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Zeros of the Riemann zeta-function and its universality

by

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In honour of Professors Vasili Ivanovich Bernik and Yuri Valentinovich Nesterenko on the occasion of their 70th birthdays

1. Introduction. The Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, has an interesting universality property which, roughly speaking, means that a wide class of analytic functions can be approximated by shifts $\zeta(s+i\tau)$ with real τ. That property was discovered by S. M. Voronin [26]. Let $0 < r < 1/4$, and let $f(s)$ be a continuous and non-vanishing function in the disc $|s| \le r$, and analytic in the interior of this disc. Voronin proved that, for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that

$$\max_{|s|\le r} |\zeta(s + 3/4 + i\tau) - f(s)| < \varepsilon.$$

Voronin's theorem turned out to be interesting for number-theorists. A. Reich, S. M. Gonek, B. Bagchi, A. Good and others proposed new methods for proving the universality, improved Voronin's theorem and extended the universality property to other zeta and L-functions. B. Bagchi [1] created a new original method based on probabilistic limit theorems for weakly convergent probability measures in the space of analytic functions. This method is comparatively simple, and is applicable for all zeta-functions defined by Dirichlet series or Euler products. In this paper, we will also use Bagchi's probabilistic approach as in [11].

Let $\mathcal{K}$ be the class of compact subsets of the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complements, and $H_0(K)$, $K \in \mathcal{K}$, be the class of continuous non-vanishing functions on K which are analytic in the interior

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of K. Then the modern version of the Voronin theorem states that if $K \in \mathcal{K}$ and $f \in H_0(K)$, then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\Big\} > 0. \tag{1.1}$$

Thus, the set of shifts $\zeta(\cdot + i\tau)$ approximating the function $f \in H_0(K)$ is infinite, and even has a positive lower density. The proof of the above statement can be found in [11].

In the above inequality, the shift τ can take arbitrary real values, and this type of universality is called *continuous*. If τ takes values from a certain discrete set, then the universality of ζ is called *discrete*. The discrete universality was proposed by A. Reich [21], and developed in [1], [22], [23], [3]–[5] and [13]. Arithmetic progressions are examples of simple discrete sets. Let $h > 0$ be a fixed number, and $K \in \mathcal{K}$ and $f \in H_0(K)$. Then, in a slightly different form, it was proved in [1] that, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon\Big\} > 0. \tag{1.2}$$

We observe that inequalities (1.1) and (1.2) are not related, though their proofs have a lot in common. In both cases, the uniform distribution modulo 1 of certain curves plays an important role. In the discrete case, additionally, Gallagher's lemma (Lemma 2.6 below) is applied.

The problem arises to describe the sequences $\{x_k\} \subset \mathbb{R}$ such that (1.2) is satisfied with x_k in place of k. The first step in this direction was made in [6] and [14], where it is proved that the sequence $x_k = k^\alpha$ with fixed $0 < \alpha < 1$ is suitable for the discrete universality of $\zeta(s)$ and of the Hurwitz zeta-function $\zeta(s,\alpha)$ for some classes of parameter α. For the same sequence x_k as in [6], a joint discrete universality theorem for Dirichlet L-functions $L(s,\chi)$ was obtained in [15]. An important extension of [6] and [15] was made by Ł. Pańkowski [19]. Assume that $\chi_1, \dots, \chi_r$ are Dirichlet characters, $\alpha_1, \dots, \alpha_r \in \mathbb{R}$, $a_1, \dots, a_r \in \mathbb{R}^+$ and $b_1, \dots, b_r$ are such that

$$b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \notin \mathbb{Z}, \\ (-\infty, 0] \cup (1,\infty) & \text{if } a_j \in \mathbb{N}, \end{cases}$$

and $a_j \ne a_k$ or $b_j \ne b_k$ if $k \ne j$. Moreover, let $K \in \mathcal{K}$ and $f_1, \dots, f_r \in H_0(K)$. Then, for every $\epsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \#\Big\{2 \le k \le N : \max_{1\le j\le r} \max_{s\in K} |L(s + i\alpha_j k^{a_j} \log^{b_j} k, \chi_j) - f_j(s)| < \epsilon\Big\} > 0.$$

Let $0 < \gamma_1 \le \gamma_2 \le \cdots$ be the imaginary parts of non-trivial zeros of the Riemann zeta-function. Our aim is to use the sequence $\{\gamma_k : k \in \mathbb{N}\}$ for the discrete universality of $\zeta(s)$.

Considering the pair correlation of zeros of $\zeta(s)$, H. L. Montgomery [17] conjectured the asymptotic relation

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ 2\pi\alpha_1/\log T\le\gamma-\gamma'\le 2\pi\alpha_2/\log T}} 1 \sim \left(\int\limits_{\alpha_1}^{\alpha_2}\left(1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right)du+\delta(\alpha_1,\alpha_2)\right)\frac{T}{2\pi}\log T \tag{1.3}$$

as $T\to\infty$. Here $\alpha_1<\alpha_2$ are fixed numbers, and $\delta(\alpha_1,\alpha_2)=1$ if $0\in[\alpha_1,\alpha_2]$, and $\delta(\alpha_1,\alpha_2)=0$ otherwise.

Remark 1.1. We observe that all statements of [17] are valid under RH, but the above conjecture is stated without RH. Therefore, we believe that the Montgomery conjecture may be independent of RH.

For our aims, a weaker conjecture is sufficient. We suppose that, as $T\to\infty$,

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ |\gamma-\gamma'|<c/\log T}} 1 \ll T\log T \tag{1.4}$$

with a certain constant $c>0$. Clearly, (1.4) follows from (1.3). Therefore, we will call (1.4) the *weak Montgomery conjecture*, and in all theorems we suppose that this conjecture is true.

Theorem 1.2. *Let $K\in\mathcal{K}$ and $f\in H_0(K)$. Then, for every $\varepsilon>0$ and $h>0$,*

$$\liminf_{N\to\infty}\frac{1}{N}\#\left\{1\le k\le N:\sup_{s\in K}|\zeta(s+i\gamma_k h)-f(s)|<\varepsilon\right\}>0. \tag{1.5}$$

Theorem 1.3. *Let $K\in\mathcal{K}$ and $f\in H_0(K)$. Then, for every $h>0$, the limit*

$$\lim_{N\to\infty}\frac{1}{N}\#\left\{1\le k\le N:\sup_{s\in K}|\zeta(s+i\gamma_k h)-f(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

Theorems 1.2 and 1.3 can be generalized to composite functions. Let $H(G)$ be the space of analytic functions on a region G equipped with the topology of uniform convergence on compacta. We consider some classes of operators $F:H(D)\to H(D)$.

Let $\beta>0$ be a fixed number. We say that the operator F belongs to the class $\operatorname{Lip}(\beta)$ if:

(i) for every $K\in\mathcal{K}$ and every polynomial p, there exists a function $g\in F^{-1}\{p\}$ such that $g(s)\ne 0$ for all $s\in K$;

(ii) for every $K \in \mathcal{K}$, there exist $K_1 \in \mathcal{K}$ and a constant $c > 0$ such that

$$\sup_{s \in K} |F(g_1(s)) - F(g_2(s))| \le c \sup_{s \in K_1} |g_1(s) - g_2(s)|^\beta$$

for all $g_1, g_2 \in H(D)$.

Let $H(K)$, $K \in \mathcal{K}$, be the class of continuous functions on K which are analytic in the interior of K.

THEOREM 1.4. *Suppose that $F \in \mathrm{Lip}(\beta)$. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon, h > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N} \#\Big\{1 \le k \le N : \sup_{s \in K} |F(\zeta(s + i\gamma_k h)) - f(s)| < \varepsilon\Big\} > 0.$$

THEOREM 1.5. *Suppose that $F \in \mathrm{Lip}(\beta)$. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $h > 0$, the limit*

$$\lim_{N \to \infty} \frac{1}{N} \#\Big\{1 \le k \le N : \sup_{s \in K} |F(\zeta(s + i\gamma_k h)) - f(s)| < \varepsilon\Big\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Define

$$S = \{g \in H(D) : \text{either } g(s) \ne 0 \text{ for all } s \in D, \text{ or } g(s) \equiv 0\}.$$

THEOREM 1.6. *Suppose that F is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then the conclusion of Theorem 1.4 is true.*

THEOREM 1.7. *Under the hypotheses of Theorem 1.6, the conclusion of Theorem 1.5 is true.*

The set G in Theorems 1.6 and 1.7 can be replaced by a single polynomial. Thus we have the following statements.

THEOREM 1.8. *Suppose that F is a continuous operator such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then the conclusion of Theorem 1.4 is true.*

THEOREM 1.9. *Under the hypotheses of Theorem 1.8, the conclusion of Theorem 1.5 is true.*

Now let $a_1, \ldots, a_r$ be distinct complex numbers. For the operator $F : H(D) \to H(D)$, define the set

$$H_{a_1,\ldots,a_r;F}(D) = \{g \in H(D) : g(s) \ne a_j \text{ for all } s \in D \text{ and } j = 1, \ldots, r\} \cup \{F(0)\}.$$

THEOREM 1.10. *Suppose that F is a continuous operator such that $F(S) \supset H_{a_1,\ldots,a_r;F}(D)$. For $r = 1$, let $K \in \mathcal{K}$ and $f \in H(K)$ with $f(s) \ne a_1$*

for all $s \in K$. *For* $r \geq 2$, *let* K *be an arbitrary compact subset of* D, *and* $f \in H_{a_1,\ldots,a_r;F}(D)$. *Then the conclusion of Theorem* 1.4 *is true.*

THEOREM 1.11. *Under the hypotheses of Theorem* 1.10, *the conclusion of Theorem* 1.5 *is true.*

The proofs of all these universality theorems are based on probabilistic limit theorems on weakly convergent probability measures in the space $H(D)$. Theorems of that type will be given in the next section.

2. Limit theorems. First we recall the definition of uniformly distributed sequences of real numbers, and some of their properties. A sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called *uniformly distributed modulo* 1 if, for each interval $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{x_k\}) = \operatorname{length} I,$$

where χ_I is the indicator function of I, and $\{u\}$ denotes the fractional part of $u \in \mathbb{R}$.

In the theory of uniformly distributed sequences modulo 1, the *Weyl criterion* plays an important role.

LEMMA 2.1. *A sequence* $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ *is uniformly distributed modulo* 1 *if and only if, for all* $m \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

The proof can be found, for example, in [10].

Let, as above, $\{\gamma_k : k \in \mathbb{N}\}$ be a sequence of imaginary parts of non-trivial zeros of $\zeta(s)$. H. Rademacher [20], under the Riemann hypothesis, proved that the sequence $\{\gamma_k\}$ is uniformly distributed modulo 1, and P. D. T. A. Elliott [7] and E. Hlawka [9] independently removed the requirement of RH. However, we need a bit more.

LEMMA 2.2. *The sequence* $\{a\gamma_k\}$ *with* $a \neq 0$ *is uniformly distributed modulo* 1.

Proof. Let $N(T)$ denote the number of zeros $\beta + i\gamma$ of ζ with $0 < \gamma \leq T$. Then, in [24], it was shown that, for every positive $x \neq 1$,

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} x^{1/2+i\gamma} \ll \frac{1}{\log\log T}.$$

Hence, taking $x = e^{2\pi a m}$, $m \in \mathbb{Z} \setminus \{0\}$, we find that

$$\lim_{T\to\infty} \frac{1}{N(T)} \sum_{0<\gamma\leq T} e^{2\pi a\gamma m i} = 0.$$

Therefore, by Lemma 2.1, the sequence $\{a\gamma_k\}$, $a \neq 0$, is uniformly distributed modulo 1. ∎

Now, as usual in limit theorems for Dirichlet series, we consider a certain topological structure. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all primes p. By the classical Tikhonov theorem, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(X)$ denotes the Borel σ-field of the space X), the probability Haar measure m_H can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the circle γ_p. Now on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, an $H(D)$-valued random element $\zeta(s, \omega)$ can be defined by the formula

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1} = \sum_{m=1}^{\infty} \frac{\omega(m)}{m^s},$$

where $\omega(p)$ is extended to the set $\mathbb{N}$ by

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad \omega \in \Omega,$$

and the infinite product and series are uniformly convergent on compact subsets of D for almost all $\omega \in \Omega$. Let P_ζ be the distribution of $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega) \in A), \qquad A \in \mathcal{B}(H(D)).$$

The aim of this section is the following limit theorem. For $A \in \mathcal{B}(H(D))$ and $h > 0$, let

$$P_N(A) = \frac{1}{N} \#\{1 \leq k \leq N : \zeta(s + i\gamma_k h) \in A\}.$$

THEOREM 2.3. *P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the set S.*

The proof of Theorem 2.3 is divided into several lemmas.

For $A \in \mathcal{B}(\Omega)$, define

$$Q_N(A) = \frac{1}{N} \#\{1 \leq k \leq N : (p^{-i\gamma_k h} : p \in \mathbb{P}) \in A\},$$

where $\mathbb{P}$ is the set of all prime numbers.

LEMMA 2.4. *Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. We consider the Fourier transform $g_N(\underline{k})$, $\underline{k} = (k_2, k_3, \dots)$, of Q_N. It is well known that

$$g_N(\underline{k}) = \int_{\Omega} \prod_p \omega^{k_p}(p)\, dQ_N = \frac{1}{N}\sum_{k=1}^{N}\prod_p p^{-ik_p\gamma_k h},$$

where only a finite number of integers k_p are distinct from zero. Thus,

$$g_N(\underline{k}) = \frac{1}{N}\sum_{k=1}^{N} \exp\Big\{-ih\gamma_k \sum_p k_p \log p\Big\}. \tag{2.1}$$

Clearly,

$$g_N(\underline{0}) = 1. \tag{2.2}$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers, we have

$$h\sum_p k_p \log p \neq 0$$

for $\underline{k} \neq \underline{0}$. Hence, in view of Lemma 2.2, the sequence

$$\Big\{-\frac{h}{2\pi}\gamma_k \sum_p k_p \log p\Big\}$$

is also uniformly distributed modulo 1 in the case $\underline{k} \neq \underline{0}$. Therefore, by Lemma 2.1 and (2.1),

$$\lim_{N\to\infty} g_N(\underline{k}) = 0$$

for $\underline{k} \neq \underline{0}$. Now (2.2) shows that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Since the Fourier transform g_N of Q_N converges to that of the measure m_H, the lemma follows by a continuity theorem for probability measures on compact topological groups (see, for example, [8]). ■

Lemma 2.4 implies a limit theorem for a certain function given by absolutely convergent Dirichlet series.

Let $\sigma_0 > 1/2$ be a fixed number. For $m, n \in \mathbb{N}$, we set

$$v_n(m) = \exp\{-(m/n)^{\sigma_0}\},$$

and define

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}.$$

It is well known [11] that the above series is absolutely convergent for $\sigma > 1/2$.

For $A \in \mathcal{B}(H(D))$, define

$$T_{N,n}(A) = \frac{1}{N}\#\{1 \le k \le N : \zeta_n(s + i\gamma_k h) \in A\}.$$

Moreover, let $u_n : \Omega \to H(D)$ be given by the formula

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{v_n(m)\omega(m)}{m^s}.$$

Since the latter series is absolutely convergent for $\sigma > 1/2$, the function u_n is continuous.

Lemma 2.5. *$T_{N,n}$ converges weakly to the measure $T_n = m_H u_n^{-1}$ as $N \to \infty$, where*

$$T_n(A) = m_H(u_n^{-1}A), \quad A \in \mathcal{B}(H(D)).$$

Proof. This is a corollary of Lemma 2.4, the continuity of u_n and properties of weakly convergent probability measures under continuous mappings [2, Theorem 5.1]. ∎

The next problem is related to approximation of ζ by ζ_n. For this, we need estimates for discrete mean values of ζ, and now we will use the weak Montgomery conjecture. It is known [25] that

$$\gamma_n \sim 2\pi n/\log n$$

as $n \to \infty$. Thus,

$$\gamma_n \ll n/\log n. \tag{2.3}$$

Now we state the Gallagher lemma which connects discrete and continuous mean squares of certain continuous functions.

Lemma 2.6 (Gallagher). *Let T_0 and $T \geq \delta > 0$ be real numbers, and $\mathcal{T}$ be a finite set in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$. Define*

$$N_\delta(x) = \sum_{\substack{t\in\mathcal{T}\\ |t-x|<\delta}} 1,$$

and let S be a complex-valued continuous function on $[T_0, T+T_0]$ having a continuous derivative on $(T_0, T+T_0)$. Then

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta}\int_{T_0}^{T_0+T} |S(x)|^2\,dx + \Big(\int_{T_0}^{T_0+T} |S(x)|^2\,dx \int_{T_0}^{T_0+T} |S'(x)|^2\,dx\Big)^{1/2}.$$

The proof can be found in [18, Lemma 1.4].

For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if K is a compact subset of D, then $K \subset K_l$ for some $l \in \mathbb{N}$. Then ρ is a metric on $H(D)$ inducing the topology of uniform convergence on compacta.

LEMMA 2.7. *We have*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=1}^{N} \rho\big(\zeta(\cdot + i\gamma_k h), \zeta_n(\cdot + i\gamma_k h)\big) = 0.$$

Proof. We start with estimation of the discrete mean

$$\sum_{k=1}^{N} |\zeta(\sigma + i\gamma_k h)|$$

for $\sigma \in (1/2, 1)$. For this, we will apply the weak Montgomery conjecture and Lemma 2.6.

By the definition of $N_\delta(x)$, (2.3) and (1.4), taking $\delta = c_2/\log\frac{c_1 N}{\log N}$ and $c_1, c_2 > 0$, we find that

$$\sum_{k=1}^{N} N_\delta(\gamma_k) = \sum_{k=1}^{N} \sum_{\substack{l=1 \\ |\gamma_k - \gamma_l| < \delta}}^{N} 1 \ll \sum_{\substack{0<\gamma,\gamma' \le c_1 N/\log N \\ |\gamma-\gamma'| < \frac{c_2}{\log c_1 N/\log N}}} 1 \ll N. \tag{2.4}$$

It is well known that, for fixed $\sigma \in (1/2, 1)$,

$$\int_1^T |\zeta(\sigma + it)|^2 \, dt \ll T, \qquad \int_1^T |\zeta'(\sigma + it)|^2 \, dt \ll T.$$

Therefore, an application of Lemma 2.6 and (2.4) give the estimate

$$\begin{aligned} \sum_{k=1}^{N} |\zeta(\sigma + i\gamma_k h)| &= \sum_{k=1}^{N} \sqrt{N_\delta(\gamma_k h) N_\delta^{-1}(\gamma_k h)} \; |\zeta(\sigma + i\gamma_k h)| \\ &\le \Big(\sum_{k=1}^{N} N_\delta(\gamma_k h) \sum_{k=1}^{N} N_\delta^{-1}(\gamma_k h) |\zeta(\sigma + i\gamma_k h)|^2 \Big)^{1/2} \\ &\ll \sqrt{N} \Big(\log N \int_{\gamma_1}^{\hat{c}N/\log N} |\zeta(\sigma + it)|^2 \, dt \\ &\quad + \Big(\int_{\gamma_1}^{\hat{c}N/\log N} |\zeta(\sigma + it)|^2 \, dt \int_{\gamma_1}^{\hat{c}N/\log N} |\zeta'(\sigma + it)|^2 \, dt \Big)^{1/2} \Big)^{1/2} \\ &\ll N, \end{aligned} \tag{2.5}$$

where $\hat{c} = \hat{c}(h)$ is a certain positive constant.

Let σ_0 be as in the definition of ζ_n, and

$$l_n(s) = \frac{s}{\sigma_0}\Gamma\left(\frac{s}{\sigma_0}\right)n^s,$$

where Γ is the gamma-function. Then it is known [11] that

$$\zeta_n(s) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \zeta(s+z)l_n(z)\,\frac{dz}{z}. \tag{2.6}$$

Let K be an arbitrary compact subset of D. Then (2.6) and the residue theorem give the estimate

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+i\gamma_k h) - \zeta_n(s+i\gamma_k h)| \\ \ll \int_{-\infty}^{\infty} |l_n(\sigma_1+i\tau)|\frac{1}{N}\sum_{k=1}^{N}|\zeta(\sigma+i\gamma_k h+it+i\tau)|\,d\tau,\end{aligned} \tag{2.7}$$

where $\sigma_1 < 0$, $\sigma \in (1/2, 1)$, and t is bounded by a constant depending on K. In view of (2.5), for such t and $\sigma \in (1/2, 1)$,

$$\frac{1}{N}\sum_{k=1}^{N}|\zeta(\sigma+i\gamma_k h+it+i\tau)| \ll 1+|\tau|.$$

Therefore, the definition of $l_n(s)$ and (2.7) show that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|\zeta(s+i\gamma_k h) - \zeta_n(s+i\gamma_k h)| = 0.$$

This and the definition of the metric ρ prove the lemma. ■

We recall that a family $\{P\}$ of probability measures on $(X, \mathcal{B}(X))$ is *tight* if, for every $\varepsilon > 0$, there exists a compact subset K of X such that

$$P(K) > 1-\varepsilon \quad \text{for all } P \in \{P\}.$$

Let T_n be as in Lemma 2.5.

LEMMA 2.8. *The family $\{T_n : n \in \mathbb{N}\}$ of probability mesures is tight.*

Proof. On a certain probability space $(\hat{\Omega}, \mathcal{A}, \mu)$, define a random variable θ_N by the formula

$$\mu(\theta_N = \gamma_k h) = 1/N, \qquad k = 1, \dots, N.$$

Consider the $H(D)$-valued random element

$$X_{N,n} = X_{N,n}(s) = \zeta_n(s+i\theta_N).$$

Let X_n be an $H(D)$-valued random element with distribution T_n. Then, denoting by $\xrightarrow{\mathcal{D}}$ convergence in distribution, we deduce from Lemma 2.5

that

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n. \tag{2.8}$$

Since the series for ζ_n is absolutely convergent for $\sigma > 1/2$, we see, for $\sigma > 1/2$, that

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma+it)|^2\, dt = \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{v_n^2(m)}{m^{2\sigma}} \le \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \le C < \infty.$$

Lemma 2.6, similarly to the proof of Lemma 2.7, now yields

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\zeta_n(s+i\gamma_k h)| \le C_1 < \infty.$$

Hence, for compact sets K_l from the definition of the metric ρ,

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K_l} |\zeta_n(s+i\gamma_k h)| \le R_l < \infty.$$

Taking an arbitrary $\varepsilon > 0$ and setting $M_l = M_l(\varepsilon) = 2^l R_l \varepsilon^{-1}$, from (2.8) we now deduce that, for all $l \in \mathbb{N}$,

$$\mu\Big(\sup_{s\in K_l} |X_n(s)| > M_l\Big) \le \frac{\varepsilon}{2^l}.$$

Therefore, for

$$K = K(\varepsilon) = \Big\{g \in H(D) : \sup_{s\in K_l} |g(s)| \le M_l,\ l \in \mathbb{N}\Big\},$$

we obtain

$$\mu(X_n \in K) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}$, and the definition of X_n proves the lemma. ∎

Proof of Theorem 2.3. By Lemma 2.8 and the Prokhorov theorem [2, Theorem 6.1], the family $\{T_n : n \in \mathbb{N}\}$ is relatively compact. Thus, there exists a subsequence T_{n_r} that converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$. In other words,

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{2.9}$$

Define an $H(D)$-valued random element Y_N by

$$Y_N(s) = \zeta(s + i\theta_N).$$

Then Lemma 2.7 implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\big(\rho(Y_N(\cdot), X_{N,n}(\cdot)) \ge \varepsilon\big) = 0.$$

Now (2.8), (2.9) and [2, Theorem 4.2] show that

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P,$$

hence P_N converges weakly to P as $N \to \infty$.

The latter relation proves that the measure P is independent of the sequence T_{n_r}. Thus,

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

or T_n converges weakly to P as $n \to \infty$. However, it is known [1], [11] that

$$\frac{1}{T}\,\mathrm{meas}\{\tau \in [0,T] : \zeta(\cdot + i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

also converges weakly to the limit measure P of T_n as $T \to \infty$, and that $P = P_\zeta$. Therefore, P_N also converges weakly to P_ζ as $N \to \infty$. Moreover, it is known [1], [11] that the support of P_ζ is S. ∎

LEMMA 2.9. *Suppose that $F : H(D) \to H(D)$ is a continuous operator. Then*

$$P_{N,F} := \frac{1}{N}\#\{1 \le k \le N : F(\zeta(\cdot + i\gamma_k h)) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_\zeta F^{-1}$ as $N \to \infty$.

Proof. This follows from Theorem 2.3, continuity of F and [2, Theorem 5.1]. ∎

3. Proofs of universality. We start with some lemmas which are usually needed in proving universality.

LEMMA 3.1. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and let f be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial p such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

This is a version of the Mergelyan theorem [16].

Let ∂A denote the boundary of the set A.

LEMMA 3.2. *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(X, \mathcal{B}(X))$. Then each of the following assertions is equivalent to weak convergence of P_n to P as $n \to \infty$:*

(i) $\liminf_{n\to\infty} P_n(G) \ge P(G)$ *for all open sets $G \subset X$;*
(ii) $\lim_{n\to\infty} P_n(A) = P(A)$ *for all continuity sets A of P, i.e., $P(\partial A) = 0$.*

The lemma is a part of [2, Theorem 2.1].

Proof of Theorem 1.2. By Lemma 3.1, there exists a polynomial p such that

$$\sup_{s\in K} |f(s) - e^{p(s)}| < \varepsilon/2. \tag{3.1}$$

Then, in view of Theorem 2.3, the set

$$\hat{G}_\varepsilon := \Big\{g \in H(D) : \sup_{s\in K} |g(s) - e^{p(s)}| < \varepsilon/2\Big\}$$

is an open neighbourhood of the element $e^{p(\cdot)}$ of the support of P_ζ. Therefore, Theorem 2.3 and Lemma 3.2 show that

$$\liminf_{N\to\infty} P_N(\hat{G}_\varepsilon) \geq P_\zeta(\hat{G}_\varepsilon) > 0.$$

Now (3.1) and the definition of $\hat{G}_\varepsilon$ lead to the assertion of the theorem. ∎

Proof of Theorem 1.3. For $\varepsilon > 0$, let

$$G_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon\Big\}.$$

Then

$$\partial G_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon\Big\}.$$

Therefore, $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ for distinct positive $\varepsilon_1, \varepsilon_2$. Hence, $P_\zeta(\partial G_\varepsilon) > 0$ for at most a countable set of values of ε. Thus, G_ε is a continuity set of P_ζ for all but at most countably many $\varepsilon > 0$. Therefore, in view of Theorem 2.3, Lemma 3.2 and the definition of G_ε,

$$\lim_{N\to\infty} \frac{1}{N} \#\Big\{1 \leq k \leq N : \sup_{s\in K} |\zeta(s + i\gamma_k h) - f(s)| < \varepsilon\Big\} = P_\zeta(G_\varepsilon) \tag{3.2}$$

for all but at most countably many $\varepsilon > 0$.

Let p be a polynomial satisfying (3.1), and $\hat{G}_\varepsilon$ be as defined in the proof of Theorem 1.2. Then $P_\zeta(\hat{G}_\varepsilon) > 0$. Moreover, in view of (3.1), we have $\hat{G}_\varepsilon \subset G_\varepsilon$. Hence, $P_\zeta(G_\varepsilon) \geq P_\zeta(\hat{G}_\varepsilon) > 0$, and (3.2) proves the theorem. ∎

Proof of Theorem 1.4. The conclusion is a consequence of Theorem 1.2, Lemma 3.1 and the definition of $\mathrm{Lip}(\beta)$. The details in the case of continuous universality can be found in [12]. ∎

Proof of Theorem 1.5. We argue similarly to the proof of Theorem 1.4, applying Theorem 1.3 in place of Theorem 1.2. ∎

Proof of Theorem 1.6. First we observe that, under the hypotheses of the theorem, the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$. Indeed, let $g \in H(D)$, and G be its open neighbourhood. The continuity of F implies that $F^{-1}G$ is open, too. Therefore, by hypothesis, $F^{-1}G$ is an open neighbourhood of a certain element of S. Hence, by Theorem 2.3, we have $P_\zeta(F^{-1}G) > 0$. Therefore, $P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$. Since g and G are arbitrary, this shows that the support of $P_\zeta F^{-1}$ is the whole $H(D)$.

The rest of the proof repeats that of Theorem 1.2. By Lemma 3.1, there exists a polynomial p such that

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon/2. \tag{3.3}$$

Then, by the above remark, the set

$$G := \Big\{g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \varepsilon/2\Big\}$$

is an open neighbourhood of the element p of the support of the measure $P_\zeta F^{-1}$. Therefore, Lemmas 2.9 and 3.2 imply

$$\liminf_{N\to\infty} P_{N,F}(G) \geq P_\zeta F^{-1}(G) > 0.$$

Now, (3.3) and the definition of G prove the theorem. ∎

Proof of Theorem 1.7. We apply, with obvious changes, arguments analogous to those used in the proof of Theorem 1.3. ∎

Proof of Theorem 1.8. We observe that the hypothesis $(F^{-1}\{p\})\cap S \neq \emptyset$ for every polynomial p implies that $(F^{-1}G)\cap S \neq \emptyset$ for every non-empty open set $G \subset H(D)$. Indeed, let G be such a set. It is well known that approximation in the space $H(D)$ coincides with uniform approximation on compact sets with connected complements. Therefore, by Lemma 3.1, there exists a polynomial p such that $p \in G$. Since $(F^{-1}\{p\})\cap S \neq \emptyset$, we also have $(F^{-1}G)\cap S \neq \emptyset$. Therefore, the conclusion follows from Theorem 1.6. ∎

Proof of Theorem 1.9. By the same remark as in the proof of Theorem 1.8, the conclusion is a consequence of Theorem 1.7. ∎

We notice that, in the cases of Theorems 1.8 and 1.9, it is more convenient to replace the space $H(D)$ by $H(D_V)$, where $D_V = \{s \in \mathbb{C} : 1/2 < \sigma < 1, |t| < V\}$ with $V > 0$, because non-vanishing of polynomials in a bounded region can be controlled by their constant terms. This remark can be applied, for example, to the operator $F(g) = g^{(k)}$, $k \in \mathbb{N}$, where $g^{(k)}$ means the kth derivative of g.

Proof of Theorem 1.10. As in the proofs of the previous theorems, we start with the support S_F of the measure $P_\zeta F^{-1}$. We will prove that S_F contains the closure of the set $H_{a_1,\dots,a_r;F}(D)$. Since $F(S) \supset H_{a_1,\dots,a_r;F}(D)$, for each $g \in H_{a_1,\dots,a_r;F}(D)$ there exists $g_1 \in S$ such that $F(g_1) = g$. Let G be an open neighbourhood of g. Then the open set $F^{-1}G$ is an open neighbourhood of a certain element of S. Therefore, in virtue of Theorem 2.3, we have $P_\zeta(F^{-1}G) > 0$. Hence, $P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$. Consequently, $g \in S_F$. Thus, $S_F \supset H_{a_1,\dots,a_r;F}(D)$. However, S_F is a closed set, so it contains the closure of $H_{a_1,\dots,a_r;F}(D)$.

We consider the cases $r = 1$ and $r \geq 2$ separately.

Let $r = 1$. Then, by Lemma 3.1, there exists a polynomial p such that

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon/4. \tag{3.4}$$

Since $f(s) \neq a_1$ for all $s \in K$, in view of (3.4) we have $p(s) \neq 0$ for all $s \in K$ as well if $\varepsilon > 0$ is small enough. Therefore, there exists a continuous branch of $\log(p(s) - a_1)$ which is analytic in the interior of K. Applying Lemma 3.1 once more, we find a polynomial p_1 such that

$$\sup_{s\in K} |p(s) - e^{p_1(s)}| < \varepsilon/4. \tag{3.5}$$

Let, for brevity, $f_1(s) = e^{p_1(s)} + a_1$. Then $f_1 \in H(D)$ and $f_1(s) \neq a_1$ for all $s \in K$. Therefore, by the beginning of the proof, $f_1 \in S_F$. Setting

$$\mathcal{G}_1 = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f_1(s)| < \varepsilon/2 \Big\},$$

we have $P_\zeta F^{-1}(\mathcal{G}_1) > 0$. Therefore, combining Lemmas 2.9 and 3.2, inequalities (3.4) and (3.5), and the definition of $\mathcal{G}_1$ gives the assertion of the theorem in the case $r = 1$.

Let $r \geq 2$. We set

$$\mathcal{G}_2 = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \Big\}.$$

Since $f \in H_{a_1,\dots,a_r;F}(D)$, we have $f \in S_F$. Thus, $P_\zeta(\mathcal{G}_2) > 0$, and it remains to use Lemmas 2.9 and 3.2. ■

Proof of Theorem 1.11. We use the fact that S_F contains the closure of $H_{a_1,\dots,a_r;F}(D)$, and follow the proofs of Theorems 1.3 and 1.10 with obvious changes. ■

For example, in Theorems 1.10 and 1.11 we can take $F(g) = g^n$, $n \in \mathbb{N}$, ($r = 1$, $a_1 = 0$), $F(g) = \sin g$ ($r = 2$, $a_1 = 1$, $a_2 = -1$), etc.

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