution is

$$\varphi(x) = - \sum_{k=0}^{l-1} \gamma(F^k x) + \frac{1}{p} \sum_{k=1}^{p-1} k \gamma(F^{k+l} x). \quad (4.9)$$

Proof. The first statement follows from Theorem 3.1 since the condition (4.8) is equivalent to (3.3). Indeed, for every $x \in X$ the point $F^l x$ is periodic of period p , thus (3.3) implies (4.8). Conversely, let (4.8) be fulfilled, and let $z \in X$ be a periodic point of a period q . Then $F^{nq}z = z$ for all $n \in \mathbb{N}$. With $n \geq l/q$ we get $z = F^l x$ where $x = F^{nq-l}z$. Obviously, p is a period of z and

$$\sum_{k=0}^{p-1} \gamma(F^k z) = 0.$$

Thus, (3.3) is fulfilled.

Now assuming (4.8) we derive (4.9) via the Cesàro summation. Let φ be a solution to (1.1). With $N > l$ and q such that $l + pq \leq N < l + p(q+1)$ we have

$$\sum_{n=0}^N \varphi(F^n x) = \sum_{n=0}^{l-1} \varphi(F^n x) + q \sum_{n=l}^{l+p-1} \varphi(F^n x) + \sum_{n=l}^{N-pq} \varphi(F^n x).$$

Hence, for all $x \in X$

$$\tau(x) = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N \varphi(F^n x) = \frac{1}{p} \sum_{n=l}^{l+p-1} \varphi(F^n x).$$

Thus, $E_\varphi = X$, so $C_\gamma = X$ by Proposition 4.2, and $-\sigma(x)$ satisfies (1.1) by Proposition 4.1.

Now we take $N = l - 1 + mp$ in (4.1), $m \rightarrow \infty$. This yields

$$-\sigma(x) = -\sum_{k=0}^{l-1} \gamma(F^k x) + \lim_{m \rightarrow \infty} \sum_{k=l}^{l-1+mp} \left(\frac{k}{l+mp} - 1 \right) \gamma(F^k x). \quad (4.10)$$

It remains to show that the limit in (4.10) is equal to the second summand in (4.9). To this end we parametrize the index k in the second sum in (4.10) as $k = l + i + pj$, where $0 \leq i \leq p-1$, $0 \leq j \leq m-1$. Since $F^{l+i+pj} = F^{l+i}$, this sum reduces to

$$\sum_{j=0}^{m-1} \sum_{i=0}^{p-1} \left(\frac{l+i+pj}{l+mp} - 1 \right) \gamma(F^{l+i} x) = \Sigma_1 + \Sigma_2$$

where

$$\Sigma_1 = \sum_{j=0}^{m-1} \left(\frac{l+pj}{l+mp} - 1 \right) \sum_{i=0}^{p-1} \gamma(F^{l+i} x)$$

and

$$\Sigma_2 = \frac{m}{l+mp} \sum_{i=1}^{p-1} i \gamma(F^{l+i} x).$$

However, $\Sigma_1 = 0$ by (4.8), while Σ_2 tends to the second summand in (4.9). ■

COROLLARY 4.4. *Let F be preperiodic, and let (4.8) be fulfilled for a function γ from a T_F -invariant subspace $E \subset \Phi(X)$. Then the c.e. (1.1) has a solution in E .*

Proof. The required solution is that of (4.9). ■

By Theorem 3.1 this can be reformulated as follows.

COROLLARY 4.5. *Let F be preperiodic, and let γ belongs to a T_F -invariant subspace $E \subset \Phi(X)$. Then if the c.e. (1.1) is solvable in $\Phi(X)$ then it is solvable in E .*

For example, if (X, F) is a preperiodic topological dynamical system and γ is a continuous function on X then there exists a continuous solution as long as a solution exists at all.

For $l = 0$ the first summand in (4.9) vanishes and we get the following

COROLLARY 4.6. *Let F be periodic of a period p , i.e. $F^p = I$. If the condition (3.3) is fulfilled for all $x \in X$ then the function*

$$\varphi(x) = \frac{1}{p} \sum_{k=1}^{p-1} k \gamma(F^k x) = -\frac{1}{p} \sum_{n=0}^{p-1} s_n(x) \quad (4.11)$$

is a solution to the c.e. (1.1).

EXAMPLE 4.7. Consider the equation (1.6) with rational $\alpha = r/p$, $\text{g.c.d.}(r, p) = 1$, and 1-periodic h . The shift $Fx = x + \alpha$ modulo 1 is periodic of period p . Under conditions

$$\sum_{k=0}^{p-1} h(x + k\alpha) = 0, \quad 0 \leq x < 1, \quad (4.12)$$

Corollary 4.6 yields the 1-periodic solution

$$f(x) = \frac{1}{p} \sum_{k=1}^{p-1} kh(x + k\alpha). \quad (4.13)$$

Note that (4.12) is equivalent to its restriction to $0 \leq x < 1/p$. Indeed, for every $x \in [0, 1)$ there exists a unique pair of integers s, t such that the integral part of px is equal to $rs + pt$ with $0 \leq s \leq p-1$. Then in (4.12) one can change x to $z = x - s\alpha - t \in [0, 1/p)$. In fact, we see that the interval $[0, 1/p)$ is a transversal. Hence, the general solution to the corresponding homogeneous equation is $g(z)$ where $z = z(x)$ is as above and g is an arbitrary function on $[0, 1/p)$.

REMARK 4.8. If (X, F) is an invertible dynamical system and X_0 is a transversal then

$$X = \bigcup \{F^n X_0 : n \in \mathbb{Z}\}. \quad (4.14)$$

It turns out that if F has no periodic points then X_0 is nonmeasurable with respect to any finite invariant measure on X . Indeed, the constituents in (4.14) are pairwise disjoint. If X_0 is measurable then $\mu(F^n X_0) = \mu(X_0)$ for all n . This implies $\mu(X_0) = 0$ since $\mu(X) < \infty$, thus, $\mu(X) = 0$.

For example, for the irrational rotation of the unit circle every transversal is nonmeasurable, in contrast to the rational case. This is a counterpart of the classical example of a nonmeasurable set, see e.g., [18], Chapter 8, Example 11.

5. Resolving functionals. In some important situations the c.e. (1.1) can be solved by a “nonlinear summation” of the resolving series (1.10).

THEOREM 5.1. *Given a dynamical system (X, F) and a function γ on X , let Λ be a set of scalar sequences $(\eta_n)_{n \geq 0}$ containing all $(-s_n(x))$, $x \in X$, and let ω be a functional $\Lambda \rightarrow \mathbb{R}$. Assume that*

- a) Λ is invariant with respect to the shift $n \mapsto n+1$ and to all translations $(\eta_n) \mapsto (\eta_n + \eta)$,
- b) ω is shift invariant and translation covariant, i.e.

$$\omega[(\eta_{n+1})] = \omega[(\eta_n)], \quad \omega[(\eta_n + \eta)] = \omega[(\eta_n)] + \eta.$$

Then ω is a resolving functional for the c.e. (1.1) in the sense that

$$\varphi(x) = \omega[(-s_n(x))] \quad (5.1)$$

is a solution to the c.e. (1.1).

Proof. We have

$$\varphi(Fx) = \omega[(-s_n(Fx))] = \omega[(-s_{n+1}(x) + \gamma(x))] = \omega[(-s_{n+1}(x))] + \gamma(x)$$

by (4.2) and the translation covariance of ω . Hence,

$$\varphi(Fx) - \varphi(x) = \gamma(x) - \{\omega[(-s_n(x))] - \omega[(-s_{n+1}(x))]\}. \quad (5.2)$$

This reduces to (1.1) by the shift invariance of ω . ■

Obviously, for every finite family $\{\omega_k\}$ of resolving functionals all linear combinations $\sum_k \alpha_k \omega_k$ with $\sum_k \alpha_k = 1$ are resolving functionals.

The conditions a) and b) of Theorem 5.1 are fulfilled for $\Lambda = B(\mathbb{N})$ (i.e. for the space of all bounded sequences (η_n)) and

$$\omega[(\eta_n)] = \overline{\lim}_n (\eta_n). \quad (5.3)$$

Other resolving functionals on this space are: the lower limit, the upper or lower limit of the arithmetic means of $(\eta_n)_{n=0}^N$, etc. All of them are nonlinear. A linear example is the Banach limit.

COROLLARY 5.2. *If the sequence $(s_n(x))$ is bounded at every $x \in X$ then the function*

$$\varphi(x) = \overline{\lim}_n [-s_n(x)] \quad (5.4)$$

is a solution to the c.e. (1.1).

COROLLARY 5.3. *The c.e. (1.1) has a solution $\varphi \in B(X)$ if and only if the sums $s_n(x)$ ($n \geq 0$) are uniformly bounded. (In particular, $\gamma(x) \equiv s_0(x)$ is bounded.) Under this condition a bounded solution is (5.4). Moreover,*

$$\frac{1}{2} \sup_n \|s_n\| \leq \|\varphi\| \leq \sup_n \|s_n\|. \quad (5.5)$$

Proof. The “if” part with the right-hand inequality in (5.5) follows from Theorem 5.1 with the upper limit in role of ω . The “only if” part with the left-hand inequality follows from (3.1) for any bounded solution φ . ■

REMARK 5.4. The “if” part of Corollary 5.3 can also be extracted from the proof of Theorem 3.1. Indeed, let the sums $s_n(x)$ be uniformly bounded. For every $p \geq 1$ and every periodic point x of period p we have $s_{p-1}(x) = n^{-1}s_{np-1}(x)$ that yields (3.3) as $n \rightarrow \infty$. Now if x_0 runs over a transversal X_0 and $\varphi(x_0)$ is bounded then we obtain a bounded solution $\varphi(x)$ by (3.4), (3.7) and (3.10).

THEOREM 5.5. *For any measurable dynamical system (X, F) the c.e. (1.1) has a bounded measurable solution if and only if the function $\gamma(x)$ is measurable and the sums $s_n(x)$ are uniformly bounded. In this case the function (5.4) is such a solution.*

This is a version of Proposition 8 from [45] where a bounded solution is determined as the upper limit of the arithmetic means of $(-s_n(x))_{n=0}^N$ as $N \rightarrow \infty$.

THEOREM 5.6. *Let (X, F) be a measurable dynamical system with a measure μ , and let $\gamma \in L_\infty(X, \mu)$. If the sequence (s_n) is L_∞ -bounded then the c.e. (1.1) is solvable in this space. If the measure μ is invariant then the boundedness condition is necessary.*

Proof. The necessity follows from (3.1). Now assume that in $L_\infty(X, \mu)$ we have $\sup_n \|s_n\| < \infty$. However, $\|s_n\| = \sup_{x \in X_n} |s_n(x)|$ for a subset $X_n \subset X$ such that $\mu(X \setminus X_n) = 0$. For the intersection Y of all X_n we have $\mu(X \setminus Y) = 0$ and

$$\sup_{x \in Y} \sup_n |s_n(x)| = \sup_n \sup_{x \in Y} |s_n(x)| \leq \sup_n \|s_n\| < \infty. \quad (5.6)$$

Now let us consider the set $Z = \{x \in X : \sup_n |s_n(x)| < \infty\}$. Since $Y \subset Z \subset X$, we have $\mu(X \setminus Z) = 0$ and $\mu(Z \setminus Y) = 0$. The set Z is T_F -invariant because of (4.2). For $x \in Z$ a solution $\varphi(x)$ can be determined by (5.4), and $\sup_{x \in Y} |\varphi(x)| < \infty$ by (5.6), thus $\varphi \in L_\infty(X, \mu)$. ■

Theorem 5.6 generalizes Theorem 2 from [14] where the L_p -solutions are constructed for $1 < p < \infty$ and then an L_∞ -solution is obtained as $p \rightarrow \infty$. For other results concerning L_∞ -solutions see [45] and [58].

The question arises: *is the supremum (or the infimum) a resolving functional?* In general, *the answer is negative*. The reason is that these functionals are not shift invariant, so the subtrahend in formula (5.2) may not vanish.

EXAMPLE 5.7. Let $X = \mathbb{N}$, and let $Fn = n + 1$, $n \geq 0$. Then the c.e. (1.1) turns into the simplest difference equation

$$\varphi(n+1) - \varphi(n) = \gamma(n). \quad (5.7)$$

Its general solution is

$$\varphi(n) = \varphi(0) + \sum_{k=0}^{n-1} \gamma(k) = \varphi(0) + s_{n-1}(0), \quad n \geq 1, \quad (5.8)$$

with an arbitrary $\varphi(0)$. The difference of any two solutions is a constant.

From (5.8) it follows that if the sums $s_n(0)$ are bounded then all solutions are bounded, therefore, the sums $s_n(m)$ are uniformly bounded. However, the function $\varphi(m) = \sup_n [-s_n(m)]$ is not a solution, in general. Indeed, since

$$s_n(m) = s_{n+m}(0) - s_{m-1}(0), \quad m \geq 1,$$

we have

$$\varphi(m) = s_{m-1}(0) - \inf_{n \geq m} [s_n(0)]. \quad (5.9)$$

This is a solution if and only if the infimum in (5.9) is a constant since $s_{m-1}(0)$ is a solution. However, if, for instance, $\gamma(n) > 0$ for all n then the infimum is equal to $s_m(0)$ that increases along with m .

Nevertheless, *the supremum and infimum are resolving functionals in some special situations*. For definiteness we consider the supremum. Then

$$\sup_n (\eta_n) \geq \sup_n (\eta_{n+1}), \quad (\eta_n) \in B(\mathbb{N}), \quad (5.10)$$

instead of the shift invariance. On this base we establish a useful lemma concerning the *residual function*

$$\delta(x) = \gamma(x) + \varphi(x) - \varphi(Fx), \quad x \in X. \quad (5.11)$$

This one is defined for any pair of functions φ, γ , and $\delta = 0$ if and only if φ is a solution to the c.e. (1.1).

LEMMA 5.8. *Let $s_n(x)$ be uniformly bounded, and let $\varphi(x) = \sup_n [-s_n(x)]$. Then*

- 1) *The residual function $\delta(x)$ is nonnegative.*
- 2) *The series*

$$\delta(x) + \delta(Fx) + \dots + \delta(F^n x) + \dots \quad (5.12)$$

converges for all x to a bounded nonnegative function $\Delta(x)$.

Proof. 1) From (5.11) and (5.2) with $\omega \equiv \sup$ it follows that

$$\delta(x) = \sup_n [-s_n(x)] - \sup_n [-s_{n+1}(x)] \geq 0. \quad (5.13)$$

2) From (5.11) it follows that

$$\sum_{k=0}^n \delta(F^k x) = s_n(x) + \varphi(x) - \varphi(F^{n+1}x). \quad (5.14)$$

The right-hand side of (5.14) is uniformly bounded. Hence, 1) \Rightarrow 2) with the inequality $0 \leq \Delta(x) \leq 3 \sup_n \|s_n\|$. ■

COROLLARY 5.9. *Under conditions of Lemma 5.8 $\delta(F^n x) \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 5.8 allows us to construct a *regularization* of the resolving series (1.10) in the case of its divergence. Namely, let us consider the difference

$$d_N(x) = \sup_n [-s_n(F^{N+1}x)] - s_N(x). \quad (5.15)$$

THEOREM 5.10. *If $s_n(x)$ are uniformly bounded then the sequence $(d_N(x))$ is nonincreasing and it tends to a solution to the c.e. (1.1) as $N \rightarrow \infty$.*

Proof. Let $\varphi(x) = \sup_n [-s_n(x)]$ as before. The function $\Delta(x)$ from Lemma 5.8 satisfies the c.e. $\Delta(Fx) - \Delta(x) = -\delta(x)$. By (5.11) the difference $\hat{\varphi} = \varphi - \Delta$ is a solution to the c.e. (1.1). From (5.14) it follows that $d_N(x)$ monotonically tends to $\hat{\varphi}(x)$ from above. ■

If the resolving series converges to a function $s(x)$ for all $x \in X$ then the supremum in (5.15) tends to zero as $N \rightarrow \infty$, thus $d_N(x)$ monotonically tends to the solution $-s(x)$ from above.

Now we consider a situation where the supremum is a resolving functional.

THEOREM 5.11. *Let (X, F) be a measurable dynamical system with a finite invariant measure μ , and let γ be a measurable function on X such that the sums $s_n(x)$ are uniformly bounded. Then the function*

$$\varphi(x) = \sup_n [-s_n(x)] \quad (5.16)$$

is a bounded measurable solution to the c.e. (1.1) a.e.

Proof. We only need to prove that φ is a solution a.e. By Lemma 5.8

$$\sum_{k=0}^n \delta(F^k x) \leq \Delta(x), \quad x \in X, \quad (5.17)$$

for all n . Therefore,

$$(n+1) \int_X \delta \, d\mu \leq \int_X \Delta \, d\mu < \infty$$

since the measure is invariant and finite and the function Δ is measurable and bounded. For $n \rightarrow \infty$ we get

$$\int_X \delta \, d\mu \leq 0,$$

so $\delta(x) = 0$ a.e. since $\delta(x) \geq 0$ for all $x \in X$. ■

REMARK 5.12. Under conditions of Theorem 5.11 the TNC (1.4) is fulfilled. To show this directly note that

$$\int_X s_n(x) \, d\mu = (n+1) \int_X \gamma \, d\mu$$

and pass to the limit as $n \rightarrow \infty$.

The conditions of solvability in the space $M(X, \mu)$ of all measurable functions are more subtle. Note that if μ is a finite invariant measure and φ is a measurable solution then for every $\varepsilon > 0$ there exists $C > 0$ such that $\mu\{x : |\varphi(x)| > C\} < \varepsilon$, and then $\mu\{x : |\varphi(F^{n+1}x)| > C\} < \varepsilon$. By (3.1) we get $\mu\{x : |s_n(x)| > 2C\} < 2\varepsilon$ for all n . In this sense the sums $s_n(x)$ are “uniformly bounded in measure” in the case of solvability in $M(X, \mu)$. For μ ergodic the converse is due to Schmidt, see [60], p. 181.

The following general criterion was established by Helson [27] using the harmonic analysis of the corresponding *multiplicative cohomological equation* $\chi(Fx)/\chi(x) = \xi(x)$ in \mathbb{T} -valued functions. This kind of c.e. is important and interesting (see e.g., [17], [30]), but we do not touch it in the present paper.

THEOREM 5.13. *Let (X, F) be an invertible measurable dynamical system with a finite invariant measure μ , and let $\gamma \in M(X, \mu)$. Then the c.e. (1.1) has a solution $\varphi \in M(X, \mu)$ if and only if $s_n(x)$ is bounded a.e. on a sequence of n having positive upper density.*

Other criteria of solvability in $M(X, \mu)$ are due to Krzyżewski [42] and Sato [59].

6. The Gottschalk–Hedlund Theorem (GHT). In this section (X, F) is a topological dynamical system. Accordingly, we look for the continuous solutions to the c.e. (1.1) with continuous $\gamma(x)$.

LEMMA 6.1. *If (X, F) is topologically transitive then every continuous solution φ to the homogeneous equation*

$$\varphi(Fx) - \varphi(x) = 0, \quad x \in X, \quad (6.1)$$

is a constant.

Proof. The restriction of φ to the closure of any orbit is a constant. ■

Further we focus on the minimal systems (X, F) . Note that if X is an infinite Hausdorff (or, at least, T_1 -) space and F is minimal then F has no periodic points, therefore, the c.e. (1.1) is solvable with any γ .

THEOREM 6.2. *Let (X, F) be minimal. The c.e. (1.1) has a solution $\varphi \in CB(X)$ if and only if $\gamma \in CB(X)$ and the sums $s_n(x)$ are uniformly bounded. Under these conditions the continuous bounded solution is*

$$\varphi(x) = \sup_n [-s_n(x)] \quad (6.2)$$

up to an arbitrary additive constant.

This is a constructive version (cf. [45]) of Browder’s generalization [14] of the GHT. The latter is a principal part of Theorem 14.11 from the book [21]. In its original form the space X is compact metric and F is invertible, while in [14] and [45] F is any continuous self-mapping of a Hausdorff space X . In Theorem 6.2 X is any topological space admitting a minimal F , cf. [54]. (In particular, X must be separable). We preserve the name GHT for all these generalizations. Our proof of the GHT is a modification of that of [45]. Its the only nontrivial “if” part consists of two statements:

- a) the function (6.2) is continuous;
- b) this is a solution to the c.e. (1.1).

The statement a) is equivalent to that the oscillation $\Omega_\varphi(x)$ equals 0 for all x . Recall that for every bounded function ϕ its *oscillation* at a point x is

$$\Omega_\phi(x) = \lim_U (\sup_{u \in U} \phi(u) - \inf_{u \in U} \phi(u)) = \inf_U (\sup_{u \in U} \phi(u) - \inf_{u \in U} \phi(u)) \geq 0, \quad (6.3)$$

where U runs over the directed set of neighborhoods of x . The function Ω_ϕ is bounded and upper semicontinuous.

In general, a function ψ is called *upper (lower) semicontinuous* if for every $\rho \in \mathbb{R}$ the set $\{x : \psi(x) < \rho\}$ (the set $\{x : \psi(x) > \rho\}$, respectively) is open. The continuous functions are just those which are upper and lower semicontinuous simultaneously. The supremum (the infimum) of any pointwise bounded family of continuous functions is lower (upper, respectively) semicontinuous.

LEMMA 6.3. *If an upper (lower) bounded function ϕ is lower (upper) semicontinuous then $\inf \Omega_\phi = 0$.*

Proof. For definiteness, let ϕ be upper bounded. With $c = \sup \phi$ and $\varepsilon > 0$ let x be such that $\phi(x) > c - \varepsilon$. Since ϕ is lower semicontinuous, there exists a neighborhood U of x such that $\phi(u) > c - \varepsilon$ for $u \in U$. Since $\phi(u) \leq c$ for all u , we get $\Omega_\phi(x) \leq \varepsilon$. ■

COROLLARY 6.4. *Under conditions of Lemma 6.3, if Ω_ϕ is a constant then the function ϕ is continuous.*

After this preparation we can proceed to the proof of Theorem 6.2.

Proof. To prove a) we note that

$$\varphi(x) = \max\{-\gamma(x), \sup_n [-s_{n+1}(x)]\},$$

whence

$$\varphi(x) + \gamma(x) = \max\{0, \sup_n [\gamma(x) - s_{n+1}(x)]\}.$$

By (4.2) we obtain

$$\varphi(x) + \gamma(x) = \max\{0, \sup_n [-s_n(Fx)]\} = \max\{0, \varphi(Fx)\} = (\varphi \circ F)_+(x). \quad (6.4)$$

From (6.4) it follows that $\Omega_\varphi(x) = \Omega_{(\varphi \circ F)_+}(x)$ since γ is continuous. However, $\Omega_{\phi_+}(x) \leq \Omega_\phi(x)$ for any bounded function ϕ . Thus, $\Omega_\varphi(x) \leq \Omega_{\varphi \circ F}(x)$. In turn, $\Omega_{\varphi \circ F}(x) \leq \Omega_\varphi(Fx)$ by continuity of F . As a result, $\Omega_\varphi(x) \leq \Omega_\varphi(Fx)$ and then

$$\Omega_\varphi(x) \leq \Omega_\varphi(F^n x), \quad n \geq 1. \quad (6.5)$$

Since the orbit $O_F(x)$ is dense and the function Ω_φ is upper semicontinuous, the inequality (6.5) yields $\Omega_\varphi(x) \leq \Omega_\varphi(y)$ for all $y \in X$. By alternation $x \leftrightarrow y$ we obtain the equality $\Omega_\varphi(x) = \Omega_\varphi(y)$ for all x, y , i.e. Ω_φ is a constant. Therefore, $\Omega_\varphi = 0$ by Corollary 6.4, i.e. φ is continuous.

Now it remains to prove that a) \Rightarrow b). We have to show that $\delta = 0$ where δ is the residual function defined by (5.11) with φ from Lemma 5.8. By this lemma the function δ is nonnegative, and it is continuous by a). Suppose to the contrary that $\delta(x) > 0$ at a point x . Then for any $\varepsilon > 0$ there exists a neighborhood W of x such that $\delta(w) \geq \varepsilon$ when $w \in W$. On the other hand, by Corollary 5.9 there is $m \in \mathbb{N}$ such that $\delta(F^n x) < \varepsilon$ for all $n \geq m$. Since the orbit of $F^m x$ is dense, there is $n \geq m$ such that $F^n x \in W$, so $\delta(F^n x) \geq \varepsilon$, a contradiction. ■

REMARK 6.5. In [54] the proof of the GHT is given in the form $b) \& [b] \Rightarrow a)$ (in our notation). The main argument for $b)$ is that from the minimality of (X, F) it follows that

$$\sup_n [-s_{n+1}(x)] = \sup_n [-s_n(x)], \quad x \in X, \quad (6.6)$$

i.e. the shift invariance of the supremum on the set of sequences $(s_n(x))$.

REMARK 6.6. *The solution (6.2) in Theorem 6.2 is nonnegative.* Indeed, from (6.4) and (1.1) we see that $\varphi(Fx) = \varphi_+(Fx)$, i.e. $\varphi(Fx) \geq 0$ for all x . This inequality implies $\varphi \geq 0$, since the image of F is dense and φ is continuous.

REMARK 6.7 ([45]). *For minimal (X, F) and continuous γ if the sums $s_n(x)$ are bounded at a point x_0 then they are uniformly bounded.* Indeed, let $r_{n,m}(x) = s_n(x) - s_m(x)$, $n > m \geq -1$, $s_{-1}(x) = 0$, and let $\sup_n |s_n(x_0)| = C < \infty$. Then the subset $M = \{x \in X : \sup_{n,m} |r_{n,m}(x)| \leq 2C\}$ is nonempty (since $x_0 \in M$), closed (since γ and F are continuous) and invariant (since $r_{n,m}(Fx) = r_{n+1,m+1}(x)$). Therefore, $M = X$ by minimality. Since $r_{n,-1}(x) = s_n(x)$, we get $\sup_n |s_n(x)| \leq 2C$ for all $x \in X$.

The minimality is essential for all conclusions in Theorem 6.2.

EXAMPLE 6.8. The system (\mathbb{N}, F) from Example 5.7 is topologically transitive but not minimal. Indeed, since the topological space \mathbb{N} is discrete, we have $\overline{O_F(0)} = \mathbb{N}$ and $\overline{O_F(1)} = \mathbb{N} \setminus \{1\}$. The c.e. (5.7) is solvable in $B(\mathbb{N}) = CB(\mathbb{N})$ if and only if the sums $s_n(0)$ are bounded, however, the sup-formula fails if $\gamma_n > 0$.

EXAMPLE 6.9. Consider the one-point compactification $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $\overline{F}(\infty) = \infty$, $\overline{\gamma}(\infty) = 0$. The dynamical system $(\overline{X}, \overline{F})$ is topologically transitive but not minimal for the reason like that in Example 6.8. If the resolving series

$$\gamma(0) + \gamma(1) + \dots + \gamma(n) + \dots \quad (6.7)$$

diverges but the sums $s_n(0)$ are bounded then there are no continuous solutions. Actually, in this case all solutions are bounded but discontinuous at infinity.

Nevertheless, the minimality can be replaced by some other assumptions. In [61] the construction (5.4) was used to prove a counterpart of the GHT such that X is a compact metric space but, instead of the minimality, F is uniquely ergodic and the invariant measure μ is such that $\mu(U) > 0$ for all open $U \subset X$. In Section 8 we consider another important class of dynamical systems where the minimality is not needed due to an assumption on $s_n(x)$ stronger than in Theorem 6.2.

Let us emphasize that the class of underlying spaces for the minimal systems is very special. For example, if such a space is locally compact then it is compact [20]. In particular, all minimal discrete dynamical system (X, F) are finite cycles: $X = Z_x = \{x, Fx, \dots, F^{p-1}x\}$ for an x such that $F^p x = x$. Indeed, let $O_F(x)$ be a dense orbit. Then $X = O_F(x)$ since X is discrete. Therefore, X has to be finite, otherwise, the orbit $O_F(Fx)$ is not dense. Hence, the point x is preperiodic, thus it is periodic by minimality.

For any topological dynamical system (X, F) one can consider in the space $CB(X)$ the linear (nonclosed, in general) subspace

$$E_F = \{\gamma : \|\gamma\|_F \equiv \sup_n \|s_n\| < \infty\}.$$

Obviously, $\|\gamma\|_F \geq \|\gamma\|$. With this stronger norm E_F is a Banach space. According to (1.2) and (3.1) we have $\text{Im}(T_F - I) \subset E_F$. Theorem 6.2 states that if (X, F) is minimal then $\text{Im}(T_F - I) = E_F$. Hence, *in this case the image $\text{Im}(T_F - I)$ is a Banach space with respect to the norm $\|\cdot\|_F$* . As a consequence, we have

PROPOSITION 6.10. *Let (X, F) be minimal. Then $\text{Im}(T_F - I)$ is closed in $CB(X)$ if and only if the norms $\|\gamma\|_F$ and $\|\gamma\|$ are equivalent on this subspace.*

Theorem 6.2 determines a mapping $R : E_F \rightarrow CB(X)$ such that $\varphi = R\gamma$ is a solution to the c.e. (1.1). This equation is linear but the mapping R is not linear! However, this collision can be removed by a small perturbation of R .

THEOREM 6.11. *Let f be a linear functional on $CB(X)$ such that $f[\mathbf{1}] = 1$. Under conditions of Theorem 6.2 the formula*

$$\varphi_0(x) = \sup_n [-s_n(x)] - f\left[\sup_n [-s_n(\cdot)]\right] \quad (6.8)$$

determines a continuous bounded solution to the c.e. (1.1) such that the mapping $\varphi_0 = R_0\gamma$ is linear. In addition, if $\|f\| = 1$ then

$$\frac{1}{2} \|\gamma\|_F \leq \|R_0\gamma\| \leq \|\gamma\|_F. \quad (6.9)$$

Thus, R_0 is a linear topological isomorphism between $\|\cdot\|_F$ -normed space $\text{Im}(T_F - I)$ and the closed hyperplane $\ker f \subset CB(X)$.

Proof. In (6.8) the first summand is a solution to (1.1) in $CB(X)$, while the second summand is a constant. Hence, φ_0 is a solution in $CB(X)$ as well. Moreover, by (6.8) the mapping $R_0 : E_F \rightarrow CB(X)$ is such that $\text{Im } R_0 \subset \ker f$. This allows us to consider R_0 as a mapping $E_F \rightarrow \ker f$. Let us also consider the linear mapping $D = T_F - I : CB(X) \rightarrow E_F$. This is a left inverse to R by definition of the latter.

Now let $D_0 = D|_{\ker f}$. The mapping D_0 is injective. Indeed, $\ker D = \text{Span}\{\mathbf{1}\}$ by Lemma 6.1, so $\ker D_0 = \text{Span}\{\mathbf{1}\} \cap \ker f = 0$ since $f[\mathbf{1}] = 1$. On the other hand, D_0 is surjective since $D_0 R_0 = D R = I|_{E_F}$. Eventually, D_0 is bijective, and R_0 is its inverse mapping. Hence, R_0 is linear.

Now assume $\|f\| = 1$. Then the functional f is nonnegative. In view of Remark 6.6 formula (6.8) yields the inequality

$$-f\left[\sup_n [-s_n(\cdot)]\right] \leq \varphi_0(x) \leq \sup_n [-s_n(x)],$$

and the right-hand part of (6.9) follows. The left-hand part follows from (3.1). ■

REMARK 6.12. The solution (6.8) is uniquely determined by the additional condition $f[\varphi_0] = 0$.

EXAMPLE 6.13. Let X be a compact space with a regular Borel measure μ such that $\mu(X) = 1$. Then formula (6.8) can be realized as

$$\varphi_0(x) = \sup_n [-s_n(x)] - \int_X \sup_n [-s_n(\cdot)] d\mu. \quad (6.10)$$

This solution is uniquely determined by the condition

$$\int_X \varphi_0 \, d\mu = 0.$$

In particular, for any $x_0 \in X$ we can take the corresponding Dirac measure and get

$$\varphi_0(x) = \sup_n [-s_n(x)] - \sup_n [-s_n(x_0)]. \quad (6.11)$$

This formula is applicable to any topological space X and yields the unique solution such that $\varphi_0(x_0) = 0$.

Now let us compare Theorem 6.2 to Corollary 5.3.

COROLLARY 6.14. *Let (X, F) be minimal, and let $\gamma \in C(X)$. Then if the c.e. (1.1) is solvable in $B(X)$ then it is solvable in $CB(X)$.*

Actually, the bounded solution (5.4) coincides with φ from Theorem 6.2 under conditions of the latter. This follows from (6.6) as $m \rightarrow \infty$.

7. The cohomological equations on topological groups. Let us start with some general remarks. Let G be a topological group. For any $g \in G$ the subsemigroup

$$[g] = \{g^n : n \in \mathbb{N}\}$$

is the τ_g -orbit of the unit e . If this is dense (thus, the dynamical system (G, τ_g) is topologically transitive) then this system is minimal. Indeed, for any $x \in G$ its orbit is $[g]x$, hence the closure of the latter is $Gx = G$.

The two-sided orbit of e is the subgroup

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}.$$

A group G such that $\langle g \rangle$ is dense for a $g \in G$ is called *monothetic*, and the element g is called its *generator*. For example, the unit circle \mathbb{T} is monothetic: $\mathbb{T} = \langle \zeta \rangle$ where $\zeta = e^{2\pi i \alpha}$ and α is any irrational number. Moreover, $\mathbb{T} = [\zeta]$, thus the dynamical system (\mathbb{T}, τ_ζ) is minimal.

In general, if $[g]$ is dense in a group G then such is $\langle g \rangle$, thus G is monothetic. The converse is not true. For example, the discrete additive group \mathbb{Z} is monothetic since $\mathbb{Z} = \langle 1 \rangle$, while $[z] = \{nz : n \in \mathbb{N}\} \neq \mathbb{Z}$. However, if G is compact monothetic then $[g]$ is dense for any generator g , see Lemma 7.1 below.

Obviously, every monothetic group is commutative and separable. On the other hand, every connected compact separable commutative group is monothetic [25].

In this section we systematically use the classical character theory. For the reader convenience let us recall some basic definitions and results. For more detail see e.g. [28], [51], [66].

Let G be a commutative topological group, and let e be its unit. A continuous function $\chi : G \rightarrow \mathbb{T}$ is called a *character of G* if $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$. In particular, $\chi(e) = 1$ and $\chi(g^{-1}) = \chi(g)^{-1} = \overline{\chi(g)}$, where the bar means the complex conjugation. The characters constitute a commutative group G' with respect to the pointwise multiplication. Its unit is $\mathbf{1}$, and it may happen that $G' = \{\mathbf{1}\}$, in spite of $G \neq \{e\}$. For

example, if G is the additive group of a linear topological space without nonzero continuous linear functionals (say, $G = L_p(0, 1)$, $0 < p < 1$) then $G' = \{1\}$. In contrast, if G is locally compact and $G' = \{1\}$ then $G = \{e\}$. A natural topology on G' is discrete being determined by the inclusion $G \subset CB(G)$. Indeed, let $\chi_1, \chi_2 \in G'$, and let $\chi_1 \neq \chi_2$. Then for every $g \in G$ we have $|\chi_1(g) - \chi_2(g)| = |\chi(g) - 1|$ where $\chi(g) = \chi_1(g)\overline{\chi_2(g)} \neq 1$. If g is such that $\chi(g) \neq 1$ then $|\chi(g^n) - 1| = |\chi(g)^n - 1| \geq \sqrt{2}$ for some $n \in \mathbb{N}$. A fortiori, $\|\chi_1 - \chi_2\| \geq \sqrt{2}$.

If G is locally compact then the classical Pontrjagin–van Kampen duality theory prescribes to endow G' with the compact-open topology and get the *dual group* G^* . This one is locally compact but not discrete, except for the case of compact G . Accordingly, although G^* coincides with G' algebraically but they are different topological groups if (and only if) G is not compact. The Duality Theorem states that topological groups G^{**} and G can be identified by the topological isomorphism $G \rightarrow G^{**}$ which is as $g \mapsto g^{**}$ such that $g^{**}(\chi) = \chi(g)$.

In the rest of this section the group G is compact, unless otherwise stated.

LEMMA 7.1. *The closures $\overline{[g]}$ and $\overline{\langle g \rangle}$ coincide for every $g \in G$.*

Proof. It suffices to show that $g^{-1} \in \overline{[g]}$. Suppose the contrary. Then the set $U = G \setminus \overline{[g]}$ is a neighborhood of g^{-1} separating this element from $\overline{[g]}$. Hence, $V = gU$ is a neighborhood of e such that $g^r \notin V$ for all $r \geq 1$. Consider the intersection

$$\Omega = \bigcap_{n \in \mathbb{N}} \overline{\{g^k : k \geq n\}}. \quad (7.1)$$

This is not empty since G is compact. Let $x \in \Omega$, and let W be a neighborhood of x such that $WW^{-1} \subset V$. By definition of Ω there exists $n \in \mathbb{N}$ such that $g^n \in W$, and there is $m > n$ such that $g^m \in W$. Then $g^r \in V$ for $r = m - n$, a contradiction. ■

COROLLARY 7.2. *The closure $\overline{[g]}$ of the semigroup $[g]$ is a group.*

COROLLARY 7.3. *The sets $[g]$ and $\langle g \rangle$ are dense or not dense simultaneously.*

LEMMA 7.4. *The semigroup $[g]$ is dense (thus, the group G is monothetic, generated by g) if and only if 1 is the only character χ such that $\chi(g) = 1$.*

Proof. If $\chi(g) = 1$ then $\chi(h) = 1$ for $h \in \overline{[g]}$. Therefore, $\chi = 1$ if $[g]$ is dense. Now let $[g]$ be not dense. By Corollary 7.3 the quotient group $\Gamma = G/\overline{\langle g \rangle}$ is nontrivial. The natural epimorphism $p : G \rightarrow \Gamma$ is continuous with respect to the standard topology on Γ . Hence, the group Γ is compact. If ξ is a nonunity character of Γ then $\chi = \xi \circ p$ is a character of G such that $\chi(g) = 1$ but $\chi \neq 1$. ■

EXAMPLE 7.5. The characters of the d -dimensional torus

$$\mathbb{T}^d = \{g = (e^{2\pi i \alpha_k}) : 0 \leq \alpha_k < 2\pi, 1 \leq k \leq d\}$$

are

$$\chi_{n_1, \dots, n_d}(g) = e^{2\pi i \sum_{k=1}^d n_k \alpha_k}, \quad (n_1, \dots, n_d) \in \mathbb{Z}^d.$$

Hence,

$$\chi(g) \neq 1 \iff \sum_{k=1}^d n_k \alpha_k \notin \mathbb{Z}.$$

Therefore, $\chi(g) \neq 1$ for all $\chi \neq \mathbf{1}$ if and only if the real numbers $1, \alpha_1, \dots, \alpha_d$ are linearly independent over the field \mathbb{Q} of rational numbers. By Lemma 7.4 this condition is necessary and sufficient for the density of the subsemigroup $[g]$ in \mathbb{T}^d . This is the famous Kronecker theorem. As a consequence, *the group \mathbb{T}^d is monothetic*. Note that for $d = 1$ the Kronecker condition just means that α_1 is irrational.

Now we immediately obtain

THEOREM 7.6. *Let G be a compact commutative group, and let $g \in G$ be such that $\chi(g) \neq 1$ for all nonunity characters of G . Then the c.e.*

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G, \quad (7.2)$$

has a solution $\varphi \in C(G)$ if and only if $\gamma \in C(G)$ and the sums

$$s_n(x) = \sum_{k=0}^n \gamma(g^k x), \quad n \geq 0,$$

are uniformly bounded. Under these conditions with real γ the continuous solution is

$$\varphi(x) = \sup_n [-s_n(x)] \quad (7.3)$$

up to an arbitrary additive constant.

Proof. By Lemma 7.4 the system (G, τ_g) is minimal. Hence, Theorem 6.2 is applicable. ■

REMARK 7.7. According to Remark 6.7 it suffices to require that the set of sums

$$s_n(e) = \sum_{k=0}^n \gamma(g^k), \quad n \geq 0,$$

is bounded.

COROLLARY 7.8. *The equation (1.6) with continuous 1-periodic h and irrational α has a continuous 1-periodic solution f if and only if the sums*

$$s_n(x) = \sum_{k=0}^n h(x + k\alpha), \quad 0 \leq x < 1,$$

are uniformly bounded.

By Remark 7.7 it is sufficient that these sums are bounded at $x = 0$.

The nonlinear construction (7.3) can be transformed into a linear one according to Theorem 6.11. For instance, such is

$$\varphi_0(x) = \sup_n [-s_n(x)] - \int_G \sup_n [-s_n(\cdot)] d\nu \quad (7.4)$$

where ν is the Haar measure, $\nu(G) = 1$, see Example 6.13. This solution is determined by the condition

$$\int_G \varphi_0 d\nu = 0. \quad (7.5)$$

Now let us analyze the c.e. (7.2) by means of harmonic analysis in the complex space $L_2(G, \nu)$. In this space the set G^* of the characters is orthonormal and complete. Thus,

for every $\psi \in L_2(G, \nu)$ we have the *Fourier decomposition*

$$\psi \sim \sum_{\chi \in G^*} c_\chi[\psi] \chi \quad (7.6)$$

where

$$c_\chi[\psi] = \int_G \psi \bar{\chi} d\nu \quad (7.7)$$

are the *Fourier coefficients*. The set of those $\chi \in G^*$ for which $c_\chi[\psi] \neq 0$ is at most countable. This set is called the *spectrum* of ψ and denoted by $\text{spec } \psi$. After omitting the vanishing summands in (7.6) and an arbitrary ordering of the rest we get the *reduced Fourier series*

$$\psi = \sum_{\chi \in \text{spec } \psi} c_\chi[\psi] \chi \quad (7.8)$$

convergent to ψ in L_2 -norm. As a consequence, $\text{spec } \psi = \emptyset$ if and only if $\psi = 0$. If $\psi \in C(G)$ then its Fourier series can be divergent in $C(G)$ but ψ is the uniform limit of a sequence of linear combinations of the characters from $\text{spec } \psi$.

The member corresponding to the unity character in (7.6) is the constant

$$c_1[\psi] = \int_G \psi d\nu \quad (7.9)$$

according to the equality

$$\int_G \chi d\nu = 0, \quad \chi \in G^* \setminus \{1\}. \quad (7.10)$$

Therefore,

$$\int_G \psi d\nu = 0$$

if and only if $1 \notin \text{spec } \psi$.

The Fourier coefficients of the shifted function $\psi_g(x) = \psi(gx)$ are

$$c_\chi[\psi_g] = \chi(g) c_\chi[\psi]. \quad (7.11)$$

Indeed,

$$\int_G \psi(gx) \overline{\chi(x)} d\nu = \chi(g) \int_G \psi(x) \overline{\chi(x)} d\nu$$

since the measure ν is invariant and $\overline{\chi(g^{-1}x)} = \chi(g) \overline{\chi(x)}$. By (7.11) the Fourier image of the c.e. (7.2) is

$$(\chi(g) - 1) c_\chi[\varphi] = c_\chi[\gamma], \quad \chi \in G^*. \quad (7.12)$$

For $\chi = 1$ this yields the TNC $c_1[\gamma] = 0$, i.e. $1 \notin \text{spec } \gamma$. Also, from (7.12) it follows that $\chi(g) \neq 1$ if $\chi \in \text{spec } \gamma$. Thus, the Fourier decomposition of the solution (7.4) is

$$\varphi_0 = \sum_{\chi \in \text{spec } \gamma} \frac{c_\chi[\gamma]}{\chi(g) - 1} \chi. \quad (7.13)$$

Obviously, $\text{spec } \varphi_0 = \text{spec } \gamma$, while $\text{spec } \varphi = \text{spec } \gamma \cup \{1\}$ for all solutions $\varphi \neq \varphi_0$.

An interesting consequence of formula (7.13) is the inequality

$$\sum_{\chi \in \text{spec } \gamma} \left| \frac{c_\chi[\gamma]}{\chi(g) - 1} \right|^2 \leq \sup_n \|s_n\|^2 \quad (7.14)$$

which is valid for all continuous coboundaries. This follows by changing the sup-norm on the left of (6.9) to the L_2 -norm which, in turn, can be found from (7.13) by the Parseval equality.

After these preliminaries we can generalize Theorem 7.6 as follows. For any $g \in G$ we consider its *annihilator* $g^\perp = \{\chi \in G^* : \chi(g) = 1\}$. It is a subgroup of the group G^* . In Theorem 7.6 we actually assume $g^\perp = \{\mathbf{1}\}$. In order to relax this condition we introduce the subgroup $H_\gamma \subset G^*$ generated by the subset $\text{spec } \gamma$. (For $\gamma = 0$ we let $H_\gamma = \{\mathbf{1}\}$.)

THEOREM 7.9. *Theorem 7.6 remains true under the condition*

$$g^\perp \cap H_\gamma = \{\mathbf{1}\} \quad (7.15)$$

instead of $g^\perp = \{\mathbf{1}\}$.

Proof. The embedding $H_\gamma \subset G^*$ induces (by duality) a continuous epimorphism $j : G \rightarrow H_\gamma^*$. The group H_γ^* is compact since such is G . Every $\theta \in C(H_\gamma^*)$ can be lifted to G as $\theta \circ j$. This is a linear isometric mapping $C(H_\gamma^*) \rightarrow C(G)$. Its image is the uniform closure $\overline{\text{Span } H_\gamma}$ since $C(H_\gamma^*) = \overline{\text{Span } H_\gamma^{**}}$.

Now in $C(H_\gamma^*)$ we consider the c.e.

$$\theta(hz) - \theta(z) = \beta(z), \quad z \in H_\gamma^*, \quad (7.16)$$

where $h = jg \in H_\gamma^*$ and $\beta \in C(H_\gamma^*)$ is such that $\beta(jx) = \gamma(x)$, $x \in G$. The only character ξ of the group H_γ^* such that $\xi(h) = 1$ is $\xi = \mathbf{1}$. Indeed, $\chi = \xi \circ j$ is a character of G such that $\chi(g) = 1$, i.e. $\chi \in g^\perp$. On the other hand, $\chi \in H_\gamma$. By (7.15) we get $\chi = \mathbf{1}$. Therefore, $\xi = \mathbf{1}$ since j is surjective.

Thus, we have $h^\perp = \mathbf{1}$. Furthermore,

$$\hat{s}_n(z) \equiv \sum_{k=0}^n \beta(h^k z) = \sum_{k=0}^n \beta(j(g^k x)) = \sum_{k=0}^n \gamma(g^k x) = s_n(x)$$

for $z = jx$, $x \in G$. Since these sums are uniformly bounded, the c.e. (7.16) has a continuous solution θ by Theorem 7.6. Then $\varphi(x) = \theta(jx)$ is a continuous solution to the c.e. (7.2). With real γ one can take

$$\theta(z) = \sup_n [-\hat{s}_n(z)] = \sup_n [-s_n(x)]. \quad \blacksquare \quad (7.17)$$

REMARK 7.10. With $\gamma = 0$ the general continuous solution to (7.2) is $\overline{\text{Span } g^\perp}$. This subspace of $C(G)$ consists of constants if and only if $g^\perp = \{\mathbf{1}\}$.

Now let G be an arbitrary commutative topological group. An *almost periodic function* (a.p.f.) on G is a complex-valued function $\psi \in CB(G)$ such that the set $\{\tau_g \psi : g \in G\}$ of its shifts is precompact in $CB(G)$. The a.p.f. constitute a subspace $AP(G) \subset CB(G)$. All characters of G are almost periodic, so $G' \subset AP(G)$. Moreover, the linear span of G' is dense in $AP(G)$. This Approximation Theorem is central in the theory of a.p.f., see e.g. [66] and the references therein. On the additive group \mathbb{R} the theory of a.p.f. was

created by Bohr [12], [13] and Bochner [11]. The characters on \mathbb{R} are $\chi_\lambda(x) = e^{i\lambda x}$ ($x, \lambda \in \mathbb{R}$), so in this case the Approximation Theorem means that the a.p.f. on \mathbb{R} are just the uniform limits of the linear combinations of these exponents. In particular, all periodic functions on \mathbb{R} are almost periodic. In this case the Approximation Theorem turns into the Weierstrass theorem from the classical Fourier analysis.

On any compact group G all continuous functions are almost periodic, i.e. $AP(G) = C(G)$ in this case. Remarkably, the general case reduces to this one as follows. (See e.g. [51] for more detail.)

Let K be a compact group such that there exists a continuous homomorphism $j: G \rightarrow K$ then for every $\psi \in C(K)$ the function $\psi \circ j$ is an a.p.f. on G . In particular, one can take the *Bohr compact* $K = bG \equiv G'^*$ and $(jg)(\chi) = \chi(g)$, $g \in G$, $\chi \in G'$. Although this j is not surjective, its image is dense. It turns out that this construction yields all a.p.f. on G . The natural extension of j to $AP(G)$ is a bijective linear isometry $AP(G) \rightarrow C(bG)$. This allows one to translate the harmonic analysis from $C(bG)$ to $AP(G)$. In particular, the *spectrum* of $\varphi \in AP(G)$ is defined as the spectrum of the corresponding $\psi \in C(bG)$, the Fourier coefficients of φ are defined as the corresponding ones for ψ , etc.

By the way, in the case $G = \mathbb{R}$ the Fourier coefficients can be introduced with no reference to the Bohr compactification, namely,

$$c^{(\lambda)}[\varphi] = \lim_{a \rightarrow \infty} \frac{1}{2a} \int_{-a}^a \varphi(x) e^{-i\lambda x} dx. \quad (7.18)$$

We call the set of those $\lambda \in \mathbb{R}$ for which $c^{(\lambda)}[\varphi] \neq 0$ the *frequency spectrum* of $\varphi \in AP(\mathbb{R})$. If φ is periodic, say, 1-periodic, then its frequency spectrum is a subset of $2\pi\mathbb{Z}$ and

$$c^{(2\pi n)}[\varphi] = c_n[\varphi] \equiv \int_0^1 \varphi(x) e^{-2\pi i n x} dx, \quad n \in \mathbb{Z}.$$

THEOREM 7.11. *Let G be a commutative topological group, and let $\gamma \in AP(G)$. Assume that the condition (7.15) is fulfilled for the subgroups g^\perp and H_γ of G' defined as before. Then the c.e.*

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G, \quad (7.19)$$

has a solution $\varphi \in AP(G)$ if and only if the sums

$$s_n(x) = \sum_{k=0}^n \gamma(g^k x), \quad n \geq 0,$$

are uniformly bounded. Under this condition and with real γ an almost periodic (a.p.) solution is

$$\varphi(x) = \sup_n [-s_n(x)]. \quad (7.20)$$

Proof. One can identify G' with $(bG)^*$. Thus, Theorem 7.9 is applicable to that c.e. in $C(bG)$ which corresponds to (7.19). ■

REMARK 7.12. According to Remark 7.10 the a.p. solution to (7.19) is determined up to an arbitrary summand $\omega \in \overline{\text{Span } g^\perp}$. The latter means that $\text{spec } \omega \subset g^\perp$.

Let us apply the Theorem 7.11 to the c.e.

$$\varphi(x + \alpha) - \varphi(x) = \gamma(x), \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R} \setminus \{0\}, \quad (7.21)$$

in $AP(\mathbb{R})$. Note that in this case the TNC (1.7) should be changed to

$$\lim_{a \rightarrow \infty} \frac{1}{2a} \int_{-a}^a \gamma(x) dx = 0 \quad (7.22)$$

according to (7.18).

COROLLARY 7.13. *Let $\gamma \in AP(\mathbb{R})$, and let the frequency spectrum of γ be Λ . If $2\pi/\alpha$ does not belong to the rational linear span of Λ then the c.e. (7.21) has an a.p. solution if and only if the sums*

$$s_n(x) = \sum_{k=0}^n \gamma(x + k\alpha)$$

are uniformly bounded. Under this condition and with real γ an a.p. solution is

$$\varphi(x) = \sup_n [-s_n(x)]. \quad (7.23)$$

Proof. We have

$$\alpha^\perp = \{\chi_\lambda : e^{i\lambda\alpha} = 1\} = \{e^{i\lambda x} : \lambda = 2\pi n/\alpha, n \in \mathbb{Z}\},$$

while $H_\gamma = \{\chi_\lambda : \lambda \in \Delta_\Lambda\}$ where Δ_Λ is the subgroup of \mathbb{R} generated by the subset Λ . The assumption in Corollary 7.13 just means that $\alpha^\perp \cap H_\gamma = \{1\}$. ■

Under conditions of Corollary 7.13 the a.p. solution is determined up to an arbitrary continuous α -periodic summand. Indeed, an a.p.f. ω with $\text{spec } \omega \subset \alpha^\perp$ is α -periodic, and vice versa.

8. An almost periodic counterpart of GHT. In the GHT the minimality condition is essential, see Example 6.9. However, in an almost periodic context this condition does not appear. Let us start this topic with some known general definitions, cf. [51].

Let T be a bounded linear operator in a Banach space B . A vector $v \in B$ is called *almost periodic (a.p.)* if its orbit $(T^n v)_{n \geq 0}$ is precompact. The set of all a.p. vectors is denoted by $AP(B, T)$. This is a linear subspace of B containing all eigenvectors of T with eigenvalues λ such that $|\lambda| \leq 1$. The operator T is called *a.p.* if $AP(B, T) = B$. By the Banach–Steinhaus theorem every a.p. operator is power bounded, i.e. $\sup_{n \geq 0} \|T^n\| < \infty$.

THEOREM 8.1. *With $h \in B$ the equation*

$$Tf - f = h \quad (8.1)$$

has an a.p. solution f if and only if the set of sums

$$s_n = \sum_{k=0}^n T^k h, \quad n \geq 0,$$

is precompact. Under this condition all solutions are a.p.

Proof. The last sentence is true since the solutions of the corresponding homogeneous equation are fixed vectors. Now let f be a solution to (8.1). Then

$$s_n = T^{n+1}f - f. \quad (8.2)$$

Thus, the precompactness of $\{s_n\}$ is equivalent to the almost periodicity of the vector f . Therefore, it suffices to prove that if $\{s_n\}$ is precompact then the equation (8.1) is solvable. To this end we consider the closure K of the set $\{-s_n\}$. By assumption, K is compact. Furthermore, K is invariant for the affine mapping $V : B \rightarrow B$ defined as $Vf = Tf - h$. Indeed, let $f \in K$, and let for a given $\varepsilon > 0$ a number n be such that $\|f + s_n\| < \varepsilon/\|T\|$. Then $\|Tf + Ts_n\| < \varepsilon$. However,

$$Ts_n = s_{n+1} - h.$$

Therefore, $\|Tf - h + s_{n+1}\| < \varepsilon$, i.e. $\|Vf + s_{n+1}\| < \varepsilon$. As a result, $Vf \in K$.

Now we consider the closure Q of the convex hull of K . This is a convex compact V -invariant set. By the Schauder fixed point theorem there exists $f \in Q$ such that $Vf = f$. This f is a solution to (8.1) by definition of V . ■

REMARK 8.2. *The precompactness of the set $\{s_n\}$ implies that h is a.p.* Indeed,

$$T^n h = s_n - s_{n-1}, \quad n \geq 1.$$

COROLLARY 8.3. *Let (X, F) be a metric compact dynamical system. Then with $\gamma \in C(X)$ the c.e. (1.1) has a solution $\varphi \in AP(C(X), T_F)$ if and only if the subset $\{s_n\} \subset C(X)$ is uniformly bounded and equicontinuous. In this case all continuous solutions are a.p.*

Proof. By the Arzela–Ascoli theorem the subset $\{s_n\}$ is precompact in $C(X)$ if and only if it is bounded and equicontinuous. ■

Now let X be any metric space with a distance d , and let F be a mapping $X \rightarrow X$. The dynamical system (X, F) is called *uniformly stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies d(F^n x, F^n y) < \varepsilon \quad (8.3)$$

for all $x, y \in X$, $n \in \mathbb{N}$. For $n = 1$ this means that F is uniformly continuous.

In fact, the uniform stability is topologically equivalent to a more elementary property. Namely, the dynamical system (X, F) is called *dissipative* if

$$d(Fx, Fy) \leq d(x, y) \quad (x, y \in X), \quad (8.4)$$

i.e. if F is a contraction. Obviously, every dissipative system is uniformly stable. In the converse direction we have

PROPOSITION 8.4. *If a dynamical system (X, F) is uniformly stable then it is dissipative with respect to a distance topologically equivalent to the original one.*

Proof. Let us introduce the distance

$$\tilde{d}(x, y) = \sup_{n \geq 0} \frac{d(F^n x, F^n y)}{1 + d(F^n x, F^n y)}$$

instead of the original distance d . Obviously, $\tilde{d}(Fx, Fy) \leq \tilde{d}(x, y)$. Since

$$\tilde{d}(x, y) \geq \frac{d(x, y)}{1 + d(x, y)},$$

the \tilde{d} -topology on X is stronger than the d -topology. On the other hand, by (8.3)

$$d(x, y) < \delta \implies \tilde{d}(x, y) < \varepsilon,$$

i.e. the d -topology is stronger than the \tilde{d} -topology. ■

An important example of a dissipative system (X, F) is a contraction $F : X \rightarrow X$ of a convex set X in a normed space:

$$\|Fx - Fy\| \leq \|x - y\| \quad (x, y \in X). \quad (8.5)$$

For (8.5) it is sufficient for F to be Gateaux differentiable with $\|F'(x)\| \leq 1$, $x \in X$.

LEMMA 8.5. *The operator T_F in $C(X)$ associated with a uniformly stable compact dynamical system (X, F) is a.p.*

Proof. It suffices to show that for every $f \in C(X)$ the set $\{T_F^n f\}$ is bounded and equicontinuous. The boundedness is obvious since T_F is a contraction. On the other hand, the function f is uniformly continuous since X is compact. Let $\eta > 0$, and let $\varepsilon > 0$ be such that

$$d(x, y) < \varepsilon \Rightarrow |f(x) - f(y)| < \eta$$

for all $x, y \in X$. A fortiori,

$$d(F^n x, F^n y) < \varepsilon \Rightarrow |f(F^n x) - f(F^n y)| < \eta \quad (8.6)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Combining (8.6) and (8.3) we get what we need. ■

In the following counterpart of GHT the minimality is not assumed.

THEOREM 8.6. *Let (X, F) be a uniformly stable compact dynamical system. The c.e. (1.1) is solvable in $C(X)$ if and only if $\gamma \in C(X)$ and the subset $\{s_n\} \subset C(X)$ is bounded and equicontinuous.*

Proof. “If” directly follows from Corollary 8.3. (The uniform stability is not needed in this part.)

“Only if”. Note that for any $\varphi \in C(X)$ the subset $\{T_F^n \varphi\} \subset C(X)$ is precompact by Lemma 8.5. If, in addition, φ satisfies (1.1) then formula (3.1) shows that $\{s_n\}$ is precompact in $C(X)$. Hence, this subset is bounded and equicontinuous. ■

If X is a convex compact in a Banach space then every continuous self-mapping of X is not minimal because it has a fixed point. The GHT is not applicable to this case, while Theorem 8.6 works immediately.

THEOREM 8.7. *Let X be a convex compact in a Banach space, and let $F : X \rightarrow X$ be a contraction. The c.e. (1.1) has a solution $\varphi \in C(X)$ if and only if $\gamma \in C(X)$ and the subset $\{s_n\} \subset C(X)$ is bounded and equicontinuous.*

In the following example all conditions of Theorem 8.7 except for the equicontinuity are fulfilled. Accordingly, there are no continuous solutions in this example.

EXAMPLE 8.8. Let $X = [0, \sqrt{2/3}]$, and let $Fx = x - x^3$, $x \in X$. This mapping $X \rightarrow X$ is a contraction since $|F'(x)| \leq 1$, $x \in X$. Note that $Fx \neq 0$ for $x \neq 0$ but $F(0) = 0$. We consider the c.e. (1.1) in $C(X)$ with

$$\gamma(x) = \sin \frac{1}{Fx} - \sin \frac{1}{x} \quad (x \neq 0), \quad \gamma(0) = 0. \quad (8.7)$$

This function is indeed continuous since we have

$$|\gamma(x)| \leq \left| \sin \frac{1}{Fx} - \sin \frac{1}{x} \right| \leq \left| \frac{1}{Fx} - \frac{1}{x} \right| = \frac{x}{1-x^2} \rightarrow 0$$

as $x \rightarrow 0$. Also, from (8.7) it follows that

$$s_n(x) = \sin \frac{1}{F^{n+1}x} - \sin \frac{1}{x} \quad (x \neq 0), \quad s_n(0) = 0,$$

hence, $|s_n(x)| \leq 2$ for all $x \in X$. However, the sums $s_n(x)$ are not equicontinuous. Indeed, suppose the contrary. Then there is $\delta > 0$ such that

$$\left| \sin \frac{1}{F^{n+1}x} - \sin \frac{1}{x} \right| = |s_n(x) - s_n(0)| < \frac{1}{2}, \quad 0 < x < \delta, \quad n \geq 0,$$

whence

$$\left| \sin \frac{1}{F^n x_0} \right| < \frac{1}{2}, \quad n \geq 1, \quad (8.8)$$

where $x_0 = 1/m\pi$ with an integer $m > 1/\pi\delta$. The inequality (8.8) yields

$$\text{dist}(z_n, \pi\mathbb{Z}) < \pi/6$$

where $z_n = 1/x_n$, $x_n = F^n x_0$. This is a contradiction. Indeed, z_n monotonically tends to infinity, while

$$z_{n+1} - z_n = \frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{1}{F x_n} - \frac{1}{x_n} = \frac{x_n}{1 - x_n^2} \rightarrow 0.$$

Hence, the sequence of fractional parts of z_n/π is dense in $[0/1]$.

REMARK 8.9. In Example 8.8 the function $\varphi(x)$ which equals $\sin(1/x) + 1$ for $x \neq 0$ and $\varphi(0) = 0$ is a bounded discontinuous solution such that $\varphi(x) = \sup_n [-s_n(x)]$ for all $x \in X$.

If on a compact metric space X a dynamical system (X, F) is uniformly stable and minimal then for every $\gamma \in C(X)$ the uniform boundedness of the sums $s_n(x)$ implies their equicontinuity. This immediately follows from the GHT combined with Corollary 8.3. In particular, this is true for the sums

$$s_n(x) = \sum_{k=0}^n h(x + k\alpha), \quad 0 \leq x < 1,$$

where h is a continuous 1-periodic function and α is irrational. For rational α this is trivial since in this case if the set $\{s_n\}$ is bounded then it is finite, see Remark 5.4. Let us emphasize that the rational rotation of \mathbb{T} is not minimal because of its periodicity.

A dynamical system (X, F) on a metric space X with a distance d is called *conservative* if

$$d(Fx, Fy) = d(x, y) \quad (x, y \in X), \quad (8.9)$$

i.e. if F is isometric. For example, such is any rotation of \mathbb{T} .

THEOREM 8.10. *Any compact topologically transitive conservative dynamical system (X, F) is homeomorphic to (G, τ_g) , where G is a monothetic compact group and g is its generator.*

This is a particular case of Theorem 6 from [50]. Its proof in [50] is based on the theory of almost periodic representations of topological semigroups [48], [51]. We use this result in Section 12. However, in this case the system is minimal. Below we briefly consider a group situation without the minimality.

By the Birkhoff–Kakutani theorem (see e.g. [28], Theorem 8.3) on any metric group G there exists a topologically equivalent distance d which is invariant in the sense that

$$d(gx, gy) = d(x, y), \quad (x, y \in X, g \in G). \quad (8.10)$$

With respect to this distance all shifts τ_g are conservative. Theorem 8.6 yields

COROLLARY 8.11. *Let G be a compact metric group, and let $g \in G$. The c.e.*

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G, \quad (8.11)$$

has a solution $\varphi \in C(G)$ if and only if $\gamma \in C(G)$ and the subset $\{s_n\} \subset C(G)$ is bounded and equicontinuous.

Here the group G can be noncommutative.

9. Recurrence and ergodicity. Some applications. In this section (X, F) is a dynamical system with an invariant measure μ . We assume that μ is finite, though in some cases (in particular, in Theorem 9.5 without the equality (9.5)) the σ -finiteness is sufficient.

The following is a modern form of the Poincaré Recurrence Theorem (RT).

THEOREM 9.1. *Let M be a measurable subset of X , $\mu(M) > 0$. Then for a.e. $x \in M$ there exists a subsequence $(n_i)_{i=1}^\infty \subset \mathbb{N}$ such that all $F^{n_i}x \in M$.*

COROLLARY 9.2. *There is a subsequence $(m_i)_{i=1}^\infty \subset \mathbb{N}$ such that $\mu(F^{-m_i}M \cap M) > 0$ for all i .*

Actually, a more general theorem can be easily derived from Theorem 9.1, see [24]. Below we give a “cohomological” proof of this generalization.

THEOREM 9.3. *For every nonnegative measurable function γ on X the resolving series (1.10) diverges a.e. on the set $M_\gamma^+ = \{x : \gamma(x) > 0\}$.*

Theorem 9.1 is just that case of Theorem 9.3 where γ is the indicator function of the subset M .

Proof. Denote by A the set of convergence of the series (1.10), and let $s(x)$ be its sum on A . The set A and the function s are measurable. Furthermore, A is completely invariant and s satisfies the c.e.

$$s(x) - s(Fx) = \gamma(x), \quad x \in A. \quad (9.1)$$

We have to prove that

$$\mu(A \cap M_\gamma^+) = 0. \quad (9.2)$$

Assume $s \in L_1(A, \mu)$. Then we have TNC

$$\int_A \gamma \, d\mu = 0, \quad (9.3)$$

and (9.2) follows since $\gamma \geq 0$ and $\gamma(x) > 0$ on M_γ^+ .

With a small modification the same argument works in general. Note that one can assume $\gamma(x) \leq 1$ without loss of generality. Indeed, if the theorem is true for the function $\min(1, \gamma(x))$ then it is true for $\gamma(x)$.

Now we consider the sequence of measurable sets $A_n = \{x \in A : s(x) \leq n\}$. They are invariant since $s(Fx) \leq s(x)$. On the other hand, $A_n \subset F^{-1}A_n \subset A_{n+1}$. (The first inclusion follows from $FA_n \subset A_n$, the second from $s(x) \leq s(Fx) + 1$.) Furthermore, $\mu(F^{-1}A_n \setminus A_n) = 0$ since $\mu(F^{-1}A_n) = \mu(A_n)$.

Since $s|_{A_n}$ is bounded and μ is finite, one can integrate (9.1) over A_n :

$$\int_{A_n} \gamma \, d\mu = \int_{A_n} s(x) \, d\mu - \int_{A_n} s(Fx) \, d\mu = \int_{A_n} s(x) \, d\mu - \int_{F^{-1}A_n} s(x) \, d\mu = 0.$$

Hence, $\mu(A_n \cap M_\gamma^+) = 0$. Passing to the limit as $n \rightarrow \infty$ we get (9.2). ■

By formula (3.1) we obtain

COROLLARY 9.4. *Under conditions of Theorem 9.3, if φ is a solution to the c.e. (1.1) then $\varphi(F^n x) \rightarrow +\infty$ a.e. on the set M_γ^+ .*

How fast can be this growth? If γ is bounded then $\varphi(F^n x) = O(n)$ by (3.1). This simple observation can be refined by the following Individual Ergodic Theorem (IET) established by Birkhoff [10] and Khinchin [32]. (See e.g. [24], [31], [38] for the proofs and numerous applications of the IET.)

THEOREM 9.5. *For every $\gamma \in L_1(X, \mu)$ there exists*

$$\tau_\gamma(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \gamma(F^k x) \quad (a.e.). \quad (9.4)$$

The function τ_γ belongs to $L_1(X, \mu)$, and

$$\int_X \tau_\gamma \, d\mu = \int_X \gamma \, d\mu. \quad (9.5)$$

Note that by Proposition 4.2 the set E_γ of convergence in (9.4) is completely invariant, and the function τ_γ is invariant. The latter is originally defined on E_γ only, however, all its continuations to X are invariant and coincide as the elements of $L_1(X, \mu)$ since $\mu(X \setminus E_\gamma) = 0$.

COROLLARY 9.6. *If in the IET the measure μ is ergodic then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \gamma(F^k x) = \frac{1}{\mu(X)} \int_X \gamma \, d\mu \quad (a.e.). \quad (9.6)$$

Proof. This follows from (9.5) since $\tau_\gamma(x)$ is a constant a.e. by ergodicity. ■

In physical terms formula (9.6) is a mathematically correct form of the famous Boltzmann Hypothesis about the asymptotic behavior of a physical system consisting of a large number of particles. The equality (9.6) states that for any admissible function γ (an “observable quantity”) its average over the space of states coincides with its average over time along almost every orbit.

COROLLARY 9.7. *If $\gamma \in L_1(X, \mu)$ and φ is a solution to the c.e. (1.1) then there exists*

$$\varepsilon_\varphi(x) = \lim_{n \rightarrow \infty} \frac{\varphi(F^n x)}{n} \quad (a.e.). \quad (9.7)$$

Under conditions of Corollary 9.7 the set of convergence in (9.7) coincides with E_γ , and $\varepsilon_\varphi(x) = \tau_\gamma(x)$ for all $x \in E_\gamma$. This equality can be extended to X by any common continuation of $\varepsilon_\varphi(x)$ and $\tau_\gamma(x)$. By (9.5) we get

$$\int_X \varepsilon_\varphi d\mu = \int_X \gamma d\mu. \quad (9.8)$$

The function $\varepsilon_\varphi(x)$ is invariant. Therefore, if the measure μ is ergodic then

$$\lim_{n \rightarrow \infty} \frac{\varphi(F^n x)}{n} = \frac{1}{\mu(X)} \int_X \gamma d\mu \quad (a.e.) \quad (9.9)$$

by (9.8) and (9.7).

THEOREM 9.8. *Let $\gamma \in L_1(X, \mu)$, and let φ be a measurable solution to the c.e. (1.1) then $\varepsilon_\varphi(x) = 0$ a.e., so $\varphi(F^n x) = o(n)$ a.e.*

Proof. It suffices to prove that $\varepsilon_\varphi(x) = 0$ a.e. on the set $M_l = \{x \in X : |\varphi(x)| \leq l\}$ for any $l > 0$. The RT yields for a.e. $x \in M_l$ a subsequence $(n_i)_1^\infty \subset \mathbb{N}$ such that $F^{n_i}x \in M_l$, i.e. $|\varphi(F^{n_i}x)| \leq l$. If, in addition, $\varepsilon_\varphi(x)$ exists then

$$\varepsilon_\varphi(x) = \lim_{i \rightarrow \infty} \frac{\varphi(F^{n_i}x)}{n_i} = 0. \quad \blacksquare$$

In the corollaries of Theorem 9.8 we assume $\gamma \in L_1(X, \mu)$.

COROLLARY 9.9. *If the c.e. (1.1) has a measurable solution then γ satisfies the TNC*

$$\int_X \gamma d\mu = 0. \quad (9.10)$$

This is the Anosov Theorem 1 from [3]. In our way to this important result we have used the same arguments (IET, RT) as in [3] but in a wider context.

COROLLARY 9.10. *Let the TNC (9.10) be not fulfilled, but let the conditions (3.3) be valid. Then the c.e. (1.1) is solvable but all its solutions are nonmeasurable.*

In particular, we have

COROLLARY 9.11. *Let the TNC (1.4) be not fulfilled, but let F have no periodic points. Then the c.e. (1.1) is solvable but all its solutions are nonmeasurable.*

For example, with irrational α the equation (1.6) is solvable in 1-periodic functions but all its solutions are nonmeasurable if the TNC (1.7) is not fulfilled.

The existence of measurable solutions was briefly discussed at the end of Section 5. Also, in this section the L_∞ -solvability was considered (Theorem 5.6). Now let us consider the L_p -solvability starting with $p = 1$.

THEOREM 9.12. *With $\gamma \in L_1(X, \mu)$ the c.e. (1.1) is L_1 -solvable if and only if the resolving series (1.10) is Cesàro summable a.e. to a function $\sigma(x) \in L_1(X, \mu)$. In this case $-\sigma(x)$ is an L_1 -solution.*

Proof. “If” follows from Proposition 4.1 where now $\mu(X \setminus C_\gamma) = 0$. “Only if” follows from Proposition 4.2 where now $\mu(X \setminus E_\varphi) = 0$ by IET. \blacksquare

COROLLARY 9.13. *Let the measure μ be ergodic. With $\gamma \in L_p(X, \mu)$, $1 \leq p \leq \infty$, the c.e. (1.1) is L_p -solvable if and only if the resolving series (1.10) is Cesàro summable a.e. to a function $\sigma(x) \in L_p(X, \mu)$. In this case $-\sigma(x)$ is an L_p -solution.*

Proof. “If” is the same as in the proof of Theorem 9.12 even without assumptions about μ . For the “only if” note that $L_p(X, \mu) \subset L_1(X, \mu)$ since the measure μ is finite. Let φ be an L_p -solution to (1.1), and let σ be that of Theorem 9.12. Then $\sigma \in L_1(X, \mu)$ and $\sigma(x)$ is the Cesàro sum of the series (1.10) a.e. and, finally, $-\sigma(x)$ is a solution to (1.1). The sum $\sigma(x) + \varphi(x)$ satisfies the corresponding homogeneous equation. By ergodicity this is a constant a.e., thus $\sigma \in L_p(X, \mu)$. ■

Other results concerning the L_p -solvability were obtained in [41] and [2].

The IET states the convergence a.e. for the sequence $(S_{n;F}\gamma)(x)$, where $S_{n;F}$ are the operators in $L_1(X, \mu)$ defined as

$$S_{n;F} = \frac{1}{n} \sum_{k=0}^{n-1} T_F^k, \quad n \geq 1.$$

The Koopman operator $T_F\varphi = \varphi \circ F$ acts isometrically (not bijectively, in general) in every $L_p(X, \mu)$, $1 \leq p < \infty$, since the measure μ is invariant. In $L_\infty(X, \mu)$ and in $B(X)$ the operator T_F is a contraction. A fortiori, T_F is power bounded in all of these cases. Therefore, the asymptotic behavior of the operator sequence $(S_{n;F})$ can be studied by the following well known Mean Ergodic Theorem (MET). For completeness we give a proof of that.

THEOREM 9.14. *Let T be a bounded linear operator in a Banach space B , and let*

$$S_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k, \quad n \geq 1. \quad (9.11)$$

Assume that $n^{-1}T^n$ strongly tends to zero as $n \rightarrow \infty$ and $\sup_n \|S_n\| < \infty$. Then for a vector $h \in B$ the sequence $(S_n h)$ converges if and only if h belongs to

$$L = \overline{\text{Im}(T - I)} + \ker(T - I).$$

The limit operator is a bounded projection from L onto $\ker(T - I)$ annihilating $\overline{\text{Im}(T - I)}$.

Proof. “If”. Consider two cases.

1. $Th = h$. Then $S_n h = h$. Thus, on $\ker(T - I)$ the sequence (S_n) strongly converges to I .

2. $h = Tf - f$, $f \in B$. Then

$$S_n h = \frac{T^n f - f}{n} \rightarrow 0 \quad (n \rightarrow \infty). \quad (9.12)$$

Thus, on $\text{Im}(T - I)$ the sequence (S_n) strongly converges to 0. This result extends to the closure $\overline{\text{Im}(T - I)}$ since the sequence $(\|S_n\|)$ is bounded. On the whole of L the limit operator is as required.

“Only if”. Let $\lim_{n \rightarrow \infty} S_n h = g$. Then

$$(T - I)g = \lim_{n \rightarrow \infty} \frac{T^n h - h}{n} = 0,$$

i.e. $g \in \ker(T - I)$. Now let $g' = h - g$. Then

$$g' = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (h - T^k h) = \lim_{n \rightarrow \infty} \frac{I - T}{n} \sum_{k=0}^{n-1} \sum_{i=0}^{k-1} T^i h,$$

i.e. $g' \in \overline{\text{Im}(T - I)}$. Hence, $h = g' + g \in L$. ■

COROLLARY 9.15. *Under conditions of MET L is a closed subspace and the topological direct decomposition*

$$L = \overline{\text{Im}(T - I)} \oplus \ker(T - I) \quad (9.13)$$

holds.

In general, $L \neq B$. The equality $L = B$ just means that the sequence (S_n) strongly converges on the whole space B .

COROLLARY 9.16. *Let the space B be reflexive, and let the operator T be such that $n^{-1}T^n$ uniformly tends to zero as $n \rightarrow \infty$ and $\sup_n \|S_n\| < \infty$. Then $L = B$.*

Proof. The operator T^* in the space B^* satisfies the same conditions. Applying MET to T^* we conclude that $\ker(T^* - I) \cap \overline{\text{Im}(T^* - I)} = 0$. Since this subspace of B^* is the annihilator of L , we get $L = B$. ■

COROLLARY 9.17. *If the space B is reflexive and the operator T is power bounded then $L = B$.*

This classical case of MET is due to Lorch [47]. For the unitary operator in Hilbert space this fact was discovered by von Neumann [64] who proved it via the spectral theorem. The following corollary of MET is due to Lin and Sine, cf. [45], Theorem 1.

COROLLARY 9.18. *Under conditions of Corollary 9.16 the equation*

$$Tf - f = h \quad (9.14)$$

in the space B is solvable if and only if the resolving series

$$h + Th + \dots + T^n h + \dots$$

is Cesàro summable in the norm-topology. If the Cesàro sum is s then $f = -s$ is a solution to (9.14).

The proof is similar to that of Theorem 9.12. In particular, we have the following counterpart of Corollary 9.13.

COROLLARY 9.19. *With $\gamma \in L_p(X, \mu)$, $1 < p < \infty$, the c.e. (1.1) is solvable in $L_p(X, \mu)$ if and only if the resolving series (1.10) is Cesàro summable in the L_p -norm to a function $\sigma(x)$. In this case $-\sigma(x)$ is an L_p -solution.*

REMARK 9.20. For the “if” part of Corollary 9.18 the reflexivity of B is not required. Thus, the cases $p = 1, \infty$ can be included into this part of Corollary 9.19.

Comparing Corollaries 9.19 and 9.19 we obtain

COROLLARY 9.21. *Let the measure μ be ergodic. If the resolving series (1.10) is Cesàro summable a.e. to a function $\sigma \in L_p(X, \mu)$, $1 < p < \infty$, then it is Cesàro summable to σ in the L_p -norm.*

Now recall that the equation (9.14) in a Banach space B is called *normally solvable* if $\text{Im}(T - I)$ is closed or, equivalently, if with h running over B it is solvable if and only if h is annihilated by all linear functionals from $\ker(T^* - I)$. For this property the following ergodic criterion is due to Dunford [15] in the “if” part and due to Lin [44] in the “only if” part. The paper [44] also contains a proof of the “if” part different from that of [15].

THEOREM 9.22. *Let T be a bounded linear operator in a Banach space B such that $n^{-1}T^n$ uniformly tends to zero as $n \rightarrow \infty$. Then $\text{Im}(T - I)$ is closed if and only if the sequence (S_n) converges uniformly. Moreover, in this case $L = B$ and the limit operator is the projection onto $\ker(T - I)$ annihilating $\text{Im}(T - I)$.*

Being considered as a criterion of the uniform convergence of (S_n) this result is called the Uniform Ergodic Theorem (UET).

10. The total attractor. This short section contains a useful information from the general theory of dynamical systems. Let (X, F) be a compact dynamical system, and

$$\Omega = \bigcap_{n \geq 1} F^n X, \quad (10.1)$$

cf. (7.1). Obviously, Ω is an invariant nonempty compact subset of X , and $\Omega \neq X$, except for the case of surjective F .

LEMMA 10.1. *The mapping $F|_{\Omega}$ is surjective.*

Proof. Let $F\Omega \neq \Omega$, and let $x \in \Omega \setminus F\Omega$. Then $F^{-1}\{x\} \cap \Omega = \emptyset$ because Ω is invariant. Therefore,

$$F^{-1}\{x\} \subset \bigcup_{n \geq 1} G_n, \quad G_n = X \setminus F^n X.$$

Since G_n is open and $G_n \subset G_{n+1}$, the compact set $F^{-1}\{x\}$ is contained in some G_m . Thus, $F^{-1}\{x\} \cap F^m X = \emptyset$, i.e. $x \notin F^{m+1} X$, a fortiori, $x \notin \Omega$, a contradiction. ■

In addition, we have

LEMMA 10.2. *Every invariant set Y such that $F|_Y$ is surjective is contained in Ω .*

Proof. $Y = F^n Y \subset F^n X$ for all n , so $Y \subset \Omega$. ■

Thus, Ω is the largest invariant subset of X with surjective restriction of F .

LEMMA 10.3. *For any neighborhood $G \supset \Omega$ there exists n such that $F^n X \subset G$.*

Proof. We have

$$\bigcap_{n \geq 1} (F^n X \setminus G) = \Omega \setminus G = \emptyset.$$

Since all $F^n X \setminus G$ are compact, there is n such that $F^n X \setminus G = \emptyset$. ■

In view of Lemma 10.3 let us call Ω the *total attractor* for the dynamical system (X, F) . Actually, this construction is classical, coming back to Birkhoff [9].

COROLLARY 10.4. *The ω -limit sets of all orbits are contained in Ω .*

Now let μ be a regular Borel measure on X . We denote by $\text{supp } \mu$ its *support*, i.e. the set of $x \in X$ such that $\mu(U) > 0$ for every neighborhood U of x . It is easy to see that the set $\text{supp } \mu$ is closed, hence it is compact. For every function $\psi \in L_1(X, \mu)$ we have

$$\int_X \psi(x) d\mu = \int_{\text{supp } \mu} \psi(x) d\mu.$$

LEMMA 10.5. *For any invariant regular Borel measure μ the set $M = \text{supp } \mu$ is invariant and the restriction $F|_M$ is surjective.*

Proof. Let $x \in M$, $y = Fx$. For any neighborhood V of y the preimage $F^{-1}V$ is a neighborhood of x and $\mu(V) = \mu(F^{-1}V) > 0$, hence $y \in M$. Thus, M is invariant. Now assume $FM \neq M$. By the Urysohn Lemma there exists a nonnegative continuous function ϕ such that $\phi|_{FM} = 0$, $\phi \neq 0$. However,

$$\int_X \phi(x) d\mu = \int_X \phi(Fx) d\mu = \int_M \phi(Fx) d\mu = 0,$$

a contradiction. ■

Denote by Ω_0 the union of supports of all invariant regular Borel measures. Combining Lemmas 10.2 and 10.5 we obtain

COROLLARY 10.6. $\Omega_0 \subset \Omega$.

In general, $\Omega_0 \neq \Omega$. For example, if X is the closed interval $[0, 1] \subset \mathbb{R}$ and $Fx = x^2$, then $\Omega = X$, but Ω_0 is the two-point set $\{0, 1\}$.

The concept of total attractor plays an important role in our theory [50] of dissipative systems. In particular, we have

THEOREM 10.7. *If a compact dynamical system (X, F) is dissipative then its restriction to the total attractor Ω is conservative.*

A compact dynamical system (X, F) with the total attractor Ω is called *topologically Ω -transitive* if the restriction $(\Omega, F|_\Omega)$ is topologically transitive.

COROLLARY 10.8. *For any compact topologically Ω -transitive dissipative dynamical system (X, F) the invariant regular Borel measure μ such that $\mu(X) = 1$ is unique.*

Proof. By Corollary 10.6 $\text{supp } \mu \subset \Omega$. Therefore, the measure $\mu|_\Omega$ is invariant for $F|_\Omega$. This restriction is conservative (Theorem 10.7). Hence, the dynamical system $(\Omega, F|_\Omega)$ is homeomorphic to (G, τ_g) where G is a monothetic compact group and g is its generator (Theorem 8.10). By Proposition 2.5 the image of the measure $\mu|_\Omega$ is the Haar measure ν on G . ■

11. The normal solvability in $C(X)$. The following result was obtained in [6].

THEOREM 11.1. *Let (X, F) be a compact dynamical system. The c.e. (1.1) is normally solvable in $C(X)$ if and only if F is preperiodic.*

Thus, as long as F is not preperiodic, there is a continuous function γ such that the TNC (1.4) is fulfilled for all invariant regular Borel measures μ , but the c.e. (1.1) has no continuous solutions.

Proof. The “if” part follows directly from Theorem 4.3. Conversely, assume that the c.e. (1.1) is normally solvable in $C(X)$, i.e. $\text{Im}(T - I)$ is closed for $T = T_F$. Since T is a contraction, the UET (i.e., Theorem 9.22) yields

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k - P \right\| = 0$$

where P is the projection onto $\ker(T - I)$ annihilating $\text{Im}(T - I)$. Hence, for every $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \gamma(F^k x) - (P\gamma)(x) \right| < \varepsilon \|\gamma\|, \quad n \geq n(\varepsilon), \quad x \in X, \quad (11.1)$$

for all $\gamma \in C(X)$, $\gamma \neq 0$.

Let Ω be the total attractor of our system, and let $\gamma|_{\Omega} = 0$. Then $\gamma(F^k x) = 0$ for $x \in \Omega$ and for all k . Therefore, $|(P\gamma)(x)| < \varepsilon \|\gamma\|$, $x \in \Omega$, thus $P\gamma|_{\Omega} = 0$. Since $P = PT^m$ for every m , we have $(P\gamma)(x) = (P\gamma)(F^m x)$ for all $x \in X$. Hence, $(P\gamma)(x) = 0$ by Corollary 10.4, and (11.1) reduces to

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \gamma(F^k x) \right| < \varepsilon \|\gamma\|, \quad n \geq n(\varepsilon), \quad x \in X.$$

For $\varepsilon = 1$ and $l = n(1) - 1$ we get

$$\left| \frac{1}{l+1} \sum_{k=0}^l \gamma(F^k x) \right| < \|\gamma\|, \quad x \in X. \quad (11.2)$$

The inequality (11.2) implies $F^l X = \Omega$. Indeed, let $x \in X$ be such that $F^l x \notin \Omega$, a fortiori, $F^k x \notin \Omega$ for $0 \leq k \leq l$. By the Urysohn Lemma there exists $\gamma \in C(X)$ such that $\gamma|_{\Omega} = 0$, $\gamma(F^k x) = 1$, $0 \leq k \leq l$, and $\|\gamma\| = 1$. This contradicts (11.2).

It remains to prove that $F|_{\Omega}$ is periodic. First, we show that the c.e. (1.1) remains normally solvable after restriction to $C(\Omega)$. Let $\gamma \in C(\Omega)$ be annihilated by all invariant regular Borel measures on Ω . Denote by $\tilde{\gamma}$ a continuous extension of γ to X , and let μ be an invariant regular Borel measure on X . By Corollary 10.6 the measure $\mu|_{\Omega}$ is invariant for $F|_{\Omega}$ and

$$\int_X \tilde{\gamma} d\mu = \int_{\Omega} \gamma d(\mu|_{\Omega}) = 0.$$

By assumption, the equation $\varphi(Fx) - \varphi(x) = \tilde{\gamma}(x)$ on X has a continuous solution. Its restriction to Ω satisfies (1.1).

Now one can assume $\Omega = X$. Then F is surjective by Lemma 10.1. Accordingly, the operator T is isometric. Its spectrum $\sigma(T)$ contains the point $\lambda = 1$ (as an eigenvalue with the eigenfunction $\mathbf{1}$). This point is isolated in $\sigma(T)$ since $\text{Im}(I - T)$ is closed, see [63], p. 330. Hence, T is invertible, otherwise $\sigma(T)$ would be the unit disk $\{x \in \mathbb{C} : |\lambda| \leq 1\}$, see [52], p. 185. On the other hand, T is an endomorphism of the Banach algebra $C(X)$. Therefore, T is an automorphism. By the Kamowitz–Scheinberg theorem [29] either T is periodic or $\sigma(T)$ is the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Thus, T is periodic, so the underlying mapping F is periodic as well. ■

Theorem 11.1 can be extended to a wide class of noncompact spaces X by the Čech–Stone compactification \overline{X} .

COROLLARY 11.2. *Let X be a completely regular topological space. Then c.e. (1.1) is normally solvable in $CB(X)$ if and only if F is preperiodic.*

Proof. The space X can be considered as a dense subset in \overline{X} and then F is the restriction to X of a continuous mapping $\overline{F} : \overline{X} \rightarrow \overline{X}$. Since the Banach spaces $CB(X)$ and $C(\overline{X})$ are isometric, the normal solvability of (1.1) in $CB(X)$ is equivalent to the same property of the c.e. $\psi(\overline{F}z) - \psi(z) = \theta(z)$ in $C(\overline{X})$. On the other hand, the preperiodicity of F is equivalent to that of \overline{F} . ■

COROLLARY 11.3. *For any set X the c.e. (1.1) is normally solvable in $B(X)$ if and only if F is preperiodic.*

COROLLARY 11.4. *Let (X, F) be a minimal topological dynamical system with infinite completely regular X . Then the c.e. (1.1) is not normally solvable in $CB(X)$.*

Proof. The only minimal preperiodic system is the rotation of a cycle. ■

COROLLARY 11.5. *Under conditions of Corollary 11.4 the norms $\|\cdot\|_F$ and $\|\cdot\|$ on $\text{Im}(T_F - I)$ are not equivalent.*

Proof. Combine Corollary 11.4 with Proposition 6.10. ■

The applications below relate to the shifts of topological semigroups and groups. For any semigroup S we denote by $[[s]]$ the subsemigroup of S generated by s , i.e. $[[s]] = \{s^n : n \geq 1\}$. We do not assume that S has a unit. If S is a group and e is its unit then $[[s]] \cup \{e\} = [s]$ according to the notation from Section 7.

COROLLARY 11.6. *Let S be a completely regular topological semigroup. With $s \in S$ the c.e.*

$$\varphi(sx) - \varphi(x) = \gamma(x), \quad x \in S, \quad (11.3)$$

is normally solvable in $CB(S)$ if and only if $[[s]]$ is finite.

Proof. If $[[s]]$ is finite then $s^{p+l} = s^p$ for some $p \geq 1$ and $l \geq 0$. Then the shift $x \mapsto sx$, $x \in S$, is preperiodic with the period p and preperiod l . Conversely, if $s^{p+l}x = s^px$ for all $x \in S$ then $s^{p+l+1} = s^{p+1}$, so $[[s]]$ is finite. ■

COROLLARY 11.7. *Let G be a topological group. With $g \in G$ the c.e.*

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G, \quad (11.4)$$

is normally solvable in $CB(G)$ if and only if the element g is of a finite order.

Proof. Every topological group is completely regular, see e.g. [28], Theorem 8.4. ■

For example, the c.e. corresponding to a rotation of the unit circle \mathbb{T} is normally solvable in $C(\mathbb{T})$ if and only if the rotation is rational.

For the almost periodic solutions we use the Bohr compactification.

COROLLARY 11.8. *If G is a commutative topological group such that its points are separated by characters (in particular, if G is locally compact) then with $g \in G$ the cohomological equation*

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G, \quad (11.5)$$

is normally solvable in $AP(G)$ if and only if the element g is of a finite order.

Proof. The normal solvability of (11.5) in $AP(G)$ is equivalent to the same property of the corresponding c.e. in $C(bG)$ where bG is the Bohr compact of the group G . Since the characters separate points of G , the canonical homomorphism $G \rightarrow bG$ is injective. Hence, g is of a finite order if and only if such is its image in bG . ■

COROLLARY 11.9. *With any real $\alpha \neq 0$ the c.e.*

$$\varphi(x + \alpha) - \varphi(x) = \gamma(x), \quad x \in \mathbb{R}, \quad (11.6)$$

is not normally solvable in $AP(\mathbb{R})$.

Proof. There are no nonzero elements of finite order in \mathbb{R} . ■

We conclude this section with the question: *is the L_p -counterpart of Theorem 11.1 true?*

12. Absence of measurable solutions. Let us start with a group situation.

THEOREM 12.1. *Let G be an infinite monothetic compact group, and let g be its generator. Then there exists a function $\gamma \in C(G)$ satisfying the TNC*

$$\int_G \gamma d\nu = 0 \quad (12.1)$$

for the Haar measure ν but such that the c.e.

$$\varphi(gx) - \varphi(x) = \gamma(x), \quad x \in G, \quad (12.2)$$

has no measurable solutions.

Let us stress that by Proposition 2.5 the only TNC is (12.1). The absence of continuous solutions with a $\gamma \in C(G)$ satisfying (12.1) follows from Corollary 11.7.

In the proof of Theorem 12.1 we use the ergodicity of the dynamical system (G, τ_g) . This property is stated by the following lemma.

LEMMA 12.2. *Under conditions of Theorem 12.1 every measurable solution φ to the homogeneous equation*

$$\varphi(gx) - \varphi(x) = 0, \quad x \in G, \quad (12.3)$$

is a constant a.e.

Proof. One can assume $\varphi \neq 0$. If the measure of the set $\{x \in G : \varphi(x) \neq 0\}$ is not zero then the measure of the subset $M_l = \{x \in G : 0 < |\varphi(x)| \leq l\}$ is not zero for an $l > 0$. By (12.3) M_l is completely invariant for the shift τ_g . We prove that $\varphi(x)$ is a constant a.e. on M_l and $\mu(G \setminus M_l) = 0$.

For any character χ of the group G we have

$$(\chi(g) - 1) \int_{M_l} \varphi(x) \overline{\chi(x)} d\nu = 0,$$

see (7.12). If $\chi \neq \mathbf{1}$ then $\chi(g) \neq 1$ since g is a generator of G . Hence,

$$\int_G \omega_l(x) \varphi(x) \overline{\chi(x)} d\nu = 0, \quad (12.4)$$

where ω_l is the indicator of M_l . The function $\theta_l(x) = \omega_l(x) \varphi(x)$ is measurable and bounded, a fortiori, $\theta_l \in L_2(G, \nu)$. The relation (12.4) states that all its Fourier coefficients corresponding to the nonunity character vanish. Hence, $\theta_l(x)$ is a constant η a.e. on G . However, $\theta_l(x) = \varphi(x)$ on M_l . Therefore, $\varphi(x) = \eta$ a.e. on M_l . On the other hand, $\theta_l(x) = 0$ on $G \setminus M_l$. If the measure of this set is not zero then $\eta = 0$ and then $\varphi(x) = 0$ a.e. on M_l , a contradiction. ■

Another lemma we need is the following.

LEMMA 12.3. *Under conditions of Theorem 12.1 the set $\Gamma = \{\chi(g) : \chi \in G^*\}$ is dense in \mathbb{T} .*

Proof. The set Γ is the image of the mapping $\chi \mapsto \chi(g)$, $\chi \in G^*$. Since the latter is a homomorphism $G^* \rightarrow \mathbb{T}$, the set Γ is a subgroup of \mathbb{T} . It is well known (and can be easily proven) that every nondense subgroup of \mathbb{T} is finite. However, if Γ is finite then all its elements are roots of 1 of a power m . Then $\chi(g^m) = \chi(g)^m = 1$ for all $\chi \in G^*$ that results in $g^m = e$ according to the duality theory. Since g is a generator of G , the latter turns out to be finite that contradicts our assumption. ■

Now we proceed to the proof of Theorem 12.1, cf. [6].

Proof. Suppose to the contrary that the equation (12.2) has a measurable solution for every γ satisfying (12.1). This TNC determines a closed subspace $C^0(G) \subset C(G)$. On the other hand, we consider the space $M = M(G, \nu)$ of measurable functions on G . As usual, we identify the functions equal a.e. Then the formula

$$\rho(\varphi_1, \varphi_2) = \int_G \frac{|\varphi_1 - \varphi_2|}{1 + |\varphi_1 - \varphi_2|} d\nu$$

determines a metric on M corresponding to the convergence in measure:

$$\lim_n \rho(\varphi_n, \varphi) = 0 \iff \forall \varepsilon > 0 \lim_n \nu \{x : |\varphi_n(x) - \varphi(x)| \geq \varepsilon\} = 0.$$

This space is not Banach but it is an F -space. (See e.g. [57] for information on this class of linear metric spaces.) In what follows we deal with the quotient space $\widehat{M} = M/Q$ where Q is the subspace of constants. This \widehat{M} is also an F -space.

For a given $\gamma \in C^0(G)$ a measurable solution φ is unique up to an additive constant by Lemma 12.2. Therefore, its coset $\widehat{\varphi} \in \widehat{M}$ is uniquely determined, i.e. $\widehat{\varphi} = R\gamma$ where R is a mapping $C^0(G) \rightarrow \widehat{M}$. Obviously, R is linear. Show that the graph of R is closed.

Let a sequence $(\gamma_n) \subset C^0(G)$ converge to a $\gamma \in C^0(G)$ uniformly, and let $(R\gamma_n) = (\widehat{\varphi}_n)$ converge to a $\widehat{\varphi} \in \widehat{M}$. By adding of suitable constants the functions φ_n can be chosen so that φ_n tends to φ in measure. By restriction to a subsequence one can assume $\lim_n \varphi_n(x) = \varphi(x)$ a.e. on G . The limit pair (γ, φ) satisfies (12.2) a.e., thus $R\gamma = \widehat{\varphi}$.

By the Closed Graph Theorem the operator R is continuous. To disprove this conclusion we apply Lemma 12.3 to get a sequence $(\chi_n) \subset G^* \setminus \{\mathbf{1}\}$ such that $\lim_n \chi_n(g) = 1$.

With

$$\gamma_n(x) = (\chi_n(g) - 1)\chi_n(x)$$

the functions χ_n satisfy (12.2) for $\gamma = \gamma_n$. By (7.10) we have

$$\int_G \gamma_n \, d\nu = (\chi_n(g) - 1) \int_G \chi_n \, d\nu = 0$$

i.e. $\gamma_n \in C^0(G)$. Since $\lim_n \gamma_n = 0$ in $C^0(G)$ and R is continuous, the sequence $\widehat{\chi}_n = R\gamma_n$ tends to zero in \widehat{M} . This means that there are constants c_n such that $(\chi_n - c_n)$ tends to zero in measure. Therefore, the sequence (c_n) is bounded and

$$\lim_n \int_G (\chi_n - c_n) \, d\nu = 0,$$

whence $\lim_n c_n = 0$, and then $\lim_n \chi_n = 0$ in measure. This contradicts the equality

$$\int_G |\chi_n| \, d\nu = \int_G d\nu = 1. \quad \blacksquare$$

COROLLARY 12.4. *In $C^0(G)$ the set of γ mentioned in Theorem 12.1 has the second Baire category.*

Proof. This set is the complement to $\text{Im}(\tau_g - I : M \rightarrow C^0(G)) \neq C^0(G)$. This image is the set of first category by the classical Banach theorem. \blacksquare

Now we are in position to obtain the following general result.

THEOREM 12.5. *Let a compact dynamical system (X, F) be uniformly stable and topologically Ω -transitive. If the total attractor Ω is infinite, then there is an invariant regular Borel measure μ on X and a function $\gamma \in C(X)$ such that the TNC (1.4) is fulfilled but the c.e. (1.1) has no measurable solutions.*

Proof. One can assume that (X, F) is dissipative. Then $(\Omega, F|_\Omega)$ is conservative by Theorem 10.7, and it is topologically transitive by definition. By Theorem 8.10 the system $(\Omega, F|_\Omega)$ is homeomorphic to (G, τ_g) where G is a monothetic compact group and g is a generator. The group is infinite since such is Ω . Thus, Theorem 12.1 is applicable. Let γ_0 be the function appearing there, and let γ_1 be the corresponding function from $C(\Omega)$, and finally, let μ_1 be the measure on Ω corresponding to the Haar measure ν on G . Then

$$\int_\Omega \gamma_1 \, d\mu_1 = 0 \tag{12.5}$$

but the equation

$$\phi(Fx) - \phi(x) = \gamma_1(x), \quad x \in \Omega, \tag{12.6}$$

has no measurable solutions. To finish the proof it suffices to trivially extend μ_1 from Ω to X and γ_1 to a $\gamma \in C(X)$. \blacksquare

REMARK 12.6. By Corollaries 10.6 and 10.8 the measure μ is unique up to normalization. Therefore, in the context of Theorem 12.5 there are no TNC's essentially different from (1.4).

In [49] the Closed Graph Theorem has been applied to prove the following Gordon theorem [19] the original proof of which is purely analytic. In the proof below we follow [49]. Actually, this way is also a prototype for the proof of Theorem 12.1.

THEOREM 12.7. *Let $\theta(t)$ ($0 < t \leq 1$) be a positive nondecreasing function such that $t^{-1}\theta(t) \rightarrow \infty$ as $t \rightarrow 0$. Let α be an irrational number. Then there exists a continuous 1-periodic function $h(x)$, $x \in \mathbb{R}$, satisfying the TNC (1.7) and such that*

$$\|h\|_{\theta} \equiv \sup_{0 \leq y < x \leq 1} \frac{|h(x) - h(y)|}{\theta(x - y)} < \infty, \quad (12.7)$$

but the c.e. (1.6) has no measurable solutions.

Proof. Let us transfer the situation to the group \mathbb{T} and consider the space \widehat{M} introduced in the proof of Theorem 12.1. On the other hand, the condition (12.7) together with (1.7) determines a Banach space $C_{\theta}^0(\mathbb{T})$ of functions on \mathbb{T} . This is a linear nonclosed subspace of $C(\mathbb{T})$ containing all characters

$$\chi_n(x) = e^{2\pi i n x}, \quad x \in \mathbb{R}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

Indeed,

$$\|\chi_n\|_{\theta} = \sup_{0 < t \leq 1} \frac{2|\sin \pi n t|}{\theta(t)} \leq 2\pi |n| \sup_{0 < t \leq 1} \frac{t}{\theta(t)} < \infty. \quad (12.8)$$

Further we use $n > 0$ only.

First of all, we show that

$$\lim_{n \rightarrow \infty} \frac{\|\chi_n\|_{\theta}}{n} = 0. \quad (12.9)$$

To this end note that for every $\tau \in (0, 1]$ we have

$$\frac{2|\sin \pi n t|}{\theta(t)} \leq \frac{2}{\theta(\tau)}, \quad t \geq \tau,$$

since $\theta(t)$ is nondecreasing. On the other hand,

$$\frac{2|\sin \pi n t|}{\theta(t)} \leq 2\pi n \sup_{0 < s < \tau} \frac{s}{\theta(s)}, \quad 0 < t < \tau.$$

By (12.8)

$$\frac{\|\chi_n\|_{\theta}}{n} \leq \max \left\{ \frac{2}{n\theta(\tau)}, 2\pi \sup_{0 < s < \tau} \frac{s}{\theta(s)} \right\} \rightarrow 0$$

as $n \rightarrow \infty$ and $\tau = 1/n$.

The character χ_n is a solution to the c.e. (1.6) with

$$h(x) = h_n(x) = (e^{2\pi i n \alpha} - 1)\chi_n(x).$$

Obviously,

$$\|h_n\|_{\theta} = 2|\sin \pi n \alpha| \|\chi_n\|_{\theta} \leq 2\pi d_n \|\chi_n\|_{\theta}$$

where $d_n = \text{dist}(n\alpha, \mathbb{Z})$. For irrational α the classical Dirichlet theorem states that $d_{n_k} \leq 1/n_k$ for a sequence $(n_k) \subset \mathbb{N}$. By (12.9) the sequence (h_{n_k}) converges to 0 in $C_{\theta}^0(\mathbb{T})$.

Now let us assume that with an irrational α the c.e. (1.6) has a measurable solution f for every $h \in C_{\theta}^0(\mathbb{T})$. Then the Closed Graph Theorem is applicable as in the proof of Theorem 12.1. As a result, the sequence (χ_{n_k}) converges to 0 in measure, that is impossible. ■

REMARK 12.8. In [19] the continuity modulus θ is supposed to be subadditive, i.e. $\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2)$. We have seen that this condition is redundant.

A nice particular case of Theorem 12.7 is $\theta(t) = t^\lambda$, $0 \leq \lambda < 1$, that means the Hölder condition for h . The case $\lambda = 0$, i.e. $\theta = 1$, is just that of Theorem 12.1 for the irrational rotation of \mathbb{T} . On the other hand, Theorem 12.7 cannot be extended to the case $\lambda = 1$, i.e. $\theta(t) = t$, that means the Lipschitz condition for h .

THEOREM 12.9. *Let α be irrational such that in the corresponding continued fraction the set of elements is bounded. Then for every 1-periodic function h satisfying the Lipschitz condition and the TNC (1.7) the c.e. (1.6) has a 1-periodic solution $f \in L_2(0, 1)$.*

Proof. In the Fourier decomposition

$$h(x) = \sum_{n=-\infty}^{\infty} h_n e^{2\pi i n x}$$

we have $h_0 = 0$ by (1.7). Consider the formal solution

$$\sum_{n \neq 0} \frac{h_n}{e^{2\pi i n \alpha} - 1} e^{2\pi i n x}. \quad (12.10)$$

By our assumption on α there exists $c > 0$ such that $\text{dist}(n\alpha, \mathbb{Z}) > c/|n|$, $n \neq 0$, see e.g. [33], Theorem 23. Therefore,

$$|e^{2\pi i n \alpha} - 1| = 2|\sin \pi n \alpha| > \frac{4c}{|n|}$$

since $\sin \pi t > 2t$ for $0 < t < 1/2$ and $\text{dist}(n\alpha, \mathbb{Z}) < 1/2$. On the other hand, $h_n = -h'_n/n$ where h'_n are the Fourier coefficients of the derivative $h'(x)$. The latter exists a.e. and bounded, a fortiori, $h' \in L_2(0, 1)$. As a result,

$$\sum_{n \neq 0} \left| \frac{h_n}{e^{2\pi i n \alpha} - 1} \right|^2 < \frac{1}{16c^2} \sum_{n \neq 0} |h'_n|^2 < \infty,$$

hence the series (12.10) converges in $L_2(0, 1)$. ■

In particular, all quadratic irrationalities satisfy the condition of Theorem 12.9 since in this case the continued fractions are periodic. The simplest example is the Fibonacci irrationality $\alpha = \frac{1+\sqrt{5}}{2}$ where all elements of the corresponding continued fraction are equal to 1.

13. Summation of divergent series. The material of this section is basically extracted from our paper [53]. This starts with the following general definition of summation of numerical series which is a modern form of the bringing together Hardy–Kolmogorov axioms [26], [35].

Let \mathbf{s} be the linear space of all scalar sequences $\boldsymbol{\xi} = (\xi_n)$, and let τ be the shift operator in \mathbf{s} , i.e. $\tau \boldsymbol{\xi} = (\xi_{n+1})$, and, finally, let $L \subset \mathbf{s}$ be a τ -invariant subspace. A linear functional σ on L is called a *summation of the series*

$$\boldsymbol{\xi} = \xi_0 + \xi_1 + \dots + \xi_n + \dots, \quad \boldsymbol{\xi} \in L, \quad (13.1)$$

if

$$\sigma[\xi] - \sigma[\tau\xi] = \xi_0 \quad (13.2)$$

accordingly to the formal relation

$$\xi_0 + \xi_1 + \dots + \xi_n + \dots = \xi_0 + (\xi_1 + \dots + \xi_n + \dots).$$

In this situation we say that the subspace L *admits summation*. Also, we say that a series ξ is *summable* if it belongs to a subspace admitting summation or, equivalently, if the subspace $\text{Span}\{\tau^k\xi : k \geq 0\}$ admits summation. If this summation is σ we say that ξ is σ -*summable*. Let us emphasize that the equation (13.2) is a c.e. with given ξ_0 and unknown σ , both are linear functionals on L . Most of classical summations (for instance, the Cesàro summation) satisfy (13.2).

LEMMA 13.1. *Let (X, F) be a dynamical system. If there exists a summation σ such that the resolving series γ_x of a function $\gamma \in \Phi(X)$ is σ -summable for all $x \in X$ then the c.e. (1.1) is solvable. A solution is $\varphi(x) = -\sigma[\gamma_x]$.*

Proof. The summation σ is defined on a T -invariant subspace $L \in \mathbf{s}$ containing all γ_x , $x \in X$. Since $\gamma_{Fx} = \tau\gamma_x$, we have

$$\varphi(Fx) - \varphi(x) = \sigma[\gamma_x] - \sigma[\gamma_{Fx}] = \sigma[\gamma_x] - \sigma[\tau\gamma_x] = (\gamma_x)_0 = \gamma(x). \blacksquare$$

The nonlinear functionals satisfying (13.2) are also interesting because of their relation to the resolving functionals introduced in Section 5. However, the resolving functional ω from Theorem 5.1 is defined on a set Λ of sequences (η_n) containing all $(-s_n(x))$, $x \in X$, while the summation σ from (13.2) is defined on a space L of sequences whose coordinates ξ_n are the members of the series, not the partial sums. Nevertheless, these situations are connected by means of the linear operator $V : \mathbf{s} \mapsto \mathbf{s}$ such that $V(\xi_n) = (\eta_n)$ where $\eta_n = -\sum_{k=0}^n \xi_k$, $n \geq 0$. Obviously, the operator V is invertible: $V^{-1}(\eta_n) = (\xi_n)$ where $\xi_n = \eta_{n-1} - \eta_n$, $n \geq 1$, $\xi_0 = -\eta_0$. Let $L = V^{-1}\Lambda$, and let $\sigma(\xi) = \omega(V\xi)$, $\xi \in L$. In general, the set L and the functional σ are nonlinear since such are Λ and ω . It is easy to prove the following

PROPOSITION 13.2. *Under conditions of Theorem 5.1 the set L is τ -invariant and the functional σ satisfies the equation (13.2). Every such a nonlinear summation appears in this way.*

In what follows all summations are linear. We start with a crucial example of a nonsummable series.

EXAMPLE 13.3. The series

$$\varepsilon = 1 + 1 + \dots + 1 + \dots \quad (13.3)$$

is not summable. Indeed, the substitution $\xi = \varepsilon$ into (13.2) yields the classical contradiction $0 = 1$.

Thus, if a subspace L admits summation then the series ε does not belong to L . Remarkably, the converse is also true.

THEOREM 13.4. *If a τ -invariant subspace L does not contain the series ε then L admits summation.*

Proof. The series ε is a fixed point of the operator τ , i.e. it belongs to the subspace $\ker(I - \tau)$. The latter consists of the series of form $\xi + \xi \dots + \xi + \dots$, so $\ker(I - \tau) = \text{Span}\{\varepsilon\}$. Hence, under condition of Theorem 13.4 we have $L \cap \ker(I - \tau) = 0$, i.e. the operator $R = (I - \tau)|L$ is injective. Therefore, R maps L onto $\text{Im } R$ bijectively. We denote the corresponding inverse operator by S . Since the subspace L is τ -invariant, we have $\text{Im } R \subset L$. Let $Q : L \rightarrow L$ be a projection onto $\text{Im } R$. Then the linear functional $\sigma(\xi) = \xi_0(SQ\xi)$, $\xi \in L$, satisfies (13.2), i.e. this is a summation on L . Indeed,

$$\sigma(\xi) - \sigma(\tau\xi) = \sigma(R\xi) = \xi_0(SQR\xi) = \xi_0(SR\xi) = \xi_0(\xi). \quad \blacksquare$$

An important application of this criterion is the following.

THEOREM 13.5. *Let (X, F) be a dynamical system with a finite ergodic invariant measure μ , and let a function $\gamma \in L_1(X, \mu)$ satisfy the TNC (1.4). Then there exists a summation σ such that the resolving series γ_x is σ -summable a.e.*

Proof. By Corollary 9.6 of the IET we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \gamma(F^k x) = 0 \quad (13.4)$$

for $x \in X_0$, where X_0 is a subset of X such that $\mu(X \setminus X_0) = 0$. Since X_0 is F -invariant, the space $L = \text{Span}\{\gamma_x : x \in X_0\}$ is τ -invariant. We prove that there exists a required summation σ on L .

Suppose the contrary. Then by Theorem 13.4 the series ε belongs to L , i.e.

$$\sum_{i=1}^m \alpha_i \gamma_{x_i} = \varepsilon$$

with some $x_i \in X_0$ and some coefficients α_i . In other words,

$$\sum_{i=1}^m \alpha_i \gamma(F^k x_i) = 1, \quad k \geq 0,$$

whence

$$\sum_{i=1}^m \alpha_i \left(\frac{1}{n} \sum_{k=0}^{n-1} \gamma(F^k x_i) \right) = 1, \quad n \geq 1,$$

Passing to the limit as $n \rightarrow \infty$ we get the contradiction $0 = 1$ by (13.4). \blacksquare

By Lemma 13.1 we obtain

COROLLARY 13.6. *Under conditions of Theorem 13.5 the c.e. (1.1) has a solution a.e. The solution is $\varphi(x) = -\sigma(\gamma_x)$.*

In particular, this yields a solution a.e. to the c.e.

$$f(qx) - f(x) = h(x), \quad x \in \mathbb{R}, \quad q \in \mathbb{N}, \quad q \geq 2, \quad (13.5)$$

in 2π -periodic functions with $h \in L_1(0, 2\pi)$. Indeed, this equation is equivalent to

$$\varphi(z^q) - \varphi(z) = \gamma(z), \quad z \in \mathbb{T}, \quad (13.6)$$

with $z = e^{ix}$. The mapping $F_q z = z^q$ is ergodic with respect to the standard measure on \mathbb{T} . By Corollary 13.6 the c.e. (13.5) with $h \in L_1(0, 2\pi)$ such that

$$\int_0^{2\pi} h(x) dx = 0 \quad (13.7)$$

is solvable a.e. In fact, the solvability follows from Example 3.7 even without the assumption (13.7). However, Corollary 13.6 yields a solution a.e. as a result of a summation of the resolving series

$$h(x) + h(qx) + \dots + h(q^n x) + \dots \quad (13.8)$$

The existence of such a summation for $q = 3$ and $h(x) = \sin x$ was conjectured by Kolmogorov [35].

Note that, in general, the summability of the resolving series of a c.e. is not necessary for the solvability. For instance, the resolving series of the Abel equation (1.5) is ε but this equation can be solvable, see Corollary 3.5.

THEOREM 13.7. *Let $h(x)$ be a trigonometric polynomial,*

$$h(x) = \sum_{k=1}^m (a_k \cos \nu_k x + b_k \sin \nu_k x),$$

with $0 < \nu_1 < \dots < \nu_m$ such that all ratios ν_k/ν_j , $k \neq j$, are not powers of q . Then all 2π -periodic solutions to the c.e. (1.12) are nonmeasurable.

This is part 3) of our Theorem 4.8 from [53]. For $q = 3$ and $h(x) = \sin x$ this was announced by Kolmogorov [35]. For $q = 2$ and $h(x) = \cos x$ this was proven by Zygmund ([68], Chapter 5, Problem 26).

COROLLARY 13.8. *Let $\gamma(z)$ be a polynomial,*

$$\gamma(z) = \sum_{k=1}^m c_k z^{\nu_k},$$

with ν_k same as in Theorem 13.7. Then all solutions to the c.e. (13.6) are nonmeasurable.

REMARK 13.9. The self-mapping $F_q z = z^q$ of \mathbb{T} is not minimal since $z = 1$ is the fixed point. Also, this is not uniformly stable. Indeed, by ergodicity of F_q the sequence $(F_q^n z)$ is dense in \mathbb{T} for a.e. z , while $F_q^n 1 = 1$ for all n .

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