ACTA ARITHMETICA
181.3 (2017)

The first moment of cusp form L-functions in weight aspect on average

by

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1. Introduction. The mollifier method allows proving strictly positive non-vanishing results for different families of L-functions. See, for example, [1, 2, 4, 6–10, 12]. The exact proportion of non-zero L-values depends on a parameter M, called mollifier length. Therefore, it is of crucial importance to optimize the value of M.

We consider the family $H_{2k}(1)$ of primitive forms of level 1 and weight $2k \geq 12$. Every $f \in H_{2k}(1)$ has a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(2k-1)/2} e(nz). \tag{1.1}$$

The associated L-function is defined by

$$L_f(s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s}, \qquad \Re s > 1. \tag{1.2}$$

Let $\Gamma(s)$ be the Gamma function. The completed L-function

$$\Lambda_f(s) = \left(\frac{1}{2\pi}\right)^s \Gamma\left(s + \frac{2k-1}{2}\right) L_f(s) \tag{1.3}$$

satisfies the functional equation

$$\Lambda_f(s) = \epsilon_f \Lambda_f(1-s), \qquad \epsilon_f = i^{2k}, \tag{1.4}$$

and can be analytically continued onto the whole complex plane.

2010 *Mathematics Subject Classification*: Primary 11F11.
Key words and phrases: primitive forms, L-functions, weight aspect, Legendre polynomials.
Received 7 July 2016; revised 3 August 2017.
Published online 20 November 2017.

DOI: 10.4064/aa8569-8-2017

The *harmonic summation* is defined by

$$\sum_{f\in H_{2k}(1)}^{h} \alpha_f := \sum_{f\in H_{2k}(1)} \alpha_f \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}\langle f,f\rangle_1}, \tag{1.5}$$

where $\langle f,f\rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

The usual choice of mollifier is

$$M(f) = \sum_{m\le M} \frac{x_m\lambda_f(m)}{m^{1/2}}, \qquad x_m\in\mathbb{R}. \tag{1.6}$$

Let h be a suitable test function (see Section 5 for details) and

$$H := \int_0^\infty h(y)\,dy. \tag{1.7}$$

Let $\mu(m)$ be the Möbius function and $\sigma(m)$ be the sum of divisors function. Iwaniec and Sarnak [6, Theorem 3] proved that for the mollifier with

$$x_m \sim \frac{\mu(m)m(\log M/\log m)^2}{\sigma(m)2\zeta(2)\log M} \tag{1.8}$$

of length $M\le K(\log K)^{-20}$, one has

$$\sum_k h\left(\frac{2k}{K}\right) \sum_{f\in H_{2k}(1)}^{h} L_f(1/2)M(f) \sim HK, \tag{1.9}$$

$$\sum_k h\left(\frac{2k}{K}\right) \sum_{f\in H_{2k}(1)}^{h} L_f^2(1/2)M^2(f) \sim 2HK\left(1+\frac{\log K}{\log M}\right). \tag{1.10}$$

Let $M := K^{\Delta}$, where Δ is called the *logarithmic mollifier length*. Note that if $\epsilon_f = i^{2k} = -1$, then it follows from the functional equation (1.4) that $L_f(1/2)$ is identically zero. For $\epsilon_f = 1$ equations (1.9), (1.10) imply (see [2] for details) that at least

$$\frac{\Delta}{\Delta+1} \tag{1.11}$$

of central L-values do not vanish on average as $K\to\infty$.

For the largest admissible $\Delta = 1-\epsilon$, the percentage of non-vanishing L-values is no less than 50%. Furthermore, according to [6, Proposition 16] any improvement over 50% with an additional lower bound on $L_f(1/2)$ would imply the non-existence of Landau–Siegel zeros for Dirichlet L-functions of real primitive characters.

In order to break the 50% barrier one needs to increase the mollifier length for both first and second moments.

In the present paper we consider only the first moment and show that equation (1.9) holds for the mollifier length $M \le K^{2-\epsilon}$ for any $\epsilon > 0$. This extension follows from the asymptotic formula for the twisted first moment.

THEOREM 1.1. *For all l one has*

$$(1.12)\qquad M_1(l) := \sum_k h\left(\frac{4k}{K}\right) \sideset{}{^h}\sum_{f\in H_{4k}(1)} \lambda_f(l) L_f(1/2) = \frac{2}{\sqrt{l}}\frac{HK}{4} + O\left(K\frac{l^{1/2+\epsilon}}{K^2}\right).$$

More precisely, the mollified moment can be expressed in terms of the twisted moment,

$$\sum_k h\left(\frac{4k}{K}\right) \sideset{}{^h}\sum_{f\in H_{4k}(1)} L_f(1/2) M(f) = \sum_{m\le M} \frac{x_m}{\sqrt{m}} M_1(m).$$

Then the mollifier length is the largest admissible M such that

$$K \sum_{m\le M} \frac{|x_m|}{\sqrt{m}} \frac{m^{1/2+\epsilon}}{K^2} \ll K^{1-\epsilon}.$$

Therefore, for any mollifier with $|x_m| \le \log M$ and any $\epsilon > 0$ one can take $M \le K^{2-\epsilon}$. Consequently, the logarithmic mollifier length Δ can be extended up to 2.

A detailed description of the mollifier method and analogous results for an individual weight can be found in [2].

2. Special functions. For $z \in \mathbb{C}$, $\Re z > 0$, the Gamma function is defined by

$$(2.1)\qquad \Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, dt.$$

By [11, Eq. 5.5.5], for $2z \ne 0, -1, -2, \ldots$ one has

$$(2.2)\qquad \Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z)\Gamma(z+1/2).$$

Let

$$e(x) := \exp(2\pi i x).$$

The confluent hypergeometric function

$$(2.3)\qquad {}_1F_1(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^\infty \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{x^k}{k!}$$

can be expressed in terms of the Bessel function of the first kind

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+v+1)}\left(\frac{x}{2}\right)^{2m+v}. \tag{2.4}$$

LEMMA 2.1. *For $\epsilon = \pm 1$ one has*

$$\begin{aligned} {}_1F_1(k, 2k; 2z) = \Gamma(k+1/2)\exp(z)\left(\frac{z}{2}\right)^{1/2-k} \\ \times e\left(\epsilon\frac{1/2-k}{4}\right) J_{k-1/2}\left(ze\left(\frac{\epsilon}{4}\right)\right). \end{aligned} \tag{2.5}$$

Proof. Using [11, Eqs. 13.2.2 and 13.6.9], we write the confluent hypergeometric function in terms of the I-Bessel function. Further, applying [11, Eq. 10.27.6], we prove the required result. ∎

Legendre polynomials are nth degree polynomials given by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]. \tag{2.6}$$

Note that by [11, Eq. 14.7.17],

$$P_n(-x) = (-1)^n P_n(x). \tag{2.7}$$

LEMMA 2.2. *For any non-negative integer n one has*

$$J_{n+1/2}(z) = (-i)^n\sqrt{\frac{z}{2\pi}}\int_0^{\pi} \exp(iz\cos\theta)P_n(\cos\theta)\sin\theta\, d\theta. \tag{2.8}$$

Proof. The assertion follows from [11, Eqs. 10.47.3 and 10.54.2]. ∎

LEMMA 2.3. *Let $|\arg z| < \pi$. For a fixed $\nu \geq 0$ and $d \geq \max(\nu/2-1/4, 1)$, as $z \to \infty$ we have*

$$\begin{aligned} J_\nu(z) = \sqrt{\frac{2}{\pi z}}\Bigg(\cos(z-\pi\nu/2-\pi/4)\left[\sum_{j=0}^{d-1}(-1)^s\frac{a_{2j}(\nu)}{z^{2j}} + R_1\right] \\ - \sin(z-\pi\nu/2-\pi/4)\left[\sum_{j=0}^{d-1}(-1)^s\frac{a_{2j+1}(\nu)}{z^{2j+1}} + R_2\right]\Bigg), \end{aligned} \tag{2.9}$$

where

$$a_j(\nu) = \frac{\Gamma(j+1/2+\nu)}{2^j j!\Gamma(-j+1/2+\nu)} \quad \textit{for } j \geq 0, \tag{2.10}$$

$$R_1 = O\left(\frac{1}{(2z)^{2d}}\right), \quad R_2 = O\left(\frac{1}{(2z)^{2d+1}}\right). \tag{2.11}$$

Proof. See [5, Eq. 8.451.1] and [11, 10.17(iii), Eq. 10.17.3]. ∎

3. Exact formula for the first moment. In this section we consider the first moment of primitive L-functions and show how to express the error in terms of special functions. For $\epsilon_1 = \pm 1$ define

$$I_{\epsilon_1}(u,v,k;x) := e\left(\frac{\epsilon_1}{8} - \frac{\epsilon_1 k}{4}\right) x^{1/2-k} \frac{\Gamma(k-v-u)}{\Gamma(2k)} \times {}_1F_1\left(k-v-u, 2k, -\frac{e(-\epsilon_1/4)}{x}\right). \tag{3.1}$$

As a consequence of the Petersson trace formula we obtain the exact formula for the twisted first moment.

THEOREM 3.1. *For $2k \geq 12$, $\Re v = 0$, $|\Re u| < k-1$ we have*

$$\sum_{f\in H_{2k}(1)}^{h} \lambda_f(l) L_f(1/2+u+v) = \frac{1}{l^{1/2+u+v}} + i^{2k}\frac{(2\pi)^{2u+2v}\Gamma(k-u-v)}{l^{1/2-u-v}\Gamma(k+u+v)} + 2\pi i^{2k} V_1(l;u,v,k). \tag{3.2}$$

The error term is given by

$$V_1(l;u,v,k) = \sum_{c=1}^{\infty}\frac{1}{c^{1/2+u+v}} \sum_{\substack{n=1\\(n,c)=1}}^{\infty} \frac{e(n^* l c^{-1})}{n^{1/2-u-v}} (2\pi)^{u+v-1/2} \times e\left(\frac{1/2-u-v}{4}\right) I_{-1}\left(u,v,k;\frac{cn}{2\pi l}\right) + \sum_{c=1}^{\infty}\frac{1}{c^{1/2+u+v}} \sum_{\substack{n=1\\(n,c)=1}}^{\infty} \frac{e(-n^* l c^{-1})}{n^{1/2-u-v}} \times (2\pi)^{u+v-1/2} e\left(-\frac{1/2-u-v}{4}\right) I_{+1}\left(u,v,k;\frac{cn}{2\pi l}\right), \tag{3.3}$$

where $nn^* \equiv 1 \pmod c$.

Proof. This formula was proved in [1, Sections 4–5] for prime power level $N = p^v$, p prime, $v \geq 2$. When $N = 1$, the integrand in [1, Eq. 4.16] has a pole in view of [1, Eq. 4.15]. Consequently, we cross this pole while shifting the contour of integration in the proof of [1, Lemma 4.8]. This yields the additional main term

$$i^{2k}\frac{(2\pi)^{2u+2v}\Gamma(k-u-v)}{l^{1/2-u-v}\Gamma(k+u+v)}$$

in (3.2). The rest of the proof is exactly the same. ∎

We are interested in the behavior of the first moment at the critical point 1/2 and therefore we can let $u = v = 0$.

LEMMA 3.2. *For $\epsilon_1 = \pm 1$ one has*

$$e(-\epsilon_1/8)I_{\epsilon_1}(0,0,k;x) = \sqrt{\pi}\, e\left(\frac{\epsilon_1}{4\pi x}\right) e\left(\frac{-\epsilon_1 k}{4}\right) J_{k-1/2}\left(\frac{1}{2x}\right). \tag{3.4}$$

Proof. Substituting the representation (2.5) in (3.1), we obtain

$$\begin{aligned} e(-\epsilon_1/8)I_{\epsilon_1}(0,0,k;x) = e\left(\frac{-\epsilon_1 k}{4}\right) x^{1/2-k} \frac{\Gamma(k)\Gamma(k+1/2)}{\Gamma(2k)} \\ \times 2^{2k-1} \exp\left(-\frac{e(-\epsilon_1/4)}{2x}\right)\left(-\frac{e(-\epsilon_1/4)}{x}\right)^{1/2-k} \\ \times e\left(\epsilon_2 \frac{1/2-k}{4}\right) J_{k-1/2}\left(-\frac{e(-\epsilon_1/4)e(\epsilon_2/4)}{2x}\right), \end{aligned}$$

where $\epsilon_2 = \pm 1$. Note that $-e(-\epsilon_1/4) = \epsilon_1 i$. Choosing $\epsilon_2 = -\epsilon_1$ yields

$$-e(-\epsilon_1/4)e(\epsilon_2/4) = -\exp(\pi i) = 1.$$

Thus

$$\begin{aligned} e(-\epsilon_1/8)&I_{\epsilon_1}(0,0,k;x) \\ &= \frac{\Gamma(k)\Gamma(k+1/2)}{\Gamma(2k)} 2^{2k-1} e\left(\frac{\epsilon_1}{4\pi x}\right) e\left(\frac{-\epsilon_1 k}{4}\right) J_{k-1/2}\left(\frac{1}{2x}\right). \end{aligned}$$

It follows by (2.2) that

$$e(-\epsilon_1/8)I_{\epsilon_1}(0,0,k;x) = \sqrt{\pi}\, e\left(\frac{\epsilon_1}{4\pi x}\right) e\left(\frac{-\epsilon_1 k}{4}\right) J_{k-1/2}\left(\frac{1}{2x}\right). \quad \blacksquare$$

COROLLARY 3.3. *For $\epsilon_1 = \pm 1$ one has*

$$I_{\epsilon_1}(0,0,k;x) \ll \frac{(4x)^{-k+1/2}}{\Gamma(k+1/2)}. \tag{3.5}$$

Proof. By [11, Eq. 10.14.4],

$$|J_{k-1/2}(z)| \le \frac{(z/2)^{k-1/2}}{\Gamma(k+1/2)}.$$

Taking $z = 1/(2x)$ we get the assertion. ■

Furthermore, I_{ϵ_1} has an integral representation in terms of Legendre polynomials.

LEMMA 3.4. *For $k \equiv 0 \pmod 2$ and $\epsilon_1 = \pm 1$ one has*

$$\begin{aligned} I_{\epsilon_1}\left(0,0,k;\frac{1}{2z}\right) = -e\left(\frac{\epsilon_1}{8}\right) e\left(\frac{\epsilon_1 z}{2\pi}\right) \sqrt{2z} \\ \times \int_0^{\pi/2} \sin(z\cos\theta) P_{k-1}(\cos\theta) \sin\theta \, d\theta. \end{aligned} \tag{3.6}$$

Proof. Consider the representation (3.4) with $z := (2x)^{-1}$. Applying (2.8) with $n = k-1$, we obtain

$$e(-\epsilon_1/8)I_{\epsilon_1}\left(0,0,k;\frac{1}{2z}\right) = e\left(\frac{-\epsilon_1 k}{4}\right)e\left(-\frac{k}{4}+\frac{1}{4}\right)e\left(\frac{\epsilon_1 z}{2\pi}\right) \times \sqrt{\frac{z}{2}}\int_0^{\pi}\exp(iz\cos\theta)P_{k-1}(\cos\theta)\sin\theta\,d\theta.$$

Since k is an even integer, one has $e(-\epsilon_1 k/4)e(-k/4) = 1$ and

$$e(-\epsilon_1/8)I_{\epsilon_1}\left(0,0,k;\frac{1}{2z}\right) = e(1/4)e\left(\frac{\epsilon_1 z}{2\pi}\right) \times \sqrt{\frac{z}{2}}\int_0^{\pi}\exp(iz\cos\theta)P_{k-1}(\cos\theta)\sin\theta\,d\theta.$$

Now we split the integral as $\int_0^{\pi} = \int_0^{\pi/2} + \int_{\pi/2}^{\pi}$ and make the change of variables $\phi := \pi - \theta$ in the second integral. Property (2.7) yields

$$P_{k-1}(-\cos\phi) = (-1)^{k-1}P_{k-1}(\cos\phi) = -P_{k-1}(\cos\phi)$$

for even k. Finally, since $e(1/4) = \exp(\pi i/2) = i$ we have

$$\begin{aligned} e(1/4)\int_0^{\pi}&\exp(iz\cos\theta)P_{k-1}(\cos\theta)\sin\theta\,d\theta \\ &= i\int_0^{\pi/2} P_{k-1}(\cos\theta)\sin\theta[\exp(iz\cos\theta)-\exp(-iz\cos\theta)]\,d\theta \\ &= -2\int_0^{\pi/2} \sin(z\cos\theta)P_{k-1}(\cos\theta)\sin\theta\,d\theta. \end{aligned}$$

The assertion follows. ■

4. Asymptotic approximation of Legendre polynomials. The following theorem is obtained by taking $\alpha = \beta = 0$ in [3, Eqs. 1.1–1.3].

THEOREM 4.1 (Baratella, Gatteschi, [3]). *Let $N := n + 1/2$. Then*

$$P_n(\cos\theta)=\sqrt{\frac{\theta}{\sin\theta}}\left(J_0(N\theta)\sum_{s=0}^{m}\frac{A_s(\theta)}{N^{2s}}+\theta J_1(N\theta)\sum_{s=0}^{m-1}\frac{B_s(\theta)}{N^{2s+1}}+E_m\right), \tag{4.1}$$

where for fixed positive constants c and δ one has

$$E_m \ll \begin{cases}\theta^{1/2}N^{-2m-3/2} & \text{if } c/N \le \theta \le \pi-\delta,\\ \theta^2 N^{-2m} & \text{if } 0<\theta\le c/N.\end{cases} \tag{4.2}$$

The functions $A_s(\theta)$, $B_s(\theta)$ are analytic for $0 \le \theta < \pi$ and defined recursively, starting from $A_0(\theta) = 1$, by

$$\theta B_s(\theta) = -\frac{1}{2}A_s'(\theta) - \frac{1}{2}\int_0^\theta \frac{A_s'(t)}{t}\,dt + \frac{1}{2}\int_0^\theta f(t)A_s(t)\,dt, \tag{4.3}$$

$$A_{s+1}(\theta) = \frac{1}{2}\theta B_s'(\theta) - \frac{1}{2}\int_0^\theta t f(t)B_s(t)\,dt + \lambda_{s+1}, \tag{4.4}$$

with

$$f(t) = \frac{1}{4t^2} - \frac{1}{16\sin^2(t/2)} - \frac{1}{16\cos^2(t/2)}. \tag{4.5}$$

5. Averaging over weight. Let $h \in C_0^\infty(\mathbb{R}^+)$ be a non-negative function, compactly supported on an interval $[\theta_1, \theta_2]$ such that $\theta_2 > \theta_1 > 0$ and

$$\|h^{(n)}\|_1 \ll 1 \qquad \text{for all } n \ge 0. \tag{5.1}$$

Applying the Poisson summation and integrating by parts $a \ge 2$ times, we obtain

$$\sum_k h\left(\frac{4k}{K}\right) = \frac{HK}{4} + O\left(\frac{1}{K^a}\right), \tag{5.2}$$

where

$$H = \int_0^\infty h(y)dy.$$

In this section we prove Theorem 1.1. Namely we show that for all l one has

$$\sum_k h\left(\frac{4k}{K}\right) \sum_{f\in H_{4k}(1)}^{\mathrm{h}} \lambda_f(l)L_f(1/2) = \frac{2}{\sqrt{l}}\frac{HK}{4} + O\left(K\frac{l^{1/2+\epsilon}}{K^2}\right). \tag{5.3}$$

The main term in (5.3) is obtained by taking $u = v = 0$ in Theorem 3.1 and averaging the main terms with respect to k. Note that in (5.3) the summation is over elements of weight $4k$, and therefore Theorem 3.1 should be used with k replaced by $2k$. The same applies to all other results of Section 3.

Consider (3.3) with $u = v = 0$. Let $z := \pi l/(cn)$. We split the error term as

$$V_1(l; 0, 0, 2k) = W_1(l, k) + W_2(l, k),$$

where the summation in $W_1(l, k)$ is over c, n such that $z < k/5$ and in $W_2(l, k)$ such that $z \ge k/5$.

LEMMA 5.1. *There exists an absolute constant $C > 1$ such that*

$$\sum_k h\left(\frac{4k}{K}\right) W_1(l,k) \ll \frac{l^{1/2+\epsilon} K^\epsilon}{C^K}. \tag{5.4}$$

Proof. Let $d := cn$. Since $z < k/5$, one has $d > d_0$ with $d_0 := 5\pi l/k$. By Corollary 3.3,

$$\begin{aligned} W_1(l,k) &\ll \frac{1}{\Gamma(2k+1/2)} \sum_{d>d_0} d^{-1/2+\epsilon} \left(\frac{\pi l}{2d}\right)^{2k-1/2} \\ &\ll \frac{e^{2k}}{(2k)^{2k}} \left(\frac{\pi l}{2}\right)^{2k-1/2} \int_{d_0}^{\infty} x^{-2k+\epsilon}\, dx \\ &\ll d_0^{1/2+\epsilon} \frac{e^{2k}}{(2k)^{2k}} \left(\frac{\pi l}{2d_0}\right)^{2k-1/2} \ll l^{1/2+\epsilon} k^{-1} \left(\frac{e}{20}\right)^{2k}. \end{aligned}$$

Summing the result over k with the test function yields the assertion. ■

If $l \ll K$ with a sufficiently small implied constant, the sums over c and n in $W_2(l,k)$ are empty and the error term in (5.3) can be estimated using Lemma 5.1. Otherwise, the main contribution comes from the term involving $W_2(l,k)$, as we now show.

LEMMA 5.2. *For any $\epsilon > 0$ one has*

$$\sum_k h\left(\frac{4k}{K}\right) W_2(l,k) \ll K \frac{l^{1/2+\epsilon}}{K^2}. \tag{5.5}$$

Proof. It follows from Lemma 3.4 that

$$\begin{aligned} \sum_k h\left(\frac{4k}{K}\right) W_2(l,k) &\ll \sum_{cn \ll l/k} \frac{\sqrt{l}}{cn} \int_0^{\pi/2} \left| \sum_k h\left(\frac{4k}{K}\right) P_{2k-1}(\cos\theta) \right| \sin\theta\, d\theta \\ &\ll l^{1/2+\epsilon} \int_0^{\pi/2} \left| \sum_k h\left(\frac{4k}{K}\right) P_{2k-1}(\cos\theta) \right| \sin\theta\, d\theta. \end{aligned}$$

To approximate the Legendre polynomials we apply Theorem 4.1 with $m = 1$ and $N = 2k - 1/2$:

$$P_{2k-1}(\cos\theta) = \sqrt{\frac{\theta}{\sin\theta}} \left(J_0(N\theta)\left(1 + \frac{A_1(\theta)}{N^2}\right) + \theta J_1(N\theta) \frac{B_0(\theta)}{N} + E_1 \right),$$

where $B_0(\theta)$ and $A_1(\theta)$ are defined by (4.3), (4.4).

First, we estimate the contribution of the error term E_1 as follows:

$$l^{1/2+\epsilon}\sum_k h\left(\frac{4k}{K}\right)\int\limits_0^{\pi/2}\sqrt{\theta\sin\theta}\,|E_1|\,d\theta$$
$$\ll l^{1/2+\epsilon}\sum_k h\left(\frac{4k}{K}\right)\left(\int\limits_0^{c/N}\frac{\theta^3\,d\theta}{N^2}+\int\limits_{c/N}^{\pi/2}\frac{\theta^{3/2}\,d\theta}{N^{7/2}}\right)$$
$$\ll l^{1/2+\epsilon}\sum_k h\left(\frac{4k}{K}\right)\frac{1}{k^{7/2}}\ll K\frac{l^{1/2+\epsilon}}{K^{7/2}}.$$

For the main terms the largest contribution comes from the first summand, namely

$$\mathrm{MT}:=l^{1/2+\epsilon}\int\limits_0^{\pi/2}\left|\sum_k h\left(\frac{4k}{K}\right)J_0(N\theta)\right|\sqrt{\theta\sin\theta}\,d\theta.$$

Indeed, the other two summands have similar oscillation (see (2.9)) but they are smaller in absolute value because

$$\theta B_0(\theta)=O(\theta),\qquad A_1(\theta)=O(\theta^2).$$

Note that the oscillation in MT is only possible when $N\theta\gg 1$. Hence let us split the integral over θ into two parts at the point $t:=C/K$ for some absolute constant $C>1$. The first part is bounded by

$$M_1:=l^{1/2+\epsilon}\sum_k h\left(\frac{4k}{K}\right)\int\limits_0^t\theta\,d\theta\ll K\frac{l^{1/2+\epsilon}}{K^2}.$$

Now we estimate the second part

$$M_2:=l^{1/2+\epsilon}\int\limits_t^{\pi/2}\left|\sum_k h\left(\frac{4k}{K}\right)J_0(N\theta)\right|\sqrt{\theta\sin\theta}\,d\theta.$$

For the J-Bessel function we apply the representation (2.9). For $d\geq 1$ the contribution of R_1 and R_2 is majorized by

$$M_{2,1}:=l^{1/2+\epsilon}\sum_k h\left(\frac{4k}{K}\right)\frac{1}{k^{1/2+2d}}\int\limits_t^{\pi/2}\frac{\theta}{\theta^{1/2+2d}}\,d\theta$$
$$\ll l^{1/2+\epsilon}\frac{K}{K^{1/2+2d}}\,\frac{t^2}{t^{1/2+2d}}\ll l^{1/2+\epsilon}\frac{K}{K^2}.$$

Since $N\theta\gg 1$ it is sufficient to consider only the contribution of the first summand in (2.9), which is bounded by

$$M_{2,2}:=l^{1/2+\epsilon}\int\limits_t^{\pi/2}\sqrt{\theta}\left|\sum_k\frac{h(4k/K)}{\sqrt{2k-1/2}}\cos\left((2k-1/2)\theta-\frac{\pi}{4}\right)\right|d\theta.$$

Other summands in (2.9) have similar oscillation but are smaller in absolute value. By Poisson's summation formula the sum over k is majorized by a linear combination of expressions of the form

$$\sum_{u\in\mathbb{Z}}\int_{-\infty}^{\infty} h\left(\frac{4y}{K}\right)\frac{\exp(i2y\theta)}{\sqrt{y}}\exp(-2\pi iyu)\,dy \\ = \frac{K}{\sqrt{K}}\sum_{u\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{h(4y)}{\sqrt{y}}\exp(iyK(2\theta-2\pi u))\,dy.$$

Since $0<\theta\le\pi/2$ one has $|2\theta-2\pi u|\gg u$ for $u\neq 0$. Integrating by parts $a\ge 2$ times, we obtain

$$M_{2,2}\ll l^{1/2+\epsilon}\int_{t}^{\pi/2}\sqrt{\theta}\,\frac{K}{\sqrt{K}}\left(\frac{1}{(\theta K)^a}+\sum_{u\neq 0}\frac{1}{(Ku)^a}\right)\int_{-\infty}^{\infty}\left|\frac{\partial^a}{\partial y^a}\left(\frac{h(y)}{\sqrt{y}}\right)\right|dy\,d\theta.$$

It follows from the definition of h that

$$\int_{-\infty}^{\infty}\left|\frac{\partial^a}{\partial y^a}\left(\frac{h(y)}{\sqrt{y}}\right)\right|dy\ll 1.$$

Thus

$$M_{2,2}\ll l^{1/2+\epsilon}\frac{K}{\sqrt{K}}\int_{t}^{\pi/2}\sqrt{\theta}\,\frac{d\theta}{(\theta K)^a}\ll l^{1/2+\epsilon}\frac{K}{K^{a+1/2}}\,\frac{t^{3/2}}{t^a}\ll K\frac{l^{1/2+\epsilon}}{K^2}.$$

Finally,

$$\mathrm{MT}\ll M_1+M_2\ll M_1+M_{2,1}+M_{2,2}\ll K\frac{l^{1/2+\epsilon}}{K^2}. \blacksquare$$

Acknowledgements. The authors thank the referee for careful reading of the manuscript and for recommending several improvements in exposition.

Research of Olga Balkanova is supported by Academy of Finland project no. 293876.

Research of Dmitry Frolenkov is supported by the Russian Science Foundation under grant [14-11-00335] and performed in Khabarovsk Division of the Institute for Applied Mathematics, Far Eastern Branch, Russian Academy of Sciences.

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