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Zeta functions and complex dimensions of relative fractal drums: Theory, examples and applications

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Abstract

In 2009, the first author introduced a new class of zeta functions, called “distance zeta functions”, associated with arbitrary compact fractal subsets of Euclidean spaces of arbitrary dimension. It represents a natural, but nontrivial extension of the existing theory of “geometric zeta functions” of bounded fractal strings. In this work, we introduce the class of “relative fractal drums” (or RFDs), which contains the classes of bounded fractal strings and of compact fractal subsets of Euclidean spaces as special cases. Furthermore, the associated (relative) distance zeta functions of RFDs extend the aforementioned classes of fractal zeta functions. The abscissa of (absolute) convergence of any relative fractal drum is equal to the relative box dimension of the RFD. We pay particular attention to constructing meromorphic extensions of the distance zeta functions of RFDs, as well as to the construction of transcendentally ∞-quasiperiodic RFDs. We also describe a class of RFDs, called *maximal hyperfractals*, such that the critical line of convergence consists solely of nonremovable singularities of the associated relative distance zeta functions. Finally, we also describe a class of Minkowski measurable RFDs which possess an infinite sequence of complex dimensions of arbitrary multiplicity $m \geq 1$, and even an infinite sequence of essential singularities along the critical line.

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Key words and phrases: fractal set, fractal string, relative fractal drum (RFD), fractal zeta functions, relative distance zeta function, relative tube zeta function, geometric zeta function of a fractal string, relative Minkowski content, relative Minkowski measurability, relative upper box (or Minkowski) dimension, relative complex dimensions of an RFD, holomorphic and meromorphic functions, abscissa of absolute and meromorphic convergence, transcendentally ∞-quasiperiodic function, transcendentally ∞-quasiperiodic RFD, a-string of higher order.

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Glossary

$a_k \sim b_k$ as $k \to \infty$, asymptotically equivalent sequences of complex numbers 19

$f \sim g$, equivalence of the DTI f and the meromorphic function g 20

$A_{\mathcal{L}} := \{a_j := \sum_{k=j}^{\infty} \ell_k : j \in \mathbb{N}\}$, canonical geometric representation of a bounded fractal string $\mathcal{L} = (\ell_k)_{k \geq 1}$ 12

$A_t := \{x \in \mathbb{R}^N : d(x, A) < t\}$, the t-neighborhood of a subset A of $\mathbb{R}^N$ $(t > 0)$ 12

A_N, the inhomogeneous Sierpiński N-gasket 68

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(A, Ω), relative fractal drum 12

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$\mathrm{B}_x(a, b) := \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$, the incomplete beta function 58

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$\dim_{PC}(A, \Omega)$, the set of principal complex dimensions of the RFD (A, Ω) 19

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$D_{\mathrm{hol}}(f)$, the abscissa of holomorphic continuation of f 19

$D_{\mathrm{mer}}(f)$, the abscissa of meromorphic continuation of f 18

$\partial\Omega$, the boundary of a subset Ω of $\mathbb{R}^N$ 19

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$\mathcal{D}_{\mathrm{aqp}}$, the family of algebraically quasiperiodic relative fractal drums 43

$\mathcal{D}_{\mathrm{tqp}}$, the family of transcendentally quasiperiodic relative fractal drums 43

DTI, Dirichlet-type integral 18

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$\mathbf{e}(m)$, the exponent sequence of an integer $m \geq 2$ 41

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$\mathbb{i} = \sqrt{-1}$, the imaginary unit 36

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$(\ell_j)_{j=1}^{\infty}$, sequence of lengths of a fractal string $\mathcal{L}$ written in nonincreasing order 12

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$\log x := \log_{\mathrm{e}} x$, the natural logarithm of x; $y = \log x \iff x = \mathrm{e}^y$ 18

$\ell^{\infty}(\mathbb{R})$, the Banach space of bounded sequences of real numbers $(\tau_j)_{j \geq 1}$ 42

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$\otimes$, tensor product of a base relative fractal drum and a fractal string ... 61

$\mathbf{p}$, the oscillatory period of a lattice self-similar set (or string or spray or RFD) ... 41

$\mathcal{P}(f)$, the set of poles of a meromorphic function f ... 19

$\mathcal{P}(f, W)$, the set of poles of a meromorphic function f contained in the interior of the set $W \subseteq \mathbb{C}$ ... 19

$\mathcal{P}_c(f)$, the set of poles of a (tamed) Dirichlet-type integral (i.e., DTI) f on the critical line ... 19

RFD, relative fractal drum (A, Ω) in $\mathbb{R}^N$ ... 12

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$\tilde{\zeta}_A(s, \Omega) = \tilde{\zeta}_{A,\Omega}(s) := \int_0^\delta t^{s-N-1} |A_t \cap \Omega| \, \mathrm{d}t$, the relative tube zeta function of a relative fractal drum (A, Ω) ... 31

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Preface

The purpose of this work is to develop the theory of complex dimensions for arbitrary compact subsets A of Euclidean spaces $\mathbb{R}^N$, of arbitrary dimension $N \geq 1$. To this end, in 2009, the first author introduced a new class of zeta functions, called *distance zeta functions* ζ_A of fractal sets A, the poles of which (after ζ_A has been suitably meromorphically extended) are defined as the *complex dimensions* of A. This notion establishes an important bridge between the geometry of fractal sets, number theory and complex analysis.

The development of the higher-dimensional theory of complex dimensions of fractal sets has led us to the discovery of the *tube zeta functions* $\tilde{\zeta}_A$ of fractal sets, which are not only a valuable technical tool, but a natural companion of the distance zeta functions ζ_A. These two fractal zeta functions are connected by a simple functional equation, which shows that, in this generality, the theory of complex dimensions can be developed either from the point of view of the distance zeta function or of the tube zeta function. Both the distance and tube zeta functions enable us to extend in a nontrivial way the existing theory of geometric zeta functions $\zeta_{\mathcal{L}}$ of bounded fractal strings $\mathcal{L}$. An even broader perspective is achieved by introducing the so-called *relative fractal drums* (RFDs) (A, Ω) in Euclidean spaces, which extend the notions of bounded fractal sets in $\mathbb{R}^N$, as well as of bounded fractal strings. The associated *relative fractal zeta functions* $\zeta_{A,\Omega}$ enable us to consider the theory of fractal zeta functions from a unified perspective. An unexpected novelty is that a relative fractal drum (A, Ω) can have a (naturally defined) *Minkowski* (or *box*) *dimension* $\dim_B(A, \Omega)$ of *negative* value (and even of value $-\infty$), or more generally, that its principal complex dimensions (i.e., the poles of $\zeta_{A,\Omega}$ on the critical line $\{\mathrm{Re}\, s = D\}$, where $D = \overline{\dim}_B(A, \Omega)$ is the upper Minkowski dimension of (A, Ω)) can have negative real parts.

The residue of a fractal zeta function, computed at the value D of the abscissa of (absolute) convergence of the zeta function (i.e., at the Minkowski dimension), is very closely related to the *Minkowski content* of the corresponding bounded set or RFD. Furthermore, we also study the quasiperiodicity of relative fractal drums, by using a classical result from (transcendental) analytic number theory, due to Alan Baker. Roughly, for any given positive integer n, it is possible to construct a fractal set with n algebraically independent quasiperiods; as a result, we obtain a *transcendentally n-quasiperiodic set.* Moreover, we can even construct transcendentally ∞-quasiperiodic sets, i.e., fractal sets with infinitely many algebraically independent quasiperiods.

Towards the end of this article, special emphasis is given to the construction of fractal sets A which have principal complex dimensions (i.e., the poles of the distance zeta function ζ_A with real part equal to $D = \overline{\dim}_B A$) of any given multiplicity $m \geq 2$, and even with "infinite multiplicity" $m = \infty$; i.e., in this case, the principal complex dimensions of A are, in fact, essential singularities of its distance zeta function ζ_A.

Finally, we also construct fractal sets A in $\mathbb{R}^N$, which we call *maximal hyperfractals*, such that the corresponding distance zeta function has the entire critical line of (absolute) convergence $\{\operatorname{Re} s = D\}$ as the set of its nonremovable singularities.

We conclude this paper by a discussion of the notion of "fractality", formulated in terms of the present higher-dimensional theory of complex dimensions. Furthermore, we illustrate this discussion by means of an RFD suitably associated with the Cantor graph (or the "devil's staircase").

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1. Introduction

1.1. The development of the idea of dimension: From integers to complex numbers. The development of the mathematical ideas behind the concept of *dimension* started in the 19th century, with the need to precisely define some basic notions like the "line" and "surface". Its history can be very roughly subdivided into the following three parts, all of them deeply interlaced: the history of integer dimensions, fractal dimensions, and complex dimensions.

1.1.1. Integer dimensions. Until the beginning of the 20th century, the notion of "dimension" had in use exclusively as a *nonnegative integer*. It was rigorously defined in the 19th century, first for linear objects, and then for manifolds; i.e., in the area of linear algebra (where it was defined as the number of elements of any base of a given linear space), as well as in differential and algebraic geometry. Soon, several other integer dimensional quantities were introduced, in order to study arbitrary subsets of Euclidean spaces (and, more generally, of topological spaces). These basic dimensional quantities are known as the small inductive dimension (Menger–Urysohn), the large inductive dimension (Brouwer–Čech) and the covering dimension (Čech–Lebesgue). The history of the extremely complex subject of integer dimensions appearing in topology is presented in the survey article [CJ].

1.1.2. Fractal dimensions. The foundations of the theory of *fractal dimensions* were already laid out in the 1920s, in the works of Minkowski, Hausdorff, Besicovitch and Bouligand, by introducing (suitably defined) dimensions which can assume *noninteger* (more specifically, nonnegative real) values, in order to better understand the geometric properties of very general subsets of Euclidean spaces. These developments resulted in the *Hausdorff dimension* and the *Minkowski dimension* or the *Minkowski–Bouligand dimension* (also called the *box dimension*, the term that we adopt in this paper), which have become essential tools of modern *fractal geometry.*

Many distinguished scholars contributed in various ways to popularizing and developing these ideas, and thereby, in particular, to the introduction of the seemingly counterintuitive concept of fractal dimension; there are too many of them to name them all here. (See, for example, [Man, Chapter XI] or [F1].) In addition, the methods of fractal geometry are today extremely developed and frequently used in various specialized scientific fields, both from the theoretical and applied points of view. An overview of the early history of fractal geometry and the development of its main ideas can be found in [Man]. (See also [L7].)

1.1.3. Complex dimensions. The idea of introducing *complex dimensions* (more specifically, of *complex numbers* as dimensions) as a quantification of the inner (oscillatory) geometric properties of objects called *bounded fractal strings* $\mathcal{L}$, was proposed in the early 1990s by the first author of this paper, based in part on earlier work in [L1]–[L3], [LPo1], [LM1, LM2]. Very roughly, bounded fractal strings can be identified with certain bounded subsets of the real line. In order to define the complex dimensions of a given bounded fractal string $\mathcal{L}$, one has to assign to it the corresponding (geometric) *zeta function* $\zeta_{\mathcal{L}}$. The "complex dimensions" of bounded fractal strings are then defined as the poles of a suitable meromorphic extension of the geometric zeta function in question. The development of the main ideas and results behind the mathematical theory of complex dimensions of fractal strings can be found in [LF3].

It is natural to ask the following question: Is it possible to define the "complex dimensions" for *any* (nonempty) bounded subset A of Euclidean space? In other words, is there a natural zeta function ζ_A such that its poles can be considered as the "complex dimensions" of a given set A (assuming that a suitable meromorphic extension of ζ_A is possible)? The answer to this question was obtained by the first author in 2009, by introducing a class of *distance zeta functions* ζ_A, as we call them in this work.

As the result of a collaboration between the authors of this paper, initiated by the first author in 2009, it soon became clear that the notion of "complex dimensions" can be introduced not only for bounded subsets of Euclidean spaces, but even for much more general geometric objects, denoted by (A,Ω), which we call *relative fractal drums* (RFDs). (An example of relative fractal drum is given by $(\partial\Omega,\Omega)$, where Ω is a bounded open subset of some Euclidean space $\mathbb{R}^N$, with $N \geq 1$, while $\partial\Omega$ is its topological boundary. The special case when $N = 1$ precisely corresponds to a bounded fractal string.) The study of the *complex dimensions of relative fractal drums* is the main goal of the present paper. The flexibility of this notion, as well as of the corresponding notion of relative distance zeta function $\zeta_{A,\Omega}$, has enabled us to view the existing theory of complex dimensions of fractal strings (and their generalizations to fractal sprays) from a unified perspective and to extend it beyond recognition. An unexpected novelty was the possibility for the relative Minkowski dimensions of some classes of relative fractal drums to take negative values (including $-\infty$).

We should mention that the set of complex dimensions of a given RFD (and, in particular, of a given bounded set of $\mathbb{R}^N$) is always a discrete subset of $\mathbb{C}$, and hence consists of a (finite or countable) sequence of complex numbers, with finite multiplicities (as poles of the corresponding fractal zeta functions). In the future, we may extend this notion to also include the possible essential singularities of the corresponding fractal zeta functions. Indeed, in this paper, we construct fractal sets and RFDs whose fractal zeta functions have infinitely many essential singularities.

The theory of complex dimensions of relative fractal drums, developed in this work, provides a useful bridge between fractal geometry, number theory and complex analysis. It has brought to light numerous interesting questions and new challenging problems for further research; see, especially, [LRŽ1, Chapter 6] and [LRŽ7, §8].

1.2. Relative fractal drums and their distance zeta functions. In 2009, the first author has introduced a new class of zeta functions ζ_A, called "distance zeta functions", associated with arbitrary compact subsets A of a given Euclidean space $\mathbb{R}^N$ of arbitrary dimension N. More specifically, the *distance zeta function* ζ_A of a bounded set $A \subset \mathbb{R}^N$ is defined by

$$\zeta_A(s) := \int_{A_\delta} d(x,A)^{s-N}\,\mathrm{d}x \tag{1.2.1}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large, where δ is a fixed positive real number and A_δ is the Euclidean δ-neighborhood of A. (Here, $d(x,A) := \inf\{|x-a| : a \in A\}$ denotes the Euclidean distance from x to A and the integral is understood in the sense of Lebesgue and is therefore absolutely convergent.) These new fractal zeta functions have been studied in [LRŽ2, LRŽ3], as well as in the research monograph [LRŽ1, Chapter 2].

We extend the class of distance zeta functions from the family of compact subset of $\mathbb{R}^N$ to a new class of objects that we call "relative fractal drums" (RFDs) in $\mathbb{R}^N$ (still for any $N \geq 1$); see Definition 1.2 below. This enables us to provide a unified approach to the study of fractal zeta functions. An unexpected novelty is that RFDs may have an upper box (or Minkowski) dimension (defined by (1.4.3) below), which is *negative*, or even $-\infty$; see Proposition 2.12 and Corollary 2.14 below.

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^N$, possibly unbounded, but of finite N-dimensional Lebesgue measure. Assume that A is a subset of $\mathbb{R}^N$ and

$$\text{there exists } \delta > 0 \text{ such that } \Omega \subseteq A_\delta. \tag{1.2.2}$$

We then say that the ordered pair (A,Ω) is a *relative fractal drum* (or an RFD, for short) in $\mathbb{R}^N$.

We stress that when working with an RFD (A,Ω), we always assume that both A and Ω are nonempty.

Relative fractal drums represent a natural simultaneous extension of the following classes of objects:

(a) The class $\mathtt{STR}_b$ of (nonempty) *bounded fractal strings* $\mathcal{L} := (\ell_j)_{j\in\mathbb{N}}$; indeed, for any given bounded fractal string (1) $\mathcal{L} := (\ell_j)_{j\in\mathbb{N}}$, we can define a disjoint union

$$\Omega := \bigcup_{j=1}^{\infty} I_j \tag{1.2.3}$$

of open intervals I_j such that $|I_j| = \ell_j$ for each $j \geq 1$, and $A := \partial\Omega$; then $\mathcal{L}$ can be identified with any such RFD $(\partial\Omega,\Omega)$; the set Ω is referred to as a *geometric realization of the bounded fractal string* $\mathcal{L}$; using this identification, we can write $\mathcal{L} = (\partial\Omega,\Omega)$; it is sometimes convenient to deal with the *canonical geometric representation of* $\mathcal{L}$, defined by

$$A_{\mathcal{L}} := \Big\{a_j := \sum_{k\geq j} \ell_k : j \in \mathbb{N}\Big\}, \tag{1.2.4}$$

---

(1) A *bounded fractal string* $\mathcal{L} := (\ell_j)_{j\in\mathbb{N}}$ is defined as a nonincreasing sequence of positive real numbers $(\ell_j)_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty} \ell_j < \infty$; see [LF3].

and in this case, the set

$$\Omega_{\mathrm{can},\mathcal{L}} := (0,\ell_1)\setminus A_{\mathcal{L}} = \bigcup_{j=1}^{\infty}(a_k,a_{k+1}) \tag{1.2.5}$$

is obviously a geometric realization of $\mathcal{L}$, which we call the *canonical geometric realization of* $\mathcal{L}$.

(b) The class $\mathtt{COM}(\mathbb{R}^N)$ of *compact fractal sets* A in Euclidean space $\mathbb{R}^N$, by identifying A with the corresponding RFD (A,A_δ), for any fixed $\delta>0$.

Moreover, denoting by $\mathtt{RFD}(\mathbb{R}^N)$ the family of all RFDs in $\mathbb{R}^N$, we have the following natural inclusions, for any $N\geq 1$:

$$\mathtt{STR}_b \subset \mathtt{COM}(\mathbb{R}^N) \subset \mathtt{RFD}(\mathbb{R}^N). \tag{1.2.6}$$

Here, any bounded fractal string $\mathcal{L} := (\ell_j)_{j\in\mathbb{N}}$ can be identified with the compact set $\overline{A}\subset\mathbb{R}$, where $A := \{a_j=\sum_{k=j}^{\infty}\ell_k : j\in\mathbb{N}\}\subset\mathbb{R}$.

We now introduce the main definition of this paper.

Definition 1.2. Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. The relative *distance zeta function* $\zeta_{A,\Omega}$ *of* (A,Ω) is defined by

$$\zeta_{A,\Omega}(s) := \int_\Omega d(x,A)^{s-N}\,\mathrm{d}x \tag{1.2.7}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re}s$ sufficiently large.

The family of relative distance zeta functions represents a natural extension of the following classes of fractal zeta functions:

(a) The class of *geometric zeta functions* $\zeta_{\mathcal{L}}$, associated with bounded fractal strings $\mathcal{L} := (\ell_j)_{j\in\mathbb{N}}$ and defined (for all $s\in\mathbb{C}$ with $\operatorname{Re}s$ sufficiently large) by

$$\zeta_{\mathcal{L}}(s) := \sum_{j=1}^{\infty}\ell_j^s \tag{1.2.8}$$

(it has been extensively studied in [LF3], as well as in the relevant references therein); more precisely, we show that

$$\zeta_{\mathcal{L}}(s) = s2^{s-1}\zeta_{\partial\Omega,\Omega}(s) \tag{1.2.9}$$

for all $s\in\mathbb{C}$ with $\operatorname{Re}s$ sufficiently large, where Ω is any geometric realization of $\mathcal{L}$, described in (a) appearing immediately after Definition 1.1.

(b) The class of *distance zeta functions* ζ_A associated with compact fractal subsets A of Euclidean spaces, defined by (1.2.1).

We point out that some of the results of this article (especially in §2.1) can be viewed in the context of very general convolution-type integrals of the form

$$H(s) = \int_E f(s,t)\,\mathrm{d}\mu(t), \tag{1.2.10}$$

about which we cite the following well-known result. We shall need it in the proofs of Theorem 3.2 and Proposition 3.12 below.

Theorem 1.3. *Let V be an open set in $\mathbb{C}$ (or even in $\mathbb{C}^n$). Furthermore, let $(E, \mathcal{B}(E), \mu)$ be a measure space, where E is a locally compact metrizable space, $\mathcal{B}(E)$ is the Borel σ-algebra of E, and μ is a positive or complex (local, i.e., locally bounded) measure, with associated total variation measure denoted by $|\mu|$. Assume that a function $f : V \times E \to \mathbb{C}$ satisfies the following conditions*:

(1) *$f(\,\cdot\,, t)$ is holomorphic for $|\mu|$-a.e. $t \in E$,*
(2) *$f(s, \,\cdot\,)$ is μ-measurable for all $s \in V$, and*
(3) *the following growth property is fulfilled by f: for every compact subset K of V, there exists $g_K \in L^1(|\mu|)$ such that $|f(s,t)| \le g_K(t)$ for all $s \in V$ and $|\mu|$-a.e. $t \in K$.*

Then the function H defined by (1.2.10) *is holomorphic on V. Moreover, one can interchange the derivative and the integral. (The problem of complex differentiation under the integral sign is discussed, for example, in* [Mat].) *More precisely, for every $s \in V$ and every $k \in \mathbb{N}$, we have*

$$F^{(k)}(s) = \int_E \frac{\partial^k}{\partial s^k} f(s,t)\, \mathrm{d}\mu(t). \tag{1.2.11}$$

Remark 1.4. According to [Mat] and as is well known, if conditions (1) and (2) from Theorem 1.3 are satisfied, then condition (3) is equivalent to the following condition, which is generally slightly easier to verify in practice:

(3′) $\int_E |f(\,\cdot\,, t)|\, \mathrm{d}|\mu|(t)$ is locally bounded; that is, for each fixed $s_0 \in V$, there exists $\delta > 0$ such that

$$\sup_{s \in V,\, |s - s_0| < \delta} \int_E |f(s,t)|\, \mathrm{d}|\mu|(t) < \infty. \tag{1.2.12}$$

In other words, we can replace (3) with (3′) in the statement of Theorem 1.3. (This is because the notion of being holomorphic is *local.*)

1.3. Overview of the main results. We note that the notion of *complex dimensions of a relative fractal drum* (RFD), necessary for a clearer understanding of this overview, is introduced in Definition 1.6 below. The definitions of the relative Minkowski content and of the relative box (or, more accurately, Minkowski) dimension can be found in (1.4.1) and (1.4.2) below, respectively.

Overview of Chapter 2. The main result of §2.1 is contained in parts (a) and (b) of Theorem 2.1, according to which the abscissa of (absolute) convergence $D(\zeta_{A,\Omega})$ of the distance zeta function $\zeta_{A,\Omega}$ of any RFD (A, Ω) is equal to the upper box (i.e., Minkowski) dimension $\overline{\dim}_B(A, \Omega)$ of the RFD. Part (c) of that theorem provides some mild conditions under which the value of $D := \dim_B(A, \Omega)$ (assuming that $\dim_B(A, \Omega)$ exists) is a singularity of the relative distance zeta function $\zeta_{A,\Omega}$, and therefore also coincides with the abscissa of holomorphic continuation of the distance zeta function of the RFD.

Theorem 2.2 shows that for a nondegenerate RFD (A, Ω) and provided $\zeta_{A,\Omega}$ possesses a meromophic extension to an open connected neighborhood of $D := \dim_B A < N$, the residue of $\zeta_{A,\Omega}$ evaluated at D always lies (up to the multiplicative constant $N - D$) between the lower and the upper D-dimensional Minkowski contents of the RFD. In particular, if the RFD is Minkowski measurable, then the residue $\operatorname{res}(\zeta_{A,\Omega}, D)$ is, up to the same multiplicative constant, equal to the D-dimensional Minkowski content of the RFD.

In Proposition 2.10, we show that if there is at least one point $a \in \overline{A} \cap \overline{\Omega}$ at which the RFD (A, Ω) satisfies a suitable *cone property* with respect to Ω (Definition 2.6), then $D(\zeta_{A,\Omega}) \geq 0$. In general, however, $D(\zeta_{A,\Omega})$ (i.e., $\overline{\dim}_B(A, \Omega)$) can take on any prescribed negative value (Proposition 2.12), and even $-\infty$ (Corollary 2.14). Here we stress that the phenomenon of negative box dimensions (including $-\infty$) for relative fractal drums of the form $(\partial\Omega, \Omega)$ and $(\{a\}, \Omega)$, with $a \in \partial\Omega$, has been studied independently by Tricot [Tr2], where the notion of *inner box dimension of the boundary* $\partial\Omega$ and of the point a with respect to Ω is used.

In §2.3, dealing with the *scaling property of the relative distance zeta functions*, the main result is stated in Theorem 2.16. It has important applications to the study of self-similar sprays (or tilings). Corollary 2.17 states an interesting scaling property of the residues of relative distance zeta functions, evaluated at their simple poles (see (2.3.2)). The countable additivity of the relative distance zeta function with respect to a disjoint union of RFDs (a notion introduced in Definition 2.18) is established in Theorem 2.19.

In §2.4, we introduce the notion of the *relative tube zeta function* (see (2.4.1)), which is closely related to the relative distance zeta function (see the functional equation (2.4.2) connecting these two fractal zeta functions). Equation (2.4.5) in Proposition 2.21 connects the residues (evaluated at any visible complex dimension) of the relative tube and distance functions. In Example 2.22, we calculate (via a direct computation) the complex dimensions of the torus RFD. Much more generally, in Proposition 2.23, by using Federer's tube formula [Fe1], we calculate the distance zeta function (and the complex dimensions) of the boundary of any compact set C of *positive reach* (and in particular of any compact convex subset of $\mathbb{R}^N$); as a special case, we obtain a similar result for a smooth compact submanifold of $\mathbb{R}^N$ (thereby significantly extending the results of Example 2.22).

The important problem of the existence and construction of meromorphic extensions of some classes of relative (tube and distance) zeta functions is studied in §2.5. It is treated in Theorem 2.24 for a class of Minkowski measurable RFDs, and in Theorem 2.25 for a class of Minkowski nonmeasurable (but Minkowski nondegenerate) RFDs. Naturally, even though the two classes of examples dealt with here are of interest in their own right and in the applications, additional results should be obtained along these lines in the future developments of the theory.

The main result of §2.6 is stated in Theorem 2.40 and deals with the construction of ∞-*quasiperiodic relative fractal drums*, a notion introduced in Definition 2.37. Its proof makes an essential use of suitable families of *generalized Cantor sets* $C^{(m,a)}$ with two parameters m and a, introduced in Definition 2.28; some of the properties of these Cantor sets are listed in Proposition 2.29. Theorem 2.40 can be considered as a fractal set-theoretic interpretation of Baker's theorem from transcendental number theory (Theorem 2.30). It provides an explicit construction of a transcendentally ∞-quasiperiodic relative fractal drum. In particular, this RFD has infinitely many algebraically incommensurable quasiperiods. In Definition 2.38, we also introduce the new notions of *hyperfractal RFDs*, as well as of *strong hyperfractals* and of *maximal hyperfractals*. It turns out that the relative fractal drums constructed in Theorem 2.40 are not only ∞-quasiperiodic, but also maximally hyperfractal. Accordingly, the critical line $\{\operatorname{Re} s = D\}$, where $D := \overline{\dim}_B(A, \Omega)$, consists solely of nonremovable singularities of the associated fractal zeta function; a fortiori, the distance and tube zeta functions of the RFD cannot be meromorphically extended beyond this vertical line.

The *scaling property of relative tube zeta functions* is provided in Proposition 2.42. This result is analogous to the one obtained for relative distance zeta functions in Theorem 2.16.

Overview of Chapter 3. This chapter deals with the problem of *embeddings of RFDs* into higher-dimensional Euclidean spaces. Theorem 3.3 shows that the notion of complex dimensions of fractal sets does not depend on the dimension of the ambient space. In Theorem 3.10, an analogous result is obtained for general RFDs. An important role in the accompanying computations is played by the gamma function, the Euler beta function, as well as by the Mellin zeta function of an RFD, introduced in (3.2.19). In Example 3.15, we apply these results to calculate the complex dimensions of the Cantor dust.

Overview of Chapter 4. In this chapter, we study *relative fractal sprays* in $\mathbb{R}^N$, introduced in Definition 4.1. The main result is given in Theorem 4.6, which deals with the distance zeta function of relative fractal sprays.

In §4.2, we study the relative Sierpiński sprays and their complex dimensions. Example 4.12 deals with the relative Sierpiński gasket, while Example 4.14 deals with the inhomogeneous Sierpiński N-gasket RFDs, for any $N \geq 2$. Furthermore, Example 4.15 deals with the relative Sierpiński carpet, while Example 4.17 deals with the Sierpiński N-carpet, for any $N \geq 2$. Interesting new phenomena occur in this context, which are discussed throughout §4.2.

In Definition 4.18, we recall (and extend to RFDs) the notion of *self-similar sprays* (or *tilings*), defined by a suitable *ratio list* of finitely many real numbers in $(0, 1)$.

Theorem 4.21 provides an explicit form for the distance zeta function of a self-similar spray, which can be found in (4.3.12). The results obtained here are illustrated by the new examples of the 1/2-square fractal and of the 1/3-square fractal, discussed in Examples 4.24 and 4.25, respectively.

In §4.4, we describe a constructive method for generating principal complex dimensions (2) of relative fractal drums of any prescribed multiplicity $m \geq 2$, including infinite multiplicity. (The latter case when $m = \infty$ corresponds to essential singularities of the associated fractal zeta function.) In Example 4.27, we provide the construction of the *mth order a-string*, while we define the *mth order Cantor string* in (4.4.6). In Examples 4.29 and 4.30, we construct Minkowski measurable RFDs which have infinitely many principal complex dimensions of arbitrary multiplicity m, with $m \geq 2$ and even with $m = \infty$ (i.e., corresponding to essential singularities).

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(2) The *principal complex dimensions* of an RFD are the poles of the associated fractal (i.e., distance or tube) zeta function with maximal real part D, where D is both the abscissa of (absolute) convergence of the zeta function and the (relative) Minkowski dimension of the RFD; see Definition 1.6 and Theorem 2.1(b) below.

Overview of Chapter 5. This chapter is dedicated to the discussion of the notion of *fractality* (of RFDs), and its intimate relationship with the notion of the complex dimensions of RFDs. In §5.1, *fractal and subcritically fractal RFDs* are discussed. These notions are illustrated in §5.2 in the case of the *Cantor graph RFD.*

1.4. Notation. In what follows, an important role is played by the definition of the upper and lower Minkowski contents of RFDs and of the upper and lower box (or Minkowski) dimensions of RFDs. We shall follow the definitions introduced in [Ž2], but with an essential difference: the parameter r appearing below can be *any* real number, and not just a nonnegative real number. (See Remark 1.5 below.) Hence, for a given $r\in\mathbb{R}$, we define the *r-dimensional upper and lower Minkowski contents* of an RFD (A,Ω) in $\mathbb{R}^N$ as follows (3):

$$\mathcal{M}^{*r}(A,\Omega):=\limsup_{t\to0^+}\frac{|A_t\cap\Omega|}{t^{N-r}},\quad \mathcal{M}_*^{r}(A,\Omega):=\liminf_{t\to0^+}\frac{|A_t\cap\Omega|}{t^{N-r}}. \tag{1.4.1}$$

We also call them the *relative upper Minkowski content* and *lower Minkowski content* of (A,Ω), respectively. They represent a natural extension of the notions of upper and lower Minkowski contents of bounded sets in $\mathbb{R}^N$, introduced by Bouligand [Bou] and Hadwiger [H], and used by Federer [Fe2] and other researchers [Sta], [Tr1], [BC], [L1], [LPo1], [F1, F2], [HeL], [LF3], [Ž2], [W], [KK], [Ko], [RW], [LRŽ1], [HL1].

As usual, we then define the *upper box dimension* of (A,Ω) by

$$\begin{aligned}\overline{\dim}_B(A,\Omega)&:=\inf\{r\in\mathbb{R}:\mathcal{M}^{*r}(A,\Omega)=0\}\\&=\sup\{r\in\mathbb{R}:\mathcal{M}^{*r}(A,\Omega)=+\infty\},\end{aligned} \tag{1.4.2}$$

and the *lower box dimension* of (A,Ω) by

$$\begin{aligned}\underline{\dim}_B(A,\Omega)&:=\inf\{r\in\mathbb{R}:\mathcal{M}_*^{r}(A,\Omega)=0\}\\&=\sup\{r\in\mathbb{R}:\mathcal{M}_*^{r}(A,\Omega)=+\infty\}.\end{aligned} \tag{1.4.3}$$

We refer to $\overline{\dim}_B(A,\Omega)$ and $\underline{\dim}_B(A,\Omega)$ as the *relative upper* and *lower box* (or *Minkowski*) *dimension* of (A,Ω), respectively. The novelty here is that, in contrast to the usual upper and lower box dimensions, the *relative* upper and lower Minkowski dimensions can attain negative values as well, and even the value $-\infty$. More specifically, it is easy to see that

$$-\infty\le\underline{\dim}_B(A,\Omega)\le\overline{\dim}_B(A,\Omega)\le N.$$

If $\overline{\dim}_B(A,\Omega)=\underline{\dim}_B(A,\Omega)$, then the common value is denoted by $\dim_B(A,\Omega)$ and we call it the *box* (or *Minkowski*) *dimension* of (A,Ω) (4).

If there exists $D\in\mathbb{R}$ such that $0<\mathcal{M}_*^{D}(A,\Omega)\le\mathcal{M}^{*D}(A,\Omega)<\infty$, then we say that (A,Ω) is *Minkowski nondegenerate.* Clearly, in this case $D=\dim_B(A,\Omega)$.

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(3) For a given measurable set $E\subset\mathbb{R}^N$, its N-dimensional Lebesgue measure is denoted by $|E|=|E|_N$.

(4) We caution the reader, however, that unlike in the standard case of bounded subsets of $\mathbb{R}^N$, the notion of *relative Minkowski dimension* of an RFD has not yet been given a suitable geometric interpretation in terms of "box counting".

If for some $r \in \mathbb{R}$ we have $\mathcal{M}^{*r}(A,\Omega) = \mathcal{M}^r_*(A,\Omega)$, the common value is denoted by $\mathcal{M}^r(A,\Omega)$. If for some $D \in \mathbb{R}$, $\mathcal{M}^D(A,\Omega)$ exists and $\mathcal{M}^D(A,\Omega) \in (0,+\infty)$, then we say that (A,Ω) is *Minkowski measurable*. Clearly, in this case, the dimension of (A,Ω) exists and $D = \dim_B(A,\Omega)$.

For example, if the sets A and Ω are a positive distance apart (i.e., $\inf\{|x-y| : x \in A, y \in \Omega\} > 0$), then it is easy to see that $\dim_B(A,\Omega) = -\infty$. Indeed, since $|A_t \cap \Omega| = 0$ for all sufficiently small $t > 0$, we have $\mathcal{M}^r(A,\Omega) = 0$ for all $r \in \mathbb{R}$. A class of nontrivial examples for which $-\infty < \dim_B(A,\Omega) < 0$ can be found in Proposition 2.12.

When $\Omega := A_\delta$, where A is a *bounded* subset of $\mathbb{R}^N$ and $\delta > 0$, we obtain the usual (nonrelative) values of *box* (or *Minkowski*) *dimensions*, i.e., $\overline{\dim}_B A := \overline{\dim}_B(A,A_\delta)$, $\underline{\dim}_B A := \dim_B(A,A_\delta)$, $\dim_B A := \dim_B(A,A_\delta)$, which are all nonnegative in this case, as well as the values of the usual Minkowski contents of A; that is, $\mathcal{M}^{r*}(A) := \mathcal{M}^{r*}(A,A_\delta)$, $\mathcal{M}^r_*(A) := \mathcal{M}^r_*(A,A_\delta)$, $\mathcal{M}^r(A) := \mathcal{M}^r(A,A_\delta)$, for any $r \geq 0$. (It is easy to see that these values do not depend on the choice of $\delta > 0$.) Consequently, as was stated in §2.1, bounded subsets of $\mathbb{R}^N$ are special cases of RFDs in $\mathbb{R}^N$. More specifically, if A is a bounded subset of $\mathbb{R}^N$, then the associated RFD in $\mathbb{R}^N$ is (A,A_δ), for any given $\delta > 0$. This comment extends to the theory of complex dimensions of RFDs developed in this paper, which therefore includes the theory of complex dimensions developed in [LRŽ2, LRŽ3].

Remark 1.5. These definitions extend to a general RFD the definitions used in [L1] for an ordinary fractal drum (i.e., a drum with fractal boundary) in the case of Dirichlet boundary conditions (see also [LF2, §12.5] and the relevant references therein, including [BC], [L2, L3]). We then have $(A,\Omega) = (\partial\Omega,\Omega)$, where Ω is a (nonempty) bounded open subset of $\mathbb{R}^N$; it follows at once that $D := D(\zeta_{\partial\Omega,\Omega}) \geq 0$. In fact, we always have $D \in [N-1,N]$ [L1]. The special case when $N = 1$ corresponds to bounded fractal strings, for which we must have $D \in [0,1]$—see, for example, [L1]–[L3], [LPo1], [LM1, LM2], [HeL], [LLF1, LLF2]. Other references related to fractal strings include [DS], [ELMR], [F2], [Fr], [HaL], [HL1, HL2], [Ko], [LL1, LL2], [L4]–[L6], [LLu], [LRŽ1]–[LRŽ8], [LRo], [LLR], [LRoŽ], [LéMé], [MS], [MSV1, MSV2], [O1, O2], [R1, R2], [RW], [T1, T2].

Given $\alpha \in \mathbb{R} \cup \pm\infty$, we denote, for example, the open right half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$ by $\{\operatorname{Re} s > \alpha\}$, with the obvious convention if $\alpha = \pm\infty$: for $\alpha = +\infty$ we obtain the empty set, and for $\alpha = -\infty$ we have all of $\mathbb{C}$. Moreover, if $\alpha \in \mathbb{R}$, we denote the vertical line $\{s \in \mathbb{C} : \operatorname{Re} s = \alpha\}$ by $\{\operatorname{Re} s = \alpha\}$.

We also let $\mathbb{N} := \{1,2,\dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The logarithm of a positive real number x with base $a > 0$ is denoted by $\log_a x$. Furthermore, $\log x := \log_{\mathrm{e}} x$.

Let $f(s) := \int_E \varphi(x)^s \, \mathrm{d}\mu(x)$ be a *tamed* generalized Dirichlet-type integral (DTI), in the sense of [LRŽ2, Definition 2.12]; that is, $\varphi \geq 0$ $|\mu|$-a.e. on E and φ is essentially $|\mu|$-bounded on E, where $|\mu|$ is the total variation of the local complex or positive measure on the locally compact space E. Then we denote by $D(f)$ the *abscissa of* (*absolute*) *convergence* of f; i.e., $D(f) \in [-\infty,\infty]$ is the infimum of all $\alpha \in \mathbb{R}$ such that φ^α (or equivalently $\varphi(x)^{\operatorname{Re} s}$, with $\alpha := \operatorname{Re} s$) is $|\mu|$-integrable. If $D(f) \in \mathbb{R}$, the corresponding vertical line $\{\operatorname{Re} s = D(f)\}$ in the complex plane is called the *critical line* of f. Furthermore, we denote by $D_{\mathrm{mer}}(f)$ the *abscissa of meromorphic continuation* of f (i.e., $D_{\mathrm{mer}}(f) \in [-\infty,\infty]$

is the infimum of all $\alpha \in \mathbb{R}$ such that f has a meromorphic extension to $\{\operatorname{Re} s > \alpha\}$). We define $D_{\mathrm{hol}}(f)$, the *abscissa of holomorphic continuation* of f, in exactly the same way, except that "meromorphic" is replaced by "holomorphic" (^{5}). In general, for any tamed DTI f, we have

$$-\infty \leq D_{\mathrm{mer}}(f) \leq D_{\mathrm{hol}}(f) \leq D(f) \leq +\infty. \tag{1.4.4}$$

(See [LRŽ1, Theorem A.2] for the next to last inequality, and [LRŽ1, Appendix A] for the general theory of tamed DTIs.)

In order to be able to define the key notions of complex dimensions and of principal complex dimensions (see (2.1.4) below), we assume that the function f can be extended to a meromorphic function defined on $G \subseteq \mathbb{C}$, where G is an open and connected neighborhood of the *window* $\boldsymbol{W}$ defined by

$$\boldsymbol{W} = \{s \in \mathbb{C} : \operatorname{Re} s \geq S(\operatorname{Im} s)\}. \tag{1.4.5}$$

Here, the function $S : \mathbb{R} \to (-\infty, D(\zeta_A)]$, called the *screen*, is assumed to be Lipschitz continuous. Note that if $f := \zeta_{A,\Omega}$, then the closed set $\boldsymbol{W}$ contains the *critical line* $\{\operatorname{Re} s = D(\zeta_{A,\Omega})\}$; in fact, it also contains the closed half-plane $\{\operatorname{Re} s \geq D(\zeta_{A,\Omega})\}$. The boundary $\partial\boldsymbol{W}$ of the window is also called the *screen* and is denoted by $\boldsymbol{S}$; it is the graph of the function S, with the horizontal and vertical axes interchanged. More specifically,

$$\boldsymbol{S} = \{S(\tau) + \mathbb{i}\tau : \tau \in \mathbb{R}\}. \tag{1.4.6}$$

Definition 1.6 (Complex dimensions of an RFD). The set of poles of f located in a window $\boldsymbol{W}$ containing the critical line $\{\operatorname{Re} s = D(f)\}$ is denoted by $\mathcal{P}(f, W)$. When the window $\boldsymbol{W}$ is known, or when $\boldsymbol{W} := \mathbb{C}$, we often use the shorter notation $\mathcal{P}(f)$ instead. If $f := \zeta_{A,\Omega}$ and $\zeta_{A,\Omega}$ can be meromorphically extended to a connected open subset containing $\{\operatorname{Re} s \geq D(\zeta_{A,\Omega})\}$, the multiset of poles (i.e., we also take the multiplicities of the poles into account) is called the multiset of (*visible*) *complex dimensions* of (A, Ω). The multiset of complex dimensions on the critical line of $\zeta_{A,\Omega}$ is called the multiset of *principal complex dimensions* of (A, Ω). This multiset is independent of the choice of δ, as well as of the meromorphic extension of ζ. We note that analogous definitions will be used for the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$, introduced in (2.4.1) below, instead of the relative distance zeta function $\zeta_{A,\Omega}$. As we shall see, provided $\overline{\dim}_B(A, \Omega) < N$, the resulting multiset of principal complex dimensions (resp., of complex dimensions) will be the same for either $\zeta_{A,\Omega}$ or $\tilde{\zeta}_{A,\Omega}$. This will follow from the functional equation connecting $\zeta_{A,\Omega}$ and $\tilde{\zeta}_{A,\Omega}$.

If Ω is a given subset of $\mathbb{R}^N$, its closure and boundary are denoted by $\overline{\Omega}$ and $\partial\Omega$, respectively.

For sequences $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ of positive numbers, we write $a_k \sim b_k$ as $k \to \infty$ if $\lim_{k\to\infty} a_k/b_k = 1$. Analogously, if $a(\,\cdot\,)$ and $b(\,\cdot\,)$ are real-valued functions defined on an open interval $(0, t_0)$, we write $a(t) \sim b(t)$ as $t \to 0^+$ if $\lim_{t\to 0^+} a(t)/b(t) = 1$.

(^{5}) Note that $D_{\mathrm{mer}}(f)$ and $D_{\mathrm{hol}}(f)$ can be defined for any given meromorphic function f on a domain U of $\mathbb{C}$, whereas $D(f)$ is only well defined if f is a tamed DTI (see [LRŽ2] or [LRŽ1, Appendix A]).

We shall also need the relation $\sim$ between Dirichlet-type integral functions (DTIs) and meromorphic functions [LRŽ2, Definition 2.22], which we now briefly recall.

Definition 1.7. Let f and g be tamed Dirichlet-type integrals, both admitting a (necessarily unique) meromorphic extension to an open connected subset U of $\mathbb{C}$ which contains $\{\operatorname{Re} s \geq D(f)\}$. Then f is said to be *equivalent* to g, and we write $f \sim g$, if $D(f) = D(g)$ (and this common value is a real number), and furthermore the sets of poles of f and g located on the common critical line $\{\operatorname{Re} s = D(f)\}$ coincide. Here, the multiplicities of the poles should be taken into account. In other words, we view the set of principal poles $\mathcal{P}_c(f)$ of f as a multiset. More succinctly,

$$f \sim g \overset{\text{def.}}{\Longleftrightarrow} D(f) = D(g) \ (\in \mathbb{R}) \text{ and } \mathcal{P}_c(f) = \mathcal{P}_c(g). \tag{1.4.7}$$

2. Basic properties of relative distance and tube zeta functions

2.1. Holomorphicity of relative distance zeta functions. We denote by $D(\zeta_{A,\Omega})$ the abscissa of (absolute) convergence of the relative distance zeta function $\zeta_{A,\Omega}$. It is clear that $D(\zeta_{A,\Omega}) \in [-\infty, N]$. Recall from the discussion in Chapter 1 that an analogous definition can be introduced for much more general, tamed Dirichlet-type integrals, introduced in [LRŽ2], as well as in [LRŽ1, especially in Appendix A].

Some of the basic properties of distance zeta functions of RFDs are listed in the following theorem.

THEOREM 2.1. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$. Then:*

(a) *The relative distance zeta function $\zeta_{A,\Omega}$ is holomorphic in the half-plane*

$$\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\},$$

and for those values of s,

$$\zeta'_{A,\Omega}(s) = \int_\Omega d(x,A)^{s-N} \log d(x,A)\,\mathrm{d}x.$$

(b) *The lower bound on the* (*absolute*) *convergence region $\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\}$ of $\zeta_{A,\Omega}$ is optimal. In other words,*

$$D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega). \tag{2.1.1}$$

(c) *If $D := \dim_B(A,\Omega)$ exists, $D < N$ and $\mathcal{M}_*^D(A,\Omega) > 0$, then $\zeta_{A,\Omega}(s) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right. Hence, under these assumptions* (^{1}),

$$D(\zeta_{A,\Omega}) = D_{\mathrm{hol}}(\zeta_{A,\Omega}) = \dim_B(A,\Omega). \tag{2.1.2}$$

We omit the proof since it follows the same steps as in the case when $\Omega := A_\delta$ (that is, in the case of A bounded) [LRŽ2, Theorem 2.5]. In the proof of (a), we need the following result. *For any relative fractal drum (A,Ω) in $\mathbb{R}^N$,*

$$\gamma < N - \overline{\dim}_B(A,\Omega) \;\Rightarrow\; \int_\Omega d(x,A)^{-\gamma}\,\mathrm{d}x < \infty. \tag{2.1.3}$$

If $\Omega = A_\delta$, where δ is a positive real number, this implication reduces to the Harvey–Polking result [HP], since in that case, $\overline{\dim}_B(A,A_\delta) = \overline{\dim}_B A$. We note that the technical condition (1.2.2) on the RFD (A,Ω) from Definition 1.1 is needed for the integrals appearing during the computation of $\zeta_{A,\Omega}$ to be well defined for $\operatorname{Re} s$ large enough.

(^{1}) The abscissa of holomorphic continuation, denoted by $D_{\mathrm{hol}}(\zeta_{A,\Omega})$, is defined prior to (1.4.4).

It is clear that the function $\zeta_{A,\Omega}$ is a tamed generalized Dirichlet-type integral in the sense of [LRŽ2, Definition 2.12]. If (A,Ω) is such that $\zeta_{A,\Omega}$ can be meromorphically extended to an open, connected window $\boldsymbol{W}$ containing the critical line $\{\operatorname{Re} s = D(\zeta_{A,\Omega})\}$, then the poles of $\zeta_{A,\Omega}$ on that line are called the *principal complex dimensions* of (A,Ω). The corresponding multiset of complex dimensions of (A,Ω) is denoted by $\dim_{PC}(A,\Omega)$. In other words,

$$\dim_{PC}(A,\Omega) := \mathcal{P}(\zeta_{A,\Omega}, W) \cap \{\operatorname{Re} s = D(\zeta_{A,\Omega})\}. \tag{2.1.4}$$

It is easy to see that the multiset $\dim_{PC}(A,\Omega)$ does not depend on the choice of the window $\boldsymbol{W}$. If A is a *bounded* subset of $\mathbb{R}^N$ and δ a fixed positive real number, the multiset of *principal complex dimensions of* A is defined by

$$\dim_{PC} A := \dim_{PC}(A, A_\delta), \tag{2.1.5}$$

and this multiset does not depend on the choice of $\delta > 0$.

Analogously, for any bounded fractal string $\mathcal{L}$ for which $D := \overline{\dim}_B \mathcal{L} > 0$, we define the multiset of principal complex dimensions of $\mathcal{L}$ by

$$\dim_{PC} \mathcal{L} := \dim_{PC}(\partial\Omega, \Omega), \tag{2.1.6}$$

where Ω is any geometric realization of the fractal string $\mathcal{L}$; see equation (1.2.9) which connects the standard geometric zeta function $\zeta_{\mathcal{L}}$ with the relative distance zeta function $\zeta_{\partial\Omega,\Omega}$. It is clear that the multiset $\dim_{PC}(\partial\Omega,\Omega)$ does not depend on the choice of the geometric realization Ω, because the same is true for $\zeta_{\partial\Omega,\Omega}$.

In light of [LRŽ2, Theorem 3.3], we have the following result.

Theorem 2.2. *Assume that* (A,Ω) *is a Minkowski nondegenerate RFD in* $\mathbb{R}^N$, *that is,* $0 < \mathcal{M}_*^D(A,\Omega) \leq \mathcal{M}^{*D}(A,\Omega) < \infty$ (*in particular,* $\dim_B(A,\Omega) = D$), *and* $D < N$. *If* $\zeta_{A,\Omega}$ *can be meromorphically continued to a connected open neighborhood of* $s = D$, *then* D *is necessarily a simple pole of* $\zeta_{A,\Omega}$, *and*

$$(N-D)\mathcal{M}_*^D(A,\Omega) \leq \operatorname{res}(\zeta_{A,\Omega}, D) \leq (N-D)\mathcal{M}^{*D}(A,\Omega). \tag{2.1.7}$$

Furthermore, if (A,Ω) *is Minkowski measurable, then*

$$\operatorname{res}(\zeta_{A,\Omega}, D) = (N-D)\mathcal{M}^D(A,\Omega). \tag{2.1.8}$$

In the following example, we compute the relative distance zeta function of an open ball in $\mathbb{R}^N$ with respect to its boundary.

Example 2.3. Let $\Omega := B_R(0)$ be the open ball in $\mathbb{R}^N$ of radius R and let $A = \partial\Omega$. Then, introducing the new variable $\rho = R - r$ and letting $\omega_N := |B_1(0)|_N$, the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$, we have

$$\begin{aligned}
\zeta_{A,\Omega}(s) &= N\omega_N \int_0^R (R-r)^{s-N} r^{N-1}\,\mathrm{d}r = N\omega_N \int_0^R \rho^{s-N}(R-\rho)^{N-1}\,\mathrm{d}\rho \\
&= N\omega_N \int_0^R \rho^{s-N} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} R^{N-1-k}\rho^k\,\mathrm{d}\rho \\
&= N\omega_N R^s \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{(-1)^k}{s-(N-k-1)} = N\omega_N R^s \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{(-1)^{N-j-1}}{s-j}
\end{aligned}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > N-1$. It follows that $\zeta_{A,\Omega}$ can be meromorphically extended to the whole complex plane and is given by

$$\zeta_{A,\Omega}(s) = N\omega_N R^s \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{(-1)^{N-j-1}}{s-j} \tag{2.1.9}$$

for all $s \in \mathbb{C}$. Therefore,

$$\begin{aligned} &\dim_B(A,\Omega) = D(\zeta_{A,\Omega}) = N-1, \\ &\mathcal{P}(\zeta_{A,\Omega}) = \{0,1,\ldots,N-1\}, \quad \dim_{PC}(A,\Omega) = \{N-1\}. \end{aligned} \tag{2.1.10}$$

Furthermore,

$$\operatorname{res}(\zeta_{A,\Omega}, j) = (-1)^{N-j-1} N\omega_N \binom{N-1}{j} R^j \tag{2.1.11}$$

for $j = 0,1,\ldots,N-1$. As a special case of (2.1.11), for $j = D := N-1$ we obtain

$$\operatorname{res}(\zeta_{A,\Omega}, D) = N\omega_N R^{N-1} = \mathcal{M}^D(A,\Omega) = (N-D)\mathcal{M}^D(A,\Omega). \tag{2.1.12}$$

(This is a very special case of (2.1.8) above.) The second to last equality in (2.1.12) follows from the following direct computation (with $D := N-1$):

$$\begin{aligned} \mathcal{M}^D(A,\Omega) &= \lim_{t\to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D}} \\ &= \lim_{t\to 0^+} \frac{\omega_N R^N - \omega_N (R-t)^N}{t} = N\omega_N R^{N-1}. \end{aligned} \tag{2.1.13}$$

Furthermore, recall that $\mathcal{H}^D(A) = \mathcal{H}^{N-1}(\partial B_R(0)) = N\omega_N R^{N-1}$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Hence, $\mathcal{M}^D(A,\Omega) = \mathcal{H}^D(A)$.

Remark 2.4. We note that the usual notions of distance and tube zeta functions, ζ_A and $\tilde{\zeta}_A$, associated with a bounded subset A of $\mathbb{R}^N$, can be recovered by considering the RFD (A, A_δ) for some $\delta > 0$:

$$\begin{aligned} \zeta_A(s) = \zeta_{A,A_\delta}(s) &= \int_{A_\delta} d(x,A)^{s-N}\,\mathrm{d}x, \\ \tilde{\zeta}_A(s) = \tilde{\zeta}_{A,A_\delta}(s) &= \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t. \end{aligned} \tag{2.1.14}$$

Here, $\tilde{\zeta}_{A,A_\delta}$ is the relative tube zeta function of the RFD (A, A_δ), as defined in (2.4.1) below.

2.2. Cone property of relative fractal drums. We introduce the cone property of a relative fractal drum (A,Ω) at a prescribed point in order to show that the abscissa of convergence $D(\zeta_{A,\Omega})$ of the associated relative zeta function $\zeta_{A,\Omega}$ is nonnegative. The main result of this section is stated in Proposition 2.10. We also construct a class of nontrivial RFDs for which the relative box dimension is an arbitrary negative number (Proposition 2.12) or even $-\infty$ (Corollary 2.14 and Remark 2.15, along with Proposition 2.10(a)).

Definition 2.5. Let $B_r(a)$ be a ball in $\mathbb{R}^N$. Assume that G is a closed connected subset contained in a hemisphere of ∂B. Intuitively, G is a disk-like subset ("calotte") of a hemisphere in ∂B. We assume that G is open in the relative topology of ∂B. The *cone* $K = K_r(a, G)$ *with vertex at* a, and of radius r, is defined as the interior of the convex hull of the union of $\{a\}$ and G.

DEFINITION 2.6. Let (A, Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{A} \cap \overline{\Omega} \neq \emptyset$. We say that the (A, Ω) has the *cone property* at $a \in \overline{A} \cap \overline{\Omega}$ if there exists $r > 0$ such that Ω contains the cone $K_r(a, G)$.

REMARK 2.7. If $a \in \overline{A} \cap \Omega$ (hence, a is an inner point of Ω), then the cone property of (A, Ω) is obviously satisfied at a. So, this property is only interesting on the boundary of Ω, that is, at $a \in \overline{A} \cap \partial\Omega$.

EXAMPLE 2.8. Given $\alpha > 0$, let (A, Ω_α) be the relative fractal drum in $\mathbb{R}^2$ defined by $A = \{(0, 0)\}$ and $\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : 0 < y < x^\alpha,\ x \in (0, 1)\}$. If $0 < \alpha \leq 1$, then the cone property of (A, Ω) is fulfilled at $a = (0, 0)$, whereas for $\alpha > 1$ it is not satisfied (at $a = (0, 0)$). Using these domains, we can construct a one-parameter family of RFDs with negative relative box dimension (Proposition 2.12 below).

In order to prove Proposition 2.10 below, we first need an auxiliary result.

LEMMA 2.9. *Let* $K = K_r(a, G)$ *be an open cone in* $\mathbb{R}^N$, *and* $f \in L^1(0, r)$ *a nonnegative function. Then there exists a positive integer* m, *depending only on* N *and on the opening angle of the cone, such that*

$$\int_{B_r(a)} f(|x - a|)\,\mathrm{d}x \leq m \int_K f(|x - a|)\,\mathrm{d}x. \tag{2.2.1}$$

Proof. Since the sphere ∂B is compact, there exist finitely many calottes $G_1, \dots, G_m$ contained in the sphere, which are all congruent to G (that is, each G_i can be obtained from G by a rigid motion, for $i = 1, \dots, m$), and which cover ∂B. Let $K_i = K_r(a, G_i)$, with $i = 1, \dots, m$, be the corresponding cones. It is clear that

$$\int_{K_i} f(|x - a|)\,\mathrm{d}x \tag{2.2.2}$$

does not depend on i. Since $B_r(a) = \bigcup_{i=1}^m K_i$, we then have

$$\int_{B_r(a)} f(|x - a|)\,\mathrm{d}x \leq \sum_{i=1}^m \int_{K_i} f(|x - a|)\,\mathrm{d}x = m \int_K f(|x - a|)\,\mathrm{d}x, \tag{2.2.3}$$

as desired. ■

PROPOSITION 2.10. *Let* (A, Ω) *be a relative fractal drum in* $\mathbb{R}^N$.

(a) *If the sets* A *and* Ω *are a positive distance apart* (*i.e.*, $d(A, \Omega) > 0$), *then* $D(\zeta_{A,\Omega}) = -\infty$, *so* $\zeta_{A,\Omega}$ *is an entire function. Furthermore,* $\dim_B(A, \Omega) = -\infty$.

(b) *Assume that there exists at least one point* $a \in \overline{A} \cap \overline{\Omega}$ *at which* (A, Ω) *satisfies the cone property. Then* $D(\zeta_{A,\Omega}) \geq 0$.

Proof. (a) For $r > 0$ so small that $r < d(A, \Omega)$, where $d(A, \Omega)$ is the distance between A and Ω, we have $A_r \cap \Omega = \emptyset$; so $\zeta_{A, A_r \cap \Omega}(s) = 0$ for all $s \in \mathbb{C}$. Therefore, $D(\zeta_{A, A_r \cap \Omega}) = -\infty$. Since $\zeta_{A,\Omega}(s) - \zeta_{A, A_r \cap \Omega}(s)$ is an entire function, we conclude that also $D(\zeta_{A,\Omega}) = -\infty$.

Since $|A_\varepsilon \cap \Omega| = 0$ for all sufficiently small $\varepsilon > 0$, we have $\mathcal{M}^r(A, \Omega) = 0$ for all $r \in \mathbb{R}$, and therefore $\dim_B(A, \Omega) = -\infty$.

(b) To reach a contradiction, assume that $D(\zeta_{A,\Omega}) < 0$. In particular, $\zeta_{A,\Omega}(s)$ is continuous at $s = 0$ (because it must then be holomorphic at $s = 0$, according to Theorem 2.1(a)). By hypothesis, there exists an open cone $K = K_r(a, G)$ such that $K \subseteq \Omega$. Using the inequality $d(x, A) \leq |x - a|$ (valid for all $x \in \mathbb{R}^N$ since $a \in \Omega$) and Lemma 2.9, we deduce that for any $s \in (0, N)$,

$$\begin{aligned}\zeta_{A,\Omega}(s) \geq \zeta_{A,K}(s) &= \int_K d(x, A)^{s-N}\,\mathrm{d}x \geq \int_K |x - a|^{s-N}\,\mathrm{d}x \\ &\geq \frac{1}{m}\int_{B_r(a)} |x - a|^{s-N}\,\mathrm{d}x = \frac{N\omega_N}{m} r^s s^{-1},\end{aligned}$$

where m is the positive constant appearing in (2.2.1). This implies that $\zeta_{A,\Omega}(s) \to +\infty$ as $s \to 0^+$, $s \in \mathbb{R}$, which contradicts the holomorphicity (or simply the continuity) of $\zeta_{A,\Omega}(s)$ at $s = 0$. ■

The cone condition can be replaced by a much weaker condition, as we explain in the following proposition.

Proposition 2.11. *Let $(r_k)_{k\geq 0}$ be a decreasing sequence of positive real numbers, converging to zero. Define*

$$K_r(a, G, (r_k)_{k\geq 0}) = \Big\{x \in K_r(a, G) : |x - a| \in \bigcup_{k=0}^{\infty} (r_{2k}, r_{2k+1})\Big\}. \tag{2.2.4}$$

If

$$\sum_{k=0}^{\infty} (-1)^k r_k^s \to L > 0 \qquad \textit{as } s \to 0^+,\ s \in \mathbb{R}, \tag{2.2.5}$$

then the conclusion of Proposition 2.10(b) still holds, with the cone condition involving $K := K(a, G)$ replaced by the above modified cone condition, involving the set $K' := K_r(a, G, (r_k)_{k\geq 0})$ contained in K.

Proof. It suffices to use a procedure analogous to the one in the proof of Proposition 2.10:

$$\begin{aligned}\zeta_{A,\Omega}(s) &\geq \int_{K'} |x - a|^{s-N}\,\mathrm{d}x \geq \frac{1}{m}\sum_{k=0}^{\infty}\int_{B_{r_{2k}}(a)\setminus B_{r_{2k+1}}(s)} |x - a|^{s-N}\,\mathrm{d}x \\ &= \frac{N\omega_N}{m} s^{-1}\sum_{k=0}^{\infty}(r_{2k}^s - r_{2k+1}^s) = \frac{N\omega_N}{m} s^{-1}\sum_{k=0}^{\infty}(-1)^k r_k^s.\end{aligned}$$

For example, if $r_k = 2^{-k}$, then condition (2.2.5) is fulfilled since

$$\sum_{k=0}^{\infty}(-1)^k r_k^s = \sum_{k=0}^{\infty}(-1)^k 2^{-ks} = \frac{1}{1 + 2^{-s}} \to \frac{1}{2} \qquad \text{as } s \to 0^+,\ s \in \mathbb{R}.$$

This concludes the proof of the proposition. ■

The following proposition (building on Example 2.8 above) shows that the box dimension of a relative fractal drum can be negative, and even take on any prescribed negative value (see Figure 1).

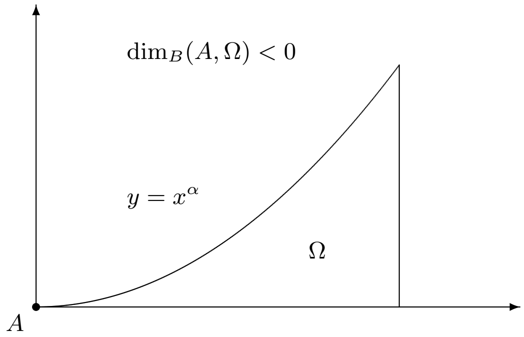

Fig. 1. A relative fractal drum (A,Ω) with negative box dimension $\dim_B(A,\Omega) = 1-\alpha < 0$ (here $\alpha > 1$), due to the "flatness" of the open set Ω at A (see Proposition 2.12). This provides a further illustration of *the drop in dimension phenomenon* (for relative box dimensions).

PROPOSITION 2.12. *Let* $A = \{(0,0)\}$ *and*

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < y < x^\alpha,\ x \in (0,1)\}, \tag{2.2.6}$$

where $\alpha > 1$ (*see Figure* 1). *Then the relative fractal drum* (A,Ω) *has a negative box dimension. More specifically,* $\dim_B(A,\Omega)$ *exists,* (A,Ω) *is Minkowski measurable and*

$$\begin{aligned} &\dim_B(A,\Omega) = D(\zeta_{A,\Omega}) = 1-\alpha < 0, \\ &\mathcal{M}^{1-\alpha}(A,\Omega) = \frac{1}{1+\alpha}, \qquad D_{\mathrm{mer}}(\zeta_{A,\Omega}) \le 3(1-\alpha). \end{aligned} \tag{2.2.7}$$

Furthermore, $s = 1-\alpha$ *is a simple pole of* $\zeta_{A,\Omega}$.

Proof. First note that $A_\varepsilon = B_\varepsilon((0,0))$. Therefore, for every $\varepsilon > 0$, we have

$$|A_\varepsilon \cap \Omega| \le \int_0^\varepsilon x^\alpha \,\mathrm{d}x = \frac{\varepsilon^{\alpha+1}}{\alpha+1}.$$

If we choose

$$(x(\varepsilon), y(\varepsilon)) \in \partial(A_\varepsilon) \cap \{(x,y) : y = x^\alpha,\, x \in (0,1)\},$$

then

$$x(\varepsilon)^2 + x(\varepsilon)^{2\alpha} = \varepsilon^2. \tag{2.2.8}$$

It is clear that

$$|A_\varepsilon \cap \Omega| \ge \int_0^{x(\varepsilon)} x^\alpha \,\mathrm{d}x = \frac{x(\varepsilon)^{\alpha+1}}{\alpha+1}.$$

Letting $D := 1-\alpha$, we conclude that

$$\frac{1}{\alpha+1}\left(\frac{x(\varepsilon)}{\varepsilon}\right)^{\alpha+1} \le \frac{|A_\varepsilon \cap \Omega|}{\varepsilon^{2-D}} \le \frac{1}{\alpha+1} \quad \text{for all } \varepsilon > 0. \tag{2.2.9}$$

We deduce from (2.2.8) that $x(\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0^+$, since

$$x(\varepsilon)/\varepsilon = (1 + x(\varepsilon)^{2(\alpha-1)})^{-1/2} \to 1 \quad \text{as } \varepsilon \to 0^+; \tag{2.2.10}$$

therefore, (2.2.9) implies that $\dim_B(A,\Omega) = D$ and $\mathcal{M}^D(A,\Omega) = 1/(\alpha+1)$.

Using (2.2.9) again, we have

$$0 \le f(\varepsilon) := \frac{1}{\alpha+1} - \frac{|A_\varepsilon \cap \Omega|}{\varepsilon^{2-D}} \le \frac{1}{\alpha+1}\left(1 - \left(\frac{x(\varepsilon)}{\varepsilon}\right)^{\alpha+1}\right). \tag{2.2.11}$$

Using (2.2.10) and the binomial expansion, we conclude that

$$\left(\frac{x(\varepsilon)}{\varepsilon}\right)^{\alpha+1} = 1 - \frac{\alpha+1}{2} x(\varepsilon)^{2\alpha-2} + o(x(\varepsilon)^{2\alpha-2}) \quad \text{as } \varepsilon \to 0^+.$$

Hence, we deduce from (2.2.11) that

$$f(\varepsilon) = O(x(\varepsilon)^{2\alpha-2}) = O(\varepsilon^{2\alpha-2}) \quad \text{as } \varepsilon \to 0^+.$$

Since $|A_\varepsilon \cap \Omega| = \varepsilon^{2-D}((\alpha+1)^{-1} + f(\varepsilon))$, we conclude that

$$D_{\mathrm{mer}}(\zeta_{A,\Omega}) \le D - (2\alpha - 2) = 3(1-\alpha).$$

Furthermore $s = D$ is a simple pole. Finally, we note that the equality $D(\zeta_{A,\Omega}) = D$ follows from (2.1.1). ∎

In the following lemma, we show that for any $\delta > 0$, the sets of principal complex dimensions of (A, Ω) and $(A, A_\delta \cap \Omega)$ coincide.

Lemma 2.13. *Assume that (A, Ω) is a relative fractal drum in $\mathbb{R}^N$. Then, for any $\delta > 0$,*

$$\zeta_{A,\Omega} \sim \zeta_{A,A_\delta\cap\Omega}, \tag{2.2.12}$$

where $\sim$ is defined in Definition 1.7. In particular,

$$\dim_{PC}(A, \Omega) = \dim_{PC}(A, A_\delta \cap \Omega), \tag{2.2.13}$$

and therefore

$$\overline{\dim}_B(A, \Omega) = \overline{\dim}_B(A, A_\delta \cap \Omega). \tag{2.2.14}$$

Here, A_δ, the δ-neighborhood of A, can be taken with respect to any norm on $\mathbb{R}^N$. (This extra freedom will be used in Corollary 2.14 below.)

Proof. Recall that according to Definition 1.1, there exists $\delta_1 > 0$ such that $d(x, A) < \delta_1$ for all $x \in \Omega$. On the other hand, $d(x, A) > \delta$ for all $x \in \Omega \setminus A_\delta$. Therefore,

$$\zeta_{A,\Omega}(s) - \zeta_{A,A_\delta\cap\Omega}(s) = \int_{\Omega\setminus A_\delta} d(x, A)^{s-N}\, \mathrm{d}x$$

defines an entire function. This proves (2.2.12). The remaining claims follow immediately from this equivalence. Finally, the fact that any norm on $\mathbb{R}^N$ can be chosen to define A_δ follows from the equivalence of all norms on $\mathbb{R}^N$. ∎

The following result provides an example of a nontrivial relative fractal drum (A, Ω) such that $\dim_B(A, \Omega) = -\infty$. It suffices to construct a domain Ω of $\mathbb{R}^2$ which is *flat* in a neighborhood of one of its boundary points.

Corollary 2.14 (A maximally flat RFD). *Let $A = \{(0, 0)\}$ and*

$$\Omega' = \{(x, y) \in \mathbb{R}^2 : 0 < y < \mathrm{e}^{-1/x},\ 0 < x < 1\}. \tag{2.2.15}$$

Then $\dim_B(A, \Omega')$ exists and

$$\dim_B(A, \Omega') = D(\zeta_{A,\Omega'}) = -\infty. \tag{2.2.16}$$

Proof. Fix $\alpha > 1$. Then, by l'Hospital's rule,

$$\lim_{x\to 0^+} \frac{\mathrm{e}^{-1/x}}{x^\alpha} = \lim_{t\to+\infty} \frac{t^\alpha}{\mathrm{e}^t} = 0.$$

Hence, there exists $\delta = \delta(\alpha) > 0$ such that $0 < \mathrm{e}^{-1/x} < x^\alpha$ for all $x \in (0, \delta)$; that is,

$$\Omega'_{\delta(\alpha)} \subset \Omega_{\delta(\alpha)},$$

where

$$\Omega'_{\delta(\alpha)} := \{(x, y) \in \mathbb{R}^2 : 0 < y < \mathrm{e}^{-1/x},\ 0 < x < \delta(\alpha)\},$$
$$\Omega_{\delta(\alpha)} := \{(x, y) \in \mathbb{R}^2 : 0 < y < x^\alpha,\ 0 < x < \delta(\alpha)\}.$$

Using Lemma 2.13 with Ω' instead of Ω and with the ℓ^∞-norm on $\mathbb{R}^2$ instead of the usual Euclidean norm (note that $\Omega'_{\delta(\alpha)} = \Omega' \cap B_{\delta(\alpha)}(0)$, where $B_\delta(0) := \{(x, y) \in \mathbb{R}^2 : |(x, y)|_\infty < \delta\}$ and $|(x, y)|_\infty := \max\{|x|, |y|\}$), along with Proposition 2.12, we see that

$$\overline{\dim}_B(A, \Omega') = \overline{\dim}_B(A, \Omega'_{\delta(\alpha)}) \le \dim_B(A, \Omega_{\delta(\alpha)}) = 1 - \alpha.$$

The claim follows by letting $\alpha \to +\infty$, since then we obtain

$$-\infty \le \underline{\dim}_B(A, \Omega') \le \overline{\dim}_B(A, \Omega') = -\infty.$$

We conclude, as desired, that $\dim_B(A, \Omega)$ exists and is equal to $-\infty$. ∎

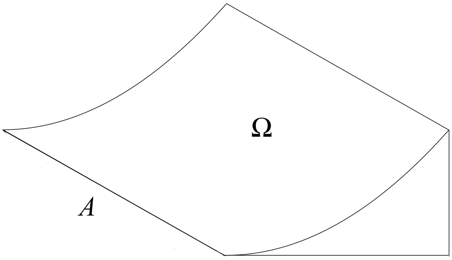

Fig. 2. A relative fractal drum (A, Ω) with infinite flatness, as described in Remark 2.15. In other words, Ω has infinite flatness near A; equivalently, $\dim_B(A, \Omega) = -\infty$, which provides an even more dramatic illustration of the *drop in dimension phenomenon* (for relative box dimensions).

Remark 2.15 (Flatness and "infinitely sharp blade"). It is easy to see that Corollary 2.14 can be significantly generalized. For example, it suffices to assume that $a \in \Omega$ is such that the *flatness property of* A (*at* a) *relative to* Ω holds. This can even be formulated in terms of subsets A of the boundary of Ω. We can imagine a bounded open set $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$, except on a subset $A \subset \partial\Omega$, which may be a line segment, near which Ω is flat (see Figure 2). A simple construction of such a set is $\Omega = \Omega' \times (0, 1)$, where Ω' is as in Corollary 2.14, and $A = \{(0, 0)\} \times (0, 1)$ (see (2.2.15)). Note that this domain is not Lipschitz near the points of A, and not even Hölderian. The *flatness of a relative fractal drum* (A, Ω) can be defined by

$$\mathrm{fl}(A, \Omega) = (\overline{\dim}_B(A, \Omega))^-,$$

where $(r)^- := \max\{0, -r\}$ is the negative part of a real number r. We say that *the flatness of* (A, Ω) *is nontrivial if* $\mathrm{fl}(A, \Omega) > 0$, that is, $\overline{\dim}_B(A, \Omega) < 0$. In the example above, we have a relative fractal drum (A, Ω) with infinite flatness, i.e., with $\mathrm{fl}(A, \Omega) = +\infty$. Intuitively, it can be viewed as an "ax" with an "infinitely sharp" blade.

2.3. Scaling property of relative distance zeta functions. We start this section with the following result, which shows that if (A, Ω) is a relative fractal drum, then for any $\lambda > 0$, the zeta function $\zeta_{\lambda A, \lambda\Omega}(s)$ of the scaled relative fractal drum $\lambda(A, \Omega) := (\lambda A, \lambda\Omega)$ is equal to the zeta function $\zeta_{A,\Omega}(s)$ of (A, Ω) multiplied by λ^s.

THEOREM 2.16 (Scaling property of relative distance zeta functions). *Let* $\zeta_{A,\Omega}(s)$ *be the relative distance zeta function of an RFD* (A, Ω). *Then, for any positive real number* λ, *we have* $D(\zeta_{\lambda A, \lambda\Omega}) = D(\zeta_{A,\Omega}) = \overline{\dim}_B(A, \Omega)$, *and*

$$\zeta_{\lambda A, \lambda\Omega}(s) = \lambda^s \zeta_{A,\Omega}(s) \tag{2.3.1}$$

for all $s \in \mathbb{C}$ *with* $\mathrm{Re}\, s > \overline{\dim}_B(A, \Omega)$ *and any* $\lambda > 0$. (*See also Corollary* 2.17 *below for a more general statement.*)

Proof. The claim is established by introducing a new variable $y = x/\lambda$, and by noting that $d(\lambda y, \lambda A) = \lambda d(y, A)$ for any $y \in \mathbb{R}^N$ (an easy consequence of the homogeneity of the Euclidean norm). Indeed, in light of part Theorem 2.1(b), for any $s \in \mathbb{C}$ with $\mathrm{Re}\, s > \overline{\dim}_B(A, \Omega) = D(\zeta_{A,\Omega})$, we have

$$\begin{aligned}\zeta_{\lambda A, \lambda\Omega}(s) &= \int_{\lambda\Omega} d(x, \lambda A)^{s-N}\, \mathrm{d}x = \int_{\Omega} d(\lambda y, \lambda A)^{s-N} \lambda^N\, \mathrm{d}y \\ &= \lambda^s \int_{\Omega} d(y, A)^{s-N}\, \mathrm{d}y = \lambda^s \zeta_{A,\Omega}(s).\end{aligned}$$

It follows that (2.3.1) holds and $\zeta_{\lambda A, \lambda\Omega}$ is holomorphic for $\mathrm{Re}\, s > \overline{\dim}_B(A, \Omega)$. Since $D(\zeta_{A,\Omega}) = \overline{\dim}_B(A, \Omega)$ (by Theorem 2.1(b)), we deduce that $D(\zeta_{\lambda A, \lambda\Omega}) \le D(\zeta_{A,\Omega})$ for every $\lambda > 0$. But then, replacing λ by its reciprocal λ^{-1} in this last inequality, we obtain the reverse inequality (more specifically, we replace (A, Ω) by $(\lambda^{-1}A, \lambda^{-1}\Omega)$ to deduce that for every $\lambda > 0$, $D(\zeta_{A,\Omega}) \le D(\zeta_{\lambda^{-1}A, \lambda^{-1}\Omega})$; we then substitute λ^{-1} for λ to obtain $D(\zeta_{A,\Omega}) \le D(\zeta_{\lambda A, \lambda\Omega})$ for every $\lambda > 0$). Hence, we conclude that

$$\overline{\dim}_B(A, \Omega) = D(\zeta_{A,\Omega}) = D(\zeta_{\lambda A, \lambda\Omega})$$

for all $\lambda > 0$, as desired. ∎

We note that if $\mathcal{L} = (\ell_j)_{j\ge1}$ is a fractal string, and λ is a positive constant, then for the scaled string $\lambda\mathcal{L} := (\lambda\ell_j)_{j\ge1}$, the corresponding claim in Theorem 2.16 is trivial: $\zeta_{\lambda\mathcal{L}}(s) = \lambda^s \zeta_{\mathcal{L}}(s)$ for every $\lambda > 0$. Indeed, by definition of the geometric zeta function of a fractal string (see (1.2.8)), we have

$$\zeta_{\lambda\mathcal{L}}(s) = \sum_{j=1}^{\infty} (\lambda\ell_j)^s = \lambda^s \sum_{j=1}^{\infty} \ell_j^s = \lambda^s \zeta_{\mathcal{L}}(s)$$

for $\mathrm{Re}\, s > D(\zeta_{\mathcal{L}})$. (The same argument as above then shows that $D(\zeta_{\mathcal{L}}) = D(\zeta_{\lambda\mathcal{L}})$.) Then, by analytic (i.e., meromorphic) continuation, the same identity continues to hold in any domain to which $\zeta_{\mathcal{L}}$ can be meromorphically extended to the left of the critical line $\{\operatorname{Re} s = D(\zeta_{\mathcal{L}})\}$.

The following result supplements Theorem 2.16 in several ways.

COROLLARY 2.17. *Fix* $\lambda > 0$. *Assume that* $\zeta_{A,\Omega}$ *admits a meromorphic continuation to some open connected neighborhood* U *of the open half-plane* $\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\}$. *Then so does* $\zeta_{\lambda A,\lambda\Omega}$ *and the identity* (2.3.1) *continues to hold for every* $s \in U$ *which is not a pole of* $\zeta_{A,\Omega}$ (*and hence not a pole of* $\zeta_{\lambda A,\lambda\Omega}$ *either*).

Moreover, if we assume, for simplicity, that ω *is a simple pole of* $\zeta_{A,\Omega}$ (*and hence also of* $\zeta_{\lambda A,\lambda\Omega}$), *then*

$$\operatorname{res}(\zeta_{\lambda(A,\Omega)},\omega) = \lambda^{\omega}\operatorname{res}(\zeta_{A,\Omega},\omega). \tag{2.3.2}$$

If s *is a multiple pole, then an analogous statement can be made about the principal parts of the zeta functions involved* (*instead of the residues*), *as the reader can easily verify.*

Proof. The fact that $\zeta_{\lambda A,\lambda\Omega}$ is holomorphic at $s \in U$ if $\zeta_{A,\Omega}$ is holomorphic at s (for example, if $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$), follows from (2.3.1) and the equality $D(\zeta_{\lambda A,\lambda\Omega}) = D(\zeta_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$. An analogous statement is true if "holomorphic" is replaced with "meromorphic". More specifically, by analytic continuation of (2.3.1), $\zeta_{\lambda A,\lambda\Omega}$ is meromorphic in an open connected set U (containing the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$) if and only if $\zeta_{A,\Omega}$ is meromorphic in U, and then, clearly, identity (2.3.1) continues to hold for every $s \in U$ which is not a pole of $\zeta_{A,\Omega}$ (and hence not a pole of $\zeta_{\lambda A,\lambda\Omega}$). Therefore, the first part of the corollary is established.

Next, assume that ω is a simple pole of $\zeta_{A,\Omega}$. Then, in light of (2.3.1) and the discussion in the previous paragraph, for all s in a punctured neighborhood of ω (contained in U but not containing any other pole of $\zeta_{A,\Omega}$),

$$(s-\omega)\zeta_{\lambda(A,\Omega)}(s) = \lambda^{s}\left((s-\omega)\zeta_{A,\Omega}(s)\right). \tag{2.3.3}$$

The fact that (2.3.2) holds now follows by letting $s \to \omega$, $s \neq \omega$ in (2.3.3). Indeed, we then have

$$\operatorname{res}(\zeta_{A,\Omega},\omega) = \lim_{s\to\omega}(s-\omega)\zeta_{A,\Omega}(s),$$

and similarly for $\operatorname{res}(\zeta_{\lambda(A,\Omega)},\omega)$. ■

The scaling property of relative zeta functions (established in Theorem 2.16 and Corollary 2.17) motivates us to introduce the notion of relative fractal spray, which is very close to (but not identical with) the usual notion of fractal spray introduced in [LPo2] (see [LF3] and the references therein, including [LP1]–[LP3], [LPW1, LPW2], [P], [PW], [DDKÜ], [DKÖÜ]). First, we define the operation of union of (disjoint) families of RFDs.

DEFINITION 2.18. Let $(A_j,\Omega_j)_{j\geq 1}$ be a countable family of relative fractal drums in $\mathbb{R}^N$, such that the family $(\Omega_j)_{j\geq 1}$ is disjoint (i.e., $\Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$), $A_j \subseteq \Omega_j$ for each $j \in \mathbb{N}$, and the set $\Omega := \bigcup_{j=1}^{\infty}\Omega_j$ is of finite N-dimensional Lebesgue measure (but may be unbounded). Then the *union of the family* $(A_j,\Omega_j)_{j\geq 1}$ is the relative fractal drum (A,Ω), where $A := \bigcup_{j=1}^{\infty} A_j$ and $\Omega := \bigcup_{j=1}^{\infty}\Omega_j$. We write

$$(A,\Omega) = \bigsqcup_{j=1}^{\infty}(A_j,\Omega_j). \tag{2.3.4}$$

It is easy to derive the following countable additivity property of the distance zeta functions.

THEOREM 2.19. *Assume that* $(A_j,\Omega_j)_{j\ge1}$ *is a family of RFDs satisfying the conditions of Definition* 2.18*, and let* (A,Ω) *be its union. Furthermore, assume that*

$$d(x,A)=d(x,A_j)\quad \text{for any } j\in\mathbb{N} \text{ and } x\in\Omega_j. \tag{2.3.5}$$

Then, for $\operatorname{Re}s>\overline{\dim}_B(A,\Omega)$,

$$\zeta_{A,\Omega}(s)=\sum_{j=1}^{\infty}\zeta_{A_j,\Omega_j}(s). \tag{2.3.6}$$

Condition (2.3.5) *is satisfied, for example, if* $A_j:=\partial\Omega_j$ *for every* $j\in\mathbb{N}$.

Proof. The claim follows from the following computation, valid for $\operatorname{Re}s>\overline{\dim}_B(A,\Omega)$:

$$\begin{aligned}\zeta_{A,\Omega}(s)&=\int_\Omega d(x,A)^{s-N}\,\mathrm{d}x=\sum_{j=1}^{\infty}\int_{\Omega_j}d(x,A)^{s-N}\,\mathrm{d}x\\&=\sum_{j=1}^{\infty}\int_{\Omega_j}d(x,A_j)^{s-N}\,\mathrm{d}x=\sum_{j=1}^{\infty}\zeta_{A_j,\Omega_j}(s).\end{aligned} \tag{2.3.7}$$

More specifically, (2.3.7) clearly holds for every real $s>\overline{\dim}_B(A,\Omega)\ge D(\zeta_{A,\Omega})$. Therefore, for such s,

$$\zeta_{A,\Omega_j}(s)=\int_{\Omega_j}d(x,A)^{s-N}\,\mathrm{d}x\le\int_\Omega d(x,A)^{s-N}\,\mathrm{d}x=\zeta_{A,\Omega}(s)<\infty$$

for every $j\ge1$. Hence,

$$\sup_{j\ge1}\{D(\zeta_{A,\Omega_j})\}\le D(\zeta_{A,\Omega})\le\overline{\dim}_B(A,\Omega), \tag{2.3.8}$$

from which (2.3.7) now follows for all $s\in\mathbb{C}$ with $\operatorname{Re}s>\overline{\dim}_B(A,\Omega)$, in light of the countable additivity of the local complex Borel measure (and hence, locally bounded measure) on Ω, given by $\mathrm{d}\gamma(x):=d(x,A)^{s-N}\,\mathrm{d}x$. (Note that according to the hypothesis of Definition 2.18 we have $|\Omega|<\infty$, so that $\mathrm{d}\gamma$ is indeed a local complex Borel measure; see, e.g., [Fo] or [Ru], along with [DF], [JL], [JLN] and [LRŽ1, Appendix A] for the notion of a local measure.) ■

REMARK 2.20. In (2.3.6), the numerical series on the right-hand side converges absolutely (and hence in $\mathbb{C}$) for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>\overline{\dim}_B(A,\Omega)$. In particular, for every real $s>\overline{\dim}_B(A,\Omega)$, it is a convergent series of positive terms (i.e., it has a finite sum). It remains to be investigated whether (and under which hypotheses) equation (2.3.6) continues to hold for all $s\in\mathbb{C}$ in a common domain of meromorphicity of $\zeta_{A,\Omega}$ and ζ_{A,Ω_j} for $j\ge1$ (away from the poles). At the poles, an analogous question could be raised for the corresponding residues (assuming, for simplicity, that the poles are simple).

2.4. Relative tube zeta functions. We begin this section by introducing the *relative tube zeta function* associated with the relative fractal drum (A,Ω) in $\mathbb{R}^N$. It is defined by

$$\tilde\zeta_{A,\Omega}(s):=\int_0^\delta t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t \tag{2.4.1}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large, where $\delta > 0$ is fixed. As we see, $\tilde{\zeta}_{A,\Omega}$ involves the *relative tube function* $t \mapsto |A_t \cap \Omega|$. As was noted in Remark 2.4, if $\Omega := A_\delta$ with $A \subset \mathbb{R}^N$ bounded, we recover the tube zeta function $\tilde{\zeta}_A(s) := \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t$.

The abscissa of convergence of $\tilde{\zeta}_{A,\Omega}$ is given by $D(\tilde{\zeta}_{A,\Omega}) = \overline{\dim}_B(A,\Omega)$. This follows from the following fundamental identity (or *functional equation*), which connects $\tilde{\zeta}_{A,\Omega}$ and the relative distance zeta function $\zeta_{A,\Omega}$, defined by (1.2.7):

$$\zeta_{A,A_\delta\cap\Omega}(s) = \delta^{s-N}|A_\delta \cap \Omega| + (N-s)\tilde{\zeta}_{A,\Omega}(s) \tag{2.4.2}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$. Its proof is based on the the known identity

$$\int_{A_\delta\cap\Omega} d(x,A)^{-\gamma}\,\mathrm{d}x = \delta^{-\gamma}|A_\delta \cap \Omega| + \gamma \int_0^\delta t^{-\gamma-1}|A_t \cap \Omega|\,\mathrm{d}t, \tag{2.4.3}$$

where $\gamma > 0$ (see [Ž1, Theorem 2.9(a)], or a more general form in [Ž2, Lemma 3.1]). As a special case, when $\Omega := A_\delta$ with $A \subset \mathbb{R}^N$ bounded, (2.4.2) reduces to

$$\zeta_A(s) = \delta^{s-N}|A_\delta| + (N-s)\tilde{\zeta}_A(s) \tag{2.4.4}$$

for $\operatorname{Re} s > \overline{\dim}_B A$, which has been obtained in [LRŽ2].

The following proposition connects the residues of the relative tube and distance zeta functions.

Proposition 2.21. *Assume that (A,Ω) is an RFD in $\mathbb{R}^N$. Let U be a connected open subset of $\mathbb{C}$ which contains the critical line $\{\operatorname{Re} s = D(\zeta_{A,\Omega})\}$ and to which the relative distance zeta function $\zeta_{A,\Omega}$ can be meromorphically extended. Then the relative tube function $\tilde{\zeta}_{A,\Omega}$ can be meromorphically extended to U as well. Furthermore, if $\omega \in U$ is a simple pole of $\zeta_{A,\Omega}$, then it is also a simple pole of $\tilde{\zeta}_{A,\Omega}$ and*

$$\operatorname{res}(\zeta_{A,\Omega},\omega) = (N-\omega)\cdot\operatorname{res}(\tilde{\zeta}_{A,\Omega},\omega). \tag{2.4.5}$$

Moreover, the functional equation (2.4.2) continues to hold for all $s \in U$.

The proposition also holds if we interchange the relative distance function and the relative tube function.

Proof. Since $\zeta_{A,\Omega}(s) - \zeta_{A,A_\delta\cap\Omega}(s) = \zeta_{A,\Omega\setminus A_\delta\cap\Omega}(s)$ is an entire function (note that $\delta < d(x,A) < c$, where $c := \sup_{x\in A} d(x,A) < \infty$; see (1.2.2)), it suffices to prove the proposition for $(A, A_\delta \cap \Omega)$ instead of (A,Ω). The claim now follows from the functional equation (2.4.2). ■

Example 2.22 (Torus relative fractal drum). Let Ω be an open solid torus in $\mathbb{R}^3$ defined by two radii r and R, where $0 < r < R < \infty$, and let $A := \partial\Omega$ be its topological boundary. In order to compute the tube zeta function of the *torus RFD* (A,Ω), we first compute its tube function. Let $\delta \in (0,r)$. Using Cavalieri's principle, we have

$$|A_t \cap \Omega|_3 = 2\pi R(r^2 - (r-t)^2) = 2\pi R(2rt - t^2) \tag{2.4.6}$$

for all $t \in (0,\delta)$, from which it follows that

$$\tilde{\zeta}_{A,\Omega}(s) := \int_0^\delta t^{s-4}|A_t \cap \Omega|_3\,\mathrm{d}t = 2\pi R\left(2r\frac{\delta^{s-2}}{s-2} - \frac{\delta^{s-1}}{s-1}\right) \tag{2.4.7}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > 2$. The right-hand side defines a meromorphic function on the entire complex plane, so that, using the principle of analytic continuation, $\tilde{\zeta}_{A,\Omega}$ can be (uniquely) meromorphically extended to the whole of $\mathbb{C}$. In particular, the multiset of complex dimensions of the torus RFD (A,Ω) is given by $\mathcal{P}(A,\Omega)=\{1,2\}$. Each of the complex dimensions 1 and 2 is simple. In particular,

$$\dim_{PC}(A,\Omega)=\{2\} \quad \text{and} \quad \operatorname{res}(\tilde{\zeta}_{A,\Omega},2)=4\pi Rr. \tag{2.4.8}$$

Also, $\overline{\dim}_B(A,\Omega)=D(\tilde{\zeta}_A)=2$. From (2.5.8) below, we conclude that the 2-dimensional Minkowski content of (A,Ω) is given by

$$\mathcal{M}^2(A,\Omega)=4\pi Rr. \tag{2.4.9}$$

Since $|A_t|_3=2\pi R\big((r+t)^2-(r-t)^2\big)$, we can also easily compute the "ordinary" tube zeta function $\tilde{\zeta}_A$ of the torus surface A in $\mathbb{R}^3$:

$$\tilde{\zeta}_A(s)=8\pi Rr\frac{\delta^{s-2}}{s-2} \tag{2.4.10}$$

for all $s\in\mathbb{C}$. In particular, $\operatorname{res}(\tilde{\zeta}_A,2)=8\pi Rr$. Using (2.4.2) and (2.4.4), we deduce from (2.4.10) the corresponding expressions for the distance zeta functions, valid for all $s\in\mathbb{C}$:

$$\zeta_{A,\Omega}(s)=2\pi R\left(2r\frac{\delta^{s-1}}{s-2}-\frac{2}{s-1}\right), \quad \zeta_A(s)=8\pi Rr\frac{\delta^{s-2}}{s-2}. \tag{2.4.11}$$

Also,

$$\mathcal{P}(\zeta_{A,\Omega})=\mathcal{P}(\tilde{\zeta}_{A,\Omega})=\{1,2\} \quad \text{and} \quad \mathcal{P}_c(\zeta_{A,\Omega})=\mathcal{P}_c(\tilde{\zeta}_{A,\Omega})=\{2\}$$

(with each pole 1 and 2 being simple) and

$$\overline{\dim}_B(A,\Omega)=D(\zeta_{A,\Omega})=D(\tilde{\zeta}_{A,\Omega})=2.$$

Furthermore, $\operatorname{res}(\zeta_{A,\Omega},2)=4\pi Rr$ and $\operatorname{res}(\zeta_A,2)=8\pi Rr$, in agreement with (2.4.5).

One can easily extend the example of the 2-torus to any (smooth) closed submanifold of $\mathbb{R}^N$ (in particular to the n-torus with $n\ge 2$). This can be done by using Federer's tube formula [Fe1] for sets of positive reach, which extends and unifies Weyl's tube formula [We] for (proper) smooth submanifolds of $\mathbb{R}^N$ and Steiner's formula (obtained by Steiner [Ste] and his successors) for compact convex subsets of $\mathbb{R}^N$. The global form of Federer's tube formula expresses the volume of t-neighborhoods of a (compact) set of positive reach (2) $A\subset\mathbb{R}^N$ as a polynomial of degree at most N in t, whose coefficients are (essentially) the so-called *Federer curvatures* and which generalize Weyl's curvatures [We] (see [BG] for an exposition) and Steiner's curvatures [Ste] (see [S1, Chapter 4] for a detailed exposition) in the case of submanifolds and compact convex sets, respectively. We also draw the attention of the reader to the related notions of "fractal curvatures" and "fractal curvature measures" for fractal sets, introduced by Winter [W] and Winter and Zahle [WZ]; for information on closely related topics in integral geometry and on tube formulas, see also, for example, [Ste], [Mi], [We], [Bl], [Fe1], [KR], [Z1]–[Z3], [S1, S2], [HLW], [LP3], [LPW1], [LRŽ1] and [LRŽ4]–[LRŽ6], along with the many relevant references therein.

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(2) A closed subset C of $\mathbb{R}^N$ is said to be of *positive reach* if there exists $\delta_0>0$ such that every point $x\in\mathbb{R}^N$ less than δ_0 away from C has a unique metric projection onto C [Fe1]. The *reach* of C is defined as the supremum of all such δ_0. Clearly, every closed convex subset of $\mathbb{R}^N$ is of infinite (and hence positive) reach.

In the present context, for a compact set $C \subset \mathbb{R}^N$ of positive reach, it is easy to deduce from the tube formula in [Fe1] an explicit expression for $\tilde{\zeta}_A(s)$ (with $A := \partial C$) (³):

Proposition 2.23. *Let $A = \partial C$ be the boundary of a* (*nonempty*) *compact set C of positive reach in $\mathbb{R}^N$. Then, for any $\delta > 0$ sufficiently small* (*less than the reach of A, in particular*), *we have*

$$\tilde{\zeta}_A(s) := \tilde{\zeta}_A(s;\delta) = \sum_{k=0}^{N-1} c_k \frac{\delta^{s-k}}{s-k}, \tag{2.4.12}$$

where $|A_t| = \sum_{k=0}^{N-1} c_k t^{N-k}$ for all $t \in (0,\delta)$ and the coefficients c_k are the (*normalized*) *Federer curvatures.* (*From the functional equation* (2.4.2), *one then deduces at once the corresponding explicit expression for $\zeta_A(s) := \zeta_A(s;\delta)$.*)

Hence, $\dim_B A$ exists and

$$D := D(\tilde{\zeta}_A) = D(\zeta_A) = \dim_B A = \max\{k \in \{0,1,\dots,N-1\} : c_k \neq 0\}, \tag{2.4.13}$$

and (*since $D \le N-1 < N$*)

$$\mathcal{P} := \mathcal{P}(\tilde{\zeta}_A) = \mathcal{P}(\zeta_A) \subseteq \{0,1,\dots,N-1\}. \tag{2.4.14}$$

In fact,

$$\mathcal{P} = \big\{k \in \{0,1,\dots,N-1\} : c_k \neq 0\big\} \subseteq \{k_0,\dots,D\}, \tag{2.4.15}$$

where $k_0 := \min\{k \in \{0,1,\dots,D\} : c_k \neq 0\}$. Furthermore, each of the complex dimensions of A is simple.

Finally, if the affine hull of C is all of $\mathbb{R}^N$ (*which is the case when the interior of C is nonempty, in particular if C is a convex body*), *then $D = N-1$, while if C is a* (*smooth*) *submanifold with boundary the closed d-dimensional smooth submanifold A* (*with $0 \le d \le N-1$*), *then $D = d$* (⁴).

For the 2-torus A, we have $N = 3$, $D = 2$ (since the Euler characteristic of A is equal to zero), $c_2 \neq 0$ (⁵), $c_1 \neq 0$, and hence $c_0 = 0$, $k_0 = 1$ and $\mathcal{P} = \{1,2\}$, as was found in Example 2.22 via a direct computation.

We note that much more general tube formulas called "fractal tube formulas" are obtained in [LRŽ5] (as well as in [LRŽ1, Chapter 5], see also [LRŽ4]) for arbitrary bounded sets (and even more generally, RFDs) in $\mathbb{R}^N$, under mild growth assumptions on the associated fractal zeta functions.

(³) Relative versions are also possible, for example for the RFD $(A, \mathring{C})$, where $\mathring{C}$ is the interior of C, assumed to be nonempty. Under appropriate assumptions, the associated expression for $\tilde{\zeta}_A$ may take a slightly different form (because the corresponding tube formula would be of pluriphase type in the sense of [LP1]–[LP3], [LPW1], that is, piecewise polynomial of degree $\le N-1$), but (2.4.14) would remain valid in this case.

(⁴) One could also work with a closed (i.e., boundaryless) d-dimensional submanifold of $\mathbb{R}^N$, with $0 \le d \le N-1$.

(⁵) Note that c_2 is just proportional to the area of the 2-torus, with the proportionality constant being a standard positive constant.

2.5. Meromorphic extensions of relative zeta functions. We shall use the following assumption on the asymptotics of the relative tube function $t \mapsto |A_t \cap \Omega|$:

$$|A_t \cap \Omega| = t^{N-D} h(t)(\mathcal{M} + O(t^\alpha)) \qquad \text{as } t \to 0^+, \tag{2.5.1}$$

where $\mathcal{M}, \alpha > 0$ and $D \le N$ are given in advance. Here, we assume that the function $h(t)$ is positive and has a sufficiently slow growth near the origin, in the sense that for any $c > 0$, $h(t) = O(t^c)$ as $t \to 0^+$. Typical examples of such functions are $h(t) = (\log t^{-1})^m$, $m \ge 1$, or more generally,

$$h(t) = \big(\underbrace{\log \ldots \log}_{n}(t^{-1})\big)^m$$

for $n \ge 1$, and in these cases obviously $\mathcal{M}^D(A, \Omega) = +\infty$. For this and other examples, see [HeL]. The function $t \mapsto t^D h(t)^{-1}$ is usually called a *gauge function*, but for simplicity we shall use this name only for $h(t)$.

Assuming that a relative fractal drum (A, Ω) in $\mathbb{R}^N$ is such that $D = \dim_B(A, \Omega)$ exists, and $\mathcal{M}^D_*(A, \Omega) = 0$ or $+\infty$ (or $\mathcal{M}^{*D}(A, \Omega) = 0$ or $+\infty$), it makes sense to define a new class of relative lower and upper Minkowski contents of (A, Ω), associated with a suitably chosen gauge function $h(t)$ as follows:

$$\begin{aligned} \mathcal{M}^D_*(A, \Omega, h) &= \liminf_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} h(t)}, \\ \mathcal{M}^{*D}(A, \Omega, h) &= \limsup_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} h(t)}. \end{aligned} \tag{2.5.2}$$

The aim is to find an *explicit* gauge function so that these two contents are in $(0, +\infty)$, and the functions $r \mapsto \mathcal{M}^r_*(A, \Omega, h)$ and $r \mapsto \mathcal{M}^{*r}(A, \Omega, h)$, $r \in \mathbb{R}$, defined exactly as in (2.5.2), except for D replaced with r, have a jump from $+\infty$ to 0 when r crosses the value of D. In this generality, the above contents are called *gauge relative Minkowski contents* (with respect to h).

If for some gauge function h we have $\mathcal{M}^D(A, \Omega, h) \in (0, +\infty)$ (which means, as usual, that $\mathcal{M}^D_*(A, \Omega, h) = \mathcal{M}^{*D}(A, \Omega, h)$ and that this common value, denoted by $\mathcal{M}^D(A, \Omega, h)$, lies in $(0, +\infty)$), we say (as in [HeL]) that the fractal drum (A, Ω) is *h-Minkowski measurable.*

In what follows, we denote the Laurent expansion of a meromorphic extension (assumed to exist) of the relative tube zeta function $\tilde{\zeta}_{A,\Omega}$ to an open, connected neighborhood of $s = D$ (more specifically, an open punctured disk centered at $s = D$) by

$$\tilde{\zeta}_{A,\Omega}(s) = \sum_{j=-\infty}^{\infty} c_j (s - D)^j, \tag{2.5.3}$$

where, of course, $c_j = 0$ for all $j \ll 0$ (that is, there exists $j_0 \in \mathbb{Z}$ such that $c_j = 0$ for all $j < j_0$).

Let us first introduce some notation. Given a T-periodic function $G : \mathbb{R} \to \mathbb{R}$, we denote by G_0 its truncation to $[0, T]$:

$$G_0(\tau) = \begin{cases} G(t) & \text{if } \tau \in [0, T], \\ 0 & \text{if } \tau \notin [0, T], \end{cases} \tag{2.5.4}$$

while the Fourier transform of G_0 is denoted by $\hat{G}_0$:

$$\hat{G}_0(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi \mathbb{i}\, t\tau} G_0(\tau)\,\mathrm{d}\tau = \int_0^T \mathrm{e}^{-2\pi \mathbb{i}\, t\tau} G(\tau)\,\mathrm{d}\tau, \tag{2.5.5}$$

where $\mathbb{i} := \sqrt{-1}$ is the imaginary unit.

The following theorem shows that, in order to obtain a meromorphic extension of the zeta function to the left of the abscissa of convergence, it is important to have some information about the second term in the asymptotic expansion of the relative tube function $t \mapsto |A_\delta \cap \Omega|$ near $t = 0$. We stress that the presence (in Theorem 2.24) of the gauge function $h(t) := (\log t^{-1})^m$ is closely related to the multiplicity of the principal complex dimension D, which is equal to $m + 1$. Theorem 2.24 extends [LRŽ3, Theorem 4.24] to the general setting of RFDs.

Observe that since $m = 0$ is allowed in Theorem 2.24 below, that theorem enables us to deal, in particular, with the usual Minkowski measurable RFDs (for which the gauge function h is trivial, i.e., $h(t) \equiv 1$).

Theorem 2.24 (Minkowski measurable RFDs). *Let* (A, Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *such that* (2.5.1) *holds for some* $D \le N$, $\mathcal{M}, \alpha > 0$ *and with* $h(t) := (\log t^{-1})^m$ *for all* $t \in (0, 1)$, *where* m *is a nonnegative integer. Then* (A, Ω) *is h-Minkowski measurable,* $\dim_B(A, \Omega) = D$, *and* $\mathcal{M}^D(A, \Omega, h) = \mathcal{M}$. *Furthermore, the relative tube zeta function* $\tilde{\zeta}_{A,\Omega}$ *has abscissa of convergence* $D(\tilde{\zeta}_{A,\Omega}) = D$, *and it possesses a* (*necessarily unique*) *meromorphic extension* (*at least*) *to the open right half-plane* $\{\mathrm{Re}\, s > D - \alpha\}$; *that is, the abscissa of meromorphic continuation* $D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega})$ *can be estimated as follows*:

$$D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) \le D - \alpha. \tag{2.5.6}$$

Moreover, $s = D$ *is the unique pole in this half-plane, and it is of order* $m+1$. *In addition, the coefficients of the Laurent series expansion* (2.5.3) *corresponding to the principal part of* $\tilde{\zeta}_{A,\Omega}$ *at* $s = D$ *are given by*

$$\begin{aligned} c_{-m-1} &= m!\mathcal{M}, \\ c_{-m} &= \cdots = c_{-1} = 0 \quad (\textit{provided } m \ge 1). \end{aligned} \tag{2.5.7}$$

If $m = 0$, *then* D *is a simple pole of* $\tilde{\zeta}_{A,\Omega}$ *and*

$$\mathrm{res}(\tilde{\zeta}_{A,\Omega}, D) = \mathcal{M}. \tag{2.5.8}$$

Proof. Set

$$\begin{aligned} \zeta_1(s) &= \mathcal{M} z_m(s), \qquad z_m(s) = \int_0^\delta t^{s-D-1}(\log t^{-1})^m\,\mathrm{d}t, \\ \zeta_2(s) &= \int_0^\delta t^{s-N-1}(\log t^{-1})^m O(t^{N-D+\alpha})\,\mathrm{d}t. \end{aligned} \tag{2.5.9}$$

Since $\tilde{\zeta}_{A,\Omega}(s) = \zeta_1(s) + \zeta_2(s)$, we can proceed as follows. For each $\varepsilon > 0$, we have $(\log t^{-1})^m = O(t^{-\varepsilon})$ as $t \to 0^+$; hence,

$$|\zeta_2(s)| \le \int_0^\delta O(t^{\mathrm{Re}\, s - 1 - D + (\alpha - \varepsilon)})\,\mathrm{d}t.$$

Since the integral is well defined for $\operatorname{Re} s > D-(\alpha-\varepsilon)$, we deduce that $D(\zeta_2) \leq D-(\alpha-\varepsilon)$. Letting $\varepsilon \to 0^+$, we obtain the desired inequality $D(\zeta_2) \leq D-\alpha$.

By the change of variable $\tau = \log t^{-1}$ (for $0 < t \leq \delta$), it is easy to see that

$$z_m(s) = \int_{\log \delta^{-1}}^{\infty} \mathrm{e}^{-\tau(s-D)} \tau^m \, \mathrm{d}\tau. \tag{2.5.10}$$

Integration by parts yields the following recursion relation, where we have to assume (at first) that $\operatorname{Re} s > D$:

$$z_m(s) = \frac{1}{s-D}\big((\log \delta^{-1})^m \delta^{s-D} + m z_{m-1}(s)\big) \quad \text{for } m \geq 1, \tag{2.5.11}$$

and $z_0(s) = (s-D)^{-1}\delta^{s-D}$. Since $D(\zeta_2) \leq D-\alpha$, it is clear that the coefficients c_j, $j < 0$, of the Laurent series expansion (2.5.3) of $\tilde{\zeta}_{A,\Omega}(s) = \zeta_1(s) + \zeta_2(s)$ in a connected open neighborhood of $s = D$ do not depend on $\delta > 0$. Indeed, changing $\delta > 0$ to $\delta_1 > 0$ in (2.4.1) is equivalent to adding $\int_{\delta}^{\delta_1} t^{s-N-1}|A_t \cap \Omega| \, \mathrm{d}t$, which is an entire function of s. Therefore, without loss of generality, we may take $\delta = 1$ in (2.5.11):

$$z_m(s) = \frac{m}{s-D} z_{m-1}(s) = \cdots = \frac{m!}{(s-D)^m} z_0(s) = \frac{m!}{(s-D)^{m+1}}. \tag{2.5.12}$$

In this way, we obtain

$$\zeta_1(s) = \frac{m!}{(s-D)^{m+1}} \mathcal{M}, \tag{2.5.13}$$

and we can meromorphically continue ζ_1 from the half-plane $\{\operatorname{Re} s > D\}$ to the entire complex plane. The claim then follows from the equality $\tilde{\zeta}_{A,\Omega}(s) = \zeta_1(s) + \zeta_2(s)$. ■

A large class of examples of RFDs satisfying condition (2.5.1), involving power logarithmic gauge functions, can be found in Example 4.27 of §4.4 below, based on [LRŽ6, Theorem 5.4]. (In fact, [LRŽ6, Theorem 5.4] can be understood as a partial converse of Theorem 2.24.) These RFDs are constructed by using consecutive tensor products of a suitable bounded fractal string $\mathcal{L}$, i.e., by an iterated spraying of $\mathcal{L}$; see [LRŽ5] for details. A nontrivial class of examples is already obtained when $\mathcal{L}$ is the ternary Cantor string. A similar comment can be made about the analogous condition (2.5.14) appearing in the following theorem.

THEOREM 2.25 (Minkowski nonmeasurable RFDs). *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that there exist $D \leq N$, a nonconstant periodic function $G : \mathbb{R} \to \mathbb{R}$ with minimal period $T > 0$, and a nonnegative integer m, satisfying*

$$|A_t \cap \Omega| = t^{N-D}(\log t^{-1})^m \big(G(\log t^{-1}) + O(t^\alpha)\big) \quad \text{as } t \to 0^+. \tag{2.5.14}$$

Then $\dim_B(A,\Omega)$ exists and $\dim_B(A,\Omega) = D$, G is continuous, and

$$\mathcal{M}_*^D(A,\Omega,h) = \min G, \qquad \mathcal{M}^{*D}(A,\Omega,h) = \max G,$$

where $h(t) := (\log t^{-1})^m$ for all $t \in (0,1)$. Furthermore, the tube zeta function $\tilde{\zeta}_{A,\Omega}$ has $D(\tilde{\zeta}_{A,\Omega}) = D$, and it has a (necessarily unique) meromorphic extension (at least) to the half-plane $\{\operatorname{Re} s > D-\alpha\}$; that is,

$$D_{\mathrm{mer}}(\tilde{\zeta}_{A,\Omega}) \leq D-\alpha. \tag{2.5.15}$$

Moreover, all of its poles located in this half-plane are of order $m+1$, *and the set of poles* $\mathcal{P}(\tilde{\zeta}_{A,\Omega})$ *is contained in the vertical line* $\{\mathrm{Re}\, s = D\}$. *More precisely,*

$$\begin{aligned}\mathcal{P}(\tilde{\zeta}_{A,\Omega})) &= \mathcal{P}_c(\tilde{\zeta}_{A,\Omega}) \\ &= \left\{ s_k = D + \frac{2\pi}{T} k\mathbb{i} \in \mathbb{C} : \hat{G}_0\left(\frac{k}{T}\right) \neq 0,\, k \in \mathbb{Z} \right\},\end{aligned} \tag{2.5.16}$$

where $s_0 = D \in \mathcal{P}(\tilde{\zeta}_{A,\Omega})$ *and* $\hat{G}_0$ *is the Fourier transform of* G_0 (*as given by* (2.5.5)). *The nonreal poles come in complex conjugate pairs: for each* $k \geq 1$, *if* s_k *is a pole, then* s_{-k} *is a pole as well.*

In addition, for any given $k \in \mathbb{Z}$, *if* $\tilde{\zeta}_{A,\Omega}(s) = \sum_{j=-\infty}^{\infty} c_j^{(k)}(s - s_k)^j$ *is the Laurent expansion in a connected open neighborhood of* $s = s_k$, *then*

$$\begin{aligned} c_j^{(k)} &= 0 \quad \textit{for } j < 0,\ j \neq -m-1, \\ c_{-m-1}^{(k)} &= \frac{m!}{T}\hat{G}_0\left(\frac{k}{T}\right), \end{aligned} \tag{2.5.17}$$

where G_0 *is the restriction of* G *to* $[0,T]$, *and* $\hat{G}_0$ *is given by* (2.5.5) *as above. Also,*

$$|c_{-m-1}^{(k)}| \leq \frac{m!}{T}\int_0^T G(\tau)\,\mathrm{d}\tau, \qquad \lim_{k\to\infty} c_{-m-1}^{(k)} = 0. \tag{2.5.18}$$

In particular, for $k = 0$, *that is, for* $s_0 = D$, *we have*

$$\begin{aligned} c_{-m-1}^{(0)} &= \frac{m!}{T}\int_0^T G(\tau)\,\mathrm{d}\tau, \\ m!\,\mathcal{M}_*^D(A,\Omega,h) < c_{-m-1}^{(0)} &< m!\,\mathcal{M}^{*D}(A,\Omega,h). \end{aligned} \tag{2.5.19}$$

If $m = 0$ (*i.e.,* $h(t) = 1$ *for all* $t \in (0,1)$), *then* D *is a simple pole of* $\tilde{\zeta}_{A,\Omega}$ *and*

$$\mathrm{res}(\tilde{\zeta}_{A,\Omega}, D) = \frac{1}{T}\int_0^T G(\tau)\,\mathrm{d}\tau = \widetilde{\mathcal{M}} \tag{2.5.20}$$

and

$$\mathcal{M}_*^D(A,\Omega) < \mathrm{res}(\tilde{\zeta}_{A,\Omega}, D) < \mathcal{M}^{*D}(A,\Omega), \tag{2.5.21}$$

where $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}^D(A,\Omega)$ *denotes the average Minkowski content of* (A,Ω). (*See Remark* 2.26 *below.*)

Proof. For $m \in \mathbb{N}_0$, define

$$z_m(s) = \int_0^\delta t^{s-D-1}(\log t^{-1})^m G(\log t^{-1})\,\mathrm{d}t.$$

The function $z_0(s)$ is the exact counterpart of $\zeta_1(s)$ from the proof of [LRŽ3, Theorem 4.24], with $|A_t|$ changed to $|A_t \cap \Omega|$ and where, much as in that proof, $\tilde{\zeta}_{A,\Omega} = \zeta_1 + \zeta_2$ and ζ_2 is an entire function. It is easy to see that $z_m(s) = (-1)^m z_0^{(m)}(s)$; therefore, $z_m(s)$ and $z_0(s)$ have the same meromorphic extension and the same sets of poles. This proves that $\tilde{\zeta}_{A,\Omega}(s)$ can be meromorphically extended from $\{\mathrm{Re}\, s > D\}$ to $\{\mathrm{Re}\, s > D - \alpha\}$. The set of poles (complex dimensions) of the relative zeta function, belonging to this half-plane, is given by

$$\mathcal{P}(\tilde{\zeta}_{A,\Omega}) = \mathcal{P}(z_m) = \mathcal{P}(z_0) = \left\{ s_k = D + \frac{2\pi}{T} k\mathbb{i} \in \mathbb{C} : \hat{G}_0\left(\frac{k}{T}\right) \neq 0,\, k \in \mathbb{Z} \right\}.$$

Each of these poles is simple. Furthermore, if

$$z_0(s) = \sum_{j=-1}^{\infty} a_j^{(k)} (s - s_k)^j$$

is the Laurent series of $z_0(s)$ in a neighborhood of $s = s_k$, then

$$z_0^{(m)}(s) = (-1)^m m!\, a_{-1}^{(k)} (s - s_k)^{-m-1} + \sum_{j=0}^{\infty} \frac{(m+j)!}{j!} a_{m+j}^{(k)} (s - s_k)^j.$$

Hence,

$$c_{-m-1}^{(k)} = m! a_{-1}^{(k)} = m!\, \frac{1}{T} \hat{G}_0\left(\frac{k}{T}\right),$$

where, in the last equality, we have used [LRŽ3, (4.32)]. The remaining claims are proved much as the corresponding ones in [LRŽ3, Theorem 4.24]. ■

REMARK 2.26. In (2.5.20), $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}^D(A,\Omega)$, the *average Minkowski content* of (A,Ω), is defined as the multiplicative Cesàro average of $t^{-(N-D)}|A_t \cap \Omega|$:

$$\widetilde{\mathcal{M}}^D(A,\Omega) := \lim_{\tau \to +\infty} \frac{1}{\log \tau} \int_{1/\tau}^{1} \frac{|A_t \cap \Omega|}{t^{N-D}} \frac{\mathrm{d}t}{t}, \tag{2.5.22}$$

provided the limit exists in $[0, +\infty]$. (See (2.5.1) and compare with [LF3, Definition 8.29, (8.55)].)

REMARK 2.27. In light of the functional equation (2.4.2) connecting $\zeta_{A,\Omega}$ and $\tilde{\zeta}_{A,\Omega}$, Theorems 2.24 and 2.25 also hold for relative *distance* zeta functions, provided $D < N$, and in that case, all of the expressions for the residues and the Laurent coefficients must be multiplied by $N - D$.

2.6. Construction of ∞-quasiperiodic relative fractal drums. Our construction of quasiperiodic RFDs (see Definition 2.37 below) is based on a certain two-parameter family of generalized Cantor sets, which we now describe.

DEFINITION 2.28. The generalized Cantor sets $C^{(m,a)}$ are determined by an integer $m \geq 2$ and a positive real number a such that $ma < 1$. In the first step of the analog of Cantor's construction of the standard ternary Cantor set, we start with m equidistant, closed intervals in $[0,1]$ of length a, with $m-1$ "holes", each of length $(1-ma)/(m-1)$. In the second step, we continue by scaling by the factor a each of the m intervals of length a; and so on, ad infinitum. The (two-parameter) *generalized Cantor set* $C^{(m,a)}$ is defined as the intersection of the decreasing sequence of compact sets constructed in this way. It is easy to check that $C^{(m,a)}$ is a perfect, uncountable compact subset of $\mathbb{R}$. (Recall that a *perfect set* is a closed set without any isolated points.) Furthermore, $C^{(m,a)}$ is also self-similar.

To avoid any possible confusion, we note that the generalized Cantor sets introduced here are different from the generalized Cantor strings studied in [LF3, Chapter 10]. With our present notation, the classical ternary Cantor set is $C^{(2,1/3)}$.

We note that the box dimension of $C^{(m,a)}$ exists and is equal to its Hausdorff dimension, as well as to its similarity dimension (here, $\log_{1/a} m$). The proof in the case of the classical Cantor set can be found in [F1] and is due to Moran [Mo] (in the case $N = 1$); see also [Hu]. For any pair (m, a) as above, this follows from a general result in [Hu] (described in [F1, Theorem 9.3]) because $C^{(m,a)}$ is a self-similar set satisfying the open set condition. (See also [Mo].)

It can be shown that the generalized Cantor sets $C^{(m,a)}$ have the following properties. Apart from the proof of (2.6.5), the proof of the next proposition is similar to that for the standard Cantor set (see [LF3, (1.11)]).

Proposition 2.29. *If $C^{(m,a)} \subset \mathbb{R}$ is the generalized Cantor set introduced in Definition 2.28, where m is an integer, $m \geq 2$, and $a \in (0, 1/m)$, then*

$$D := \dim_B C^{(m,a)} = D(\zeta_A) = \log_{1/a} m. \tag{2.6.1}$$

Furthermore, the tube formula associated with $C^{(m,a)}$ is

$$|C_t^{(m,a)}| = t^{1-D} G(\log t^{-1}) \tag{2.6.2}$$

for all $t \in \big(0, \frac{1-ma}{2(m-1)}\big)$, where $G = G(\tau)$ is a nonconstant periodic function, with minimal period $T = \log(1/a)$, defined by

$$G(\tau) = c^{D-1}(ma)^{g\left(\frac{\tau-c}{T}\right)} + 2\, c^D m^{g\left(\frac{\tau-c}{T}\right)}. \tag{2.6.3}$$

Here, $c = \frac{1-ma}{2(m-1)}$, and $g : \mathbb{R} \to [0, +\infty)$ is the 1-periodic function defined by $g(x) = 1 - x$ for $x \in (0, 1]$.

Moreover,

$$\begin{aligned} \mathcal{M}_*^D(C^{(m,a)}) &= \min G = \frac{1}{D}\left(\frac{2D}{1-D}\right)^{1-D}, \\ \mathcal{M}^{*D}(C^{(m,a)}) &= \max G = \left(\frac{1-ma}{2(m-1)}\right)^{D-1} \frac{m(1-a)}{m-1}. \end{aligned} \tag{2.6.4}$$

Finally, if we assume that $\delta \geq \frac{1-ma}{2(m-1)}$, then the distance zeta function of $A := C^{(m,a)}$ is given by

$$\zeta_A(s) := \int_{-\delta}^{1+\delta} d(x, A)^{s-2}\, \mathrm{d}x = \left(\frac{1-ma}{2(m-1)}\right)^{s-1} \frac{1-ma}{s(1-ma^s)} + \frac{2\delta^s}{s}. \tag{2.6.5}$$

As a result, $\zeta_A(s)$ admits a meromorphic continuation to all of $\mathbb{C}$, given by the last expression in (2.6.5). In particular, the set of poles of $\zeta_A(s)$ (in $\mathbb{C}$) and the residue of $\zeta_A(s)$ at $s = D$ are respectively

$$\begin{aligned} \mathcal{P}(\zeta_A) &= (D + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{0\}, \\ \operatorname{res}(\zeta_A, D) &= \frac{1-ma}{DT}\left(\frac{1-ma}{2(m-1)}\right)^{D-1}, \end{aligned} \tag{2.6.6}$$

where $\mathbf{p} = 2\pi/T$ *is the oscillatory period* (*in the sense of* [LF3]). *Furthermore,*

$$D = \frac{\log m}{2\pi}\,\mathbf{p},$$

and both $\mathbf{p} \to 0^+$ *and* $D \to 0^+$ *as* $a \to 0^+$. *In particular,* $\mathcal{P}(\zeta_A)$ *converges to the imaginary axis in the Hausdorff metric as* $a \to 0^+$. *Finally, each pole in* $\mathcal{P}(\zeta_A)$ *is simple.*

We shall need the following important theorem from transcendental number theory, due to Baker.

Theorem 2.30 ([B, Theorem 2.1]). *Let* $n \in \mathbb{N}$ *with* $n \geq 2$. *If* $m_1, \dots, m_n$ *are positive algebraic numbers such that* $\log m_1, \dots, \log m_n$ *are linearly independent over the rationals, then*

$$1, \log m_1, \dots, \log m_n$$

are linearly independent over the field of all algebraic numbers (*or algebraically independent, in short*). *In particular, the numbers* $\log m_1, \dots, \log m_n$ *are transcendental, as are their pairwise quotients.*

Here, we describe a general construction of quasiperiodic fractal drums possessing infinitely many algebraically incommensurable periods. It is based on properties of generalized Cantor sets, as well as on Baker's theorem above.

Let $m \geq 2$ be a given integer and $D \in (0,1)$ a real number. Then, for $a = m^{-1/D}$, we have $am = m^{1-1/D} < 1$, and hence the generalized Cantor set $A = C^{(m,a)}$ is well defined and $\dim_B A = \log_{1/a} m = D$.

Definition 2.31. A finite set of real numbers is said to be *rationally* (resp., *algebraically*) *linearly independent*, or simply *rationally* (resp., *algebraically*) *independent*, if it is linearly independent over the field of rational (resp., algebraic) real numbers.

Definition 2.32. A sequence $(T_i)_{i\geq 1}$ of real numbers is said to be *rationally* (resp., *algebraically*) *linearly independent* if any of its finite subsets is rationally (resp., algebraically) independent. We then say that $(T_i)_{i\geq 1}$ is *rationally* (resp., *algebraically*) *independent*, for short.

Definition 2.33. Let $m \geq 2$ be a positive integer. Let $\mathbf{p} = (p_i)_{i\geq 1}$ be the sequence of all prime numbers, arranged in increasing order:

$$\mathbf{p} = (2, 3, 5, 7, 11, \dots).$$

We then define the *exponent sequence* $\mathbf{e} = \mathbf{e}(m) := (\alpha_i)_{i\geq 1}$ *associated with* m, where $\alpha_i \geq 0$ is the multiplicity of p_i in the factorization of m. We also let

$$\mathbf{p}^{\mathbf{e}} := \prod_{\{i\geq 1:\alpha_i>0\}} p_i^{\alpha_i}. \tag{2.6.7}$$

The set of all sequences $\mathbf{e}$ with components in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that all but at most finitely many components are zero is denoted by $(\mathbb{N}_0)_c^\infty$.

With this definition, for any integer $m \geq 2$, we obviously have $m = \mathbf{p}^{\mathbf{e}(m)}$. Conversely, any $\mathbf{e} \in (\mathbb{N}_0)_c^\infty$ defines a unique integer $m \geq 2$ such that $m = \mathbf{p}^{\mathbf{e}}$.

Definition 2.34. Given an exponent vector $\mathbf{e} = (\alpha_i)_{i\geq 1} \in (\mathbb{N}_0)^\infty_c$, we define the *support of* $\mathbf{e}$ to be

$$S(\mathbf{e}) = \mathrm{supp}(\mathbf{e}) := \{i \geq 1 : \alpha_i > 0\}. \tag{2.6.8}$$

The *support of an integer* $m \geq 2$ is denoted by $\mathrm{supp}\, m$ and defined by $\mathrm{supp}\, m := \mathrm{supp}\, \mathbf{e}(m)$.

The following definition will be useful.

Definition 2.35. We say that a set $\{\mathbf{e}_i : i \geq 1\}$ of exponent vectors is *rationally linearly independent* if any of its finite subsets is linearly independent over $\mathbb{Q}$. We then say for short that the exponent vectors are *rationally independent.*

The following two definitions refine and extend the definition of n-quasiperiodic functions and sets introduced in [LRŽ2].

Definition 2.36. We say that a function $G : \mathbb{R} \to \mathbb{R}$ is ∞-*quasiperiodic* if it is of the form

$$G(\tau) = H(\tau, \tau, \dots),$$

where $H : \ell^\infty(\mathbb{R}) \to \mathbb{R}$ (6), $H = H(\tau_1, \tau_2, \dots)$ is a function which is T_j-periodic in its jth component, for each $j \in \mathbb{N}$, with $T_j > 0$ as minimal periods, and such that the set of periods

$$\{T_j : j \geq 1\} \tag{2.6.9}$$

is *rationally* independent. We say that the *order of quasiperiodicity* of the function G is equal to infinity (or that the function G is ∞-*quasiperiodic*).

In addition, we say that G is

(a) *transcendentally quasiperiodic of infinite order* (or *transcendentally* ∞-*quasiperiodic*) if the periods in (2.6.9) are *algebraically* independent;
(b) *algebraically quasiperiodic of infinite order* (or *algebraically* ∞-*quasiperiodic*) of infinite order if the periods in (2.6.9) are *rationally* independent and algebraically dependent.

We say that a sequence $(T_i)_{i\geq 1}$ of real numbers is *algebraically dependent* of infinite order if there exists a finite subset J of $\mathbb{N}$ such that $(T_i)_{i\in J}$ is algebraically dependent (that is, linearly dependent over the field of algebraic numbers). Recall that a finite set $\{T_1, \dots, T_k\}$ of real numbers is said to be *algebraically dependent* if there exist k algebraic real numbers $\lambda_1, \dots, \lambda_k$, not all zero, such that $\lambda_1 T_1 + \dots + \lambda_k T_k = 0$.

The notion of quasiperiodic function provided in Definition 2.36 has been motivated by [KT]. However, while in [KT], it is assumed that the reciprocals of the quasiperiods $T_1, \dots, T_n$ are rationally independent, we assume in Definition 2.36 that the quasiperiods $T_1, \dots, T_n$ themselves are rationally independent. The distinction between algebraically n-quasiperiodic and transcendentally n-quasiperiodic functions seems to be new.

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(6) Here, $\ell^\infty(\mathbb{R})$ stands for the usual Banach space of bounded sequences $(\tau_j)_{j\geq 1}$ of real numbers, endowed with the norm $\|(\tau_j)_{j\geq 1}\|_\infty := \sup_{j\geq 1} |\tau_j|$.

Definition 2.37. Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ satisfying the following tube formula:

$$|A_t\cap\Omega| = t^{N-D}\big(G(\log t^{-1})+o(1)\big) \quad \text{as } t\to 0^+, \tag{2.6.10}$$

where $D\in(-\infty,N]$, and G is a nonnegative function such that

$$0<\liminf_{\tau\to+\infty} G(\tau)\le \limsup_{\tau\to+\infty} G(\tau)<\infty.$$

(Note that it follows that $\dim_B(A,\Omega)$ exists and is equal to D. Moreover, $\mathcal{M}_*^D(A,\Omega)=\liminf_{\tau\to+\infty}G(\tau)$ and $\mathcal{M}^{*D}(A,\Omega)=\limsup_{\tau\to+\infty}G(\tau)$.)

We then say that the *relative fractal drum* (A,Ω) *in* $\mathbb{R}^N$ *is quasiperiodic* and of *infinite order of quasiperiodicity* (or, for short, ∞-*quasiperiodic*) if the function $G=G(\tau)$ is ∞-quasiperiodic (see Definition 2.36).

In addition, (A,Ω) is said to be

(a) a *transcendentally* ∞-*quasiperiodic relative fractal drum* if the corresponding function G is transcendentally ∞-quasiperiodic;
(b) an *algebraically* ∞-*quasiperiodic relative fractal drum* if G is algebraically ∞-quasiperiodic.

The following definition is closely related to the the notion of fractality (given in [LF3], §12.1.1 and §12.1.2, including Figures 12.1–12.3, along with §13.4.3).

Definition 2.38. Let A be a bounded subset of $\mathbb{R}^N$ and let $D:=\overline{\dim}_B A$.

(i) The set A is a *hyperfractal* (or is *hyperfractal*) if there is a screen $\boldsymbol{S}$ along which the associated tube (or equivalently, if $D<N$, distance) zeta function is a natural boundary. This means that the zeta function cannot be meromorphically continued to an open neighborhood of $\boldsymbol{S}$ (or, equivalently, of the associated window $\boldsymbol{W}$).
(ii) The set A is a *strong hyperfractal* (or is *strongly hyperfractal*) if the critical line $\{\operatorname{Re} s=D\}$ is a (meromorphic) natural boundary of the associated zeta function, that is, we can choose $\boldsymbol{S}=\{\operatorname{Re} s=D\}$ in (i).
(iii) Finally, A is *maximally hyperfractal* if it is strongly hyperfractal and every point of the critical line $\{\operatorname{Re} s=D\}$ is a nonremovable singularity of the zeta function.

An analogous definition can be provided (in the obvious manner) where instead of A we have a fractal string $\mathcal{L}=(\ell_j)_{j\ge1}$ in $\mathbb{R}$ or, more generally, a relative fractal drum (A,Ω) in $\mathbb{R}^N$.

Remark 2.39. Following [LF3], but now using the higher-dimensional theory of complex dimensions developed here and in [LRŽ1]–[LRŽ8], we say that a bounded set $A\subset\mathbb{R}^N$ (or, more generally, an RFD (A,Ω) in $\mathbb{R}^N$) is “fractal” if it has at least one nonreal visible complex dimension (7) (i.e., if the associated fractal zeta function has a nonreal visible pole) or if it is hyperfractal (in the sense of Definition 2.38(i)).

The following result can be considered as a fractal set-theoretic interpretation of Baker's theorem [B, Theorem 2.1] (i.e., of Theorem 2.30) from transcendental number theory. It provides a construction of a transcendentally ∞-quasiperiodic relative fractal

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(7) Then it clearly has at least two complex conjugate nonreal complex dimensions.

drum. In particular, this drum possesses infinitely many algebraically incommensurable quasiperiods T_i. In our construction, we use the two-parameter family of generalized Cantor sets $C^{(m,a)}$ introduced in Definition 2.28 and whose basic properties are described in Proposition 2.29.

Theorem 2.40. *Let* $D \in (0,1)$ *be a given real number, and let* $(m_i)_{i\geq 1}$ *be a sequence of integers such that* $m_i \geq 2$ *for each* $i \geq 1$. *For any* $i \geq 1$, *define* $a_i = m_i^{-1/D}$ *and let* $C^{(m_i,a_i)}$ *be the corresponding generalized Cantor set* (*Definition* 2.28). *Assume that* $(\Omega_i)_{i\geq 1}$ *is a family of disjoint open intervals on the real line such that* $|\Omega_i| \leq C_1 m_i^{1-1/D} c_i^{1/D}$ *for each* $i \geq 1$, *where the sequence* $(c_i)_{i\geq 1}$ *of positive real numbers is summable, and* $C_1 > 0$. *Let*

$$(A,\Omega) := \bigcup_{i\geq 1}(A_i,\Omega_i), \quad \textit{where} \quad A_i := |\Omega_i|\, C^{(m_i,a_i)} + \inf \Omega_i \ \textit{for all}\ i \geq 1.$$

Assume that the sequence of real numbers

$$\{\log m_1, \dots, \log m_n, \dots\} \ \textit{is rationally independent.} \tag{2.6.11}$$

Then the sequence of real numbers

$$\{1/D, T_1, T_2, \dots\} \tag{2.6.12}$$

is algebraically independent. In other words, the relative fractal drum (A,Ω) *is transcendentally quasiperiodic with infinite order of quasiperiodicity and associated sequence of quasiperiods* $(T_i)_{i\geq 1}$, *where* $T_i := log(1/a_i) = (\log m_i)/D$ *for each* $i \geq 1$. *Furthermore,*

$$D(\zeta_{A,\Omega}) = D_{\rm mer}(\zeta_{A,\Omega}), \tag{2.6.13}$$

and moreover all the points on the critical line $\{\mathrm{Re}\, s = D\}$ *are nonremovable singularities of* $\zeta_{A,\Omega}$; *in other words, the relative fractal drum* (A,Ω) *is also maximally hyperfractal* (*in the sense of Definition* 2.38(iii)).

Finally, the relative fractal drum (A,Ω) *is Minkowski nondegenerate, in the sense that*

$$0 < \mathcal{M}_*^D(A,\Omega) \leq \mathcal{M}^{*D}(A,\Omega) < \infty.$$

Theorem 2.40 admits a partial extension. If instead of (2.6.11) we assume that $m_i \to \infty$ as $i \to \infty$, then (2.6.13) still holds, and moreover all of the points of the critical line are nonremovable singularities of ζ_A, and hence (A,Ω) is maximally hyperfractal. Furthermore, it is Minkowski nondegenerate.

We shall need the following lemma, which states a simple scaling property of the tube functions and Minkowski contents of RFDs. We note that the identity (2.6.15) below yields a partial extension of [Ž2, Proposition 4.4]. Compare also with the scaling property of the corresponding distance zeta function $\zeta_{A,\Omega}$, obtained in Theorem 2.16.

Lemma 2.41.

(a) *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$. *Then, for any fixed* $\lambda > 0$ *and for all* $t > 0$, *we have*

$$(\lambda A)_t \cap \lambda\Omega = \lambda(A_{t/\lambda} \cap \Omega), \quad |(\lambda A)_t \cap \lambda\Omega| = \lambda^N |A_{t/\lambda} \cap \Omega|. \tag{2.6.14}$$

Furthermore, for any $r \in \mathbb{R}$, we have the following scaling (or homogeneity) properties of the relative upper and lower Minkowski contents:

$$\mathcal{M}^{*r}(\lambda A, \lambda\Omega) = \lambda^r \mathcal{M}^{*r}(A,\Omega), \qquad \mathcal{M}^r_*(\lambda A, \lambda\Omega) = \lambda^r \mathcal{M}^r_*(A,\Omega). \tag{2.6.15}$$

(b) *If A is a generalized Cantor set $C^{(m,a)}$ (as in Definition 2.28), then*

$$|(\lambda C^{(m,a)})_t \cap (0,\lambda)| = t^{1-D}(G_\lambda(\log t^{-1}) - 2t^D),$$

where

$$G_\lambda(\tau) := \lambda^D G(\tau + \log\lambda)$$

and G is the T-periodic function defined in (2.6.3).

Proof. (a) Scaling $A_t \cap \Omega$ by the factor λ, we obtain $\lambda(A_t \cap \Omega)$. On the other hand, the same result is obtained by intersecting the scaled sets $(\lambda A)_{\lambda t}$ and $\lambda\Omega$:

$$\lambda(A_t \cap \Omega) = (\lambda A)_{\lambda t} \cap \lambda\Omega.$$

The first equality in (2.6.14) now follows by replacing t with t/λ. The second is an immediate consequence of the first. We also have

$$\begin{aligned}\mathcal{M}^{*r}(\lambda A, \lambda\Omega) &= \limsup_{t\to 0^+} \frac{|(\lambda A)_t \cap \lambda\Omega|}{t^{N-r}} = \lambda^N \limsup_{t\to 0^+} \frac{|(A)_{t/\lambda} \cap \Omega|}{t^{N-r}} \\ &= \lambda^N \limsup_{\tau\to 0^+} \frac{|(A)_\tau \cap \Omega|}{(\lambda\tau)^{N-r}} = \lambda^r \mathcal{M}^{*r}(A,\Omega).\end{aligned}$$

The second equality in (2.6.15) is proved in the same way, but now using the lower limit instead of the upper limit.

(b) In the case of the generalized Cantor set, we use (2.6.14) with $N = 1$ together with Proposition 2.29:

$$\begin{aligned}|(\lambda C^{(m,a)})_t \cap (0,\lambda)| &= \lambda |C^{(m,a)}_{t/\lambda} \cap (0,1)| = \lambda \left(\frac{t}{\lambda}\right)^{1-D} \left(G\left(\log\frac{\lambda}{t}\right) - 2(t/\lambda)^D\right) \\ &= t^{1-D}\big(\lambda^D G(\log\lambda + \log t^{-1}) - 2t^D\big). \quad \blacksquare\end{aligned}$$

Relative tube zeta functions have a scaling property which is analogous to that obtained for the tube zeta functions of bounded sets [LRŽ1, Proposition 2.2.22]. We omit the corresponding simple direct proof (8).

Proposition 2.42 (Scaling property of relative tube zeta functions). *Let (A,Ω) be a relative fractal drum and let $\delta > 0$. Denote by $\tilde{\zeta}_{A,\Omega;\delta}(s)$ the associated relative fractal zeta function defined by* (2.4.1). *Then, for any $\lambda > 0$, we have $D(\tilde{\zeta}_{\lambda A,\lambda\Omega;\lambda\delta}) = D(\tilde{\zeta}_{A,\Omega;\delta}) = \overline{\dim}_B(A,\Omega)$ and*

$$\tilde{\zeta}_{\lambda A,\lambda\Omega;\lambda\delta}(s) = \lambda^s \tilde{\zeta}_{A,\Omega;\delta}(s) \tag{2.6.16}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_B(A,\Omega)$. Furthermore, if $\omega \in \mathbb{C}$ is a simple pole of $\tilde{\zeta}_{A,\Omega;\delta}$, where $\tilde{\zeta}_{A,\Omega;\delta}$ is meromophically extended to an open connected neighborhood of the critical line $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$ (as usual, we keep the same notation for the extended

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(8) An alternative proof of Proposition 2.42 would rely on the functional equation (2.4.2) combined with Theorem 2.16, the scaling property for distance zeta functions.

function), then

$$\operatorname{res}(\tilde{\zeta}_{\lambda A,\lambda\Omega;\lambda\delta}\,,\,\omega)=\lambda^{\omega}\operatorname{res}(\tilde{\zeta}_{A,\Omega;\delta}\,,\,\omega). \tag{2.6.17}$$

In the proof of Theorem 2.40, we shall use the following simple fact. If a function $G(\tau):=H(\tau,\tau,\dots)$ is transcendentally quasiperiodic with respect to a sequence of quasiperiods $(T_i)_{i\geq 1}$, it is clear that for any fixed sequence $\mathbf{d}=(d_i)_{i\geq 1}$ of real numbers, the corresponding function

$$G_{\mathbf{d}}(\tau):=H(d_1+\tau,d_2+\tau,\dots)$$

is also quasiperiodic with respect to $(T_i)_{i\geq 1}$.

Proof of Theorem 2.40. The proof is divided into three steps.

Step 1. First of all, note that the generalized Cantor sets $C^{(m_i,a_i)}$ are well defined, since $m_ia_i=m_i^{1-1/D}<1$. Furthermore,

$$|\Omega|=\sum_{i=1}^{\infty}|\Omega_i|\leq C_1\sum_{i=1}^{\infty}m_i^{1-1/D}c_i^{1/D}\leq C_1\sum_{i=1}^{\infty}c_i^{1/D}\leq C_1\sum_{i=1}^{\infty}c_i<\infty,$$

where we have assumed without loss of generality that $c_i\leq 1$ for all $i\geq 1$. Using Lemma 2.41, we have

$$\begin{aligned}|A_t\cap\Omega|&=\sum_{i=1}^{\infty}|(A_i)_t\cap\Omega_i|=t^{1-D}\sum_{i=1}^{\infty}|\Omega_i|^D\bigg(G_i\bigg(\log|\Omega_i|+\log\frac{1}{t}\bigg)-2t^D\bigg)\\&=t^{1-D}\bigg(G\bigg(\log\frac{1}{t}\bigg)-2|\Omega|\,t^D\bigg),\end{aligned}$$

where

$$G(\tau):=\sum_{i=1}^{\infty}|\Omega_i|^DG_i(\log|\Omega_i|+\tau),$$

and the functions $G_i=G_i(\tau)$ are T_i-periodic with $T_i:=\log(1/a_i)$ for all $i\geq 1$. This shows that $G(\tau)=H(\tau,\tau,\dots)$, where

$$H((\tau_i)_{i\geq 1}):=\sum_{i=1}^{\infty}|\Omega_i|^D\,G_i(\log|\Omega_i|+\tau_i).$$

Note that the last series is well defined, and that so is the series defining $G(\tau)$. Indeed, letting $\mathcal{M}_i=\mathcal{M}^{*D}(C^{(m_i,a_i)})$ and using Proposition 2.29, we see that

$$0<G_i(\tau)\leq\mathcal{M}_i=\left(\frac{2(m_i-1)}{1-m_ia_i}\right)^{1-D}\frac{m_i}{m_i-1}(1-a_i)\leq Cm_i^{1-D}, \tag{2.6.18}$$

where C is a positive constant independent of i, since $m_i\to\infty$ and $m_ia_i\to 0$ as $i\to\infty$. Therefore,

$$\sum_{i=1}^{\infty}|\Omega_i|^DG_i(\tau_i)\leq\sum_{i=1}^{\infty}(C_1^Dm_i^{D-1}c_i)(Cm_i^{1-D})=CC_1^D\sum_{i=1}^{\infty}c_i<\infty.$$

In particular,

$$\mathcal{M}^{*D}(A,\Omega)\leq CC_1^D\sum_{i=1}^{\infty}c_i<\infty.$$

On the other hand, since $(A_1,\Omega_1)\supset(A,\Omega)$, we can use Lemma 2.41(a) (with $r:=D$) and Proposition 2.29 to obtain

$$\mathcal{M}_*^D(A,\Omega)\geq\mathcal{M}_*^D(A_1,\Omega_1)=|\Omega_1|^D\mathcal{M}_*^D(\text{`}C^{(m_1,a_1)})=|\Omega_1|^D\frac{1}{D}\left(\frac{2D}{1-D}\right)^{1-D}>0.$$

Step 2. Let n be any fixed positive integer. Since the set of real numbers

$$\{\log m_1,\dots,\log m_n\}$$

is rationally independent, we conclude from Baker's Theorem 2.30 that the set $\{1,\log m_1,\dots,\log m_n\}$ is algebraically independent. Dividing all of these numbers by D, and using $D=(\log m_i)/T_i$, where $T_i=\log(1/a_i)$ for all $i\geq 1$ (see Proposition 2.29), we deduce that

$$\left\{\frac{1}{D},\frac{\log m_1}{D},\dots,\frac{\log m_n}{D}\right\}=\left\{\frac{1}{D},T_1,\dots,T_n\right\}$$

is algebraically independent as well. Since n is arbitrary, this proves that (A,Ω) is transcendentally ∞-quasiperiodic, in the sense of Definition 2.37.

Step 3. To prove the last claim, note that the critical line $\{\operatorname{Re}s=D\}$ contains the union of the set of poles $\mathcal{P}_i:=\mathcal{P}(\tilde{\zeta}_{A_i,\Omega_i},\mathbb{C})=D+\mathbf{p}_i\mathbb{i}\mathbb{Z}$ of the tube zeta functions $\tilde{\zeta}_{A_i,\Omega_i}$, $i\geq 1$. Since the integers m_i are all distinct, we have $m_i\to\infty$ as $i\to\infty$, and therefore $\mathbf{p}_i=2\pi/T_i=2\pi D/\log m_i\to 0$. This proves that $\bigcup_{i\geq 1}\mathcal{P}_i$, as a set of nonisolated singularities of $\tilde{\zeta}_{A,\Omega}=\sum_{i\geq 1}\tilde{\zeta}_{A_i,\Omega_i}$, is dense in $\{\operatorname{Re}s=D\}$. (Indeed, it is easy to deduce from the definitions that the subset of nonremovable singularities of $\zeta_{A,\Omega}$ along $L:=\{\operatorname{Re}s=D\}$ is closed in L, and hence must coincide with L since it is also dense in L; see [LRŽ2, proof of Theorem 5.3].) It follows in particular that (2.6.13) holds, as desired. ∎

It is noteworthy that the sequence $\mathcal{M}^{*D}(C^{(m_i,a_i)},(0,1))$ appearing in Theorem 2.40 is divergent. More precisely, it is easy to deduce from (2.6.18) that

$$\mathcal{M}^{*D}(C^{(m_i,a_i)},(0,1))\sim(2m_i)^{1-D}\quad\text{as } i\to\infty.$$

The conditions of Theorem 2.40 are satisfied if, for example, $m_i:=p_i$ for all $i\geq 1$ (that is, $(m_i)_{i\geq 1}$ is the sequence of prime numbers $(p_i)_{i\geq 1}$, written in increasing order), and $C_1:=1$ and $c_i:=2^{-i}$ for every $i\geq 1$.

3. Embeddings into higher-dimensional spaces

In this chapter, we obtain useful results concerning relative fractal drums and bounded subsets of $\mathbb{R}^N$ embedded into higher-dimensional spaces. In particular, we show that the complex dimensions (and their multiplicities) of a bounded set (or, more generally, of a relative fractal drum) are independent of the dimension of the ambient space (Theorems 3.3 and 3.10). In addition, we apply some of these results in order to calculate the complex dimensions of the Cantor dust (Example 3.15).

3.1. Embeddings of bounded sets. We begin this section by stating a result which (along with the subsequent result, Theorem 3.2) will be key to the developments in this chapter.

PROPOSITION 3.1. *Let $A \subset \mathbb{R}^N$ be a bounded set and let $\overline{D} := \overline{\dim}_B A$. Then, for the tube zeta functions of A and $A \times \{0\} \subset \mathbb{R}^{N+1}$, the following equality holds*:

$$\tilde{\zeta}_{A\times\{0\}}(s;\delta) = 2\int_0^{\pi/2} \frac{\tilde{\zeta}_A(s;\delta\sin\tau)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau \tag{3.1.1}$$

for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{D}$.

Proof. First of all, it is well known and easy to check that $\overline{\dim}_B(A\times\{0\}) = \overline{\dim}_B A$, from which we conclude that the tube zeta functions of A and $A\times\{0\}$ are both holomorphic in $\{\operatorname{Re} s > \overline{D}\}$. Furthermore, we use the fact [Re, Proposition 6] that for every $t > 0$,

$$|(A\times\{0\})_t|_{N+1} = 2\int_0^t |A_{\sqrt{t^2-u^2}}|_N\,\mathrm{d}u, \tag{3.1.2}$$

where, as before, $|\cdot|_N$ denotes the N-dimensional Lebesgue measure. (See also the proof of Lemma 3.5 in §3.2 below.) After the change of variable $u := t\cos v$, this yields

$$|(A\times\{0\})_t|_{N+1} = 2t\int_0^{\pi/2} |A_{t\sin v}|_N \sin v\,\mathrm{d}v. \tag{3.1.3}$$

Finally, for the tube zeta function of $A\times\{0\}$, we can write successively

$$\begin{aligned}\tilde{\zeta}_{A\times\{0\}}(s;\delta) &= \int_0^\delta t^{s-N-2}|(A\times\{0\})_t|_{N+1}\,\mathrm{d}t = 2\int_0^\delta t^{s-N-1}\,\mathrm{d}t\int_0^{\pi/2}|A_{t\sin v}|_N\sin v\,\mathrm{d}v\\ &= 2\int_0^{\pi/2}\sin v\,\mathrm{d}v\int_0^\delta t^{s-N-1}|A_{t\sin v}|_N\,\mathrm{d}t\\ &= 2\int_0^{\pi/2}\sin^{N+1-s} v\,\mathrm{d}v\int_0^{\delta\sin v}\tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau = 2\int_0^{\pi/2}\frac{\tilde{\zeta}_A(s;\delta\sin v)}{\sin^{s-N-1}v}\,\mathrm{d}v,\end{aligned}$$

where we have used the Fubini–Tonelli theorem in order to justify the interchange of integrals (in the third equality), as well as the change of variable $\tau := t\sin v$. ■

In the following theorem, $\Gamma(t) := \int_0^{+\infty} x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x$, initially defined by this integral for $t > 0$, is the usual gamma function, meromorphically extended to all of $\mathbb{C}$.

THEOREM 3.2. *Let* $A \subset \mathbb{R}^N$ *be a bounded set and let* $\overline{D} := \overline{\dim}_B A$. *Then we have the following equality between* $\tilde{\zeta}_A$, *the tube zeta function of* A, *and* $\tilde{\zeta}_{A_M}$, *the tube zeta function of* $A_M := A \times \{0\} \times \cdots \times \{0\} \times \subset \mathbb{R}^{N+M}$, *with* $M \in \mathbb{N}$ *arbitrary*:

$$\tilde{\zeta}_{A_M}(s;\delta) = \frac{(\sqrt{\pi})^M \Gamma\left(\frac{N-s}{2}+1\right)}{\Gamma\left(\frac{N+M-s}{2}+1\right)} \tilde{\zeta}_A(s;\delta) + E(s;\delta), \tag{3.1.4}$$

initially valid for all $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > \overline{D}$. *Here, the error function* $E(s) := E(s;\delta)$ (*initially defined for* $M = 1$ *by the integral on the right-hand side of* (3.1.7) *below*) *admits a meromorphic extension to all of* $\mathbb{C}$. *The possible poles* (*in* $\mathbb{C}$) *of* $E(s;\delta)$ *are located at* $s_k := N + 2 + 2k$ *for every* $k \in \mathbb{N}_0$, *and all of them are simple.* (*It follows that* $\tilde{\zeta}_A$ *is well defined at each* s_k.) *Moreover, for each* $k \in \mathbb{N}_0$,

$$\operatorname{res}(E(\,\cdot\,;\delta), s_k) = \frac{(-1)^{k+1}\left(\sqrt{\pi}\right)^M}{k!\,\Gamma\left(\frac{M}{2}-k\right)} \tilde{\zeta}_A(s_k;\delta). \tag{3.1.5}$$

(*We refer to Theorem* 3.3 *below for more precise information about the domain of validity of* (3.1.4), *and to Corollary* 3.4 *for the relationship between the visible poles of* $\tilde{\zeta}_A$ *and* $\tilde{\zeta}_{A_M}$.) *More specifically, if* M *is even, then all of the poles* s_k *of* $E(s;\delta)$ *are canceled for* $k \geq M/2$, *i.e., the corresponding residues in* (3.1.5) *are equal to zero. On the other hand, if* M *is odd, then there are no such cancellations, so that all the* s_k*'s are* (*simple*) *poles of* $E(s;\delta)$ *in that case.*

Proof. We will prove the theorem in the case when $M = 1$. The general case then follows immediately by induction. From Proposition 3.1, formula (3.1.1) holds for $\operatorname{Re} s > \overline{\dim}_B A$. In turn, this latter identity can be written as follows:

$$\begin{aligned}
\tilde{\zeta}_{A\times\{0\}}(s;\delta) &= 2\tilde{\zeta}_A(s;\delta)\int_0^{\pi/2} \frac{\mathrm{d}\tau}{\sin^{s-N-1}\tau} \\
&\quad - 2\int_0^{\pi/2} \frac{\mathrm{d}v}{\sin^{s-N-1} v} \int_{\delta\sin v}^{\delta} \tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau \\
&= \tilde{\zeta}_A(s;\delta)\cdot \mathrm{B}\left(\frac{N-s}{2}+1, \frac{1}{2}\right) + E(s;\delta),
\end{aligned} \tag{3.1.6}$$

where B denotes the Euler beta function and

$$E(s;\delta) := -2\int_0^{\pi/2} \frac{\mathrm{d}v}{\sin^{s-N-1} v} \int_{\delta\sin v}^{\delta} \tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau. \tag{3.1.7}$$

By using the functional equation which links the beta function with the gamma function (namely, $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ for all $x, y > 0$ and hence, upon meromorphic continuation, for all $x, y \in \mathbb{C}$), we find that (3.1.4) holds (with $M = 1$) for $\operatorname{Re} s > \overline{\dim}_B A$.

By looking at $E(s;\delta)$, we see that the integrand is holomorphic for every $v \in (0, \pi/2)$ since $\int_{\delta\sin v}^{\delta} \tau^{s-N-1}|A_\tau|_N\,\mathrm{d}\tau$ is equal to $\tilde{\zeta}_A(s;\delta) - \tilde{\zeta}_A(s;\delta\sin v)$, which is an entire function.

Furthermore, if we assume that $\operatorname{Re} s < N+1$, then since $\tau \mapsto \tau^{\operatorname{Re} s - N-1}$ is decreasing, we have

$$\begin{aligned}
|E(s;\delta)| &\le 2\int_0^{\pi/2} \sin^{N+1-\operatorname{Re} s} v \,\mathrm{d}v \int_{\delta\sin v}^{\delta} \tau^{\operatorname{Re} s-N-1}|A_\tau|_N \,\mathrm{d}\tau \\
&\le 2|A_\delta|_N \int_0^{\pi/2} \sin^{N+1-\operatorname{Re} s} v \,\mathrm{d}v \int_{\delta\sin v}^{\delta} \tau^{\operatorname{Re} s-N-1} \,\mathrm{d}\tau \\
&\le 2\delta^{\operatorname{Re} s-N-1}|A_\delta|_N \int_0^{\pi/2} \sin^{N+1-\operatorname{Re} s} v \,\sin^{\operatorname{Re} s-N-1} v \int_{\delta\sin v}^{\delta} \mathrm{d}\tau \\
&= 2\delta^{\operatorname{Re} s-N}|A_\delta|_N \int_0^{\pi/2} (1-\sin v)\,\mathrm{d}v = 2\delta^{\operatorname{Re} s-N}|A_\delta|_N(\pi/2-1). \qquad (3.1.8)
\end{aligned}$$

Hence for $s_0 \in \{\operatorname{Re} s < N+1\}$, the condition (3′) of Remark 1.4 is satisfied, which implies, in light of Theorem 1.3, that $E(s;\delta)$ is holomorphic on $\{\operatorname{Re} s < N+1\}$.

On the other hand, we know that both $\tilde{\zeta}_A$ and $\tilde{\zeta}_{A_M}$ are holomorphic on $\{\operatorname{Re} s > \overline{\dim}_B A\} \supseteq \{\operatorname{Re} s > N\}$. The fact that $E(s;\delta)$ is meromorphic on $\mathbb{C}$, as well as the statement about its poles, now follow from (3.1.4) (with $M=1$) and the fact that the gamma function is nowhere vanishing in $\mathbb{C}$. (In fact, $1/\Gamma(s)$ is an entire function with zeros at the nonpositive integers.) More specifically, the poles of $E(s;\delta)$ must coincide with the poles $s_k = N+2+2k$, for $k\in\mathbb{N}_0$, of $\Gamma((N-s)/2+1)$ since the left-hand side of (3.1.4) is holomorphic on $\{\operatorname{Re} s > \overline{\dim}_B A\}$ and because $\tilde{\zeta}_A(s_k) > 0$ (since it is defined as the integral of a positive function). (Note that since $N \ge \overline{D}$, we have $s_k > \overline{D}$, and hence $\tilde{\zeta}_A$ is well defined at s_k for each $k \in \mathbb{N}_0$.)

Finally, by multiplying (3.1.4) by $s - s_k$, taking the limit as $s \to s_k$ and then using the fact that the residue of the gamma function at $-k$ is equal to $(-1)^k/k!$, we deduce that (3.1.5) holds, as desired. Furthermore, if M is odd, there are no cancellations between the poles of the numerator and of the denominator in (3.1.4) since an integer cannot be both even and odd; i.e., the residues are nonzero for each $k\in\mathbb{N}_0$. On the other hand, if M is even, then it is clear that all of the residues at s_k for $k \ge M/2$ are zero, i.e., the corresponding poles at s_k cancel out with the poles of the denominator in (3.1.4). ■

Theorem 3.2 has an important consequence, namely, the notion of complex dimensions does not depend on the dimension of the ambient space.

THEOREM 3.3. *Let $A\subset\mathbb{R}^N$ be a bounded set and A_M be its embedding into $\mathbb{R}^{N+M}$, with $M\in\mathbb{N}$ arbitrary. Then the tube zeta function $\tilde{\zeta}_A$ has a meromorphic extension to a given connected open neighborhood U of the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$ if and only if the analogous statement is true for the tube zeta function $\tilde{\zeta}_{A_M}$. Furthermore, in that case, the approximate functional equation* (3.1.4) *remains valid for all $s\in U$. In addition, the multisets of the poles of $\tilde{\zeta}_A$ and $\tilde{\zeta}_{A_M}$ located in U coincide: $\mathcal{P}(\tilde{\zeta}_A, U) = \mathcal{P}(\tilde{\zeta}_{A_M}, U)$* (1)*. Consequently, neither the values nor the multiplicities of the complex dimensions of A depend on the dimension of the ambient space.*

---

(1) Recall that the bounded sets A and A_M have the same upper Minkowski dimension, $\overline{\dim}_B A = \overline{\dim}_B A_M$, and hence the same critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$.

Proof. This is a direct consequence of Theorem 3.2 and the principle of analytic continuation. More specifically, (3.1.4) is valid for $\operatorname{Re} s > \overline{\dim}_B A$ and the function $E(s;\delta)$ is meromorphic on $\mathbb{C}$. Furthermore, according to Theorem 3.2, the poles of $E(s;\delta)$ belong to $\{\operatorname{Re} s \geq N+2\}$, which implies that $s \mapsto E(s;\delta)$ is holomorphic on $\{\operatorname{Re} s < N+2\}$. Identity (3.1.4) then remains valid if $\tilde{\zeta}_A$ or $\tilde{\zeta}_{A_M}$ has a meromorphic continuation to some connected open neighborhood of $\{\operatorname{Re} s = \overline{\dim}_B A\}$. ∎

Corollary 3.4. *Let $A \subset \mathbb{R}^N$ be a bounded set (with $\overline{D} := \overline{\dim}_B A$) such that its tube zeta function $\tilde{\zeta}_A$ has a meromorphic continuation to a connected open neighborhood U of the critical line $\{\operatorname{Re} s = \overline{\dim}_B A\}$. Furthermore, suppose that $s = \overline{D}$ is a simple pole of $\tilde{\zeta}_A$. Let $A_M \subset \mathbb{R}^{N+M}$ be the embedding of A into $\mathbb{R}^{N+M}$, as in Theorem 3.2. Then*

$$\operatorname{res}(\tilde{\zeta}_{A_M}, \overline{D}) = \frac{\big(\sqrt{\pi}\big)^M \Gamma\big(\frac{N-\overline{D}}{2}+1\big)}{\Gamma\big(\frac{N+M-\overline{D}}{2}+1\big)} \operatorname{res}(\tilde{\zeta}_A, \overline{D}). \tag{3.1.9}$$

We point out that the above corollary is compatible with the dimensional invariance of the normalized Minkowski content, established by M. Kneser [Kn] and later recovered independently in [Re]. More specifically, if in the above corollary we assume in addition that $\overline{D}$ is the only pole of $\tilde{\zeta}_A$ on $\{\operatorname{Re} s = \overline{D}\}$ (i.e., $\overline{D}$ is the only complex dimension of A with real part $\overline{D}$), then, according to [LRŽ5, Theorem 5.2], A and $A \times \{0\}$ are Minkowski measurable, with Minkowski dimension $D := \overline{D}$ and Minkowski contents satisfying

$$\frac{\mathcal{M}^D(A)}{\pi^{(D-N)/2}\Gamma\big(\frac{N-D}{2}+1\big)} = \frac{\mathcal{M}^D(A \times \{0\})}{\pi^{(D-N-1)/2}\Gamma\big(\frac{N+1-D}{2}+1\big)}. \tag{3.1.10}$$

3.2. Embeddings of relative fractal drums. The results obtained in the previous section for bounded subsets of $\mathbb{R}^N$ can also be obtained for relative fractal drums (RFDs) in $\mathbb{R}^N$. More specifically, let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and let

$$(A \times \{0\}, \Omega \times (-1,1))$$

be its natural embedding into $\mathbb{R}^{N+1}$. We want to connect the relative tube zeta functions of these two RFDs; the following lemma will be needed for this purpose.

Lemma 3.5. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ and fix $\delta \in (0,1)$. Then*

$$|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1} = 2\int_0^\delta |A_{\sqrt{\delta^2-u^2}} \cap \Omega|_N \, \mathrm{d}u. \tag{3.2.1}$$

Proof. We proceed much as in the proof of [Re, Proposition 6]. Namely, we let $(x,y) \in \mathbb{R}^N \times \mathbb{R} \equiv \mathbb{R}^{N+1}$ and define

$$V := \{(x,y) : d_{N+1}((x,y), A \times \{0\}) \leq \delta\} \cap \{(x,y) : x \in \Omega,\ |y| < 1\}, \tag{3.2.2}$$

where for any $k \in \mathbb{N}$, d_k denotes the Euclidean distance in $\mathbb{R}^k$. It is clear that $d_{N+1}((x,y), A \times \{0\}) = \sqrt{d_N(x,A)^2 + y^2}$. This implies that for a fixed $y \in [-\delta,\delta]$, we have

$$V_y := \{x \in \mathbb{R}^N : d_{N+1}((x,y), A \times \{0\}) \leq \delta\} = \{x : d_N(x,A) \leq \sqrt{\delta^2 - y^2}\}. \tag{3.2.3}$$

(Note that if $|y| > \delta$, then V_y is empty.) Finally, Fubini's theorem implies that

$$
\begin{aligned}
|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1} &= \int_V \mathrm{d}x\,\mathrm{d}y = \int_{-\delta}^{\delta} \mathrm{d}y \int_{V_y \cap \{x \in \mathbb{R}^N : x \in \Omega\}} \mathrm{d}x \\
&= 2\int_0^\delta |A_{\sqrt{\delta^2 - y^2}} \cap \Omega|_N \,\mathrm{d}y. \ \blacksquare
\end{aligned}
$$

The above lemma will eventually yield (in Theorem 3.10 below) an RFD analog of Proposition 3.1. First, however, we will show that the upper and lower relative box dimensions of an RFD are independent of the ambient space dimension.

PROPOSITION 3.6. *Let (A, Ω) be an RFD in $\mathbb{R}^N$ and let*

$$(A, \Omega)_M := (A_M, \Omega \times (-1,1)^M) \tag{3.2.4}$$

be its embedding into $\mathbb{R}^{N+M}$ for some $M \in \mathbb{N}$. Then

$$\overline{\dim}_B(A, \Omega) = \overline{\dim}_B(A, \Omega)_M, \tag{3.2.5}$$

$$\underline{\dim}_B(A, \Omega) = \underline{\dim}_B(A, \Omega)_M. \tag{3.2.6}$$

Proof. We only prove the proposition when $M = 1$, from which the general result easily follows by induction. It is clear that for $0 < \delta < 1$, we have

$$(A \times \{0\})_\delta \cap (\Omega \times (-1,1)) \subseteq (A \times \{0\})_\delta \cap (\Omega \times (-\delta,\delta)) \subseteq (A_\delta \cap \Omega) \times (-\delta, \delta),$$

so that

$$|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1} \leq 2\delta |A_\delta \cap \Omega|_N. \tag{3.2.7}$$

This observation, in turn, implies that for every $r \in \mathbb{R}$,

$$\frac{|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1}}{\delta^{N+1-r}} \leq \frac{2|A_\delta \cap \Omega|_N}{\delta^{N-r}}. \tag{3.2.8}$$

Furthermore, by successively taking the upper and lower limits as $\delta \to 0^+$ in (3.2.8), we obtain the following inequalities, involving the r-dimensional upper and lower relative Minkowski contents of the RFDs $(A, \Omega)_1$ and (A, Ω), respectively:

$$\mathcal{M}^{*r}(A, \Omega)_1 \leq 2\mathcal{M}^{*r}(A, \Omega) \quad \text{and} \quad \mathcal{M}_*^r(A, \Omega)_1 \leq 2\mathcal{M}_*^r(A, \Omega). \tag{3.2.9}$$

In light of the definition of the relative upper and lower box (or Minkowski) dimensions (see (1.4.2) and (1.4.3) and the text surrounding them), we deduce that

$$\overline{\dim}_B(A, \Omega)_1 \leq \overline{\dim}_B(A, \Omega) \quad \text{and} \quad \underline{\dim}_B(A, \Omega)_1 \leq \underline{\dim}_B(A, \Omega). \tag{3.2.10}$$

On the other hand, for geometric reasons, we have

$$(A_{\delta/2} \cap \Omega) \times (-\delta\sqrt{3}/2, \delta\sqrt{3}/2) \subseteq (A \times \{0\})_\delta \cap (\Omega \times (-1,1)),$$

so that

$$\delta\sqrt{3}\, |A_{\delta/2} \cap \Omega|_N \leq |(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1}\,. \tag{3.2.11}$$

Much as before, this inequality implies that for every $r \in \mathbb{R}$,

$$\frac{\sqrt{3}\, |A_{\delta/2} \cap \Omega|_N}{2^{N-r}(\delta/2)^{N-r}} \leq \frac{|(A \times \{0\})_\delta \cap (\Omega \times (-1,1))|_{N+1}}{\delta^{N+1-r}}, \tag{3.2.12}$$

and by successively taking the upper and lower limits as $\delta\to 0^+$, we obtain

$$\frac{\sqrt{3}\,\mathcal{M}^{*r}(A,\Omega)}{2^{N-r}}\le\mathcal{M}^{*r}(A,\Omega)_1\quad\text{and}\quad\frac{\sqrt{3}\,\mathcal{M}_*^{r}(A,\Omega)}{2^{N-r}}\le\mathcal{M}_*^{r}(A,\Omega)_1. \tag{3.2.13}$$

This completes the proof because (again in light of (1.4.2) and (1.4.3) and the text surrounding them) (3.2.13) implies the reverse inequalities for the upper and lower relative box dimensions in (3.2.10). ∎

Remark 3.7. Observe that it follows from Proposition 3.6 (combined with Theorem 2.1(b)) that (A,Ω) and $(A,\Omega)_M$ have the same upper Minkowski dimension, $\overline{\dim}_B(A,\Omega)=\overline{\dim}_B(A,\Omega)_M$, and hence the same critical line $\{\operatorname{Re}s=\overline{\dim}_B(A,\Omega)\}$. This will be used implicitly in the statements of Proposition 3.8 and of Theorems 3.9 and 3.10.

We can now state the desired results for embedded RFDs and their relative zeta functions. In light of Lemma 3.5 and Proposition 3.6, the proofs follow the same steps as in the corresponding results established in §3.1 for bounded subsets of $\mathbb{R}^N$ (Proposition 3.1 and Theorem 3.2), and for this reason we will omit them.

Proposition 3.8. *Fix $\delta\in(0,1)$ and let (A,Ω) be an RFD in $\mathbb{R}^N$, with $\overline{D}:=\overline{\dim}_B(A,\Omega)$. Then, for the relative tube zeta functions of (A,Ω) and $(A,\Omega)_1:=(A\times\{0\},\Omega\times(-1,1))$, the following equality holds*:

$$\tilde\zeta_{A\times\{0\},\Omega\times(-1,1);\delta}(s)=2\int_0^{\pi/2}\frac{\tilde\zeta_{A,\Omega;\delta\sin\tau}(s)}{\sin^{s-N-1}\tau}\,\mathrm{d}\tau \tag{3.2.14}$$

for $\operatorname{Re}s>\overline{D}$.

Theorem 3.9. *Fix $\delta\in(0,1)$ and let (A,Ω) be an RFD in $\mathbb{R}^N$ with $\overline{D}:=\overline{\dim}_B(A,\Omega)$. Then we have the following equality between $\tilde\zeta_{A,\Omega}$, the tube zeta function of (A,Ω), and $\tilde\zeta_{A_M,\Omega\times(-1,1)^M}$, the tube zeta function of the relative fractal drum $(A,\Omega)_M:=(A_M,\Omega\times(-1,1)^M)$ in $\mathbb{R}^{N+M}$ for $M\in\mathbb{N}$*:

$$\tilde\zeta_{A_M,\Omega\times(-1,1)^M;\delta}(s)=\frac{(\sqrt{\pi})^M\Gamma\big(\frac{N-s}{2}+1\big)}{\Gamma\big(\frac{N+M-s}{2}+1\big)}\tilde\zeta_{A,\Omega;\delta}(s)+E(s;\delta), \tag{3.2.15}$$

initially valid for $\operatorname{Re}s>\overline{D}$. (*See Theorem* 3.10 *for more precise information about the domain of validity of* (3.2.15).) *Here, $E(s):=E(s;\delta)$ is meromorphic on all of $\mathbb{C}$. Furthermore, the possible poles* (*in* $\mathbb{C}$) *of $E(s;\delta)$ are at $s_k:=N+2+2k$ for every $k\in\mathbb{N}_0$, and all of them are simple.* (*It follows that $\tilde\zeta_A$ is well defined at each s_k.*) *Moreover, for each $k\in\mathbb{N}_0$,*

$$\operatorname{res}(E(\,\cdot\,;\delta),s_k)=\frac{(-1)^{k+1}(\sqrt{\pi})^M}{k!\,\Gamma(M/2-k)}\tilde\zeta_{A,\Omega;\delta}(s_k). \tag{3.2.16}$$

More specifically, if M is even, then all of the poles s_k of $E(s;\delta)$ are canceled for $k\ge M/2$; i.e., the residues in (3.2.16) *are zero. On the other hand, if M is odd, then there are no such cancellations, so that all s_k's are* (*simple*) *poles of $E(s;\delta)$ in that case.*

We deduce at once from Theorem 3.9 the following key result about the invariance of the complex dimensions of a relative fractal drum with respect to the dimension of the ambient space. This result extends Theorem 3.3 to general RFDs.

THEOREM 3.10. *Let* (A,Ω) *be an RFD in* $\mathbb{R}^N$ *and let* $(A,\Omega)_M := (A_M, \Omega\times(-1,1)^M)$ *be its embedding into* $\mathbb{R}^{N+M}$ *for some* $M\in\mathbb{N}$. *Then the tube zeta function* $\tilde{\zeta}_{A,\Omega}$ *has a meromorphic extension to a given open connected neighborhood* U *of the critical line* $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$ *if and only if the analogous statement is true for* $\tilde{\zeta}_{(A,\Omega)_M} := \tilde{\zeta}_{A_M,\Omega\times(-1,1)^M}$. *(See Remark* 3.7.*) Furthermore, in that case,* (3.2.15) *remains valid for all* $s\in U$. *In addition, the multisets of the poles of* $\tilde{\zeta}_{A,\Omega}$ *and* $\tilde{\zeta}_{(A,\Omega)_M}$ *belonging to* U *coincide:*

$$\mathcal{P}(\tilde{\zeta}_{A,\Omega}, U) = \mathcal{P}(\tilde{\zeta}_{(A,\Omega)_M}, U). \tag{3.2.17}$$

Consequently, neither the values nor the multiplicities of the complex dimensions of the RFD (A,Ω) *depend on the dimension of the ambient space.*

REMARK 3.11. In the above discussion about embedding RFDs into higher-dimensional spaces, we can also make similar observations if we embed (A,Ω) as a “one-sided” RFD, for example of the form $(A\times\{0\}, \Omega\times(0,1))$, a fact which can be more useful when decomposing a relative fractal drum into a union of relative fractal subdrums in order to compute its distance (or tube) zeta function. This observation follows immediately from the above results for “two-sided” embeddings of RFDs, since by symmetry we have

$$\tilde{\zeta}_{A\times\{0\},\Omega\times(-1,1)}(s) = 2\,\tilde{\zeta}_{A\times\{0\},\Omega\times(0,1)}(s). \tag{3.2.18}$$

We note that when using the above formulas, one only has to be careful to take into account the factor 2. Furthermore, we can also embed (A,Ω) as

$$(A\times\{0\}, \Omega\times(-\alpha,\alpha)) \quad \text{or} \quad (A\times\{0\}, \Omega\times(0,\alpha)),$$

for some $\alpha>0$, but in that case the corresponding formulas will only be valid for all $\delta\in(0,\alpha)$.

We could now use the functional equation (2.4.2) connecting the tube and distance zeta functions, in order to translate the above results in terms of $\zeta_{A,\Omega}$, the (relative) distance zeta function of the RFD (A,Ω). However, we will instead use another approach because it gives some additional information about the resulting error function. More specifically, consider the *Mellin zeta function of a relative fractal drum* (A,Ω) defined by

$$\zeta^{\mathfrak{M}}_{A,\Omega}(s) := \int_0^{+\infty} t^{s-N-1}|A_t\cap\Omega|\,\mathrm{d}t \tag{3.2.19}$$

for all $s\in\mathbb{C}$ in a suitable vertical strip. In fact, in light of [LRŽ5, Theorem 5.7] (see also [LRŽ1, Theorem 5.4.7]), the above Lebesgue integral is absolutely convergent (and hence convergent) for $\operatorname{Re} s\in(\overline{\dim}_B(A,\Omega), N)$. Moreover, the relative distance and Mellin zeta functions of (A,Ω) are connected by the functional equation

$$\zeta_{A,\Omega}(s) = (N-s)\zeta^{\mathfrak{M}}_{A,\Omega}(s) \tag{3.2.20}$$

on every open connected set $U\subseteq\mathbb{C}$ to which any of the two zeta functions has a meromorphic continuation. Observe that in (3.2.20), the parameter δ is absent. Indeed, this means implicitly that the functional equation (3.2.20) is valid only for those $\delta>0$ for which $\Omega\subseteq A_\delta$, that is, when $\zeta_{A,\Omega;\delta}(s) = \int_\Omega d(x,A)^{s-N}\,\mathrm{d}x$.

We will now embed the relative fractal drum (A,Ω) of $\mathbb{R}^N$ into $\mathbb{R}^{N+1}$ as

$$(A\times\{0\},\Omega\times\mathbb{R}).$$

Strictly speaking, this is not a relative fractal drum in $\mathbb{R}^{N+1}$ since there does not exist a $\delta>0$ such that $\Omega\times\mathbb{R}\subseteq(A\times\{0\})_\delta$. On the other hand, observe that Lemma 3.5 is now valid for every $\delta>0$:

$$|(A\times\{0\})_\delta\cap(\Omega\times\mathbb{R})|_{N+1}=2\int_0^\delta|A_{\sqrt{\delta^2-u^2}}\cap\Omega|_N\,\mathrm{d}u. \tag{3.2.21}$$

Proposition 3.12. *Let (A,Ω) be an RFD in $\mathbb{R}^N$ such that $\overline{\dim}_B(A,\Omega)<N$. Then the function $F=F(s)$, defined by the integral*

$$F(s):=\int_0^{+\infty}t^{s-N-2}\big|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})\big|_{N+1}\,\mathrm{d}t, \tag{3.2.22}$$

is holomorphic inside the vertical strip $\{\overline{\dim}_B(A,\Omega)<\operatorname{Re}s<N\}$.

Proof. We split the integral into two integrals: $F(s)=\int_0^1+\int_1^{+\infty}$. According to Proposition 3.6, the first integral

$$\int_0^1 t^{s-N-2}|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})|_{N+1}\,\mathrm{d}t=\int_0^1 t^{s-N-2}|(A\times\{0\})_t\cap(\Omega\times(-1,1))|_{N+1}\,\mathrm{d}t$$

defines a holomorphic function on the right half-plane $\{\operatorname{Re}s>\overline{\dim}_B(A,\Omega)\}$.

In order to deal with the second integral, we observe that

$$|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})|_{N+1}\le 2t|\Omega|_N,$$

and consequently

$$\left|\int_1^{+\infty}t^{s-N-2}|(A\times\{0\})_t\cap(\Omega\times\mathbb{R})|_{N+1}\,\mathrm{d}t\right|\le 2|\Omega|_N\int_1^{+\infty}t^{\operatorname{Re}s-N-1}\,\mathrm{d}t=\frac{2|\Omega|_N}{N-\operatorname{Re}s}$$

for $\operatorname{Re}s<N$. In light of Theorem 1.3 and Remark 1.4, the latter inequality implies that the integral over $(1,\infty)$ defines a holomorphic function on $\{\operatorname{Re}s<N\}$. Therefore, $F(s)$ is holomorphic in $\{\overline{\dim}_B(A,\Omega)<\operatorname{Re}s<N\}$. ∎

In light of the above proposition, we continue to use the convenient notation $\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}$ for the integral on the right-hand side of (3.2.22), although, as was noted earlier, $(A\times\{0\},\Omega\times\mathbb{R})$ is not technically a relative fractal drum in $\mathbb{R}^{N+1}$ (see Remark 3.11). The following is the counterpart of Theorem 3.2 in the present, more general context.

Theorem 3.13. *Let (A,Ω) be a relative fractal drum in $\mathbb{R}^N$ such that $\overline{D}:=\overline{\dim}_B(A,\Omega)<N$. Then, for every $a>0$, the following approximate functional equation holds*:

$$\zeta_{A\times\{0\},\Omega\times(-a,a)}(s)=\frac{\sqrt{\pi}\,\Gamma\big(\frac{N-s}{2}\big)}{\Gamma\big(\frac{N+1-s}{2}\big)}\zeta_{A,\Omega}(s)+E(s;a), \tag{3.2.23}$$

initially valid for all $s\in\mathbb{C}$ such that $\operatorname{Re}s>\overline{D}$. Here, the error function $E(s):=E(s;a)$ is initially given (for all $s\in\mathbb{C}$ such that $\operatorname{Re}s<N$) by

$$E(s;a):=(s-N-1)\int_a^{+\infty}t^{s-N-2}|(A\times\{0\})_t\cap\Omega\times(\mathbb{R}\setminus(-a,a))|_{N+1}\,\mathrm{d}t, \tag{3.2.24}$$

and admits a meromorphic extension to all of $\mathbb{C}$*, with the set of simple poles equal to* $\{N+2k : k \in \mathbb{N}_0\}$*.*

Moreover, (3.2.23) *remains valid on any connected open neighborhood of the critical line* $\{\operatorname{Re} s = \overline{D}\}$ *to which* $\zeta_{A,\Omega}$ (*or equivalently* $\zeta_{A\times\{0\},\Omega\times(-a,a)}$) *can be meromorphically continued.*

Proof. In a completely analogous way as in the proof of Theorem 3.2, we obtain

$$\tilde{\zeta}_{A\times\{0\},\Omega\times\mathbb{R};\delta}(s) = \frac{\sqrt{\pi}\,\Gamma\big(\frac{N-s}{2}+1\big)}{\Gamma\big(\frac{N+1-s}{2}+1\big)}\tilde{\zeta}_{A,\Omega;\delta}(s) + \tilde{E}(s;\delta), \tag{3.2.25}$$

now for all $\delta > 0$ (see (3.2.21) and the discussion preceding it). Furthermore, the error function $\tilde{E}(s) := \tilde{E}(s;\delta)$ is holomorphic on $\{\operatorname{Re} s < N+1\}$ and

$$|\tilde{E}(s,\delta)| \le 2\delta^{\operatorname{Re} s-N}|A_\delta \cap \Omega|_N(\pi/2-1) \tag{3.2.26}$$

for $\operatorname{Re} s < N+1$. See the proof of Theorem 3.2 and (3.1.8) for the derivation of the above estimate. The estimate (3.2.26) now implies that the sequence of holomorphic functions $\tilde{E}(\,\cdot\,;n)$ tends to 0 as $n \to \infty$, uniformly on every compact subset of $\{\operatorname{Re} s < N\}$, since $|A_n \cap \Omega| = |\Omega|$ for all n sufficiently large. Furthermore, $\tilde{\zeta}_{A,\Omega;n} \to \zeta^{\mathfrak{M}}_{A,\Omega}$ and

$$\tilde{\zeta}_{A\times\{0\},\Omega\times\mathbb{R}}(s;n) \to \zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}} \quad \text{as } n \to \infty, \tag{3.2.27}$$

uniformly on every compact subset of $\{\overline{D} < \operatorname{Re} s < N\}$. This implies that by lefting $\delta \to +\infty$ in (3.2.25), we obtain the following functional equality between holomorphic functions:

$$\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}(s) = \frac{\sqrt{\pi}\,\Gamma\big(\frac{N-s}{2}+1\big)}{\Gamma\big(\frac{N+1-s}{2}+1\big)}\zeta^{\mathfrak{M}}_{A,\Omega}(s) \tag{3.2.28}$$

in $\{\overline{D} < \operatorname{Re} s < N\}$. (We can obtain this equality even more directly by applying Lebesgue's dominated convergence theorem to a counterpart of (3.2.14).)

Moreover, according to (3.2.20) and (3.2.28), we have the functional equation

$$\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}(s) = \frac{2\sqrt{\pi}\,\Gamma\big(\frac{N-s}{2}\big)}{\Gamma\big(\frac{N+1-s}{2}+1\big)}\zeta_{A,\Omega}(s), \tag{3.2.29}$$

from which we deduce that the right-hand side admits a meromorphic extension to $\{\operatorname{Re} s > \overline{D}\}$, with simple poles located at the simple poles of $\Gamma((N-s)/2)$, that is, at $s_k := N+2k$ for all $k \in \mathbb{N}_0$. (Observe that in the above ratio of gamma functions, there are no cancellations between the poles of the numerator and of the denominator, as an integer cannot be both even and odd.) By the principle of analytic continuation, the same property also holds for the left-hand side of (3.2.29), and the left-hand side has a meromorphic extension to any domain $U \subseteq \mathbb{C}$ to which the right-hand side can be meromorphically extended.

To complete the proof of the theorem, we now observe that for any $a > 0$, since

$$\begin{aligned}|(A\times\{0\})_t \cap (\Omega\times\mathbb{R})| = {} & |(A\times\{0\})_t \cap (\Omega\times(-a,a))| \\ & + |(A\times\{0\})_t \cap (\Omega\times(\mathbb{R}\setminus(-a,a)))|,\end{aligned}$$

the left-hand side of (3.2.29) can be split into two parts:

$$\begin{aligned}\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times\mathbb{R}}(s)&=\zeta^{\mathfrak{M}}_{A\times\{0\},\Omega\times(-a,a)}(s)\\&\quad+\int_a^{+\infty}t^{s-N-2}|(A\times\{0\})_t\cap(\Omega\times(\mathbb{R}\setminus[-a,a]))|\,\mathrm{d}t\\&=\frac{\zeta_{A\times\{0\},\Omega\times(-a,a)}(s)}{N+1-s}-\frac{E(s;a)}{N+1-s}.\end{aligned}$$

We then combine this observation with (3.2.29) to obtain (3.2.23). In light of Theorem 2.1(a), we know that $\zeta_{A\times\{0\},\Omega\times(-a,a)}(s)$ is holomorphic on $\{\operatorname{Re}s>\overline{D}\}$. Furthermore, much as in the proof of Proposition 3.12, we can show that $E(s):=E(s;a)$ defines a holomorphic function on $\{\operatorname{Re}s<N\}$. This fact, together with the functional equation (3.2.23), now ensures that $E(s;a)$ admits a meromorphic continuation to all of $\mathbb{C}$, with the set of simple poles equal to $\{N+2k:k\in\mathbb{N}_0\}$. (Note that $\zeta_{A,\Omega}(s)>0$ for all $s\in[N,+\infty)$, which implies that there are no zero-pole cancellations on the right-hand side of (3.2.23).) This completes the proof of Theorem 3.13. ∎

We note that in Example 3.15 below, we actually want to embed (A,Ω) into $\mathbb{R}^{N+1}$, as $(A\times\{0\},\Omega\times(0,a))$ for some $a>0$. By looking at the proof of the above theorem and using a suitable symmetry argument, we can obtain the following result, which deals with this type of embedding.

THEOREM 3.14. *Let* (A,Ω) *be a relative fractal drum in* $\mathbb{R}^N$ *such that* $\overline{D}:=\overline{\dim}_B(A,\Omega)<N$. *Then the following approximate functional equation holds*:

$$\zeta_{A\times\{0\},\Omega\times(0,a)}(s)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{N-s}{2}\right)}{2\Gamma\left(\frac{N+1-s}{2}\right)}\zeta_{A,\Omega}(s)+E(s;a),\tag{3.2.30}$$

initially valid for all $s\in\mathbb{C}$ *such that* $\operatorname{Re}s>\overline{D}$. *Here, the error function* $E(s):=E(s;a)$ *is initially given* (*for all* $s\in\mathbb{C}$ *such that* $\operatorname{Re}s<N$) *by*

$$E(s;a):=(s-N-1)\int_a^{+\infty}t^{s-N-2}|(A\times\{0\})_t\cap\Omega\times(\mathbb{R}\setminus(0,a))|_{N+1}\,\mathrm{d}t,\tag{3.2.31}$$

and admits a meromorphic continuation to all of $\mathbb{C}$, *with the set of simple poles equal to* $\{N+2k:k\in\mathbb{N}_0\}$.

Moreover, (3.2.30) *remains valid on any connected open neighborhood of the critical line* $\{\operatorname{Re}s=\overline{D}\}$ *to which* $\zeta_{A,\Omega}$ (*or, equivalently,* $\zeta_{A\times\{0\},\Omega\times(0,a)}$) *can be meromorphically continued.*

EXAMPLE 3.15 (Complex dimensions of the Cantor dust RFD). In this example, we will consider the relative fractal drum consisting of the Cantor dust contained in $[0,1]^2$ and compute its distance zeta function. More precisely, let $A:=C^{(1/3)}\times C^{(1/3)}$ be the Cantor dust and let $\Omega:=(0,1)^2$. We will not obtain for ζ_A an explicit formula in closed form, but we will use Theorem 3.14 to deduce that the distance zeta function of A has a meromorphic continuation to all of $\mathbb{C}$.

More interestingly, we will also show that the set of complex dimensions of the Cantor dust is the union of (a nontrivial subset of) a periodic set contained in the critical line $\{\operatorname{Re}s=\log_3 4\}$ and the set of complex dimensions of the Cantor set (which is a periodic set contained in the critical line $\{\operatorname{Re} s = \log_3 2\}$). *This is significant because it shows that in this case, the distance* (*or tube*) *zeta function also detects the "lower-dimensional" fractal nature of the Cantor dust.*

Note that, as is well known, the Minkowski dimension of the RFD (or Cantor string) $(C,(0,1))$ is given by $\dim_B(C,(0,1)) = \log_3 2$ (see, e.g., [LF1, §1.2.2]). Furthermore, it will follow from the discussion below that, as might be expected since $(A,\Omega) = (C,(0,1)) \times (C,(0,1))$),

$$\dim_B(A,\Omega) = 2\dim_B(C,(0,1)) = \log_3 4. \tag{3.2.32}$$

Consequently, the critical line of the RFD $(C,(0,1))$ in $\mathbb{R}$ (the *Cantor string RFD*) is the vertical line $\{\operatorname{Re} s = \log_3 2\}$, while the critical line of the Cantor dust, viewed as the RFD (A,Ω) in $\mathbb{R}^2$, is the vertical line $\{\operatorname{Re} s = \log_3 4\}$, as was stated in the previous paragraph.

The construction of the RFD (A,Ω) can be carried out by beginning with the unit square and removing the open middlethird "cross", and then iterating this procedure ad infinitum. This procedure implies that we can subdivide the Cantor dust into a countable union of RFDs which are scaled down versions of two base (or generating) RFDs, (A_1,Ω_1) and (A_2,Ω_2). Here $\Omega_1 := (0,1/3)^2$ and A_1 is the union of the four vertices of the closure of Ω_1, while $\Omega_2 := (0,1/3) \times (0,1/6)$ and A_2 is the ternary Cantor set contained in $[0,1/3] \times \{0\}$.

At the nth step of the iteration, we have exactly 4^{n-1} RFDs of the type $(a_nA_1, a_n\Omega_1)$ and $8 \cdot 4^{n-1}$ RFDs of the type $(a_nA_2, a_n\Omega_2)$, where $a_n := 3^{-n}$ for each $n \in \mathbb{N}$. Together with the scaling property of the relative distance zeta function (see Theorem 2.16), this yields successively (for $\operatorname{Re} s$ sufficiently large):

$$\begin{aligned}
\zeta_{A,\Omega}(s) &= \sum_{n=1}^{\infty} 4^{n-1}\zeta_{a_nA_1,a_n\Omega_1}(s) + 8\sum_{n=1}^{\infty} 4^{n-1}\zeta_{a_nA_2,a_n\Omega_2}(s) \\
&= (\zeta_{A_1,\Omega_1}(s) + 8\zeta_{A_2,\Omega_2}(s)) \sum_{n=1}^{\infty} 4^{n-1} \cdot 3^{-ns} \\
&= \frac{1}{3^s - 4}(\zeta_{A_1,\Omega_1}(s) + 8\zeta_{A_2,\Omega_2}(s)).
\end{aligned} \tag{3.2.33}$$

Moreover,

$$\begin{aligned}
\zeta_{A_1,\Omega_1}(s) &= 8\int_0^{1/6} \mathrm{d}x \int_0^x (\sqrt{x^2+y^2})^{s-2}\,\mathrm{d}y \\
&= 8\int_0^{\pi/4} \mathrm{d}\varphi \int_0^{1/6\cos\varphi} r^{s-1}\,\mathrm{d}r \\
&= \frac{8}{6^s s}\int_0^{\pi/4} \cos^{-s}\varphi\,\mathrm{d}\varphi = \frac{8I(s)}{6^s s},
\end{aligned} \tag{3.2.34}$$

where $I(s) := \int_0^{\pi/4} \cos^{-s}\varphi\,\mathrm{d}\varphi$ and is easily seen to be an entire function. (In fact, $I(s) = 2^{-1}\mathrm{B}_{1/2}\,(1/2,(1-s)/2)$, where $\mathrm{B}_x(a,b) := \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the incomplete beta function.) Consequently, $\zeta_{A,\Omega}$ admits a meromorphic continuation to all of $\mathbb{C}$ and

$$\zeta_{A,\Omega}(s) = \frac{8}{3^s-4}\left(\frac{I(s)}{6^s s} + \zeta_{A_2,\Omega_2}(s)\right) \tag{3.2.35}$$

for all $s \in \mathbb{C}$. Furthermore, let $\zeta_{C,(0,1)}$ be the relative distance zeta function of the Cantor middlethird set constructed inside $[0,1]$ [LRŽ5, Example 6.3]. Alternatively, use the relation $\zeta_{C,(0,1)}(s) = \zeta_{\mathcal{L}_{CS}}(s)(2^{1-s}/s)$, where $\mathcal{L}_{CS}$ is the Cantor string and (by [LF3, (1.29), p. 22]) $\zeta_{\mathcal{L}_{CS}}(s) = 1/(3^s - 2)$ for all $s \in \mathbb{C}$. From Theorem 3.14 and the scaling property of the relative distance zeta function (Theorem 2.16), we now deduce that

$$\begin{aligned}\zeta_{A_2,\Omega_2}(s) &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-s}{2}\right)}{2\Gamma\left(\frac{2-s}{2}\right)}\zeta_{3^{-1}C,3^{-1}(0,1)}(s) + E(s;6^{-1})\\ &= \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\,\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1}), \qquad (3.2.36)\end{aligned}$$

where $E(s;6^{-1})$ is meromorphic on all of $\mathbb{C}$ with the set of simple poles equal to $\{2k+1 : k \in \mathbb{N}_0\}$; so that for all $s \in \mathbb{C}$, we have

$$\zeta_{A,\Omega}(s) = \frac{8}{s(3^s-4)}\left(\frac{I(s)}{6^s} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\,\frac{\sqrt{\pi}}{6^s s(3^s-2)} + E(s;6^{-1})\right). \qquad (3.2.37)$$

Formula (3.2.37) implies that $\mathcal{P}(\zeta_{A,\Omega})$, the set of all complex dimensions (in $\mathbb{C}$) of the "relative" Cantor dust, is a subset of

$$\left(\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{0\} \qquad (3.2.38)$$

and consists of simple poles of $\zeta_{A,\Omega}$. Of course, we know that $\log_3 4 \in \mathcal{P}(\zeta_{A,\Omega})$, but we can only conjecture that the other poles on the critical line $\{\operatorname{Re} s = \log_3 4\}$ are in $\mathcal{P}(\zeta_{A,\Omega})$ since it may happen that there are some zero-pole cancellations in (3.2.37). On the other hand, since it is known that the Cantor dust is not Minkowski measurable (see [FZ]), we can deduce from [LRŽ6, Theorem 4.2] that there must exist at least two other (necessarily nonreal) poles $s_{\pm k_0} = \log_3 4 \pm \frac{2k_0\pi\mathbb{i}}{\log 3}$ of $\zeta_{A,\Omega}$ for some $k_0 \in \mathbb{N}$. (Indeed, according to the Minkowski measurability criterion established in [LRŽ6, Theorem 4.2] (see also [LRŽ1, Theorem 5.4.20]), $D := \log_3 4$ cannot be the only complex dimension of (A,Ω) on the critical line $\{\operatorname{Re} s = D\}$, since otherwise the Cantor dust would be Minkowski measurable, which is not the case.) Based on (3.2.37), we cannot even claim that $0 \in \mathcal{P}(\zeta_{A,\Omega})$ for sure, but we can see that all of the principal complex dimensions of the Cantor set are elements of $\mathcal{P}(\zeta_{A,\Omega})$, i.e., $\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \subseteq \mathcal{P}(\zeta_{A,\Omega})$. We conjecture that also $\log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z} \subseteq \mathcal{P}(\zeta_{A,\Omega})$, that is, $\mathcal{P}_c(\zeta_{A,\Omega}) = \log_3 4 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$.

The above example can be easily generalized to the case of Cartesian products of any finite number of generalized Cantor sets (as given by Definition 2.28), in which case we conjecture that the set of complex dimensions of the product is contained in the union of the sets of complex dimensions of the factors, modulo any zero-pole cancellations which may occur. In light of this and other similar examples, it would be interesting to obtain some results about zero-free regions for fractal zeta functions. We leave this problem as a possible subject for future investigations.

4. Relative fractal sprays and principal complex dimensions of any multiplicity

In this chapter, we consider a special type of RFDs, called *relative fractal sprays*, and study their distance zeta functions. We then illustrate our results by computing the corresponding complex dimensions of relative Sierpiński sprays. More specifically, we determine the complex dimensions (as well as the associated residues) of the relative Sierpiński gasket (Example 4.12) and of the relative Sierpiński carpet (Example 4.15); we also consider higher-dimensional analogs of these examples, namely, the inhomogeneous Sierpiński N-gasket RFD (Example 4.14) and the Sierpiński N-carpet RFD (Example 4.17).

4.1. Relative fractal sprays in $\mathbb{R}^N$. We now introduce the definition of relative fractal spray, which is very similar to (but more general than) the notion of fractal spray (see [LPo2], [LF3, Definition 13.2], [LP1, LP2] and [LPW1, LPW2]), itself a generalization of the notion of (ordinary) fractal string [LPo1, L1, L3, LLF2].

Definition 4.1. Let $(\partial\Omega_0,\Omega_0)$ be a fixed relative fractal drum in $\mathbb{R}^N$ (which we call the *base relative fractal drum*, or *generating relative fractal drum*), $(\lambda_j)_{j\ge0}$ a decreasing sequence of positive numbers (scaling factors), converging to zero, and $(b_j)_{j\ge0}$ a given sequence of positive integers (multiplicities). The associated *relative fractal spray* is a relative fractal drum (A,Ω) obtained as the disjoint union of a sequence of RFDs, $\mathcal{F}:=\{(\partial\Omega_i,\Omega_i):i\in\mathbb{N}_0\}$, where $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, such that each Ω_i can be obtained from $\lambda_j\Omega_0$ by a rigid motion in $\mathbb{R}^N$, and for each $j\in\mathbb{N}_0$ there are precisely b_j RFDs in the family $\mathcal{F}$ that can be obtained from $\lambda_j\Omega_0$ by a rigid motion. Any relative fractal spray (A,Ω), generated by the base relative fractal drum (or "basic shape") Ω_0 and the sequences of "scales" $(\lambda_j)_{j\ge0}$ with associated "multiplicities" $(b_j)_{\ge0}$, is denoted by

$$(A,\Omega):=\mathrm{Spray}(\Omega_0,(\lambda_j)_{j\ge0},(b_j)_{\ge0}). \tag{4.1.1}$$

The family $\mathcal{F}$ is called the *skeleton* of the spray. The distance zeta function $\zeta_{A,\Omega}$ of the relative fractal spray (A,Ω) is computed in Theorem 4.6 below.

If there exist $\lambda\in(0,1)$ and an integer $b\ge2$ such that $\lambda_j=\lambda^j$ and $b_j=b^j$ for all $j\in\mathbb{N}_0$, then we simply write

$$(A,\Omega)=\mathrm{Spray}(\Omega_0,\lambda,b).$$

Definition 4.2. The relative fractal spray $(A,\Omega)=\mathrm{Spray}(\Omega_0,(\lambda_j)_{j\ge0},(b_j)_{j\ge0})$ can be viewed as a relative fractal drum generated by $(\partial\Omega_0,\Omega_0)$ and a fractal string $\mathcal{L}=(\ell_j)_{j\ge0}$, consisting of the decreasing sequence $(\lambda_j)_{j\ge0}$ of positive real numbers, in which each λ_j has multiplicity b_j for every $j\ge0$. Thus, we can write $(A,\Omega)=\mathrm{Spray}(\Omega_0,\mathcal{L})$. It is also convenient to view the construction of (A,Ω) in Definition 4.1 as the *tensor product* of the base relative fractal drum (A_0,Ω_0) and the fractal string $\mathcal{L}$:

$$(A,\Omega)=(\partial\Omega_0,\Omega_0)\otimes\mathcal{L}. \tag{4.1.2}$$

We can also define the *tensor product of two (possibly unbounded) fractal strings* $\mathcal{L}_1=(\ell_{1j})_{j\geq 1}$ and $\mathcal{L}_2=(\ell_{2k})_{k\geq 1}$ as the following fractal string (note that here $\mathcal{L}_1$ and $\mathcal{L}_2$ are viewed as nonincreasing sequences of positive numbers tending to zero, but we may have $\sum_{j=1}^{\infty}\ell_{1j}=+\infty$ or $\sum_{k=1}^{\infty}\ell_{2k}=+\infty$):

$$\mathcal{L}_1\otimes\mathcal{L}_2:=(\ell_{1j}\ell_{2k})_{j,k\geq 1}. \tag{4.1.3}$$

By construction, the multiplicity of any $l\in\mathcal{L}_1\otimes\mathcal{L}_2$ is equal to the number of ordered pairs of (ℓ_{1j},ℓ_{2k}) in the Cartesian product $\mathcal{L}_1\times\mathcal{L}_2$ of multisets such that $l=\ell_{1j}\ell_{2k}$.

Example 4.3. Let $\Omega_0:=B_1(0)$ be the open unit disk in $\mathbb{R}^2$ and $A_0:=\partial\Omega_0$ the unit circle. Let $\mathcal{L}:=(\ell_j)_{j\geq 0}$ be the Cantor string. In other words, $\mathcal{L}$ is the multiset consisting of $l_k=3^{-k-1}$ with multiplicity 2^k for each $k\geq 0$. As in Definition 4.2, we define

$$(A,\Omega):=(A_0,\Omega_0)\otimes\mathcal{L}. \tag{4.1.4}$$

Then

$$\begin{aligned}\zeta_{A,\Omega}(s)&=\sum_{k=0}^{\infty}2^k\zeta_{3^{-k-1}(A_0,\Omega_0)}(s)=3^{-s}\sum_{k=0}^{\infty}2^k3^{-sk}\zeta_{A_0,\Omega_0}(s)\\&=\frac{3^{-s}\zeta_{A_0,\Omega_0}(s)}{1-2\cdot 3^{-s}}=\frac{2\pi(s-2)3^{-s}}{s(s-1)(1-2\cdot 3^{-s})},\end{aligned} \tag{4.1.5}$$

where in the last equality we have used (2.1.9) with $N=2$. It follows that $\zeta_{A,\Omega}$ has a meromorphic continuation to all of $\mathbb{C}$ given by the last expression of (4.1.5). Therefore,

$$\mathcal{P}(\zeta_{A,\Omega})=\{0\}\cup\left(\log_3 2+\frac{2\pi}{\log 3}\mathrm{i}\mathbb{Z}\right)\cup\{1\},\qquad \dim_{PC}(\tilde{A},\tilde{\Omega})=\{1\}, \tag{4.1.6}$$

and all the complex dimensions are simple. The RFD (A,Ω) has a vertical sequence of equidistant complex dimensions (namely, $\{\log_3 2+\frac{2\pi}{\log 3}\mathrm{i}k\}_{k\in\mathbb{Z}}$), while there is only one principal complex dimension (namely, $s=1$), and it is simple. Using the Minkowski measurability criterion obtained in [LRŽ6, Theorem 4.2] (see also [LRŽ1, Theorem 5.4.20]), we conclude that the RFD (A,Ω) is Minkowski measurable.

We point out, however, that one can also show (by using the results and techniques of [LRŽ5] and [LRŽ1, Chapter 5]) that (A,Ω) is (*strictly*) *subcritically Minkowski nonmeasurable in dimension* $d:=\log_3 2$, in a sense specified in those references. Heuristically, this means that it has *geometric oscillations of lower order* $d=\log_3 2$, but none of leading order $D=1$.

We can easily modify the notion of relative fractal spray in Definition 4.1 in order to deal with a finite collection of K basic RFDs (or generating RFDs) $(\partial\Omega_{01},\Omega_{01}),\dots,(\partial\Omega_{0K},\Omega_{0K})$, where K is an integer ≥ 1, as in [LPo2], [LF3, Definition 13.2] (and [LP1, LP2], [LPW1, LPW2]). A slightly more general notion would result from replacing $(\partial\Omega_0,\Omega_0)$ by any relative fractal drum (A_0,Ω_0) [LRŽ1].

It is important to stress that, from our point of view, the sets Ω_i in the definition of a relative fractal spray (Definition 4.1) do not have to be "densely packed". In fact, they cannot be "densely packed" in general, as indicated by Example 4.5(c) below. They can just be viewed as the union of the disjoint family $\{(\partial\Omega_i,\Omega_i)\}_{i\geq 0}$ of RFDs in $\mathbb{R}^N$. The corresponding disjoint family $\{\Omega_i\}_{i\geq 0}$ of open sets can even be unbounded in $\mathbb{R}^N$, since its union does not have to be of finite N-dimensional Lebesgue measure.

The following simple lemma provides necessary and sufficient conditions for a relative fractal spray (A,Ω) to have $|\Omega|<\infty$.

Lemma 4.4. *Assume that* $(A,\Omega):=\mathrm{Spray}(\Omega_0,(\lambda_j)_{j\geq 0},(b_j)_{\geq 0})$ *in* $\mathbb{R}^N$ *is a relative fractal spray. Then* $|\Omega|<\infty$ *if and only if* $|\Omega_0|<\infty$ *and*

$$\sum_{j=0}^{\infty} b_j\lambda_j^N<\infty. \tag{4.1.7}$$

In that case,

$$|\Omega|=|\Omega_0|\sum_{j=0}^{\infty} b_j\lambda_j^N. \tag{4.1.8}$$

In particular, the relative fractal drum (A,Ω) *is well defined and* $\overline{\dim}_B(A,\Omega)\leq N$.

Proof. Let us prove the sufficiency part. For $\Omega_j=\lambda_j\Omega_0$ we have $|\Omega_j|=|\lambda_j\Omega_0|=\lambda_j^N|\Omega_0|$, and therefore

$$|\Omega|=\sum_{j=0}^{\infty}|\Omega_j|=\sum_{j=0}^{\infty} b_j|\lambda_j\Omega_0|=|\Omega_0|\sum_{j=0}^{\infty} b_j\lambda_j^N.$$

The proof of the necessity part is also easy and is therefore omitted. ■

Example 4.5. Here, we provide a few simple examples of relative fractal sprays:

(a) The ternary Cantor set can be viewed as a relative fractal drum

$$(A,\Omega)=\mathrm{Spray}(\Omega_0,1/3,2)$$

(called the *Cantor relative fractal drum*, or the *relative Cantor fractal spray*), generated by

$$(\partial\Omega_0,\Omega_0)=(\{1/3,2/3\}\,,\,(1/3,2/3))$$

as the base relative fractal drum, $\lambda=1/3$ and $b=2$. Its relative box dimension exists and is given by $D=\log_3 2$. Of course, this is just an example of an ordinary fractal string, namely the well-known Cantor string [LF3, §1.1.2].

(b) The Sierpiński gasket can be viewed as a relative fractal drum (called the *Sierpiński relative fractal drum*, or *Sierpiński relative fractal spray*), generated by $(\partial\Omega_0,\Omega_0)$ as the basic relative fractal drum, where Ω_0 is an open equilateral triangle of sides of length $1/2$, $\lambda=1/2$ and $b=3$. (See the left part of Figure 1.) Its relative box dimension exists and is given by $D=\log_2 3$.

(c) If Ω_0 is any bounded open set in $\mathbb{R}^2$ (say, an open disk), $\lambda=1/2$ and $b=3$, we obtain a fractal spray $(A,\Omega)=\mathrm{Spray}(\Omega_0,1/2,3)$, in the sense of Definition 4.1. In Theorem 4.6, we shall see that if Ω_0 has a Lipschitz boundary, then the set of poles of the relative zeta function of this fractal spray (which is a relative fractal drum), as well as the multiplicities of the poles, do not depend on the choice of Ω_0. In this sense, examples

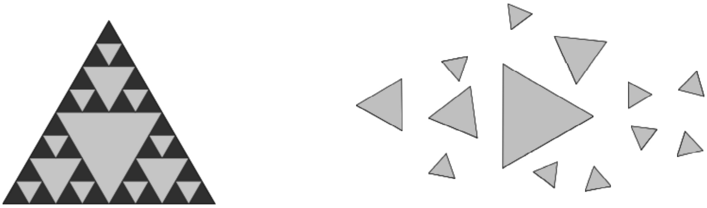

Fig. 1. Left: The Sierpiński gasket A, viewed as a relative fractal drum (A,Ω), with Ω being the countable disjoint union of open triangles contained in the unit triangle Ω_0. Right: An equivalent interpretation of the Sierpiński gasket drum (A,Ω). Here, Ω is a countable disjoint union of open equilateral triangles, and $A=\partial\Omega$. (There are 3^j triangles with sides 2^{-j-1} in the union, $j\in\mathbb{N}_0$.) Both pictures depict the first three iterations of the construction. We can also view the standard Sierpiński gasket A as a relative fractal drum (A,Ω), in which Ω is just the open unit triangle in the left picture.

(b) and (c) are equivalent. In particular, the box dimension of the *generalized Sierpiński relative fractal drum* is constant, and equal to $D=\log_2 3$.

In other words, the Sierpiński gasket $(A,\Omega)=\mathrm{Spray}(\Omega_0,1/2,3)$, appearing in Example 4.5(b), can be viewed as *any* countable disjoint collection of open triangles in the plane (which can even be an unbounded collection) and their bounding triangles, of sizes $\lambda_j=2^{-j-1}$ and multiplicities $b_j=3^j$, $j\in\mathbb{N}_0$, and not just as the standard disjoint collection of open triangles, densely packed inside the unit open triangle; see the right part of Figure 1.

By using the scaling property stated in Theorem 2.16, it is easy to explicitly compute the distance zeta function of relative fractal sprays. Note that the zeta function involves the Dirichlet series $f(s)=\sum_{j=0}^{\infty}b_j\lambda_j^s$. Theorem 4.6 just below can be considered as an extension of Theorem 2.16.

Theorem 4.6 (Distance zeta function of relative fractal sprays). *Let*

$$(A,\Omega)=\mathrm{Spray}(\Omega_0,(\lambda_j)_{j\geq 0},(b_j)_{j\geq 0})$$

be a relative fractal spray in $\mathbb{R}^N$, *in the sense of Definition* 4.1, *and such that* $|\Omega_0|<\infty$. *Assume condition* (4.1.7) *is satisfied, that is,* $|\Omega|<\infty$. *Let* Ω *be the* (*countable, disjoint*) *union of all the open sets appearing in the skeleton, corresponding to the fractal spray.* (*In other words,* Ω *is the disjoint union of the open sets* Ω_j, *each repeated with multiplicity* b_j *for* $j\in\mathbb{N}_0$.) *Let* $f(s):=\sum_{j=0}^{\infty}b_j\lambda_j^s$. (*Note that according to* (4.1.7), *this Dirichlet series converges absolutely for* $\mathrm{Re}\,s\geq N$; *hence,* $D(f)\leq N$.) *Then, for* $\mathrm{Re}\,s>\max\{\overline{\dim}_B(A,\Omega),D(f)\}$, *the distance zeta function of the relative fractal spray* (A,Ω) *is given by the* factorization formula

$$\zeta_{A,\Omega}(s)=f(s)\cdot\zeta_{\partial\Omega_0,\Omega_0}(s), \tag{4.1.9}$$

and

$$\overline{\dim}_B(A,\Omega)=\max\{\overline{\dim}_B(\partial\Omega_0,\Omega_0),D(f)\}. \tag{4.1.10}$$

Proof. Clearly, it follows from (4.1.7) that $f(N) < \infty$. Hence, $D(f) \leq N$, so that $\overline{\dim}_B(A,\Omega) \leq N$. Each open set of the skeleton of the relative fractal spray is obtained by a rigid motion of sets of the form $\lambda_j \Omega_0$, and for any fixed $j \in \mathbb{N}_0$, there are precisely b_j such sets. Identity (4.1.9) then follows immediately from Theorems 2.16 and 2.19. The remaining claims are easily derived by using this identity. ■

It follows from Definition 4.2 and relation (4.1.9) that the distance zeta function of the tensor product is equal to the product of the zeta functions of its components:

$$\zeta_{(\partial\Omega_0,\Omega_0)\otimes\mathcal{L}}(s) = \zeta_{\mathcal{L}}(s) \cdot \zeta_{\partial\Omega_0,\Omega_0}(s). \tag{4.1.11}$$

Equation (4.1.10) can therefore be written as follows:

$$\overline{\dim}_B((\partial\Omega_0,\Omega_0)\otimes\mathcal{L}) = \max\{\overline{\dim}(\partial\Omega_0,\Omega_0), \overline{\dim}_B \mathcal{L}\}. \tag{4.1.12}$$

Definition 4.7. The Dirichlet series $f(s) := \sum_{j=1}^{\infty} b_j \lambda_j^s$ (or, more generally, its meromorphic extension to a connected open subset U of $\mathbb{C}$, when it exists), is called the *scaling zeta function* of the relative fractal spray (A,Ω) and is denoted by $\zeta_{\mathfrak{S}}(s)$; hence, the factorization formula (4.1.9) can also be rewritten as follows:

$$\zeta_{A,\Omega}(s) = \zeta_{\mathfrak{S}}(s) \cdot \zeta_{\partial\Omega_0,\Omega_0}(s). \tag{4.1.13}$$

Theorem 4.8. *Assume that a relative fractal spray* $(A,\Omega) = \mathrm{Spray}(\Omega_0,\lambda,b)$, *as introduced at the end of Definition* 4.1, *is such that* $|\Omega_0| < \infty$, $\lambda \in (0,1)$, $b \geq 2$ *is an integer, and* $b\lambda^N < 1$. *Then, for* $\mathrm{Re}\, s > \max\{\overline{\dim}_B(\partial\Omega_0,\Omega_0), \log_{1/\lambda} b\})$, *we have*

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial\Omega_0,\Omega_0}(s)}{1 - b\lambda^s}, \tag{4.1.14}$$

and the lower bound for $\mathrm{Re}\, s$ *is optimal. In particular, it is equal to* $D(\zeta_{A,\Omega})$, *and hence*

$$\overline{\dim}_B(A,\Omega) = D(\zeta_{A,\Omega}) = \max\{\overline{\dim}_B(\partial\Omega_0,\Omega_0), \log_{1/\lambda} b\}.$$

If, in addition, Ω_0 *is bounded and has a Lipschitz boundary* $\partial\Omega_0$ *that can be described by finitely many Lipschitz charts, then* $\dim_B(A,\Omega)$ *exists and*

$$\dim_B(A,\Omega) = \max\{N-1, \log_{1/\lambda} b\}. \tag{4.1.15}$$

Therefore, if we assume that $\log_{1/\lambda} b \in (N-1,N)$, *then the set* $\dim_{PC}(A,\Omega) = \mathcal{P}_c(\zeta_{A,\Omega})$ *of principal complex dimensions of the relative fractal spray* (A,Ω) *is given by*

$$\dim_{PC}(A,\Omega) = \log_{1/\lambda} b + \frac{2\pi}{\log(1/\lambda)}\mathrm{i}\mathbb{Z}. \tag{4.1.16}$$

Proof. If $\lambda_j := \lambda$ and $b_j := b^j$ for all $j \in \mathbb{N}_0$, with $b\lambda^N < 1$, then $\sum_{j=0}^{\infty} b^j \lambda^{jN} = \frac{1}{1-b\lambda^N} < \infty$, so that $|\Omega| < \infty$, as desired. Identity (4.1.14) follows immediately from (4.1.9), using the fact that for Ω_0 with a Lipschitz boundary satisfying the stated assumption, we have $\dim_B(\partial\Omega_0,\Omega_0) = \dim_B \partial\Omega_0 = N-1$ (this follows, for example, from [ŽŽ, Lemma 3]; see also [L1]), together with the property of finite stability of the upper box dimension; see, e.g., [F1, p. 44]. ■

Example 4.9. Here, we construct a relative fractal spray

$$(A,\Omega) = \mathrm{Spray}(\Omega_0, (\lambda_j)_{j\geq 1}, (b_j)_{\geq 1})$$

in $\mathbb{R}^2$ such that $|\Omega_0| < \infty$, $b_j \equiv 1$, $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$ (hence, $|\Omega| < \infty$ by Lemma 4.4), and such that the base set Ω_0 is *unbounded*, as is its boundary $\partial\Omega_0$. Let Ω_0 be any unbounded Borel set of finite 2-dimensional Lebesgue measure such that both Ω_0 and $\partial\Omega_0$ are unbounded, and Ω_0 is contained in the horizontal strip

$$V_1 := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}.$$

We can explicitly construct such a set as follows:

$$\Omega_0 := \{(x, y) \in \mathbb{R}^2 : 0 < y < x^{-\alpha},\, x > 1\},$$

where $\alpha > 1$, so that $|\Omega_0| < \infty$.

Furthermore, let $(V_j)_{j\geq 1}$ be a countable, disjoint sequence of horizontal strips in the plane, defined for each $j \in \mathbb{N}$ by $V_j = V_1 + (0, j)$ (the Minkowski sum). Let $(\lambda_j)_{j\geq 1}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$. It is clear that for any λ_j, $j \geq 2$, the set $\lambda_j\Omega_0$ is congruent (up to a rigid motion) to the subset $\Omega_j := \lambda_j\Omega_0 + (0, j)$ of V_j. Then the fractal spray

$$(A, \Omega) = \bigcup_{j=1}^{\infty} (\partial\Omega_j, \Omega_j)$$

has the desired properties.

4.2. Relative Sierpiński sprays and their complex dimensions. We provide two examples of relative fractal sprays: the *relative Sierpiński gasket* and the *relative Sierpiński carpet*, respectively. It will be useful to introduce the following definition.

Definition 4.10. We say that two given relative fractal drums (A_1, Ω_1) and (A_2, Ω_2) in $\mathbb{R}^N$ are *congruent* if there exists an isometry $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $A_2 = f(A_1)$ and $\Omega_2 = f(\Omega_1)$.

It is easy to see that the congruence of RFDs is an equivalence relation.

The following lemma states, in particular, that two congruent RFDs have equal distance zeta functions. We leave its proof as a simple exercise.

Lemma 4.11. *Let (A_1, Ω_1) and (A_2, Ω_2) be congruent RFDs in $\mathbb{R}^N$. Then, for any $r \in \mathbb{R}$,*

$$\mathcal{M}_*^r(A_1, \Omega_1) = \mathcal{M}_*^r(A_2, \Omega_2), \qquad \mathcal{M}^{*r}(A_1, \Omega_1) = \mathcal{M}^{*r}(A_2, \Omega_2) \tag{4.2.1}$$

and

$$\underline{\dim}_B(A_1, \Omega_1) = \underline{\dim}_B(A_2, \Omega_2), \qquad \overline{\dim}_B(A_1, \Omega_1) = \overline{\dim}_B(A_2, \Omega_2) =: \overline{D}. \tag{4.2.2}$$

Furthermore,

$$\zeta_{A_1,\Omega_1}(s) = \zeta_{A_2,\Omega_2}(s) \tag{4.2.3}$$

for $\operatorname{Re} s > \overline{\dim}_B(A_1, \Omega_1)$.

It follows from (4.2.3) that under the hypotheses of Lemma 4.11 and given a connected open set $U \subseteq \mathbb{C}$ (containing the critical line $\{\operatorname{Re} s = \overline{D}\}$ of (A_1, Ω_1) and (A_2, Ω_2); see (4.2.2)), ζ_{A_1,Ω_1} and ζ_{A_2,Ω_2} have the exact same meromorphic continuation to U, and therefore the same poles in U and associated residues (or more generally, principal parts in the case of multiple poles). In particular, two congruent RFDs have the same (visible) complex dimensions.

EXAMPLE 4.12 (Relative Sierpiński gasket). Let A be the Sierpiński gasket in $\mathbb{R}^2$, the outer boundary of which is an equilateral triangle with unit sides. Consider the countable family of all open triangles in the standard construction of the gasket. (Namely, these are the open triangles which are deleted at each stage of the construction.) If Ω is the largest open triangle (with unit sides), then the *relative Sierpiński gasket* is defined as the ordered pair (A,Ω). Its distance zeta function $\zeta_{A,\Omega}$ can be computed as the distance zeta function of the following relative fractal spray (see the end of Definition 4.1):

$$\mathrm{Spray}(\Omega_0,\lambda=1/2,b=3),$$

where Ω_0 is the first deleted open triangle with sides $1/2$. It suffices to apply (4.1.14). Decomposing Ω_0 into the union of six congruent right triangles (determined by the heights of the triangle Ω_0, see Figure 2) with disjoint interiors, we see that

$$\begin{aligned}\zeta_{\partial\Omega_0,\Omega_0}(s)&=6\zeta_{A',\Omega'}(s)=6\int_{\Omega'}d((x,y),A')^{s-2}\,\mathrm{d}x\,\mathrm{d}y\\&=6\int_0^{1/4}\mathrm{d}x\int_0^{x/\sqrt{3}}y^{s-2}\,\mathrm{d}y=6\frac{(\sqrt{3})^{1-s}2^{-s}}{s(s-1)}\end{aligned}\tag{4.2.4}$$

for $\mathrm{Re}\,s>1$. Using (4.1.14) and appealing to Lemma 4.11, we deduce that

$$\zeta_{A,\Omega}(s)=\frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(1-3\cdot2^{-s})}\sim\frac{1}{1-3\cdot2^{-s}},\tag{4.2.5}$$

where the equality holds for $\mathrm{Re}\,s>\log_2 3$ and the equivalence $\sim$ holds in the sense of Definition 1.7. Therefore, by analytic continuation, $\zeta_{A,\Omega}$ has a meromorphic extension to the entire complex plane, given by the same closed form as in (4.2.5):

$$\zeta_{A,\Omega}(s)=\frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(1-3\cdot2^{-s})}\quad\text{for all } s\in\mathbb{C}.\tag{4.2.6}$$

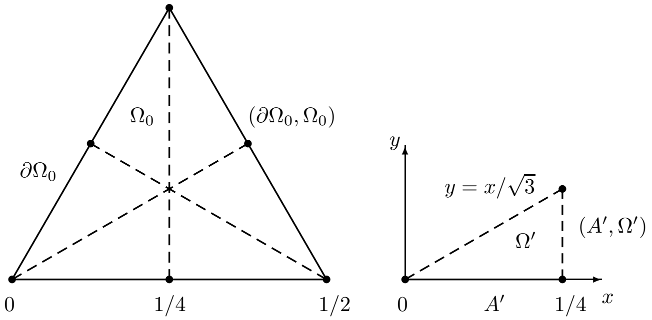

Fig. 2. On the left is depicted the base relative fractal drum $(\partial\Omega_0,\Omega_0)$ of the relative Sierpiński gasket, where Ω_0 is the associated (open) equilateral triangle with sides $1/2$. It can be viewed as the (disjoint) union of six RFDs, all of which are congruent to the relative fractal drum (A',Ω') on the right. This figure explains equation (4.2.4); see Lemma 4.11.

Hence, the set of all complex dimensions (in $\mathbb{C}$) of the relative Sierpiński gasket is

$$\mathcal{P}(\zeta_{A,\Omega}) = \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right) \cup \{0,1\}. \tag{4.2.7}$$

Each of these complex dimensions in (4.2.7) is simple (i.e., is a simple pole of $\zeta_{A,\Omega}$). Note that here $\{0,1\}$ can be interpreted as the set of *integer dimensions* of A, in the sense of [LP1]–[LP3] and [LPW1]. In particular, we deduce from (4.2.7) that $D(\zeta_{A,\Omega}) = \log_2 3$, and we thus recover a well-known result

$$\dim_{PC}(A,\Omega) = \log_2 3 + \mathbf{p}\mathbb{i}\mathbb{Z}, \tag{4.2.8}$$

where $\mathbf{p} = 2\pi/\log 2$ is the oscillatory period of the Sierpiński gasket [LF3, §6.6.1].

Note, however, that in [LF1]–[LF3], the complex dimensions are obtained in a completely different manner (via an associated spectral zeta function) and not geometrically. In addition, all of the complex dimensions of the Sierpiński gasket A are shown to be principal (that is, located on $\operatorname{Re} s = \log_2 3 = \dim_B A$), a conclusion which is slightly different from (4.2.7) above (¹). We also refer to [CIL] and [LS], as well as to [LP1]–[LP3] and [LPW1, LPW2], for different approaches (via a spectral zeta function associated to a suitable geometric Dirac operator and via a self-similar tiling associated with A, respectively) leading to the same conclusion.

In light of (4.2.6), the residue of $\zeta_{A,\Omega}$ at any principal pole $s_k := \log_2 3 + \mathbf{p}k\mathbb{i}$, $k \in \mathbb{Z}$, is

$$\operatorname{res}(\zeta_{A,\Omega}, s_k) = \frac{6(\sqrt{3})^{1-s_k}}{2^{s_k}(\log 2)s_k(s_k-1)}.$$

In particular,

$$|\operatorname{res}(\zeta_{A,\Omega}, s_k)| \sim \frac{6(\sqrt{3})^{1-D}}{D2^D \log 2}\, k^{-2} \qquad \text{as } k \to \pm\infty,$$

where $D := \log_2 3$.

The following proposition shows that the relative Sierpiński gasket can be viewed as the relative fractal spray generated by the relative fractal drum (A', Ω') appearing on the right-hand side of Figure 2.

Proposition 4.13 (Relative Sierpiński gasket). *Let (A',Ω') be the relative fractal drum defined on the right of Figure 2. Let (A,Ω) be the relative fractal spray generated by the base relative fractal drum (A',Ω'), with scaling ratio $\lambda = 1/2$ and with multiplicities $m_k = 6 \cdot 3^{k-1}$, for any positive integer k:*

$$(A,\Omega) = \operatorname{Spray}\big((A',\Omega'),\ \lambda = 1/2,\ m_k = 6\cdot 3^{k-1} \textit{ for } k \in \mathbb{N}\big). \tag{4.2.9}$$

(*Note that we assume here that the base relative fractal drum (A',Ω') has multiplicity* 8.) *Then the relative distance zeta function of the relative fractal spray (A,Ω) coincides with the relative distance zeta function of the relative Sierpiński gasket* (*see* (4.2.6)).

(¹) Analogously, for a fractal string RFD (A,Ω), we have $\mathcal{P}(\zeta_{A,\Omega}) = \mathcal{P}(\zeta_{\mathcal{L}}) \cup \{0\}$; here, of course, $N = 1$ instead of $N = 2$.

EXAMPLE 4.14 (Inhomogeneous Sierpiński N-gasket RFD). The usual Sierpiński gasket is contained in the unit triangle in the plane. Its analog in $\mathbb{R}^3$, which we call the *inhomogeneous Sierpiński* 3*-gasket* or *inhomogeneous tetrahedral gasket*, and denote by A_3, is obtained by deleting the middle open octahedron (from the initial compact, convex unit tetrahedron), defined as the interior of the convex hull of the midpoints of each of the six edges of the initial tetrahedron (thus, four subtetrahedra are left after the first step), and so on. Such sets, along with their higher-dimensional counterparts, are analogous to, but not identical with, the (homogeneous) self-similar N-gaskets discussed, for example, in [KL].

More generally, for any integer $N \geq 2$, the *inhomogeneous Sierpiński N-gasket A_N*, contained in $\mathbb{R}^N$, can be defined as follows. Let $V_N := \{P_1, \dots, P_{N+1}\}$ be a set of $N+1$ points in $\mathbb{R}^N$ such that the mutual distance of any two points from the set is equal to 1.

The set V_N, where $N \geq 2$, with the indicated property, can be constructed inductively as follows. For $N = 2$, we take V_2 to be the set of vertices of any unit triangle in $\mathbb{R}^2$. We then reason by induction. Given $N \geq 2$, we assume that the set V_N of $N+1$ points in $\mathbb{R}^N$ has been constructed. Note that V_N is contained in a sphere, whose center is denoted by O. Let us consider the line of $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ through O and perpendicular to the hyperplane $\mathbb{R}^N = \mathbb{R}^N \times \{0\}$ in $\mathbb{R}^{N+1}$. There exists a unique point P_{N+2} in the half-plane $\{x_{N+1} > 0\}$ of $\mathbb{R}^{N+1}$, which is a unit distance from all of the N vertices of V_N. (Here, we identify V_N with $V_N \times \{0\} \subset \mathbb{R}^{N+1}$.) We then define V_{N+1} by $V_{N+1} := V_N \cup \{P_{N+2}\}$.

Let us define Ω_N as the convex hull of the set V_N. As usual, we call it the *N-simplex.* Let $\Omega_{N,0}$, called the *N-plex*, be the open set defined as the interior of the convex hull of the set of midpoints of all of the $\binom{N+1}{2}$ edges of the N-simplex Ω_N. [For example, $\Omega_{2,0}$ is an open equilateral triangle in $\mathbb{R}^2$ of side lengths equal to $1/2$, while $\Omega_{3,0}$ is an open octahedron in $\mathbb{R}^3$ of side lengths equal to $1/2$.] The set $\overline{\Omega}_N \setminus \Omega_{N,0}$ is equal to the union of $N+1$ congruent N-simplices with disjoint interiors, having all side lengths equal to $1/2$. This is the first step of the construction. We proceed analogously with each of the $N+1$ compact N-simplices. The compact set A_N obtained in this way is called the *inhomogeneous Sierpiński N-gasket.* It can be identified with the relative fractal spray (A_N, Ω_N) in $\mathbb{R}^N$, called the *inhomogeneous Sierpiński N-gasket RFD* (and, for short, the *inhomogeneous N-gasket RFD*), defined by

$$(A_N, \Omega_N) = \operatorname{Spray}\big((\partial\Omega_{N,0}, \Omega_{N,0}),\, \lambda = 1/2,\, b = N+1\big). \tag{4.2.10}$$

(See the end of Definition 4.1.) According to Theorem 4.8, we have

$$\zeta_{A_N,\Omega_N}(s) = \zeta_{\mathfrak{S}}(s) \cdot \zeta_{\partial\Omega_{N,0},\Omega_{N,0}}(s), \tag{4.2.11}$$

where the *scaling zeta function* $\zeta_{\mathfrak{S}}(s)$ of the N-gasket RFD is given for $\operatorname{Re} s > \log_2(N+1)$ by

$$\zeta_{\mathfrak{S}}(s) = \sum_{k=0}^{\infty} (N+1)^k (2^{-k})^s = \frac{1}{1-(N+1)2^{-s}}. \tag{4.2.12}$$

$\zeta_{\mathfrak{S}}$ can be meromorphically continued to the whole of $\mathbb{C}$ and

$$\zeta_{\mathfrak{S}}(s) = \frac{1}{1-(N+1)2^{-s}} \quad \text{for all } s \in \mathbb{C}. \tag{4.2.13}$$

Since (by (4.2.13)) the set of poles of $\zeta_{\mathfrak{S}}$ is

$$\mathcal{P}(\zeta_{\mathfrak{S}}) = \log_2(N+1) + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z} \tag{4.2.14}$$

and the set of poles of the distance zeta function of the *relative N-plex* $(\partial\Omega_{N,0}, \Omega_{N,0})$ is

$$\mathcal{P}(\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}) = \{0, 1, \dots, N-1\}, \tag{4.2.15}$$

and $\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}(s) \neq 0$ for all $s \in \mathbb{C} \setminus \mathcal{P}(\zeta_{\partial\Omega_{N,0},\Omega_{N,0}})$, we conclude that the set of poles (complex dimensions) of the *relative Sierpiński N-gasket* (A_N, Ω_N) is

$$\mathcal{P}(\zeta_{A_N,\Omega_N}) = \{0, 1, \dots, N-1\} \cup \left\{\log_2(N+1) + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z}\right\}, \tag{4.2.16}$$

where each complex dimension is simple. In particular, the set of principal complex dimensions of the RFD (A_N, Ω_N) is (2)

$$\dim_{PC}(A_N, \Omega_N) = \begin{cases} \log_2 3 + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z} & \text{for } N = 2, \\ 2 + \frac{2\pi}{\log 2}\,\mathbb{i}\mathbb{Z} & \text{for } N = 3, \\ \{N-1\} & \text{for } N \geq 4, \end{cases} \tag{4.2.17}$$

and

$$\overline{\dim}_B(A_N, \Omega_N) = \begin{cases} \log_2 3 & \text{for } N = 2, \\ N-1 & \text{for } N \geq 3, \end{cases} \tag{4.2.18}$$

which extends the well-known results for $N = 2$ and 3, corresponding to the usual Sierpiński gasket in $\mathbb{R}^2$ and the tetrahedral gasket in $\mathbb{R}^3$, respectively. (Namely, their respective relative box dimensions are equal to $\log_2 3$ and 2.)

It can be shown that in this case, $\dim_B(A_N, \Omega_N)$ and $\dim_B A_N$ exist and

$$\dim_B(A_N, \Omega_N) = \dim_B A_N = \dim_H A_N, \tag{4.2.19}$$

as given by the right-hand side of (4.2.18), where $\dim_H(\,\cdot\,)$ denotes the Hausdorff dimension. Furthermore, it is easy to see that $\dim_{PC}(A_N, \Omega_N) = \dim_{PC} A_N$.

The relative distance zeta function $\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}$ can be explicitly computed as follows when $N = 3$. It is easy to see that $(\partial\Omega_{3,0}, \Omega_{3,0})$ can be identified with 16 copies of disjoint RFDs, each of which is congruent to the pyramidal RFD (T, Ω') in $\mathbb{R}^3$, where Ω' is the open (irregular) pyramid with vertices at $O(0,0,0)$, $A(1/4,0,0)$, $B(1/4,1/4,0)$ and $C(0,0,1/(2\sqrt{2}))$, while the triangle $T = \operatorname{conv}(A, B, C)$ is a face of the pyramid. Since for any $(x, y, z) \in \Omega'$ we have

$$d((x,y,z), T) = \frac{1}{\sqrt{3}}\left(\frac{1}{2\sqrt{2}} - \sqrt{2}\,x - z\right), \tag{4.2.20}$$

---

(2) Recall that, by definition, $\dim_{PC}(A_N, \Omega_N) = \mathcal{P}_c(\zeta_{A_N,\Omega_N})$.

we deduce that

$$\begin{aligned}\zeta_{\partial\Omega_{3,0},\Omega_{3,0}}(s)&=16\zeta_{T,\Omega'}(s)\\&=16\iiint_{\Omega'}d((x,y,z),T)^{s-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\&=16\int_0^{1/4}\mathrm{d}x\int_0^x\mathrm{d}y\int_0^{\frac{1}{2\sqrt{2}}-\sqrt{2}\,x}\left(\frac{\frac{1}{2\sqrt{2}}-\sqrt{2}x-z}{\sqrt{3}}\right)^{s-3}\mathrm{d}z.\end{aligned}\tag{4.2.21}$$

Evaluating the last integral in (4.2.21), we obtain by a direct computation

$$\begin{aligned}\zeta_{\partial\Omega_{3,0},\Omega_{3,0}}(s)&=16\frac{(\sqrt{3})^{3-s}}{s-2}\int_0^{1/4}\left(\frac{1}{2\sqrt{2}}-\sqrt{2}\,x\right)^{s-2}x\,\mathrm{d}x\\&=8\frac{(\sqrt{3})^{3-s}}{s-2}\int_0^{1/(2\sqrt{2})}u^{s-2}\left(\frac{1}{2\sqrt{2}}-u\right)\mathrm{d}u\\&=\frac{8(\sqrt{3})^{3-s}(2\sqrt{2})^{-s}}{s(s-1)(s-2)}\end{aligned}\tag{4.2.22}$$

for $\operatorname{Re}s>2$. Therefore, we deduce from (4.2.11) that the distance zeta function of the thetrahedral RFD in $\mathbb{R}^3$ can be meromorphically extended to the whole complex plane and is given for all $s\in\mathbb{C}$ by

$$\zeta_{A_3,\Omega_3}(s)=\frac{8(\sqrt{3})^{3-s}(2\sqrt{2})^{-s}}{s(s-1)(s-2)(1-4\cdot2^{-s})}.\tag{4.2.23}$$

It is worth noting that $s=2$ is the only pole of order 2, since $s=2$ is the simple pole of both $(s-2)^{-1}$ and $(1-4\cdot2^{-s})^{-1}$. More specifically, since the derivative of $1-4\cdot2^{-s}$ computed at $s=2$ is nonzero (and, in fact, is equal to $4\log2$), it follows that $s=2$ is a simple zero of $1-4\cdot2^{-s}$, that is, a simple pole of $1/(1-4\cdot2^{-s})$.

Moreover, it immediately follows from (4.2.23) that

$$\zeta_{A_3,\Omega_3}(s)\sim\frac{1}{(s-2)(1-4\cdot2^{-s})}.\tag{4.2.24}$$

In particular, as we have already seen in (4.2.17) (recall that $N:=3$ here), we have

$$\dim_{PC}(A_3,\Omega_3)=2+\frac{2\pi}{\log2}\mathrm{i}\mathbb{Z}.\tag{4.2.25}$$

Since $D=2$ is a simple pole of both $1/(s-2)$ and $1/(1-4\cdot2^{-s})$, we conclude that $D=2$ is the only complex dimension of order two of the RFD (A_3,Ω_3). Consequently, the case of the relative Sierpiński 3-gasket (A_3,Ω_3) reveals a new phenomenon: its relative box dimension $D=2$ is a complex dimension of order (i.e., multiplicity) two, while all the other complex dimensions of the relative Sierpiński 3-gasket (including the double sequence of nonreal complex dimensions on the critical line of convergence $\{\operatorname{Re}s=2\}$) are simple.

By using arguments similar to those used when $N=3$, one can show that for any $N\geq3$, the distance zeta function of the relative N-plex $(\partial\Omega_{N,0},\Omega_{N,0})$ is of the form

$$\zeta_{\partial\Omega_{N,0},\Omega_{N,0}}(s)=\frac{g(s)}{s(s-1)\cdots(s-(N-1))},\tag{4.2.26}$$

where $g(s)$ is a nonvanishing entire function. In the special case when $N = 3$, this is in accordance with (4.2.22) above. From (4.2.11) and (4.2.13), we conclude that

$$\zeta_{A_N,\Omega_N}(s) = \frac{g(s)}{s(s-1)\cdots(s-(N-1))(1-(N+1)2^{-s})}. \tag{4.2.27}$$

This extends (4.2.23) to any $N \geq 3$.

In the case when $N \geq 4$, $D = N - 1$ is the only principal complex dimension of the relative Sierpiński N-gasket. (Indeed, for $N \geq 4$ we have $\log_2(N+1) < N - 1$ (i.e., $N+1 < 2^{N-1}$), which can be easily proved, for example, by induction on N.) Also, all the other complex dimensions are simple. Furthermore, we immediately deduce from (4.2.27) that

$$\zeta_{A_N,\Omega_N}(s) \sim \frac{1}{s-(N-1)}. \tag{4.2.28}$$

Moreover, if $N \geq 4$ is of the form $N = 2^k - 1$ for some integer $k \geq 3$, then $s = k$ (smaller than $D = N - 1$) is the only complex dimension of order two (since it is a simple pole of both $(s-k)^{-1}$ and $(1-(N+1)2^{-s})^{-1}$), while all the other complex dimensions are simple.

On the other hand, if $N \geq 4$ is not of the form $N = 2^k - 1$ for any integer $k \geq 3$, then all of the complex dimensions of the relative Sierpiński N-gasket are simple.

Roughly speaking, when $N = 3$, the fact that $s = 2$ has multiplicity two can be explained geometrically as follows: firstly, $s = 2$ is a simple pole arising from the self-similarity of the RFD (A_3, Ω_3) (3), while at the same time, $s = 2$ is a simple pole arising from the geometry of the boundary of the first (deleted) octahedron, which is also 2-dimensional.

For the ordinary Sierpiński gasket, i.e., of the relative Sierpiński 2-gasket, the value of $s = \log_2 3$ (which is the simple pole arising from the self-similarity of (A_2, Ω_2)) is strictly larger than the dimension $s = 1$ of the boundary of the deleted triangle (i.e., of the 2-plex $\Omega_{2,0}$). Moreover, the relative Sierpiński 2-gasket is Minkowski nondegenerate and Minkowski nonmeasurable, while the relative Sierpiński 3-gasket is Minkowski degenerate, with 2-dimensional Minkowski content $+\infty$.

On the other hand, when $N \geq 4$, the dimension $N - 1$ of the boundary of the N-plex $\Omega_{N,0}$ is larger than the similarity dimension $\log_2(N+1)$ arising from "fractality". Since $D = \log_2(N+1)$ is the only complex dimension on the critical line (and it is simple), we conclude that for $N \geq 4$, the RFD (A_N, Ω_N) is Minkowski measurable (see [LRŽ5]). Thus, the case $N = 3$ is indeed very special among all relative Sierpiński N-gaskets.

We refer the interested reader to [LRŽ5] and [LRŽ6] (as well as to the relevant part of [LRŽ1, §5.5]) for a detailed discussion of Minkowski measurability (or nonmeasurability) of the N-gasket RFD (A_N, Ω_N) for any $N \geq 2$, and for the corresponding fractal tube formulas. Let us just mention here that for $N = 3$, a suitable gauge function can be found with respect to which A_3 is not only Minkowski nondegenerate but also Minkowki measurable. (Note that for $N \neq 3$, A_N is Minkowski nondegenerate in the usual sense, that is, relative to the trivial gauge function obeying the standard power law.)

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(3) Indeed, the similarity dimension of the 3-gasket A_3 is equal to 2.

Let σ_0 be the common *similarity dimension* of the inhomogeneous Sierpiński N-gasket A_N, the relative Sierpiński N-gasket (A_N, Ω_N) (where the latter is viewed as a self-similar fractal spray or RFD) and the classical Sierpiński N-gasket S_N (to be discussed below). Since the corresponding scaling ratios $\{r_j\}_{j=1}^{N+1}$ are $r_1 = \cdots = r_{N+1} = 1/2$, the similarity dimension σ_0, defined as being the unique real solution s of the Moran equation $\sum_{j=1}^{N+1} r_j^s = 1$ (i.e., here, $2^s = N+1$, $s \in \mathbb{R}$), is given by

$$\sigma_0 = \log_2(N+1). \tag{4.2.29}$$

In light of (4.1.10) and since $\dim_B(A_{N,0}, \Omega_{N,0}) = N - 1$ (by (4.2.15)), we see that $\dim_B(A_N, \Omega_N) = \sigma_0$ for $N = 2$ or $N = 3$, and

$$\sigma_0 = \dim_B(A_N, \Omega_N) > \dim_B(A_{N,0}, \Omega_{N,0}) \tag{4.2.30}$$

for $N = 2$,

$$\sigma_0 = \dim_B(A_N, \Omega_N) = \dim_B(A_{N,0}, \Omega_{N,0}) \ (= 2) \tag{4.2.31}$$

for $N = 3$, whereas for every $N \geq 4$,

$$\sigma_0 = \log_2(N+1) < \dim_B(A_N, \Omega_N) = \dim_B(A_{N,0}, \Omega_{N,0}) = N - 1. \tag{4.2.32}$$

(Recall from (4.2.19) that $\dim_B A_N = \dim_B(A_N, \Omega_N)$.) On the other hand, if S_N denotes the *classical Sierpiński N-gasket* in $\mathbb{R}^N$ (to be further discussed below), then for every $N \geq 2$,

$$\dim_B S_N \ (= \dim_H S_N) = \sigma_0 = \log_2(N+1). \tag{4.2.33}$$

This follows from a classical result of Hutchinson [Hu] for self-similar sets satisfying the open set condition (which is the case of S_N for every $N \geq 2$) and extending to higher dimensions the basic result of Moran [Mo] for one-dimensional self-similar sets ([4]). (See [F1, Theorem 9.3] for the statement and a detailed proof of this theorem.)

We close this discussion of the N-gasket RFD (A_N, Ω_N) by explaining the discrepancy between the results obtained in (4.2.30), (4.2.31) and, especially, (4.2.32) for the self-similar spray (A_N, Ω_N) and the usual result (4.2.33) for the self-similar set S_N, the classical Sierpiński N-gasket.

First of all, in light of (4.2.11) and (4.2.13) (see also Theorem 4.6), we must have

$$\begin{aligned} &\overline{\dim}_B(A_N, \Omega_N) \ (= \overline{\dim}_B A_N) \\ &\qquad = \max\{\sigma_0, \overline{\dim}_B(A_{N,0}, \Omega_{N,0})\} = \max\{\log_2(N+1), N-1\}, \end{aligned} \tag{4.2.34}$$

and, in the present case, the upper Minkowski dimensions can be replaced by the Minkowski dimensions in (4.2.34).

Indeed, by (4.2.11), we have

$$D(\zeta_{A_N, \Omega_N}) = \max\{D(\zeta_{\mathfrak{S}}), D(\zeta_{A_{N,0}, \Omega_{N,0}})\}, \tag{4.2.35}$$

and by Theorem 2.1(b), we have

$$D(\zeta_{A_N, \Omega_N}) = \overline{\dim}_B D(A_N, \Omega_N), \qquad D(\zeta_{A_{N,0}, \Omega_{N,0}}) = \overline{\dim}_B D(A_{N,0}, \Omega_{N,0}),$$

from which (4.2.34) follows since $\sigma_0 = \log_2(N+1)$. (See Theorem 4.6.)

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(4) Note that $S_1 \subset \mathbb{R}$ is just the unit interval, viewed as a self-similar set with scaling ratios $r_1 = r_2 = 1/2$. However, in the present discussion, we consider the more interesting case $N \geq 2$.

Identity (4.2.34) explains why (4.2.30)–(4.2.32) hold. Indeed, if we let $D_G := \dim_B(A_{N,0},\Omega_{N,0})$ (the Minkowski dimension of the base RFD $(A_{N,0},\Omega_{N,0})$ generating the self-similar RFD (A_N,Ω_N)) and $D := \dim_B(A_N,\Omega_N)$, we deduce from (4.2.34) and an elementary computation that $D=\sigma_0$ if $N=2$, $D=\sigma_0=D_G$ if $N=3$, whereas $D=D_G$ if $N\geq 4$, in agreement with (4.2.30)–(4.2.32), respectively.

From the geometric point of view, the difference between A_N and S_N can be explained as follows. As is well known (see, e.g., [KL] and the relevant references therein), the (homogeneous) Sierpiński N-gasket S_N is a self-similar set (satisfying the open set condition), associated with the iterated function system (IFS) $\{\Phi_j\}_{j=1}^{N+1}$, where (for $j=1,\dots,N+1$) each Φ_j is a contractive similitude of $\mathbb{R}^N$ with fixed point P_j and scaling ratio $r_j=1/2$; i.e., the associated scaling ratio list $\{r_j\}_{j=1}^{N+1}$ of $\{\Phi_j\}_{j=1}^{N+1}$ is $r_1=\cdots=r_{N+1}=1/2$. More specifically, S_N is the unique (nonempty) compact subset K of $\mathbb{R}^N$ which is the solution of the (homogeneous) fixed point equation

$$K=\mathbf{\Phi}(K):=\bigcup_{j=1}^{N+1}\Phi_j(K). \tag{4.2.36}$$

On the other hand, unless $N=2$, the inhomogeneous Sierpiński N-gasket A_N is *not* a self-similar set in the classical sense of [Hu] (see also [F1, Chapter 9]). (For $N=2$, A_2 coincides with the usual Sierpiński gasket S_2.) However, interestingly, it is an *inhomogeneous* self-similar set, in the sense of Barnsley and Demko [BD] (see also [BD], [Fra] and the relevant references therein for further results about such sets). More specifically, A_N is the unique (nonempty) solution K of the *inhomogeneous* fixed point equation

$$K=\mathbf{\Phi}(K)\cup B, \tag{4.2.37}$$

where $\mathbf{\Phi}$ is defined as in (4.2.36) and B is a suitable compact subset of $\mathbb{R}^N$. For $N=2$, the set $A_2=S_2$ is both homogeneous and inhomogeneous, since it satisfies (4.2.37) for both $B=\emptyset$ and $B=\partial A_{2,0}$ (the boundary of the unit triangle). By contrast, when $N\geq 3$, the compact set B is *nonempty*, and hence A_N is not self-similar for the IFS $\{\Phi_j\}_{j=1}^{N+1}$. For $N=3$, a description of several possible choices for B can be found in the caption of Figure 3. When $N\geq 3$, let us just state that we can choose B to be the boundary of $\Omega_{N,0}$. (Other choices are possible, however.)

Example 4.15 (Relative Sierpiński carpet). Let A be the Sierpiński carpet contained in the unit square Ω. Let (A,Ω) be the corresponding *relative Sierpiński carpet*. Its distance zeta function $\zeta_{A,\Omega}$ coincides with the distance zeta function of the following relative fractal spray (see the end of Definition 4.1):

$$\mathrm{Spray}(\Omega_0,\lambda=1/3,b=8),$$

where Ω_0 is the first deleted open square with sides 1/3. As in Example 4.12, using Theorem 4.8 and Lemma 4.11, we find that the relative distance zeta function $\zeta_{A,\Omega}$ has a meromorphic continuation to the entire complex plane given for all $s\in\mathbb{C}$ by

$$\zeta_{A,\Omega}(s)=\frac{8\cdot 6^{-s}}{s(s-1)(1-8\cdot 3^{-s})}. \tag{4.2.38}$$

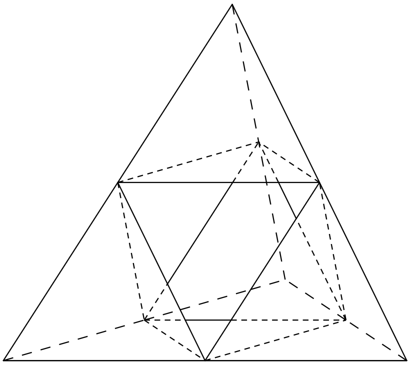

Fig. 3. The open octahedron $\Omega_{3,0}$ inscribed in the largest (compact) tetrahedron Ω_3, surrounded with four smaller (compact) tetrahedra scaled by the factor 1/2. Each of them contains analogous scaled open octahedra, etc. The countable family of all open octahedra (viewed jointly with their boundaries) constitutes the tetrahedral gasket RFD or the Sierpiński 3-gasket RFD. The complement of the union of all open octahedra, with respect to the initial tetrahedron Ω_3, is called the *inhomogeneous Sierpiński* 3*-gasket*.

Unlike the classical Sierpiński 3-gasket (also known as the Sierpiński pyramid or tetrahedron) S_3, which is a (homogeneous or standard) self-similar set in $\mathbb{R}^3$ and satisfies the usual fixed point equation, $S = \bigcup_{j=1}^4 \Phi_j(S)$, where $\{\Phi_j\}_{j=1}^4$ are suitable contractive similitudes of $\mathbb{R}^3$ with respective fixed points $\{P_j\}_{j=1}^4$ and scaling ratios $\{r_j\}_{j=1}^4$ of common value 1/2, the inhomogeneous Sierpiński 3-gasket A_3 is *not* a self-similar set. Instead, it is an *inhomogeneous self-similar set* (in the sense of [BD], see also (4.2.37) above and the discussion surrounding it). More specifically, $A := A_3$ satisfies the following *inhomogeneous* fixed point equation (of which it is the unique solution in the class of all nonempty compact subsets of $\mathbb{R}^3$): $A = \bigcup_{j=1}^4 \Phi_j(A) \cup B$, where B is the boundary of the first octahedron $\Omega_{3,0}$ (in fact, B can simply be taken as the union of four middle triangles on the boundary of the outer tetrahedron Ω_3).

Indeed, clearly, the base relative fractal drum $(\partial\Omega_0, \Omega_0)$ is the (disjoint) union of eight relative fractal drums, each congruent to a relative fractal drum (A', Ω'), where Ω' is an appropriate isosceles right triangle; see Figure 4. We then deduce from Lemma 4.11 that

$$\begin{aligned}\zeta_{\partial\Omega_0,\Omega_0}(s) &= 8\,\zeta_{A',\Omega'}(s) = 8\int_{\Omega'} d((x,y),A')^{s-2}\,\mathrm{d}x\,\mathrm{d}y \\ &= 8\int_0^{1/6}\mathrm{d}x\int_0^x y^{s-2}\,\mathrm{d}y = \frac{8\cdot 6^{-s}}{s(s-1)}\end{aligned} \tag{4.2.39}$$

for $\operatorname{Re} s > 1$, and hence, in light of Theorem 4.8, $\zeta_{A,\Omega}(s)$ is given by (4.2.38). Note that, after analytic continuation, we also have

$$\zeta_{\partial\Omega_0,\Omega_0}(s) = \frac{8\cdot 6^{-s}}{s(s-1)} \quad \text{for all } s \in \mathbb{C}. \tag{4.2.40}$$

Since by (4.2.40),

$$\zeta_{A,\Omega}(s) \sim \frac{1}{1 - 8\cdot 3^{-s}},$$

one deduces that the abscissa of convergence of $\zeta_{A,\Omega}$ is $D = \log_3 8 = \dim_B(A,\Omega)$, where the last equality follows from Theorem 2.1(b).

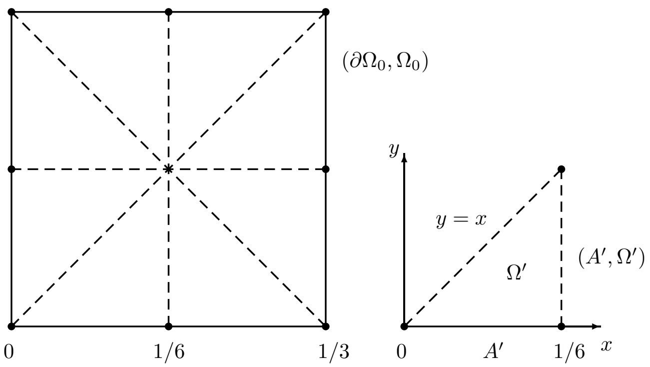

Fig. 4. On the left is the base relative fractal drum $(\partial\Omega_0, \Omega_0)$ of the relative Sierpiński carpet (A,Ω) described in Example 4.15, where Ω_0 is the associated (open) square with sides 1/3. Here $(\partial\Omega_0, \Omega_0)$ can be viewed as the (disjoint) union of eight RFDs, each congruent to the relative fractal drum (A', Ω') on the right. This figure explains (4.2.39); see Lemma 4.11.

Here, the relative box dimension of A coincides with its usual box dimension, namely, $\log_3 8$. Moreover, the set $\mathcal{P}_c(\zeta_{A,\Omega})$ of relative principal complex dimensions is given by

$$\dim_{PC}(A,\Omega) = \log_3 8 + \mathbf{p}\mathbb{i}\mathbb{Z}, \tag{4.2.41}$$

where $\mathbf{p} := 2\pi/\log 3$ is the oscillatory period of the Sierpiński carpet A.

Observe that it follows immediately from (4.2.38) that the set $\mathcal{P}(\zeta_{A,\Omega})$ of all relative complex dimensions is

$$\mathcal{P}(\zeta_{A,\Omega}) = \dim_{PC} A \cup \{0,1\} = (\log_3 8 + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{0,1\},$$

where $\{0,1\}$ can be viewed as the set of "integer dimensions" of A (in the sense of [LP1]–[LP3] and [LPW1], see also [LF3, §13.1]). Furthermore, each of these relative complex dimensions is simple (i.e., a simple pole of $\zeta_{A,\Omega}$). Interestingly, these are exactly the complex dimensions which one would expect to be associated with A, according to the theory developed in [LP1]–[LP3] and [LPW1, LPW2] (as described in [LF3, §13.1]) via self-similar tilings (or sprays) and associated tubular zeta functions.

In light of (4.2.38), the residue at any principal pole $s_k := \log_3 8 + \mathbf{p}\mathbb{i}k$, $k \in \mathbb{Z}$, is given by

$$\operatorname{res}(\zeta_{A,\Omega}, s_k) = \frac{2^{-s_k}}{(\log 3)s_k(s_k-1)}.$$

In particular,

$$|\operatorname{res}(\zeta_{A,\Omega}, s_k)| \sim \frac{2^{-D}}{D\log 3}\,k^{-2} \qquad \text{as } k \to \pm\infty,$$

where $D := \log_3 8$.

As in the case of the relative Sierpiński gasket (see Proposition 4.13), the relative Sierpiński carpet can be viewed as a fractal spray generated by the base RFD appearing on the right of Figure 4.

Proposition 4.16 (Relative Sierpiński carpet). *Let* (A', Ω') *be the RFD defined on the right of Figure* 4. *Let* (A, Ω) *be the relative fractal spray generated by the base relative fractal drum* (A', Ω')*, with scaling ratio* $\lambda = 1/3$ *and with multiplicities* $m_k = 8^k$ *for any positive integer* k*:*

$$(A, \Omega) = \operatorname{Spray}\big((A', \Omega'),\, \lambda = 1/3,\, m_k = 8^k \textit{ for } k \in \mathbb{N}\big). \tag{4.2.42}$$

(*Note that we assume here that the base relative fractal drum* (A', Ω') *has multiplicity* 8.) *Then the relative distance zeta function of the relative fractal spray* (A, Ω) *coincides with the relative distance zeta function of the relative Sierpiński carpet* (*see* (4.2.38)).

Example 4.17 (Sierpiński N-carpet). It is easy to generalize the example of the standard Sierpiński carpet (a compact subset of $[0,1]^2 \subset \mathbb{R}^2$, see Example 4.15) to the *Sierpiński* N*-carpet* (or N*-carpet*, for short), defined analogously as a compact subset A of the unit N-dimensional cube $[0,1]^N \subset \mathbb{R}^N$. More precisely, we divide $[0,1]^N$ into the union of 3^N congruent N-dimensional subcubes of length $1/3$ and with disjoint interiors and then remove the middle open subcube. The remaining compact set is denoted by F_1. We then remove the middle open N-dimensional cubes of length $1/3^2$ from the remaining $3^N - 1$ subcubes. The resulting compact subset is denoted by F_2. Proceeding analogously ad infinitum, we obtain a decreasing sequence of compact subsets F_k of $[0,1]^N$, for $k \geq 1$. The Sierpiński N-carpet A is then defined by

$$A := \bigcap_{k=1}^{\infty} F_k. \tag{4.2.43}$$

(Note that the Sierpiński 1-carpet coincides with the usual ternary Cantor set, while the Sierpiński 2-carpet coincides with the usual Sierpiński carpet discussed in Example 4.15; furthermore, the Sierpiński 3-carpet is discussed in [LRŽ7, Example 2].)

It is clear that the *Sierpiński* N*-carpet RFD* (A, Ω), where A is the Sierpiński N-carpet and $\Omega := (0,1)^N$ is the open unit cube of $\mathbb{R}^N$, can be viewed as the following relative fractal spray (see the end of Definition 4.1):

$$(A, \Omega) = \operatorname{Spray}\big((\partial\Omega_0, \Omega_0),\, \lambda = 1/3,\, b = 3^N - 1\big). \tag{4.2.44}$$

(Here, $\Omega_0 = (0, 1/3)^N$ is obtained by a suitable translation of the middle open subcube from the first step of the construction of A.) According to Theorem 4.8, we then have

$$\zeta_{A,\Omega}(s) = f(s) \cdot \zeta_{\partial\Omega_0,\Omega_0}(s) = \frac{\zeta_{\partial\Omega_0,\Omega_0}(s)}{1 - (3^N - 1)3^{-s}} \sim \frac{1}{1 - (3^N - 1)3^{-s}}. \tag{4.2.45}$$

Since Ω_0 has a Lipschitz boundary and $\log_{1/\lambda} b = \log_3(3^N - 1) \in (N-1, N)$, we deduce from (4.1.16) that the set of principal complex dimensions of the relative Sierpiński N-carpet spray is

$$\dim_{PC}(A, \Omega) = \log_3(3^N - 1) + \frac{2\pi}{\log 3}\,\mathbb{i}\mathbb{Z}, \tag{4.2.46}$$

and hence

$$\dim_{PC}(A, \Omega) \subset \{\operatorname{Re} s = \log_3(3^N - 1)\} \subset \{N - 1 < \operatorname{Re} s < N\}.$$

In particular, according to Theorem 2.1(b),

$$\overline{\dim}_B(A,\Omega) = \log_3(3^N - 1). \tag{4.2.47}$$

Furthermore, it can be shown that $\dim_B A$ and $\dim_B(A,\Omega)$ exist and

$$\overline{\dim}_B(A,\Omega) = \dim_B(A,\Omega) = \dim_B A = \log_3(3^N - 1). \tag{4.2.48}$$

It is easy to see that the set $\dim_{PC} A$ of principal complex dimensions of the Sierpiński N-carpet coincides with the set $\dim_{PC}(A,\Omega)$ appearing in (4.2.46). As simple special cases, we obtain the set of principal complex dimensions of the ternary Cantor set and of the usual Sierpiński carpet appearing in (4.2.41).

Since the set of all complex dimensions of the RFD $(\partial\Omega_0,\Omega_0)$ is $\{0,1,\dots,N-1\}$ (⁵), it follows from (4.2.45) that the set of all complex dimensions of the Sierpiński N-carpet relative fractal spray (A,Ω) is

$$\begin{aligned}\mathcal{P}(\zeta_{A,\Omega}) &= \dim_{PC}(A,\Omega) \cup \{0,1,\dots,N-1\}\\ &= \left(\log_3(3^N-1) + \frac{2\pi}{\log 3}\,\mathbb{i}\mathbb{Z}\right) \cup \{0,1,\dots,N-1\}.\end{aligned} \tag{4.2.49}$$

This concludes our study of the relative fractal drum (A,Ω) naturally associated with the N-dimensional Sierpiński carpet.

4.3. Self-similar sprays and RFDs. Let us now recall the definition of a self-similar spray or tiling (see [LP1]–[LP3], [LPW1, LPW2], [LF3, §13.1]). More precisely, let us state this definition slightly more generally and in the context of relative fractal drums.

DEFINITION 4.18 (Self-similar spray or tiling). Let G be a given open subset (*base set* or *generator*) of $\mathbb{R}^N$ of finite N-dimensional Lebesgue measure, and let $\{r_1,\dots,r_J\}$ be a finite multiset (also called a *ratio list*) of positive real numbers such that $J \in \mathbb{N}$, $J \geq 2$ and

$$\sum_{j=1}^{J} r_j^N < 1. \tag{4.3.1}$$

Furthermore, let Λ be the multiset consisting of all the possible "words" of multiples of the scaling factors $r_1,\dots,r_J$:

$$\begin{aligned}\Lambda := \{1, r_1,\dots,r_J, r_1r_1,\dots,r_1r_J, r_2r_1,\dots,r_2r_J,\dots,r_Jr_1,\dots,r_Jr_J,\\ r_1r_1r_1,\dots,r_1r_1r_J,\dots\},\end{aligned} \tag{4.3.2}$$

and arrange all of the elements of Λ into a *scaling sequence* $(\lambda_i)_{i\geq 0}$, where $\lambda_0 := 1$. (Note that $0 < \lambda_i < 1$ for every $i \geq 1$.)

A *self-similar spray* (or *tiling*) generated by the base set G and the ratio list $\{r_1,\dots,r_J\}$ is an RFD $(\partial\Omega,\Omega)$ in $\mathbb{R}^N$, where Ω is a disjoint union

$$\Omega := \bigsqcup_{i=0}^{\infty} G_i \tag{4.3.3}$$

(⁵) Note that the relative zeta function $\zeta_{A,\Omega}$ appearing in (4.2.45) can be meromophically extended in a unique way to the whole complex plane $\mathbb{C}$ since the same can be done with $\zeta_{\partial\Omega_0,\Omega_0}$. See, for example, (4.2.40) dealing with the case $N = 2$.

of open sets G_i congruent to $\lambda_i G$, for every $i \geq 0$. Here, the disjoint union can be understood as the disjoint union of RFDs given in Definition 2.18, with $(A_i, \Omega_i) := (\partial G_i, G_i)$ for each $i \geq 0$, in the notation of that definition. In what follows, $(\partial G, G)$ is also referred to as a *self-similar RFD.*

Remark 4.19. Note that in the above definition, the scaling sequence $(\lambda_i)_{i\geq 0}$ consists of all the products of ratios $r_1, \ldots, r_J$ appearing in the infinite sum

$$\sum_{n=0}^{\infty}\Big(\sum_{j=1}^{J} r_j\Big)^n, \tag{4.3.4}$$

after expanding the powers and counted with their multiplicities. More precisely, for every $\alpha = (\alpha_1, \ldots, \alpha_J) \in \mathbb{N}_0^J$, the multiplicity of $r_1^{\alpha_1}\cdots r_J^{\alpha_J}$ in Λ is equal to the multinomial coefficient

$$\binom{|\alpha|}{\alpha_1, \ldots, \alpha_J} = \frac{|\alpha|!}{\alpha_1!\cdots\alpha_J!}, \tag{4.3.5}$$

where $|\alpha| := \sum_{j=1}^{J}\alpha_j$. Of course, depending on the specific values of the ratios $r_1, \ldots, r_J$, some of the numbers $r_1^{\alpha_1}\cdots r_J^{\alpha_J}$ may be equal for different multi-indices $\alpha \in \mathbb{N}_0^J$.

Furthermore, the condition (4.3.1) ensures that the set $\Omega = \bigsqcup_{i\geq 0} G_i$ has finite N-dimensional Lebesgue measure. Indeed, we have

$$\begin{aligned} |\Omega| &= \sum_{i=0}^{\infty}|G_i| = \sum_{i=0}^{\infty}|\lambda_i G| = |G|\sum_{i=0}^{\infty}\lambda_i^N \\ &= |G|\sum_{n=0}^{\infty}\Big(\sum_{j=1}^{J} r_j^N\Big)^n = \frac{|G|}{1-\big(\sum_{j=1}^{J} r_j^N\big)^n}, \end{aligned} \tag{4.3.6}$$

since (4.3.1) is satisfied. Note that the second to last equality above follows from the construction of the scaling sequence $(\lambda_i)_{i\geq 0}$.

Consider now a self-similar spray as a relative fractal drum (A, Ω); that is, let $A := \partial\Omega$ and $\Omega := \bigsqcup_{i\geq 0} G_i$ (see Definition 4.18). The "self-similarity" of (A, Ω) is nicely exhibited by the scaling relation (4.3.7) in the following lemma.

Lemma 4.20. *Let (A, Ω) be a self-similar spray in $\mathbb{R}^N$, as in Definition 4.18. Then the relative fractal drum (A, Ω) satisfies the following* "self-similar identity":

$$(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} r_j(A, \Omega), \tag{4.3.7}$$

where $\bigsqcup_{j=1}^{J}$ indicates a disjoint union of copies of (A, Ω) scaled by factors $r_1, \ldots, r_J$ and displaced by isometries of $\mathbb{R}^N$.

Proof. Let us re-index the scaling sequence $(\lambda_i)_{i\geq 0}$ in a way that keeps track of the actual construction of the numbers λ_i out of the scaling ratios $r_1, \ldots, r_J$ (see (4.3.2)). We let

$$I := \{\emptyset\} \cup \bigcup_{m=1}^{\infty}\{1, \ldots, J\}^m \tag{4.3.8}$$

be the set of all finite sequences consisting of numbers $1, \dots, J$ (or, equivalently, of all finite words based on the alphabet $\{1, \dots, J\}$). Furthermore, for every $\alpha \in I$, define

$$\lambda_\alpha := \begin{cases} 1, & \alpha = \emptyset, \\ r_{\alpha_1} \cdots r_{\alpha_m}, & \alpha \neq \emptyset. \end{cases} \tag{4.3.9}$$

We then deduce from the construction of (A, Ω) that

$$\begin{aligned}(A, \Omega) &= \bigsqcup_{i=0}^{\infty} (\partial G_i, G_i) = \bigsqcup_{i=0}^{\infty} \lambda_i (\partial G, G) \\ &= \bigsqcup_{\alpha \in I} \lambda_\alpha (\partial G, G) = (\partial G, G) \sqcup \bigsqcup_{\alpha \in I \setminus \{\emptyset\}} \lambda_\alpha (\partial G, G).\end{aligned}$$

Observe now that in the last disjoint union above, every $\alpha \in \{1, \dots, J\}^m$ can be written as $\{j\} \times \{1, \dots, J\}^{m-1}$ for some $j \in \{1, \dots, J\}$ if we identify $\{j\}$ with $\{j\} \times \{\emptyset\}$ when $m = 1$. Note that this identification is consistent with the definition of λ_α, in the sense that $\lambda_{\{j\} \times \beta} = r_j \lambda_\beta$ for all $j \in \{1 \dots, J\}$ and $\beta \in I$. In light of this, we can next partition the last union above with respect to which number $j \in \{1, \dots, J\}$ the sequence α begins with:

$$\begin{aligned}(A, \Omega) &= (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} \bigsqcup_{\alpha \in \{j\} \times I} \lambda_\alpha (\partial G, G) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} \bigsqcup_{\beta \in I} r_j \lambda_\beta (\partial G, G) \\ &= (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} r_j \Big(\bigsqcup_{\beta \in I} \lambda_\beta (\partial G, G) \Big) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{J} r_j (A, \Omega). \quad \blacksquare\end{aligned}$$

In light of the identity (4.3.7) and a very special case of Theorem 2.19, it is now clear that the distance zeta function of (A, Ω) satisfies the following functional equation, which itself can be considered as a *self-similar identity*:

$$\zeta_{A,\Omega}(s) = \zeta_{\partial G, G}(s) + \sum_{j=1}^{J} \zeta_{r_j (A,\Omega)}(s) \tag{4.3.10}$$

for $\operatorname{Re} s$ sufficiently large (⁶). Furthermore, for such s, by using the scaling property of the relative distance zeta function (Theorem 2.16), we deduce that the above equation then becomes

$$\zeta_{A,\Omega}(s) = \zeta_{\partial G, G}(s) + \sum_{j=1}^{J} r_j^s \zeta_{A,\Omega}(s). \tag{4.3.11}$$

Finally, this last identity together with an application of the principle of analytic continuation yields the following theorem.

THEOREM 4.21. *Let G be the generator of a self-similar spray in $\mathbb{R}^N$, and let $\{r_1, \dots, r_J\}$, with $r_j > 0$ (for $j = 1, \dots, J$, $J \geq 2$) and $\sum_{j=1}^{J} r_j^N < 1$, be its scaling ratios. Furthermore,*

(⁶) For instance, it suffices to assume that $\operatorname{Re} s > N$ since, by Theorem 2.1, all the zeta functions appearing in (4.3.10) are holomorphic on $\{\operatorname{Re} s > N\}$.

let $(A,\Omega) := (\partial\Omega,\Omega)$ *be the self-similar spray generated by* G*, as in Definition* 4.18*. Then the distance zeta function of* (A,Ω) *is given by*

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1-\sum_{j=1}^{J} r_j^s} \tag{4.3.12}$$

for all $s \in \mathbb{C}$ *with* $\operatorname{Re} s$ *sufficiently large. In addition,*

$$D(\zeta_{A,\Omega}) = \max\{\overline{\dim}_B(\partial G,G), D\}, \tag{4.3.13}$$

where $D > 0$ *is the unique real solution of* $\sum_{j=1}^{J} r_j^D = 1$ (*i.e.,* D *is the* similarity dimension *of the self-similar spray* $(\partial\Omega,\Omega)$).

More specifically, given a connected open neighborhood U *of the critical line* $\{\operatorname{Re} s = D\}$, $\zeta_{A,\Omega}$ *has a meromorphic continuation to* U *if and only if* $\zeta_{\partial G,G}$ *does, and in that case,* $\zeta_{A,\Omega}(s)$ *is given by* (4.3.12) *for all* $s \in U$*. Consequently, the visible complex dimensions of* (A,Ω) *satisfy*

$$\mathcal{P}(\zeta_{A,\Omega},U) \subseteq (\mathfrak{D} \cap U) \cup \mathcal{P}(\zeta_{\partial G,G},U), \tag{4.3.14}$$

where $\mathfrak{D}$ *is the set of all the complex solutions of the Moran equation* $\sum_{j=1}^{J} r_j^s = 1$ (*i.e., the* scaling complex dimensions *of the fractal spray*)*; see Remark* 4.23 *for detailed information about* $\mathfrak{D}$*. Finally, if there are no zero-pole cancellations in* (4.3.12)*, then we have equality in* (4.3.14).

REMARK 4.22 (Complex dimensions and the definition of fractality). In [LF1]–[LF3], a geometric object is said to be “fractal” if the associated zeta function has at least one nonreal complex dimension (with positive real part; see [LF3, §12.1 and §12.2] for a detailed discussion). In [LF2, LF3], in order, in particular, to take into account some possible situations pertaining to random fractals (see [HaL], partly described in [LF3, §13.4]), the definition of fractality (within the context of the theory of complex dimensions) was extended so as to allow for the case described in Definition 2.38(i), namely, the existence of a (meromorphic) natural boundary along a screen. (See [LF3, §13.4.3].)

We note that in [LF3] (and the other aforementioned references), the term “hyperfractal” was not used. More important, except for fractal strings and in very special higher-dimensional situations (such as suitable fractal sprays), we did not have at our disposal (as we now do) [LRŽ1]–[LRŽ8] a general definition of “fractal zeta function” associated with an arbitrary bounded subset of $\mathbb{R}^N$, for every $N \geq 1$. Therefore, we can now define the “fractality” of any bounded subset of $\mathbb{R}^N$ (including Julia sets and the Mandelbrot set) and, more generally, of any relative fractal drum, by the presence of a nonreal complex dimension or else by the “hyperfractality” (in the sense of Definition 2.38(i)) of the geometric object under consideration. Here, “complex dimension” is understood as a (visible) pole of the associated fractal zeta function (the distance or tube zeta function of a bounded subset or a relative fractal drum of $\mathbb{R}^N$, or else, as was the case in most of [LF3], the geometric zeta function of a fractal string).

Much as in [LF1]–[LF3] and [L3]–[L7], this terminology (concerning fractality, hyperfractality, and complex dimensions) can be extended to “virtual geometries”, as well as to (absolute or) relative fractal drums, noncommutative geometries, dynamical systems, and arithmetic geometries, via suitably associated “fractal zeta functions”, be they absolute or relative distance or tube zeta functions, spectral zeta functions, dynamical zeta functions, or arithmetic zeta functions (or the logarithmic derivatives thereof).

We will return to the discussion of the notion of fractality in Chapter 5; see also the next remark.

Remark 4.23. The multiset $\mathfrak{D}$ of scaling complex dimensions of the self-similar spray (A, Ω) is analyzed in detail in [LF3, Chapter 3; esp., Theorem 3.6] (7). Accordingly, there is a natural lattice/nonlattice dichotomy defined as follows: (A, Ω) is *lattice* if the multiplicative subgroup G of $(0, +\infty)$ generated by the distinct values of the scaling ratios $r_1, \dots, r_J$ is of rank 1 (i.e. of the form $r^{\mathbb{Z}}$ for a unique $r \in (0,1)$, called the *multiplicative generator* of the spray). It is *nonlattice* otherwise (i.e., if the above group is of rank > 1), and *generic nonlattice* if G is of maximal rank > 1 (i.e., of rank J', the number of distinct elements in the ratio list $r_1, \dots, r_J$, and $J' > 1$).

Then, according to [LF3, Theorem 3.6], in the lattice case, all of the scaling complex dimensions are periodically distributed along finitely many vertical lines (the rightmost of which is $\{\operatorname{Re} s = D\}$) with the same period $T := 2\pi/\log(r^{-1})$, called the *oscillatory period* of the lattice self-similar spray (8). On the other hand, in the nonlatice case, $\mathfrak{D}$ is simple and is the only principal scaling complex dimension (i.e., the only scaling complex dimension with real part D). However, there is an infinite sequence of distinct scaling complex dimensions converging from the left to (but not touching) the vertical line $\{\operatorname{Re} s = D\}$.

Moreover, it was conjectured in [LF2, LF3, §3.7] (and especially in reference [LF1] of [LF3]) that in the generic nonlattice case, the set of real parts of the scaling dimensions is dense in a compact interval $[\sigma_l, D]$, with $\sigma_l \in \mathbb{R}$ and $\sigma_l < D$; i.e., the *set of "fractality"* (that is, the closure of the above set of real parts, as defined in [LF2, LF3]) is equal to $[\sigma_l, D]$, in striking contrast to the lattice case where it is a finite set. This conjecture has recently been proved in [MSV1], where it was also shown that in the nonlattice (but not necessarily generic nonlattice) case the set of "fractality" is equal to a finite (and nonempty) disjoint union of nonempty compact intervals.

Finally, via Diophantine approximation techniques, the scaling complex dimensions of a nonlattice self-similar spray can be approximated by those of a sequence of lattice sprays with larger and larger periods. (See [LF3, §3.4, esp., Theorem 3.19].) Accordingly, in the nonlattice case, the scaling complex dimensions exhibit a quasiperiodic pattern (studied in detail both numerically and theoretically in [LF3, Chapter 3]).

Example 4.24 (The 1/2-square fractal). In this planar example, we will further investigate and illustrate the new interesting phenomenon which occurs in the case of the Sierpiński 3-gasket RFD discussed in Example 4.14. Namely, we start with the closed unit square $I = [0,1]^2$ in $\mathbb{R}^2$ and subdivide it into four smaller squares by taking the centerlines of its sides. We then remove the two diagonal open smaller squares, denoted

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(7) See also [LF3, Chapter 2; esp., Theorem 2.16] for the one-dimensional case, corresponding to self-similar strings.

(8) On each of these vertical lines, the corresponding scaling complex dimensions all have the same multiplicity. In particular, along the vertical line $\{\operatorname{Re} s = D\}$, they are all simple.

by G_1 and G_2 in Figure 5, so that $G := G_1 \cup G_2$ is our generator in the sense of Definition 4.18. Next, we repeat this step with the remaining two closed smaller squares and continue this process ad infinitum. The 1/2-square fractal is then defined as the set A which remains at the end of the process; see Figure 5, where the first six iterations are shown. More precisely, A is the closure of the union of the boundaries of the disjoint family of open squares appearing in Figure 5 and packed in the unit square I. If we now let $\Omega := (0,1)^2$, then (A, Ω) is an example of a self-similar spray (or tiling), in the sense of Definition 4.18, with generator $G = G_1 \cup G_2$ and scaling ratios $r_1 = r_2 = 1/2$. Note, however, that A is not a (*homogeneous*) self-similar set in the usual sense (see, e.g., [F1, Hu]), defined via iterated function systems, but it is an *inhomogeneous* self-similar set.

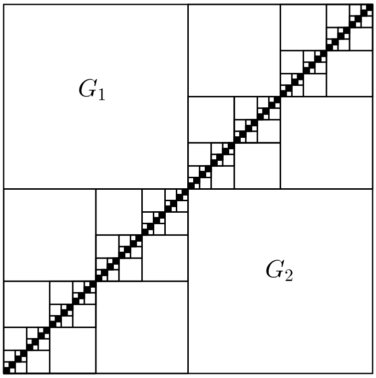

Fig. 5. The 1/2-square fractal A from Example 4.24. The first six iterations are depicted. Here, $G := G_1 \cup G_2$ is the single generator of the corresponding self-similar spray or RFD (A, Ω), in the sense of Definition 4.18. The set A is equal to the complement of the union of the disjoint family of all open squares, with respect to $\Omega = (0,1)^2$. Equivalently, the set A coincides with the closure of the union of the boundaries of all the open squares.

More specifically, A is the unique nonempty compact subset of $\mathbb{R}^2$ which is the solution of the inhomogeneous fixed point equation

$$A = \bigcup_{j=1}^{2} \Phi_j(A) \cup B, \tag{4.3.15}$$

where Φ_1 and Φ_2 are contractive similitudes of $\mathbb{R}^2$ with fixed points located at the lower left vertex and the upper right vertex of the unit square, respectively, and with a common scaling ratio equal to 1/2 (i.e., $r_1 = r_2 = 1/2$, where $\{r_j\}_{j=1}^2$ are the scaling ratios of the self-similar RFD (A, Ω)). Furthermore, the nonempty compact set B in (4.3.15) is the union of the left and upper sides of the square G_1 and the right and lower sides of the square G_2; see Figure 5. We note that here, the corresponding (classical or homogeneous) self-similar set (i.e., the unique nonempty compact subset C of $\mathbb{R}^2$ which is the solution of the homogeneous fixed point equation $C = \bigcup_{j=1}^2 \Phi_j(C)$) is the diagonal C of the unit square connecting the lower left and the upper right vertices of the unit square.

Let us now compute the distance zeta function ζ_A. Without loss of generality, we may assume that $\delta > 1/4$, so that

$$\zeta_A(s) = \zeta_{A,\Omega}(s) + \zeta_I(s), \tag{4.3.16}$$

where, intuitively, ζ_I denotes the distance zeta function corresponding to the "outer" δ-neighborhood of A. Clearly, ζ_I is the distance zeta function of $I := [0,1]^2$; it is straightforward to compute it and show that it has a meromorphic extension to all of $\mathbb{C}$ given by

$$\zeta_I(s) = \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} \tag{4.3.17}$$

for all $s \in \mathbb{C}$.

Furthermore, by using Theorem 4.21, we obtain

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1 - 2\cdot 2^{-s}} = \frac{2^s\zeta_{\partial G,G}(s)}{2^s - 2} \tag{4.3.18}$$

for $\operatorname{Re} s$ sufficiently large. Next, we compute the distance zeta function of $(\partial G, G)$ by subdividing $G = G_1 \cup G_2$ into 16 congruent triangles (see also Figure 4, which describes the way we subdivide both G_1 and G_2) and by using local Cartesian coordinates $(x,y) \in \mathbb{R}^2$ to deduce that

$$\zeta_{\partial G,G}(s) = 16\int_0^{1/4} \mathrm{d}x \int_0^x y^{s-2}\,\mathrm{d}y = \frac{4^{-s}}{s(s-1)},$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Hence,

$$\zeta_{\partial G,G}(s) = \frac{4^{-s}}{s(s-1)}, \tag{4.3.19}$$

an identity valid initially for $\operatorname{Re} s > 1$, and then, after meromorphic continuation, for all $s \in \mathbb{C}$. Finally, by combining (4.3.16)–(4.3.19), we conclude that ζ_A is meromorphic on $\mathbb{C}$ and is given by

$$\zeta_A(s) = \frac{2^{-s}}{s(s-1)(2^s-2)} + \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} \tag{4.3.20}$$

for all $s \in \mathbb{C}$.

Consequently, $\dim_B A$ exists ([9]), and

$$\begin{aligned} D(\zeta_A) &= \dim_B A = 1, \\ \mathcal{P}(\zeta_A) &:= \mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup (1 + \mathbf{p}\mathbb{i}\mathbb{Z}) \end{aligned} \tag{4.3.21}$$

and

$$\dim_{PC} A := \mathcal{P}_c(\zeta_A) = 1 + \mathbf{p}\mathbb{i}\mathbb{Z}, \tag{4.3.22}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := 2\pi/\log 2$. All of the complex dimensions in $\mathcal{P}(\zeta_A)$ are simple except for $\omega = 1$, which is a double pole of ζ_A. Finally, in light of (4.3.21) (and hence of the presence of nonreal complex dimensions), the set A is indeed *fractal* according to our proposed definition of fractality in Remark 4.22 and further discussed in Chapter 5 below. In fact, according to (4.3.22), it is *critically fractal* (i.e., fractal in dimension $d := 1 = \dim_B A$, in the sense of §5.1).

([9]) The existence of $\dim_B A$ in Example 4.24 (as well as in Examples 4.25 and 4.26 below) follows from [LRŽ1, Theorem 5.4.30] (see also [LRŽ6, Theorem 4.2]).

EXAMPLE 4.25 (The 1/3-square fractal). In the present planar example, we illustrate a situation which is similar to that of the inhomogeneous Sierpiński N-gasket RFD discussed in Example 4.14 for $N \geq 4$. Again, we start with the closed unit square $I = [0,1]^2$ in $\mathbb{R}^2$ and subdivide it into nine smaller congruent squares (as in the case of the Sierpiński carpet). Next, we remove seven of those smaller squares, only leaving the lower left and the upper right squares (see Figure 6). In other words, our generator G (in the sense of Definition 4.18) is the (nonconvex) open polygon depicted in Figure 6.

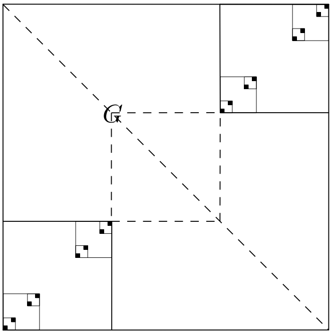

Fig. 6. The 1/3-square fractal A from Example 4.25. The first four iterations are depicted. Here, G is the single generator of the corresponding self-similar spray or RFD (A, Ω), in the sense of Definition 4.18. The set A is equal to the complement in $\Omega = (0,1)^2$ of the union of the disjoint family of all the open 8-gons. The largest 8-gon is the union of two open squares indicated by dashed sides of length 2/3, while each of the next two smaller 8-gons is obtained by scaling the first one by the factor 1/3. Any of the 2^k 8-gons of the kth generation is obtained by scaling the first one by the factor $1/3^{k-1}$, for any $k \in \mathbb{N}$. Equivalently, A coincides with the closure of the union of the boundaries of all the 8-gons.

As usual, we proceed by iterating this procedure with the remaining two closed squares and then continue this process ad infinitum. (The first four iterations are depicted in Figure 6.) The 1/3-square fractal is then defined as the set A which remains at the end of the process. We now let $\Omega := (0,1)^2$, which makes the RFD (A, Ω) a self-similar spray (or tiling), in the sense of Definition 4.18, with generator G and scaling ratios $\{r_j\}_{j=1}^2$ such that $r_1 = r_2 = 1/3$. Again, A is not a homogeneous self-similar set, but it is an inhomogeneous self-similar set.

More specifically, A is the unique nonempty compact subset of $\mathbb{R}^2$ which is the solution of the inhomogeneous equation

$$A = \bigcup_{j=1}^{2} \Phi_j(A) \cup B, \tag{4.3.23}$$

where Φ_1 and Φ_2 are contractive similitudes of $\mathbb{R}^2$ with fixed points located at the lower left vertex and the upper right vertex of the unit square, respectively, and with a common scaling ratio 1/3. Furthermore, the set B in (4.3.23) is equal to the boundary of G without the part belonging to the boundary of the two smaller squares which are left behind in the first iteration (see Figure 6). We also observe that here the corresponding (classical or homogeneous) self-similar set generated by the IFS consisting of Φ_1 and Φ_2 is the ternary Cantor set located along the diagonal of the unit square.

We now compute the distance zeta function ζ_A of the 1/3-square fractal. Without loss of generality, we may assume that $\delta > 1/4$, so that

$$\zeta_A(s) = \zeta_{A,\Omega}(s) + \zeta_I(s), \tag{4.3.24}$$

where, as before in Example 4.24, ζ_I denotes the distance zeta function corresponding to the "outer" δ-neighborhood of A and coincides with the distance zeta function of $I := [0,1]^2$. Recall that ζ_I is given in (4.3.17).

Furthermore, by using Theorem 4.21, we obtain

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1 - 2 \cdot 3^{-s}} = \frac{3^s \zeta_{\partial G,G}(s)}{3^s - 2} \tag{4.3.25}$$

for $\operatorname{Re} s$ sufficiently large.

Next, we compute the distance zeta function of $(\partial G, G)$ by subdividing G into 14 congruent triangles denoted by G_i for $i = 1, \ldots, 14$ (see Figure 6). By symmetry, we obtain the functional equation

$$\zeta_{\partial G,G}(s) = 12\zeta_{\partial G,G_1}(s) + 2\zeta_{\partial G,G_{13}}, \tag{4.3.26}$$

valid initially for $\operatorname{Re} s$ sufficiently large.

We use local Cartesian coordinates $(x, y) \in \mathbb{R}^2$ in order to compute

$$\zeta_{\partial G,G_1} = \int_0^{1/3} \mathrm{d}x \int_0^x y^{s-2}\,\mathrm{d}y = \frac{3^{-s}}{s(s-1)}.$$

Hence,

$$\zeta_{\partial G,G_1} = \frac{3^{-s}}{s(s-1)} \tag{4.3.27}$$

for $\operatorname{Re} s > 1$, and after meromorphic continuation for all $s \in \mathbb{C}$. To compute $\zeta_{\partial G,G_{13}}$, we use local polar coordinates (r, θ) and deduce that

$$\begin{aligned}\zeta_{\partial G,G_{13}}(s) &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^{3^{-1}(\sin\theta+\cos\theta)^{-1}} r^{s-1}\,\mathrm{d}r \\ &= \frac{3^{-s}}{s} \int_0^{\pi/2} (\cos\theta + \sin\theta)^{-s}\,\mathrm{d}\theta \end{aligned} \tag{4.3.28}$$

for $\operatorname{Re} s > 0$, and after meromorphic continuation for all $s \in \mathbb{C}$. It is easy to check that

$$Z(s) := \int_0^{\pi/2} (\cos\theta + \sin\theta)^{-s}\,\mathrm{d}\theta \tag{4.3.29}$$

is an entire function, since it is a generalized DTI $f(s) := \int_E \varphi(\theta)^s\,\mathrm{d}\mu(\theta)$, where $E := [0, \pi/2]$, $\varphi(\theta) := (\cos\theta + \sin\theta)^{-1}$ for all $\theta \in E$ is uniformly bounded by positive constants both from above and below, and $\mathrm{d}\mu(\theta) := \mathrm{d}\theta$.

Finally, by combining (4.3.17) and (4.3.24)–(4.3.29), we obtain

$$\zeta_A(s) = \frac{2}{s(3^s - 2)}\left(\frac{6}{s-1} + Z(s)\right) + \frac{4\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} \tag{4.3.30}$$

for $\operatorname{Re} s > 1$, and after meromorphic continuation for all $s \in \mathbb{C}$.

Consequently, $\dim_B A$ exists and

$$\begin{aligned} &D(\zeta_A) = \dim_B A = 1, \\ &\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) \subseteq \{0\} \cup (\log_3 2 + \mathbf{p}\mathbb{i}\mathbb{Z}) \cup \{1\}, \end{aligned} \tag{4.3.31}$$

and

$$\dim_{PC} A := \mathcal{P}_c(\zeta_A) = \{1\}, \tag{4.3.32}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := 2\pi/\log 3$. In (4.3.31), we only have an inclusion since, in principle, some of the complex dimensions with real part $\log_3 2$ may be canceled by the zeros of $6/(s-1) + Z(s)$. However, it can be checked numerically that $\log_3 2 \in \mathcal{P}(\zeta_A)$ and that there also exist nonreal complex dimensions with real part $\log_3 2$ in $\mathcal{P}(\zeta_A)$. All of the complex dimensions in $\mathcal{P}(\zeta_A)$ are simple. We also note that A is indeed *fractal*, according to our definition of fractality (see Remark 4.22 and Chapter 5). More precisely, in light of (4.3.31) and (4.3.32), it is *strictly subcritically fractal* and *fractal in dimension* $d = \log_3 2$, in the sense of §5.1.

Example 4.26 (A self-similar fractal nest). In the final planar example of this section, we investigate the case of a self-similar fractal nest ([10]). The set A which we now define is an inhomogeneous self-similar set. As in Example 4.25, A will be *fractal* in the sense of our definition in Remark 4.22, and moreover *strictly subcritically fractal* in the sense of §5.1.

Let $a \in (0, 1)$. We define A as the union of concentric circles with center at the origin and of radius a^k for $k \in \mathbb{N}_0$ (see Figure 7). Furthermore, let G be the open annulus such that ∂G consists of the circles of radius 1 and a, as depicted in Figure 7, and let $\Omega := B_1(0)$. We can now consider the RFD (A, Ω) as a self-similar spray with generator G in the sense of Definition 4.18.

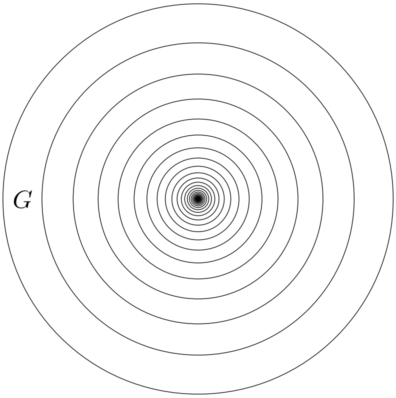

Fig. 7. The self-similar fractal nest from Example 4.26

We note that even though (A, Ω) is a fractal spray with a single generator G, it is not (strictly speaking) self-similar in the traditional sense because it only has one scaling

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[10] As we shall see, the use of "self-similar" is somewhat abusive since only one similarity transformation is involved.

ratio $r = a$ (associated with a single contractive similitude). However, we will continue using this abuse of language throughout this example. Also, a moment's reflection reveals that this does not affect any of the conclusions relevant to the distance zeta function of such an RFD. Namely, we obviously have

$$(A,\Omega) = (\partial G, G) \sqcup a(A,\Omega), \tag{4.3.33}$$

so that

$$\zeta_{A,\Omega}(s) = \zeta_{\partial G,G}(s) + \zeta_{a(A,\Omega)}(s) \tag{4.3.34}$$

for $\operatorname{Re} s$ sufficiently large. Furthermore, by using the scaling property of the relative distance zeta function (see Theorem 2.16), we conclude that

$$\zeta_{A,\Omega}(s) = \frac{\zeta_{\partial G,G}(s)}{1-a^s} \tag{4.3.35}$$

for $\operatorname{Re} s$ sufficiently large.

Next, we compute the distance zeta function of the generator by using polar coordinates (r,θ):

$$\begin{aligned}\zeta_{\partial G,G}(s) &= \int_0^{2\pi} \mathrm{d}\theta \int_a^{(1+a)/2} (r-a)^{s-2} r\,\mathrm{d}r + \int_0^{2\pi} \mathrm{d}\theta \int_{(1+a)/2}^{1} (1-r)^{s-2} r\,\mathrm{d}r \\ &= \frac{2^{2-s}\pi(1+a)(1-a)^{s-1}}{s-1},\end{aligned} \tag{4.3.36}$$

an identity valid, after meromorphic continuation, for all $s \in \mathbb{C}$.

Equation (4.3.36) combined with (4.3.35) now shows that $\zeta_{A,\Omega}$ is meromorphic on $\mathbb{C}$ and is given for all $s \in \mathbb{C}$ by

$$\zeta_{A,\Omega}(s) = \frac{2^{2-s}\pi(1+a)(1-a)^{s-1}}{(s-1)(1-a^s)}. \tag{4.3.37}$$

Finally, we fix an arbitrary $\delta > (1-a)/2$ and observe that

$$\zeta_A(s) = \zeta_{A,\Omega}(s) + \zeta_{A,B_{1+\delta}(0)\setminus\Omega}(s) \tag{4.3.38}$$

for $\operatorname{Re} s$ sufficiently large. Furthermore,

$$\zeta_{A,B_{1+\delta}(0)\setminus\Omega}(s) = \int_0^{2\pi} \mathrm{d}\theta \int_1^{1+\delta} (r-1)^{s-2} r\,\mathrm{d}r = \frac{2\pi\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} \tag{4.3.39}$$

for $\operatorname{Re} s > 1$, and after meromorphic continuation for all $s \in \mathbb{C}$.

Combining now the above equation with (4.3.38), we finally see that ζ_A is meromorphic on $\mathbb{C}$ and

$$\zeta_A(s) = \frac{2^{2-s}\pi(1+a)(1-a)^{s-1}}{(s-1)(1-a^s)} + \frac{2\pi\delta^{s-1}}{s-1} + \frac{2\pi\delta^s}{s} \tag{4.3.40}$$

for all $s \in \mathbb{C}$.

Consequently, $\dim_B A$ exists and

$$\begin{aligned} &D(\zeta_A) = \dim_B A = 1, \\ &\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \mathbf{p}\mathbb{i}\mathbb{Z} \cup \{1\},\end{aligned} \tag{4.3.41}$$

$$\dim_{PC}(A) := \mathcal{P}_c(\zeta_A) = \{1\}, \tag{4.3.42}$$

where the oscillatory period $\mathbf{p}$ of A is given by $\mathbf{p} := 2\pi/\log a^{-1}$ and all of the complex dimensions in $\mathcal{P}(\zeta_A)$ are simple.

In closing, we mention that A is indeed *fractal* according to our definition (see Remark 4.22 and Chapter 5). More specifically, in light of (4.3.41), A is *strictly subcritically fractal* and *fractal in dimension* $d := 0$, in the sense of §5.1 below.

4.4. Generating complex dimensions of RFDs of any multiplicity. A key tool in generating (principal) complex dimensions of higher multiplicities is the tensor product of bounded fractal strings, which we now briefly define; see [LRŽ3] for more details. If $\mathcal{L}_1 := (\ell_{1j})_{j\geq 1}$ and $\mathcal{L}_2 := (\ell_{2k})_{k\geq 1}$ are two bounded fractal strings, their *tensor product* $\mathcal{L}_1 \otimes \mathcal{L}_2$ is defined as the multiset consisting of all possible products of the form $\ell_{1j}\ell_{2k}$ for all ordered pairs $(j,k) \in \mathbb{N}^2$; hence, we take into account the multiplicities. It is easy to see that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is also a bounded fractal string. Furthermore, the geometric zeta function of the tensor product is the product of the component geometric zeta functions. More specifically,

$$\zeta_{\mathcal{L}_1\otimes\mathcal{L}_2}(s) = \zeta_{\mathcal{L}_1}(s)\cdot\zeta_{\mathcal{L}_2}(s) \tag{4.4.1}$$

for $\operatorname{Re} s > \max\{D(\zeta_{\mathcal{L}_1}), D(\zeta_{\mathcal{L}_2})\}$, and

$$D(\zeta_{\mathcal{L}_1\otimes\mathcal{L}_2}) = \max\{D(\zeta_{\mathcal{L}_1}), D(\zeta_{\mathcal{L}_2})\};$$

see [LRŽ3, Lemma 4.13].

The following example provides a class of bounded relative fractal drums, generated by an a-string, which illustrates Theorem 2.24 above. Note that here, we have a unique, nonsimple, principal complex dimension, D, on the critical line, and its multiplicity is equal to any prescribed positive integer m. We shall need the notion of the *disjoint union* (of an at most countable family) *of bounded fractal strings* $\mathcal{L}_m = (\ell_{mj})_{j\geq 1}$ for $m \in \mathbb{N}$:

$$\mathcal{L} := \bigsqcup_{m=1}^{\infty} \mathcal{L}_m, \tag{4.4.2}$$

defined as the multiset $\mathcal{L}$ consisting of elements of the union of the fractal strings, counting their multiplicities. Assuming, additionally, that $\ell_{m1} \to 0^+$ as $m \to \infty$, it is easy to see that the mulitplicity of each element of the multiset $\mathcal{L} := \bigsqcup_{m=1}^{\infty} \mathcal{L}_m$ is finite, so that $\mathcal{L}$ is indeed a bounded fractal string.

Example 4.27 (mth order a-string). Let $\mathcal{L}(a) := \{\ell_k := k^{-a} - (k+1)^{-a}\}_{k=1}^{\infty}$ be the a-string, where $a > 0$ (see [L1, Example 5.1] and [LF3, §6.5.1]), and let m be a positive integer. Let

$$\mathcal{L}_m(a) := \begin{cases} \mathcal{L}(a) & \text{for } m = 1, \\ \mathcal{L}(a) \otimes \overset{m}{\cdots} \otimes \mathcal{L}(a) & \text{for } m \geq 2, \end{cases} \tag{4.4.3}$$

which we call the *mth order a-string.* Then $D = 1/(1+a)$ is the only principal complex dimension of $\mathcal{L}_m(a)$, and it is of multiplicity m, since in light of (4.4.1) we have

$$\zeta_{\mathcal{L}_m(a)}(s) = [\zeta_{\mathcal{L}(a)}(s)]^m = \Big(\sum_{k=1}^{\infty} \ell_k^s\Big)^m \tag{4.4.4}$$

for $\operatorname{Re} s > 1/(1+a)$. Defining $h(t) := (\log t^{-1})^{m-1}$ for all $t \in (0,1)$, and using [LRŽ6, Theorem 5.4 and Example 3.7], we deduce that the fractal string $\mathcal{L}_m(a)$ is h-Minkowski measurable. Moreover, also according to [LRŽ6, Theorem 5.4] (see also [LRŽ1, Theorem 5.4.27 and Example 5.5.10]), $\mathcal{L}_m(a)$ has the following tube asymptotics:

$$|A_t| = t^{1-D} h(t)(\mathcal{M} + o(1)) \quad \text{as } t \to 0^+, \tag{4.4.5}$$

where $\mathcal{M} \in (0, +\infty)$ is the *h-Minkowski content* of $\mathcal{L}_m(a)$ and can be explicitly computed in terms of the $(-m)$th coefficient c_{-m} of the Laurent expansion of the tube zeta function $\tilde{\zeta}_A$ around $s = D$, as follows: $\mathcal{M} = c_{-m}/(m-1)!$ [LRŽ6, Theorem 5.4]. In particular, $\mathcal{L}_m(a)$ is *h-Minkowski measurable.*

In [LRŽ3, §4.4], we have constructed a (Cantor-type) bounded fractal string $\mathcal{L}_m$ which has infinitely many principal complex dimensions of arbitrary prescribed multiplicity $m \geq 2$. The bounded fractal string was obtained by tensoring the usual Cantor string $\mathcal{L}_{CS}$:

$$\mathcal{L}_m := \begin{cases} \mathcal{L}_{CS} & \text{for } m = 1, \\ \mathcal{L}_{CS} \otimes \overset{m-1}{\cdots} \otimes \mathcal{L}_{CS} & \text{for } m \geq 2, \end{cases} \tag{4.4.6}$$

which we call the *mth order Cantor string* or the *m-Cantor string*, for short. The corresponding multiset of principal complex dimensions is

$$\dim_{PC} \mathcal{L}_m = \log_3 2 + \frac{2\pi}{\log 3} \mathbb{i}\mathbb{Z}, \tag{4.4.7}$$

and each of its elements has multiplicity m. Note that by [LRŽ6, Theorem 3.1] (see also [LRŽ1, Theorem 5.4.20]), $\mathcal{L}_m$ is not Minkowski measurable for $m \geq 2$.

Furthermore, by letting

$$\mathcal{L}_\infty := \bigsqcup_{m=1}^{\infty} \frac{3^{-m}}{m!} \mathcal{L}_m, \tag{4.4.8}$$

we obtain a bounded fractal string $\mathcal{L}_\infty$, called the *Cantor string of infinite order* or the *∞-Cantor string*, whose geometric zeta function $\zeta_{\mathcal{L}_\infty}$ has an infinite sequence of *essential* singularities along the critical line $\{\operatorname{Re} s = D\}$, located at the points $D + \mathbb{i}\mathbf{p}k$ (with $k \in \mathbb{Z}$, $D := \log_3 2$ and $\mathbf{p} := 2\pi/\log 3$) of the periodic set defined by the right-hand side of (4.4.7).

Example 4.28. Let m be a fixed positive integer and let a be a positive real number chosen small enough, so that $D := 1/(1+a) > \log_3 2$. Consider the bounded fractal string defined by

$$\mathcal{L} := \mathcal{L}_m(a) \sqcup \mathcal{L}_\infty, \tag{4.4.9}$$

where $\mathcal{L}_m(a)$ and $\mathcal{L}_\infty$ are defined by (4.4.3) and (4.4.8), respectively, and generated by tensor products of a-strings and Cantor strings, respectively. Here, $D_{\mathrm{mer}}(\zeta_{\mathcal{L}}) = \log_3 2$, since the geometric zeta function

$$\zeta_{\mathcal{L}}(s) = \zeta_{\mathcal{L}_m(a)}(s) + \zeta_{\mathcal{L}_\infty}(s) \tag{4.4.10}$$

is holomorphic on the connected open set $\{\operatorname{Re} s > 0\} \setminus \left(\{D\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right)\right)$, where $D = \dim_B \mathcal{L}$ and is the (unique) pole of $\zeta_{\mathcal{L}}$ of order m in $\{\operatorname{Re} s > 0\}$, while $\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$ is the set of essential singularities of $\zeta_{\mathcal{L}}$ in $\{\operatorname{Re} s > 0\}$. Denoting by $A_{\mathcal{L}} := \{a_k :=$

$\sum_{j=k}^{\infty}\ell_j : k\in\mathbb{N}\}$ the canonical representation of the fractal string $\mathcal{L} := \{\ell_j\}_{j=1}^{\infty}$, and applying [LRŽ1, Theorem 5.4.27] to the RFD $(A_{\mathcal{L}}, (A_{\mathcal{L}})_\delta)$ (for any fixed positive real number δ) (^{11}), we obtain the following asymptotic expansion of the tube function of the set $A_{\mathcal{L}}$:

$$|(A_{\mathcal{L}})_t| = t^{1-D}h(t)\big(\mathcal{M} + O(t^{D-D_{\mathrm{mer}}(\zeta_{\mathcal{L}})-\varepsilon})\big) \quad \text{as } t\to 0^+, \tag{4.4.11}$$

for any $\varepsilon > 0$, where $h(t) := (\log t^{-1})^{m-1}$ for all $t\in(0,1)$; i.e.,

$$|(A_{\mathcal{L}})_t| = t^{a/(1-a)}h(t)\big(\mathcal{M} + O(t^{\frac{1}{1+a}-\log_3 2-\varepsilon})\big) \quad \text{as } t\to 0^+, \tag{4.4.12}$$

where $\mathcal{M}$ is a positive real number (the h-Minkowski content) and can be computed (see [LRŽ6, Theorem 4.5] or [LRŽ1, Theorem 5.4.27]). According to [LRŽ6, Theorem 5.6] (or [LRŽ1, Theorem 5.4.29]), the exponent $\frac{1}{1+a}-\log_3 2$ appearing on the right-hand side of (4.4.12) is optimal i.e., it cannot be increased.

In Examples 4.29 and 4.30 below, we construct *Minkowski measurable* RFDs which possess infinitely many complex dimensions of arbitrary multiplicity $m \geq 1$, or even essential singularities.

EXAMPLE 4.29. Let us first define the unit square RFD (A_0, Ω_0) by $\Omega_0 := [0,1]^2$ and $A_0 := \partial\Omega_0$. We introduce the RFD

$$(A_m', \Omega_m') := (A_0, \Omega_0) \sqcup \mathcal{L}_m, \tag{4.4.13}$$

where we embed $\mathcal{L}_m$ via its *canonical geometric representation* $A_{\mathcal{L}_m}$ into the x-axis of $\mathbb{R}^2$. Since $\zeta_{A_m',\Omega_m'}(s) = \zeta_{A_0,\Omega_0}(s) + \zeta_{\mathcal{L}_m}(s)$, we have

$$\mathcal{P}(\tilde{\zeta}_{A_m',\Omega_m'}) = \{0,1\} \cup \mathcal{P}', \tag{4.4.14}$$

where $\mathcal{P}' := \log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$ and each of the complex dimensions $D + \mathbb{i}\mathbf{p}k$ (with $k\in\mathbb{Z}$, $D := \log_3 2$) of $\mathcal{P}'$ is of multiplicity m. On the other hand, the only principal complex dimension of (A_m', Ω_m') is 1, and it is simple. Therefore, according to [LRŽ5, Theorem 5.2], the RFD (A_m', Ω_m') is Minkowski measurable.

EXAMPLE 4.30. Let us again define $\Omega_0 := [0,1]^2$ and $A_0 := \partial\Omega$. We introduce the RFD

$$(A_m, \Omega_m) := (A_0, \Omega_0) \otimes \mathcal{L}_m, \tag{4.4.15}$$

where $\mathcal{L}_m$ is defined by (4.4.6), and the tensor product is defined as in Example 4.3. Using (4.1.9) (see also (4.1.14)), we obtain

$$\zeta_{A_m,\Omega_m}(s) = \zeta_{A_0,\Omega_0}(s)\cdot\zeta_{\mathcal{L}_m}(s) = \frac{g(s)}{s(s-1)(3^s-2)^m}. \tag{4.4.16}$$

Here, $g(s)$ is an entire function without zeros at 0, 1 or at any point of the arithmetic set

$$\mathcal{P}' := \log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}. \tag{4.4.17}$$

In other words,

$$\dim_{PC}(A_m, \Omega_m) = \{1\}, \qquad \mathcal{P}(A_m, \Omega_m) = \{0,1\} \cup \mathcal{P}', \tag{4.4.18}$$

(^{11}) The half-plane $\{\operatorname{Re} s > D_{\mathrm{mer}}(\zeta_{\mathcal{L}})\}$ contains no poles of $\zeta_{\mathcal{L}}$ except for $s = D$.

and each complex dimension of (A,Ω) lying in $\mathcal{P}'$ has multiplicity m. The value of $D := \overline{\dim}_B(A_m,\Omega_m)=1$ is the only complex dimension located on the critical line $\{\operatorname{Re} s=1\}$, while the infinite set $\mathcal{P}'$ is contained in the vertical line $\{\operatorname{Re} s = \log_3 2\}$ located strictly to the left of the critical line. It follows from [LRŽ6, Theorem 5.4] (or [LRŽ1, Theorem 5.4.29]) that the RFD (A,Ω) is h-Minkowski measurable with respect to the gauge function $h(t) := (\log t^{-1})^{m-1}$ for all $t \in (0,1)$.

We can further define the RFD

$$(A_\infty,\Omega_\infty) := \bigsqcup_{m=2}^{\infty} \frac{3^{-m}}{m!} \cdot (A_m,\Omega_m). \tag{4.4.19}$$

As above, we have (with $\mathcal{P}'$ given by (4.4.17))

$$\dim_{PC}(A_\infty,\Omega_\infty) = \{1\}, \quad \mathcal{P}(A_\infty,\Omega_\infty) = \{0,1\} \cup \mathcal{P}', \tag{4.4.20}$$

and each complex dimension of (A_∞,Ω_∞) lying in $\mathcal{P}' = \log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}$ is an essential singularity of $\zeta_{A_\infty,\Omega_\infty}$.

5. Fractality, complex dimensions and singularities

We close this article by specifying, within the general theory of fractal zeta functions developed here and in [LRŽ1]–[LRŽ8], the elusive notion of "fractality". Much as in [LF1]–[LF3] (see especially [LF3, §12.1 and §13.4.3]), but now using the general higher-dimensional notion of fractal zeta function and the associated notion of complex dimensions, we say that a bounded set A (or, more generally, an RFD (A,Ω)) in $\mathbb{R}^N$ is *fractal* if it has at least one nonreal (visible) complex dimension (i.e., a nonreal pole of the associated fractal zeta function) (1), relative to some screen $\boldsymbol{S}$, or else if there exists a screen $\boldsymbol{S}$ which is a (meromorphic) natural boundary for its fractal zeta function (i.e., the fractal zeta function cannot be meromorphically extended to the left of $\boldsymbol{S}$). In the latter situation, A (or, more generally, (A,Ω)) is said to be *hyperfractal*. In particular, it is *strictly hyperfractal* if we may choose $\boldsymbol{S}=\{\operatorname{Re} s = D\}$, and *maximally hyperfractal* if the critical line $\boldsymbol{S}=\{\operatorname{Re} s = D\}$ consists entirely of nonremovable singularities of the fractal zeta function; see Definition 2.38. Here, as before, we let $D := \overline{\dim}_B A$ (or $D := \overline{\dim}_B(A,\Omega)$). Recall that in Theorem 2.40, we have constructed a family of maximally hyperfractal RFDs.

5.1. Fractal and subcritically fractal RFDs. In this work, we have seen many examples of fractals (that are not hyperfractal), for instance, the Cantor string or set, the relative Sierpiński gasket and carpet (Examples 4.12 and 4.15) or, more generally, the relative N-gasket RFD and the N-carpet RFD (Examples 4.14 and 4.17), along with the examples discussed in §4.3. Among these examples, some have nonreal complex dimensions located on the critical line (such as, for instance, the Cantor string, the inhomogeneous Sierpiński gasket and carpet RFDs, the N-carpet RFD for any $N \geq 2$, as well as the inhomogeneous N-gasket (A_N,Ω_N) when $N = 2$ or 3). These are called *critically fractal*. Yet others only have nonreal complex dimensions with real parts strictly less than D. The latter are called *subcritically fractal*. In addition, *strictly subcritical fractals* are subcritical fractals which do not have any nonreal principal complex dimensions (i.e., complex dimensions with real part D). Examples of strictly subcritical fractals include the inhomogeneous Sierpiński N-gasket RFD when $N \geq 4$ (Example 4.12), the 1/3-square fractal (Example 4.25), a self-similar fractal nest (Example 4.26), as well as the modified devil's staircase (or Cantor graph) RFD to be discussed in Example 5.1 below.

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(1) Provided $D := \overline{\dim}_B A$ (resp., $\overline{\dim}_B(A,\Omega)$) $< N$, it does not matter whether we use ζ_A or $\tilde{\zeta}_A$ (resp., $\zeta_{A,\Omega}$ or $\tilde{\zeta}_{A,\Omega}$) throughout this definition.

Finally, we complete this list of definitions by stating that, given $d \in \mathbb{R}$, the bounded set A (or, more generally, the RFD (A, Ω)) is *fractal in dimension* d if it has nonreal complex dimensions of real parts d. (In light of Theorem 2.1, we then necessarily have $d \leq N$.) Hence, a critical fractal is such that $d := D$, while a strictly subcritical fractal is one with $d < D$. For instance, with the notation of Example 4.14, for $N \geq 4$, the Sierpiński N-gasket RFD (A_N, Ω_N) is fractal in dimension

$$d = \sigma_0 = \log_2(N+1) < D = N - 1 = \dim_B(A_{N,0}, \Omega_{N,0}) \tag{5.1.1}$$

but not in dimension D, and therefore it is strictly subcritically fractal. By contrast, when $N = 2$ or 3, it is critically fractal (indeed, in those cases, it is fractal in dimension $d := D = \sigma_0$, the similarity dimension).

We point out that, much as in the one-dimensional situation of [LF3, Chapter 12], based on the general explicit formulas and fractal tube formulas obtained in [LF1]–[LF3] (see especially [LF3, Chapters 5 and 8]), the definitions of fractality, critical fractality and (strict) subcritical fractality are justified in part by the general fractal tube formulas obtained in [LRŽ5] (see also [LRŽ4] and [LRŽ1, Chapter 5]) (2). Indeed, the latter tube formulas show that, under mild assumptions, the presence of nonreal complex dimensions of real part $d \in \mathbb{R}$ corresponds to oscillations of order d in the geometry of A (or of (A, Ω)). Similarly, roughly speaking, critical fractality (along with the simplicity of D) corresponds to the Minkowski nonmeasurability of A (or (A, Ω)), while strict subcritical fractality (still assuming the simplicity of D) not only corresponds to (critical) Minkowski measurability but also to (strictly subcritical) Minkowski nonmeasurability in dimension $d < D$. (See also [LRŽ6].) This is the case, for instance, for the inhomogeneous Sierpiński N-gasket RFD (Example 4.12) whenever $N \geq 4$ (and avoiding nongeneric values of N), for the RFDs of Examples 4.25 and 4.26, as well as for the (modified) devil's staircase RFD, which we discuss in Example 5.1 below.

Finally, it follows from the discussion in Remark 4.23 (and from Theorem 4.21) that, under mild assumptions on their generators (3), self-similar sprays (in the sense of Definition 4.18) are fractal in dimension d for only a finite (but nonempty) set of values of d in the lattice case, whereas they are fractal in dimension d for an infinite countable and dense set of values of d in the nonlattice case. More specifically, the set of d's for which nonlattice (respectively, generic nonlattice) self-similar RFDs are fractal in dimension d is dense in finitely many nonempty compact intervals (respectively, in a single compact interval of the form $[D_l, D]$, where $D_l \in \mathbb{R}$ and $D_l < D$) (4). More generally, we conjecture

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(2) These fractal tube formulas generalize to any $N \geq 1$ and to arbitrary bounded sets A (or, more generally, RFDs) in $\mathbb{R}^N$ the ones obtained for fractal strings (i.e., when $N = 1$) in [LF1]–[LF3] (see especially [LF3, Chapter 8]), as well as for the very special but important higher-dimensional case of fractal sprays, in [LP1]–[LP3] and, more generally, in [LPW1, LPW2] (see [LF3, §13.1] for an exposition).

(3) It suffices to assume that the base RFD $(\partial G, G)$ is "nonfractal" (so that it does not have any nonreal complex dimensions) and "sufficiently nice" (so that $\zeta_{\partial G, G}$ has a meromorphic continuation to all of $\mathbb{C}$). Both conditions are satisfied, for instance, if G is the interior of a convex polytope, which is the case for essentially all of the classical examples.

(4) We refer to Remark 4.23 for the definitions of the terms "lattice", "nonlattice" and "generic nonlattice", as well as for the appropriate references.

that under suitable mild hypotheses, self-similar RFDs and sets satisfying the open set condition enjoy the same properties.

5.2. The Cantor graph relative fractal drum. Recall that in the introduction (i.e., in §1) we have discussed the Cantor graph (or devil's staircase) in relation with the notion of fractality. We end this chapter by considering a closely related example, namely the Cantor graph RFD.

Example 5.1 (The Cantor graph RFD). In this example, we compute the distance zeta function of the RFD (A,Ω) in $\mathbb{R}^2$, where A is the graph of the Cantor function and Ω is the union of triangles $\triangle_k$ that lie above and the triangles $\tilde{\triangle}_k$ that lie below each of the horizontal parts of the graph denoted by B_k. (At each step of the construction there are 2^{k-1} mutually congruent triangles $\triangle_k$ and $\tilde{\triangle}_k$.) Each of these triangles is isosceles, has for one of its sides a horizontal part of the Cantor function graph, and has a right angle at the left end of B_k, in the case of $\triangle_k$, or at the right end of B_k, in the case of $\tilde{\triangle}_k$. (See Figure 1.)

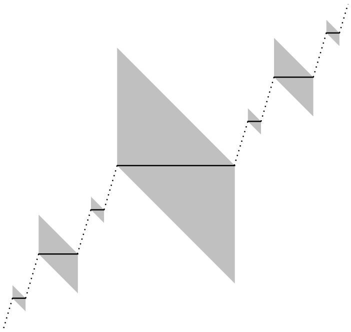

Fig. 1. The third step in the construction of the Cantor graph relative fractal drum (A,Ω) from Example 5.1. One can see, in particular, the sets B_k, $\triangle_k$ and $\tilde{\triangle}_k$ for $k=1,2,3$.

For obvious geometric reasons and by using the scaling property of the relative distance zeta function of the resulting RFD (A,Ω) (see Theorem 2.16), we then have the following identity:

$$\begin{aligned}\zeta_{A,\Omega}(s) &= \sum_{k=1}^{\infty} 2^k \zeta_{B_k,\triangle_k}(s) = \sum_{k=1}^{\infty} 2^k \zeta_{3^{-k}B_1,3^{-k}\triangle_1}(s)\\ &= \zeta_{B_1,\triangle_1}(s)\sum_{k=1}^{\infty}\frac{2^k}{3^{ks}} = \frac{2\zeta_{B_1,\triangle_1}(s)}{3^s-2},\end{aligned} \tag{5.2.1}$$

valid for $\operatorname{Re} s$ sufficiently large. Here, $(B_1,\triangle_1)$ is the relative fractal drum described above with two perpendicular sides of length 1. It is straightforward to compute its relative distance zeta function:

$$\zeta_{B_1,\triangle_1}(s) = \int_0^1 \mathrm{d}x \int_0^x y^{s-2}\,\mathrm{d}y = \frac{1}{s(s-1)} \tag{5.2.2}$$

for $\operatorname{Re} s > 1$, and after meromorphic continuation for all $s \in \mathbb{C}$. This fact, combined with (the last equality of) (5.2.1), yields the distance zeta function of (A,Ω), which is clearly meromorphic on all of $\mathbb{C}$:

$$\zeta_{A,\Omega}(s)=\frac{2}{s(3^s-2)(s-1)}\quad\text{for all } s\in\mathbb{C}. \tag{5.2.3}$$

Therefore the set of complex dimensions of the RFD (A,Ω) is given by

$$\mathcal{P}(\zeta_{A,\Omega}):=\mathcal{P}(\zeta_{A,\Omega},\mathbb{C})=\{0,1\}\cup\left(\log_3 2+\frac{2\pi}{\log 3}\mathrm{i}\mathbb{Z}\right), \tag{5.2.4}$$

with each complex dimension being simple. Hence, the set of principal complex dimensions is

$$\dim_{PC}(A,\Omega):=\mathcal{P}_c(\zeta_{A,\Omega})=\{1\}. \tag{5.2.5}$$

We conclude from Theorem 2.1(b) that $\dim_B(A,\Omega)=1$ and that (A,Ω) is Minkowski measurable. Moreover, one also deduces from [LRŽ6, Theorem 4.2] that the (one-dimensional) Minkowski content of (A,Ω) is given by

$$\mathcal{M}^1(A,\Omega)=\frac{\operatorname{res}(\zeta_{A,\Omega},1)}{2-1}=2, \tag{5.2.6}$$

which coincides with the length of the Cantor graph (i.e., the graph of the Cantor function, also called the devil's staircase in [Man]).

In what follows, we associate the RFD $(A,A_{1/3})$ in $\mathbb{R}^2$ to the classical Cantor graph. We do not know if the right-hand side of (5.2.4) coincides with the set of complex dimensions of the "full" graph of the Cantor function (i.e., the original devil's staircase), or equivalently, the RFD $(A,A_{1/3})$, but we expect that this is indeed the case since (A,Ω) is a "relative fractal subdrum" of $(A,A_{1/3})$. Moreover, it clearly follows from the construction of (A,Ω) that for the distance zeta function of $(A,A_{1/3})$ associated with the graph of the Cantor function, we have

$$\zeta_{A,A_{1/3}}(s)=\zeta_{A,\Omega}(s)+\zeta_{A,A_{1/3}\setminus\Omega}(s). \tag{5.2.7}$$

In order to prove that $\mathcal{P}(\zeta_{A,\Omega})$, given by (5.2.4), is a subset of the set of complex dimensions of the "full" Cantor graph, it would therefore remain to show that $\zeta_{A,A_{1/3}\setminus\Omega}$ has a meromorphic continuation to some connected open neighborhood U of the critical line $\{\operatorname{Re} s=1\}$ such that U contains the set of complex dimensions of (A,Ω), as given by (5.2.4), and that there are no pole-pole cancellations in the right-hand side of (5.2.7).

We now return to the RFD (A,Ω) (that is, the Cantor graph relative fractal drum). It follows from (5.2.4) that (A,Ω) is fractal in our sense. More specifically, in light of (5.2.5), *it is not critically fractal* (because its only complex dimension of real part D_{CG} $(=\overline{D}=\dim_B(A,\Omega))=1$ is 1 itself, the Minkowski dimension of the Cantor graph RFD, and it is simple) *but it is strictly subcritically fractal.* In fact, it is subcritically fractal in a single dimension, namely in dimension $d:=D_{CS}=\log_3 2$, the Minkowski dimension of the Cantor set.

We expect the exact same statements to be true for the devil's staircase itself (i.e., the "full" graph of the Cantor function), represented by the RFD $(A,A_{1/3})$ and of which (A,Ω) is a "relative fractal subdrum", as explained above. Clearly, in light of (5.2.7) and

(5.2.4), we have the following inclusions (between multisets):

$$\mathcal{P}(\zeta_{A,A_{1/3}}) \subseteq \mathcal{P}(\zeta_{A,\Omega}) \cup \mathcal{P}(\zeta_{A,A_{1/3}\setminus\Omega}) \subseteq \{0,1\} \cup \left\{D_{CS} + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right\}. \tag{5.2.8}$$

Also, we know that $\dim_B(A, A_{1/3})$ exists and

$$D(\zeta_{A,A_{1/3}}) = \dim_B(A, A_{1/3}) = 1, \tag{5.2.9}$$

so that

$$\dim_{PC}(A, A_{1/3}) := \mathcal{P}_c(\zeta_{A,A_{1/3}}) = \{1\}. \tag{5.2.10}$$

(Thus, $\{1\} \subseteq \mathcal{P}(\zeta_{A,A_{1/3}})$ in (5.2.8).) Note that (5.2.9) (and hence (5.2.10)) follows from the rectifiability of the devil's staircase, combined with a well-known result in [Fe2] and with Theorem 2.1(b).

As was conjectured in [LF3, §12.1.2 and §12.3.2], based on an "approximate tube formula", we expect that $\mathcal{P}(\zeta_{A,A_{1/3}}) = \mathcal{P}(\zeta_{A,\Omega})$, as given by (5.2.4), and hence that we actually have equalities instead of inclusions in (5.2.8), even equalities between multisets. If so, the "full" Cantor graph $(A, A_{1/3})$ is fractal, not critically fractal, but (strictly) subcritically fractal in the single dimension $d := D_{CS} = \log_3 2$.

In his celebrated book, Mandelbrot reluctantly defined "fractality" by the property that a geometric object has Hausdorff dimension strictly greater than (i.e., different from) its topological dimension [Man, p. 15]. However, he was aware of an obvious counterexample to his definition; namely, the Cantor graph (or devil's staircase, depicted in [Man, plate 83, p. 83]), for which all the notions of fractal dimensions (Hausdorff, Minkowski, etc.) coincide with the topological dimension (i.e., one). In this regard, he stated in [Man, p. 82] about the devil's staircase that "one would love to call the present curve a fractal, but to achieve this goal, we would have to define fractals less stringently, on the basis of notions other than [the Hausdorff dimension] alone."

The above paradox has puzzled the first author from the very beginning and was one of the key motivations for the development of the mathematical theory of complex dimensions, eventually in [LF1]–[LF3], for fractal strings (i.e., when $N = 1$), and in higher dimensions in [LRŽ1]–[LRŽ8]. If we replace the (full) Cantor graph $(A, A_{1/3})$ by the Cantor graph RFD (A, Ω), then the paradox is completely resolved since, as was discussed above, (A, Ω) is "fractal", in the sense of the theory of complex dimensions. Nevertheless, the fact that (A, Ω) is strictly subcritically fractal (i.e., does not have nonreal principal complex dimensions, that is, with real part 1, but has nonreal complex dimensions with real part < 1, namely, on the vertical line $\{\operatorname{Re} s = \log_3 2\}$) shows that the issue at hand is rather subtle.

Since, according to the above discussion, the (full) Cantor graph (or devil's staircase) is also expected to be fractal (as well as strictly subcritically fractal), the original paradox should itself be completely resolved in the future within the present theory of complex dimensions of relative fractal drums. Naturally, we expect that many other apparent paradoxes can be similarly resolved within the theory developed in this paper and in [LRŽ1]–[LRŽ8].

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