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On the growth of a holomorphic curve with deficient moving hyperplanes

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Abstract. We generalize some results on the growth of meromorphic functions or ν-valued algebroid functions to holomorphic curves with deficient moving hyperplanes.

1. Notation and results. We denote the n-dimensional complex projective space by $P^n(\mathbb{C})$. Let

$$f = [f_0 : f_1 : \cdots : f_n] : \mathbb{C} \to P^n(\mathbb{C})$$

be a holomorphic curve and $\mathbf{f} = (f_0, f_1, \ldots, f_n) : \mathbb{C} \to \mathbb{C}^{n+1} \setminus \{0\}$ be its *reduced representation*, i.e., $f_0(z), f_1(z), \ldots, f_n(z)$ have no common zeros.

Cartan's characteristic function $T(r, f)$ of f is defined by

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\|\, d\theta - \log\|\mathbf{f}(0)\|, \tag{1.1}$$

where $\mathbf{f}$ is a reduced representation of f and $\|\mathbf{f}(z)\| = (\sum_{j=0}^{n} |f_j(z)|^2)^{1/2}$. The *order* and *lower order* of f are defined as

$$\lambda(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \mu(f) := \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

By a *moving hyperplane* H in $P^n(\mathbb{C})$, we mean

$$H = \{[x_0 : \cdots : x_n] \in P^n(\mathbb{C}) \mid a_0x_0 + \cdots + a_nx_n = 0\},$$

where $a_0, \ldots, a_n$ are entire functions without common zeros. So H is associated with a holomorphic map $a = [a_0 : \cdots : a_n] : \mathbb{C} \to P^n(\mathbb{C})$. Write $\mathbf{a} = (a_0, \ldots, a_n)$. Given a holomorphic map $f : \mathbb{C} \to P^n(\mathbb{C})$, we say that f

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and H are *free* if $\langle \mathbf{f}(z), \mathbf{a}(z)\rangle \not\equiv 0$, where

$$\langle \mathbf{f}(z), \mathbf{a}(z)\rangle = a_0(z)f_0(z) + a_1(z)f_1(z) + \cdots + a_n(z)f_n(z).$$

Let $n(r, H, f)$ denote the number of zeros (counting multiplicities) of the entire function

$$F(z) := \langle \mathbf{f}(z), \mathbf{a}(z)\rangle = \sum_{j=0}^{n} a_j(z)f_j(z) \not\equiv 0$$

in the disk $\{|z| \le r\}$. We denote by $N(r, H, f)$ the counting function of zeros of the entire function $F(z)$, that is, $N(r, H, f) = N(r, 0, F)$.

Under the assumption that $\langle \mathbf{f}(z), \mathbf{a}(z)\rangle \not\equiv 0$, we define the approximating function $m(r, H, f)$ of f with respect to H by

$$m(r, H, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\phi})\|\,\|\mathbf{a}(re^{i\phi})\|}{|\langle \mathbf{f}(re^{i\phi}), \mathbf{a}(re^{i\phi})\rangle|}\, d\phi.$$

Set

$$T(r, H, f) = m(r, H, f) + N(r, H, f).$$

Applying the Jensen formula to the entire function $F(z)$, we have

$$N(r, 0, F) = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\, d\theta - \log|F(0)|. \tag{1.2}$$

This gives

$$\begin{aligned} T(r, H, f) &= m(r, H, f) + N(r, H, f) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log(\|\mathbf{f}(re^{i\phi})\|\,\|\mathbf{a}(re^{i\phi})\|)\, d\phi - \log|\langle \mathbf{f}(0), \mathbf{a}(0)\rangle| \\ &= T(r, f) + T(r, a) + \log\frac{\|\mathbf{f}(0)\|\,\|\mathbf{a}(0)\|}{|\langle \mathbf{f}(0), \mathbf{a}(0)\rangle|}. \end{aligned} \tag{1.3}$$

We define the *Nevanlinna deficiency* $\delta(H) = \delta(H, f)$ of H with respect to f by

$$\delta(H, f) = \liminf_{r\to\infty} \frac{m(r, H, f)}{T(r, f)} = 1 + \limsup_{r\to\infty} \frac{T(r, a) - N(r, H, f)}{T(r, f)}. \tag{1.4}$$

The moving hyperplane H is said to be *Nevanlinna deficient* if $\delta(H, f) > 0$. We notice that if a satisfies

$$T(r, a) = o(T(r, f)), \quad r \to \infty, \tag{1.5}$$

then (1.4) is reduced to

$$\delta(H, f) = 1 - \limsup_{r\to\infty} \frac{N(r, H, f)}{T(r, f)}. \tag{1.6}$$

We define the *Valiron deficiency* $\varDelta(H) = \varDelta(H,f)$ of H with respect to f by

$$\varDelta(H,f) = \limsup_{r\to\infty} \frac{m(r,H,f)}{T(r,f)} = 1 + \liminf_{r\to\infty} \frac{T(r,a) - N(r,H,f)}{T(r,f)}. \tag{1.7}$$

The moving hyperplane H is *Valiron deficient* if $\varDelta(H,f) > 0$. We notice that if H (or a) satisfies $T(r,a) = o(T(r,f))$, then (1.7) is reduced to

$$\varDelta(H,f) = 1 - \liminf_{r\to\infty} \frac{N(r,H,f)}{T(r,f)}. \tag{1.8}$$

Following [P2], we define the *Petrenko deviation* $\beta(H) = \beta(H,f)$ of H with respect to f by

$$\beta(H,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)}, \tag{1.9}$$

where

$$\mathcal{L}(r,a,f) = \max_{|z|=r} \log \frac{\|\mathbf{f}(z)\|\,\|\mathbf{a}(z)\|}{|\langle \mathbf{f}(z), \mathbf{a}(z)\rangle|}.$$

For $W = [w_0 : \cdots : w_n] \in P^n(\mathbb{C})$, let

$$\omega = dd^c \log \|W\|^2.$$

Then ω is a well defined $(1,1)$ form on $P^n(\mathbb{C})$, called the *Fubini–Study form.* Define

$$A(r,f) = \int_{|z|\le r} f^*\omega, \qquad \mathcal{T}(r,f) = \int_0^r \frac{A(t,f)}{t}\,dt.$$

$\mathcal{T}(r,f)$ is called the *Ahlfors characteristic function.* In [R, p. 101] it is pointed out that $T(r,f) = \mathcal{T}(r,f)$.

Following [B], we define the *Bergweiler deviation* $b(H) = b(H,f)$ of H with respect to f by

$$b(H,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{rT'_-(r,f)} = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{A(r,f)}, \tag{1.10}$$

where $T'_-(r,f)$ is the left derivative of $T(r,f)$ at r.

If $T(r,a) = o(T(r,f))$, then $0 \le \delta(H,f) \le \varDelta(H,f) \le 1$ and $\delta(H,f) \le \beta(H,f)$.

Let $A = \{H^j \mid j = 1,\dots,q\}$ be $q \ge n+1$ moving hyperplanes in $P^n(\mathbb{C})$, associated with q holomorphic maps $a^j = [a_0^j : \cdots : a_n^j] : \mathbb{C} \to P^n(\mathbb{C})$. Suppose that $\mathbf{a}^j = (a_0^j, a_1^j, \dots, a_n^j)$ $(j = 1,\dots,q)$ are the reduced representations of a^j $(j = 1,\dots,q)$. If $\det\{a_i^{j_k};\ 0 \le i \le n,\ 1 \le k \le n+1\} \not\equiv 0$ for any $\{j_1,\dots,j_{n+1}\} \subset \{1,\dots,q\}$, then A is said to be *in general position.* A holomorphic curve f is said to be *nondegenerate* with respect to A if $F_j(z) = \sum_{i=0}^n a_i^j(z) f_i(z) \not\equiv 0$ for all $j = 1,\dots,q$.

Edrei and Fuchs [EF] proved that if a transcendental meromorphic function $f(z)$ has two Nevanlinna deficient values, then its lower order is positive. Mori [M] generalized this conclusion: if a transcendental meromorphic function $f(z)$ has two Nevanlinna deficient small functions, then the lower order is positive. Ozawa [O1, O2] proved that if an n-valued transcendental algebroid function $f(z)$ has $n+1$ deficient values, then the lower order is positive.

In this paper we will establish an analogous conclusion for holomorphic curves (Theorem 1.1). Since an algebroid function is a special case of a holomorphic curve, our results are more general.

Theorem 1.1. *Suppose that $f(z)$ is a holomorphic curve in $P^n(\mathbb{C})$ satisfying*

$$\lim_{r\to\infty} T(r,f)/\log r = \infty$$

with $n+1$ Nevanlinna deficient moving hyperplanes $A = \{H^j \mid j = 1,\dots, n+1\}$ (or $a^1,\dots,a^{n+1}$) satisfying $T(r,a^j) = o(T(r,f))$ $(j = 1,\dots,n+1)$, where A is in general position and f is nondegenerate with respect to A. Then the lower order of $f(z)$ is positive.

Lamzina [L] studied the relation between the Petrenko deviation and the Nevanlinna deficiency of an entire curve $\vec{G}(z)$ at a vector $\vec{a}$ and obtained the following theorem.

Theorem 1.2 ([L]). *If for an entire curve $\vec{G}(z)$ of lower order $\mu < 1$ we have $\beta(\vec{a}_p,\vec{G}) > 0$, then*

$$\beta(\vec{a}_{p-1},\vec{G}) \le \pi\mu\tan\frac{\pi\mu}{2}\Big(p - \sum_{k=1}^{p}\delta(\vec{a}_k,\vec{G})\Big),$$

where the vectors $\vec{a}_k$ $(k = 1,\dots,p)$ are such that $\beta(\vec{a}_k,\vec{G}) > 0$ $(k = 1,\dots,p)$ and

$$\beta(\vec{a}_{k+1},\vec{G}) \le \beta(\vec{a}_k,\vec{G}), \qquad k = 1,\dots,p-1.$$

However, Lamzina [L] did not study the deviation of $\vec{G}(z)$ at its small entire curves $\vec{a}_1(z),\dots,a_p(z)$. In this paper we assume that $A = \{H^j \mid j = 1,\dots,n+1\}$ are moving hyperplanes with respect to $f(z)$, and we will estimate the Bergweiler deviation $b(H^{n+1},f)$ for a holomorphic curve $f(z)$ by the Nevanlinna deficiencies $\delta(H^j,f)$ $(j = 1,\dots,n+1)$. The method here is different from the method in [L].

Theorem 1.3. *Let $f(z)$ be a holomorphic curve of lower order $0 < \mu < 1$ in $P^n(\mathbb{C})$, $A = \{H^j \mid j = 1,\dots,n+1\}$ (or $a^1,\dots,a^{n+1}$) be $n+1$ moving hyperplanes in general position with reduced representations $\boldsymbol{a}^j$ $(j = 1,\dots,n+1)$ and satisfying $\max_{|z|=r} |\log\|\boldsymbol{a}^j(z)\|| = o(A(r,f))$ $(j = 1,\dots, n+1)$ and $0 < b(H^{j+1},f) \le b(H^j,f)$ $(j = 1,\dots,n)$. Suppose that f is* *nondegenerate with respect to A. Then*

$$b(H^n, f) \le \pi \tan \frac{\pi\mu}{2}\Big(n+1-\sum_{j=1}^{n+1}\delta(H^j, f)\Big).$$

2. Some lemmas and preliminaries. In this section we give some lemmas which are used in the proofs of our theorems.

LEMMA 2.1 ([EF]). *Suppose $f(z)$ is a meromorphic function with roots $\{a_\mu\}$ and poles $\{b_\nu\}$, and $q \ge 0$ is an integer. Then for any $S, R > 0$ with $2S < R/2$, for any z with $2S \le |z| = r \le R/2$, we have*

$$\begin{aligned}(2.1)\qquad \log|f(z)| = \log\Big|\prod_{S<|a_\mu|<R} E\Big(\frac{z}{a_\mu}, q\Big)\Big| - \log\Big|\prod_{S<|b_\nu|<R} E\Big(\frac{z}{b_\nu}, q\Big)\Big| \\ + W(z) + O(\log|z|),\end{aligned}$$

where

$$E(u,q) = \begin{cases} (1-u)\exp(u + u^2/2 + \cdots + u^q/q), & q > 0, \\ 1-u, & q = 0, \end{cases}$$

$$|W(z)| \le \begin{cases} A[(r/S)^q T(2S, f) + (r/R)^{q+1} T(2R, f)], & q \ge 1, \\ A[\log(r/S) T(2S, f) + (r/R) T(2R, f)], & q = 0, \end{cases}$$

and A is a positive number independent of r, S, R, f.

In order to prove Theorem 1.3, we need a lemma from Petrenko [P1] to estimate the error term (see Lemma 2.3 below). The proof of Lemma 2.3 can be found in [F, P1]. There are several versions of Lemma 2.3, without essential differences. Our version comes from [MS]. First we give the definition of Pólya peaks of the second kind.

DEFINITION 2.2 ([MS]). Suppose that $T(r)$ is an increasing function of $r > r_0$. Then $\{r_m\}$ is called a sequence of *Pólya peaks of the second kind of order* μ for $T(r)$ if $r_m \to \infty$ and there are sequences $K_m \to \infty$ and $\epsilon_m \to 0$ such that

$$T(r) > (1-\epsilon_m)(r/r_m)^\mu T(r_m) \qquad (r_m/K_m < r < K_m r_m).$$

LEMMA 2.3 ([MS]). *Let $T(r)$ be a real function with finite lower order μ, let $\{S_m\}$ and $\{R_m\}$ be sequences of real numbers such that*

$$\lim_{m\to\infty} S_m = \lim_{m\to\infty} R_m = \lim_{m\to\infty} R_m/S_m = \infty$$

and $\{4S_m\}$, $\{4R_m\}$ are sequences of Pólya peaks of the second kind of order μ for $T(r)$. Then for any $\varepsilon > 0$, there exists a number $m_0(\varepsilon) > 0$ such that for *each $m > m_0$,*

$$T(4S_m)S_m^{-\mu} + T(4R_m)R_m^{-\mu} < \varepsilon \int\limits_{2S_m}^{R_m/12} T(t)t^{-\mu-1}\,dt.$$

By modifying [Y, Lemma 2], we can obtain the following lemma.

Lemma 2.4 ([Y]). *Suppose that $f : \mathbb{C} \to P^n(\mathbb{C})$ is a holomorphic curve with reduced representation $\mathbf{f}$ and $A = \{H^j \mid j = 1, \ldots, n+1\}$ are $n+1$ moving hyperplanes associated with holomorphic maps a^j $(j = 1, \ldots, n+1)$ with reduced representations $\mathbf{a}^j$ $(j = 1, \ldots, n+1)$. Then*

$$\begin{aligned} T(r, F_i/F_{n+1}) - \max_{1\le j\le n+1} T(r, a_j) \\ \le T(r, f) \le \sum_{i=0}^{n} T(r, F_i/F_{n+1}) + \max_{1\le j\le n+1} T(r, a_j). \end{aligned}$$

3. Proof of Theorem 1.1.

Set

$$F_j(z) := \langle \mathbf{f}(z), \mathbf{a}^j(z)\rangle = \sum_{i=0}^{n} f_i(z)a_i^j(z) \qquad (j = 1, \ldots, n+1). \tag{3.1}$$

For the entire function $F_j(z)$, Edrei and Fuchs [EF] proved that

$$\log|F_j(z)| \le \frac{4}{\sigma-1} m(\sigma r, F_j) + N(\sigma r, 0, F_j) + O(\log r), \qquad \sigma > 1,\ r > 2.$$

Thus for $\sigma > 1, r > 2$, we have

$$\begin{aligned} &\max_{1\le j\le n+1} \log|F_j(z)| \\ &\quad\le \frac{4}{\sigma-1} \max_{1\le j\le n+1} m(\sigma r, F_j) + \max_{1\le j\le n+1} N(\sigma r, 0, F_j) + O(\log r). \end{aligned} \tag{3.2}$$

Set

$$F(z) = \max_{1\le j\le n+1} |F_j(z)|.$$

By the definition of $F_j(z)$, it follows that

$$\begin{aligned} \frac{1}{2\pi}\int\limits_0^{2\pi} \log|F(re^{i\phi})|\,d\phi &= \frac{1}{2\pi}\int\limits_0^{2\pi} \max_{1\le j\le n+1} \log|F_j(re^{i\phi})|\,d\phi \\ &\le \frac{1}{2\pi}\int\limits_0^{2\pi} \log\|\mathbf{f}(re^{i\phi})\|\,d\phi + o(T(r, f)). \end{aligned} \tag{3.3}$$

On the other hand, we can obtain the components $f_i(z)$ $(0 \le i \le n)$ of $f(z)$ by solving the equation (3.1), and estimate $T(r,f)$ as follows:

$$\begin{aligned} T(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \max_{0\le i\le n} \log|f_i(re^{i\phi})|\,d\phi + O(1) \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\phi})|\,d\phi + o(T(r,f)). \end{aligned} \tag{3.4}$$

Applying (3.3) and (3.4) in (3.2), we obtain

$$\begin{aligned} T(r,f) - o(T(r,f)) \le{} & \frac{4}{\sigma-1}T(\sigma r,f) + o(T(\sigma r,f)) \\ & + \max_{1\le j\le n+1} N(\sigma r,0,F_j) + O(\log r). \end{aligned}$$

Pick $\max\{1-\delta(H^j,f)\} < c' < c < 1$. Then, for $r \ge r_0 > 2$,

$$\begin{aligned} T(r,f) - o(T(r,f)) \le{} & \frac{4}{\sigma-1}T(\sigma r,f) + o(T(\sigma r,f)) \\ & + c'T(\sigma r,f) + O(\log r). \end{aligned}$$

Thus, $T(r,f) \le AT(\sigma r,f)$ for $r \ge r^* \ge r_0$, where $A = 4/(\sigma-1)+c$. Take σ large enough that $A<1$. Fixing σ and A, we can derive the following.

Letting r satisfy $\sigma^k r^* \le r < \sigma^{k+1}r^*$, we have

$$\frac{T(\sigma^k r^*,f)}{T(r^*,f)} = \prod_{j=1}^{k} \frac{T(\sigma^j r^*,f)}{T(\sigma^{j-1}r^*,f)} > A^{-k}.$$

Moreover,

$$\frac{\log T(r,f)}{\log r} > \frac{\log T(\sigma^k r^*,f)}{\log \sigma^{k+1} r^*} > \frac{-k\log A + \log T(r^*,f)}{(k+1)\log\sigma + \log r^*}.$$

Taking the limit, we have

$$\liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} > \frac{-\log A}{\log \sigma} > 0.$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.3. Set

$$F_k(z) := \langle \mathbf{f}(z), \mathbf{a}^k(z)\rangle = \sum_{i=0}^{n} f_i(z)a_i^k(z).$$

Let $\theta_k = \theta_k(r)$ be the angle defined by the relation

$$\mathcal{L}(r,a^k,f) = \max_{|z|=r}\log\frac{\|\mathbf{f}(z)\|\,\|\mathbf{a}^k(z)\|}{|\langle \mathbf{f}(z),\mathbf{a}^k(z)\rangle|} = \log\frac{\|\mathbf{f}(re^{i\theta_k})\|\,\|\mathbf{a}^k(re^{i\theta_k})\|}{|F_k(re^{i\theta_k})|}.$$

For any $r > r_0$, we have

$$F_k(re^{i\theta_{n+1}}) = \sum_{i=0}^{n} f_i(re^{i\theta_{n+1}})a_i^k(re^{i\theta_{n+1}}), \qquad k = 1, \dots, n+1. \tag{4.1}$$

Hence $f_j(re^{i\theta_{n+1}}) = D_j/D$, where D is the coefficient determinant of the linear equations (4.1), and

$$D_j(re^{i\theta_{n+1}}) = \sum_{i=1}^{n+1}(-1)^{j+i+1}D_{ij}(re^{i\theta_{n+1}})F_i(re^{i\theta_{n+1}}),$$

where $D_{ij}(re^{i\theta_{n+1}})$ is the determinant obtained from D by deleting the ith row and the jth column. From Hadamard's inequality, we have

$$|D_{ij}(re^{i\theta_{n+1}})| \le \prod_{l=1,\, l\ne i}^{n+1} \|\mathbf{a}^l(re^{i\theta_{n+1}})\|,$$

so that

$$|f_j(re^{i\theta_{n+1}})| \le \frac{1}{|D|}\sum_{i=1}^{n+1}\Big(\prod_{l=1,\, l\ne i}^{n+1} \|\mathbf{a}^l(re^{i\theta_{n+1}})\|\Big)|F_i(re^{i\theta_{n+1}})|.$$

Therefore,

$$\begin{aligned}&\|\mathbf{f}(re^{i\theta_{n+1}})\| \\ &\quad\le \frac{(n+1)\sqrt{n+1}}{|D|}\max_{1\le i\le n+1}\Big(\prod_{l=1,\, l\ne i}^{n+1} \|\mathbf{a}^l(re^{i\theta_{n+1}})\|\Big)|F_i(re^{i\theta_{n+1}})|.\end{aligned} \tag{4.2}$$

Now, we obtain

$$\begin{aligned}&\|\mathbf{f}(re^{i\theta_{n+1}})\|\,\|\mathbf{a}^{n+1}(re^{i\theta_{n+1}})\| \\ &\quad\le \frac{(n+1)\sqrt{n+1}}{|D|}\max_{1\le i\le n+1}\Big(\prod_{l=1,\, l\ne i}^{n+1} \|\mathbf{a}^l(re^{i\theta_{n+1}})\|\Big) \\ &\qquad\times |F_i(re^{i\theta_{n+1}})|\cdot\|\mathbf{a}^{n+1}(re^{i\theta_{n+1}})\| \\ &\quad\le \frac{(n+1)\sqrt{n+1}}{|D|}\max_{1\le i\le n+1}\Big(\prod_{l=1,\, l\ne i}^{n+1} \|\mathbf{a}^l(re^{i\theta_{n+1}})\|\Big) \\ &\qquad\times \max_{1\le i\le n+1}|F_i(re^{i\theta_{n+1}})|\cdot\|\mathbf{a}^{n+1}(re^{i\theta_{n+1}})\| \\ &\quad= \frac{(n+1)\sqrt{n+1}}{|D|}\max_{1\le i\le n+1}\Big(\prod_{l=1,\, l\ne i}^{n+1} \|\mathbf{a}^l(re^{i\theta_{n+1}})\|\Big) \\ &\qquad\times \|\mathbf{a}^{n+1}(re^{i\theta_{n+1}})\|\cdot|F_{\nu_n}(re^{i\theta_{n+1}})|,\end{aligned} \tag{4.3}$$

where $\nu_n = \nu(n,r) \in \{1, \dots, n+1\}$.

Next, we will prove that $\nu(n,r) \neq n+1$. Assume the opposite. Then there exists a sequence $r_m \to \infty$ for which $\nu(n,r_m) = n+1$. Then from (4.3) we find that

$$\begin{aligned}\log \frac{\|\mathbf{f}(r_m e^{i\theta_{n+1}})\| \, \|\mathbf{a}^{n+1}(r_m e^{i\theta_{n+1}})\|}{|F_{n+1}(r_m e^{i\theta_{n+1}})|} \\ \leq \tfrac{3}{2}\log(n+1) + \log^+(1/|D|) + \log^+ \|\mathbf{a}^{n+1}(r_m e^{i\theta_{n+1}})\| \\ + \max_{1\leq i\leq n+1} \sum_{l=1,\, l\neq i}^{n+1} \log^+ \|\mathbf{a}^l(r_m e^{i\theta_{n+1}})\|.\end{aligned}$$

Noticing $|D|$ is a rational function of a_j^i, we have

$$\log \frac{\|\mathbf{f}(r_m e^{i\theta_{n+1}})\| \, \|\mathbf{a}^{n+1}(r_m e^{i\theta_{n+1}})\|}{|F_n(r_m e^{i\theta_{n+1}})|} = o(A(r_m,f)).$$

Thus $b(H^{n+1},f) = 0$, which contradicts our hypothesis.

Hence ν_n may assume only the values $1,\dots,n$. Therefore,

$$b(H^n,f) \leq b(H^{\nu_n},f). \tag{4.4}$$

In addition, from the definition of θ_{n+1},

$$\log \frac{\|\mathbf{f}(re^{i\theta_{n+1}})\| \, \|\mathbf{a}^{n+1}(re^{i\theta_{n+1}})\|}{|F_{n+1}(re^{i\theta_{n+1}})|} \geq \log \frac{\|\mathbf{f}(re^{i\theta_{\nu_n}})\| \, \|\mathbf{a}^{n+1}(re^{i\theta_{\nu_n}})\|}{|F_{n+1}(re^{i\theta_{\nu_n}})|}. \tag{4.5}$$

By the definition of $b(H^n,f)$ and (4.5), for $r > r_0(\varepsilon)$ it follows that

$$\begin{aligned}b(H^n,f)(1-\varepsilon)A(r,f) &\leq b(H^{\nu_n},f)(1-\varepsilon)A(r,f) \\ &\leq \log \frac{\|\mathbf{f}(re^{i\theta_{\nu_n}})\| \, \|\mathbf{a}^{\nu_n}(re^{i\theta_{\nu_n}})\|}{|F_{\nu_n}(re^{i\theta_{\nu_n}})|} \\ \leq \log \frac{\|\mathbf{f}(re^{i\theta_{n+1}})\| \, |F_{n+1}(re^{i\theta_{\nu_n}})|}{|F_{n+1}(re^{i\theta_{n+1}})| \, |F_{\nu_n}(re^{i\theta_{\nu_n}})|} &+ \log \|\mathbf{a}^{\nu_n}(re^{i\theta_{\nu_n}})\| + \log \|\mathbf{a}^n(re^{i\theta_{n+1}})\|.\end{aligned} \tag{4.6}$$

Combining (4.3) with (4.6), we obtain

$$\begin{aligned}&b(H^n,f)(1-\varepsilon)A(r,f) \\ &\leq \log \frac{|F_{\nu_n}(re^{i\theta_{n+1}})| \, |F_{n+1}(re^{i\theta_{\nu_n}})|}{|F_n(re^{i\theta_{n+1}})| \, |F_{\nu_n}(re^{i\theta_{\nu_n}})|} + \log \|\mathbf{a}^{\nu_n}(re^{i\theta_{\nu_n}})\| + \log \|\mathbf{a}^n(re^{i\theta_{n+1}})\| \\ &\quad + \tfrac{3}{2}\log(n+1) + \log(1/|D|) + \max_{1\leq i\leq n+1}\Big(\sum_{l=1,\, l\neq i}^{n+1} \log \|\mathbf{a}^l(re^{i\theta_{n+1}})\|\Big) \\ &\leq \log \frac{|F_{\nu_n}(re^{i\theta_{n+1}})| \, |F_{n+1}(re^{i\theta_{\nu_n}})|}{|F_{n+1}(re^{i\theta_{n+1}})| \, |F_{\nu_n}(re^{i\theta_{\nu_n}})|} + o(A(r,f)).\end{aligned} \tag{4.7}$$

Applying Lemma 2.1 to the meromorphic function $f(z) = F_{\nu_n}(z)/F_{n+1}(z)$

and $q = 0$, we get

$$\begin{aligned}\log\left|\frac{F_{\nu_n}(re^{i\theta_{n+1}})}{F_{n+1}(re^{i\theta_{n+1}})}\right| \le & \sum_{S<|c_m(\nu_n)|<R} \log\left|1-\frac{re^{i\theta_{n+1}}}{c_m(\nu_n)}\right| \\ & - \sum_{S<|c_m(n+1)|<R} \log\left|1-\frac{re^{i\theta_{n+1}}}{c_m(n+1)}\right| \\ & + K_1\left[\log\left(\frac{r}{S}\right)T\left(2S,\frac{F_{\nu_n}}{F_{n+1}}\right)+\left(\frac{r}{R}\right)T\left(2R,\frac{F_{\nu_n}}{F_{n+1}}\right)\right] \\ & + O(\log r),\end{aligned}$$

where $c_m(\nu_n)$ are the roots of the entire function $F_{\nu_n}(z)$, and $c_m(n+1)$ are the roots of $F_{n+1}(z)$. A similar estimate holds for

$$\log|F_{n+1}(re^{i\theta_{\nu_n}})/F_{\nu_n}(re^{i\theta_{\nu_n}})|.$$

From (4.7), it follows that

$$\begin{aligned}& b(H^n,f)(1-\varepsilon)A(r,f) \\ & \le \sum_{S<|c_m(\nu_n)|<R} \log\left|\frac{re^{i\theta_{n+1}}-c_m(\nu_n)}{re^{i\theta_{\nu_n}}-c_m(\nu_n)}\right| + \sum_{S<|c_m(n+1)|<R} \log\left|\frac{re^{i\theta_{\nu_n}}-c_m(n+1)}{re^{i\theta_n}-c_m(n+1)}\right| \\ & \quad + K_1\left[\log\left(\frac{r}{S}\right)T\left(2S,\frac{F_{\nu_n}}{F_{n+1}}\right)+\left(\frac{r}{R}\right)T\left(2R,\frac{F_{\nu_n}}{F_{n+1}}\right)\right] \\ & \quad + O(\log r) + o(A(r,f)),\end{aligned}$$

where $c_m(n+1)$ are the roots of $F_{n+1}(z)$. Applying Lemma 2.4, we get $T(r, F_{\nu_n}/F_{n+1}) \le T(r,f) + o(A(r,f))$. By noticing that $A(r,f) \le T(2r,f) + O(1)$, it follows that

$$\begin{aligned}b(H^n,f)(1-\varepsilon)A(r,f) \le & \sum_{k=1}^{n+1}\sum_{S<|c_m(k)|<R} \log\left|\frac{r+|c_m(k)|}{r-|c_m(k)|}\right| \\ & + 2K_1\left[\log\left(\frac{r}{S}\right)T(4S,f)+\left(\frac{r}{R}\right)T(4R,f)\right] + O(\log r) + o(A(r,f)).\end{aligned} \tag{4.8}$$

From the equality (see [F])

$$\int_0^\infty \log\left|\frac{t^\gamma+|b|^\gamma}{t^\gamma-|b|^\gamma}\right| t^{-p-1}dt = \begin{cases} \frac{\pi}{p}|b|^{-p}\tan\frac{\pi p}{2\gamma}, & p \ne 0, \\ \frac{\pi^2}{2\gamma}, & p = 0, \end{cases} \tag{4.9}$$

we find

$$\int_{2S}^{R/2} r^{-\mu-1}\log\left|\frac{r+|c_m(k)|}{r-|c_m(k)|}\right| dr \le \frac{\pi}{\mu}\,\frac{1}{|c_m(k)|^\mu}\tan\frac{\pi\mu}{2} \tag{4.10}$$

for $0 < \mu < 1$. Summing over (4.10) and applying Stieltjes integration, we find that

$$\begin{aligned}
(4.11)\qquad \sum_{S<|c_m(k)|<R} \int_{2S}^{R/2} r^{-\mu-1} \log\left|\frac{r+|c_m(k)|}{r-|c_m(k)|}\right| dr \\
&\le \frac{\pi}{\mu}\tan\frac{\pi\mu}{2} \sum_{S<|c_m(k)|<R} \frac{1}{|c_m(k)|^\mu} \\
&\le (1+\mu)\frac{\pi}{\mu}\tan\frac{\pi\mu}{2}\,\frac{T(2R,f)}{R^\mu} + \pi\mu\tan\frac{\pi\mu}{2}\int_S^R \frac{N(r,0,F_k)}{r^{\mu+1}}\,dr.
\end{aligned}$$

We now divide by $r^{1+\mu}$ in (4.8) and integrate from $2S$ to $R/2$. Upon using (4.11), we see that

$$\begin{aligned}
b(H^n,f)(1-\varepsilon)\int_{2S}^{R/2} \frac{A(r,f)}{r^{1+\mu}}\,dr \\
&\le \pi\mu\tan\frac{\pi\mu}{2}\sum_{k=1}^{n+1}\int_S^R \frac{N(r,H^k,f)}{r^{1+\mu}}\,dr \\
&\quad + \left((1+\mu)\frac{\pi}{\mu}\tan\frac{\pi\mu}{2} + \frac{2^\mu\cdot 2K_1}{1-\mu}\right)\frac{T(4R,f)}{R^\mu} \\
&\quad + \left(\frac{2K_1\log 2}{\mu 2^\mu} + \frac{2K_1}{\mu^2 2^\mu}\right)\frac{T(4S,f)}{S^\mu} + o\left(\int_{2S}^{R/2}\frac{A(r,f)}{r^{\mu+1}}\,dr\right).
\end{aligned}$$

By the definition of $\delta(H^k,f)$, for $S<r<R$, it follows that

$$\begin{aligned}
N(r,H^k,f) &< (1-\delta(H^k,f)+\varepsilon)T(r,f) + T(r,a^k) \\
&= (1-\delta(H^k,f)+\varepsilon)T(r,f) + o(A(r,f)).
\end{aligned}$$

Hence,

$$\begin{aligned}
b(H^n,f)(1-\varepsilon)\int_{2S}^{R/2} \frac{A(r,f)}{r^{1+\mu}}\,dr \\
&\le \pi\mu\tan\frac{\pi\mu}{2}\sum_{k=1}^{n+1}(1-\delta(H^k,f)+\varepsilon)\cdot\int_S^R \frac{T(r,f)}{r^{1+\mu}}\,dr \\
&\quad + \left((1+\mu)\frac{\pi}{\mu}\tan\frac{\pi\mu}{2} + \frac{2^\mu\cdot 2K_1}{1-\mu}\right)\frac{T(4R,f)}{R^\mu} \\
&\quad + \left(\frac{2K_1\log 2}{\mu 2^\mu} + \frac{2K_1}{\mu^2 2^\mu}\right)\frac{T(4S,f)}{S^\mu} + o\left(\int_{2S}^{R/2}\frac{A(r,f)}{r^{\mu+1}}\,dr\right).
\end{aligned}$$

Set $S = S_m$ and $R = R_m$. From Lemma 2.3, we immediately obtain

$$\begin{aligned} (4.12) \qquad & b(H^n, f)(1-\varepsilon) \int_{2S_m}^{R_m/2} \frac{A(r,f)}{r^{1+\mu}}\,dr \\ & \le \Big(\pi\mu \tan\frac{\pi\mu}{2} \sum_{k=1}^{n+1} (1-\delta(H^k,f)+\varepsilon) + \varepsilon + o(1)\Big) \cdot \int_{2S_m}^{R_m/2} \frac{T(r,f)}{r^{1+\mu}}\,dr \\ & \quad + o\Big(\int_{2S_m}^{R_m/2} \frac{A(r,f)}{r^{\mu+1}}\,dr\Big). \end{aligned}$$

Integration by parts gives

$$\int_{2S_m}^{R_m/2} \frac{T(r,f)}{r^{1+\mu}}\,dr = \frac{T(2S_m,f)}{\mu 2^\mu S_m^\mu} - \frac{2^\mu T(R_m/2,f)}{\mu R_m^\mu} + \frac{1}{\mu} \int_{2S_m}^{R_m/2} \frac{A(r,f)}{r^{\mu+1}}\,dr.$$

Then, by Lemma 2.3,

$$(4.13) \qquad \int_{2S_m}^{R_m/2} \frac{T(r,f)}{r^{1+\mu}}\,dr < \frac{1+\varepsilon}{\mu} \int_{2S_m}^{R_m/2} \frac{A(r,f)}{r^{\mu+1}}\,dr.$$

Combining (4.12) and (4.13), we obtain the conclusion. ■

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