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Squares in Piatetski-Shapiro sequences

by

KUI LIU (Qingdao), IGOR E. SHPARLINSKI (Sydney) and
TIANPING ZHANG (Xi'an)

1. Introduction

1.1. Motivation and formulation of the problem. *Piatetski-Shapiro sequences* (PS-sequences), that is, sequences of the form

$$\mathbb{N}^c = (\lfloor n^c \rfloor)_{n\in\mathbb{N}} \quad (c > 1,\, c \notin \mathbb{N}),$$

where $\lfloor z \rfloor$ is the integer part of a real z, have been extensively studied by many authors since their introduction by Piatetski-Shapiro [17] (see [1–6, 10, 20] and the references therein).

Here we consider the distribution of perfect squares in PS-sequences, which seems to be a new, yet natural question to study. More precisely, for a real $c > 1$ and positive integers N and s, we denote by $Q_c(s;N)$ the number of solutions to the equation

$$\lfloor n^c \rfloor = sm^2, \quad 1 \le n \le N,\, m,n \in \mathbb{Z}.$$

Clearly, we have the trivial bound

$$Q_c(s;N) \le \min\{N,\, s^{-1/2}N^{c/2}\}. \tag{1.1}$$

We use a variety of different techniques to obtain asymptotic formulas or upper bounds improving (1.1). We also study $Q_c(s;N)$ on average over positive square-free integers $s \le S$, that is, the quantity

$$\mathfrak{Q}_c(S,N) = \sum_{\substack{s\le S\\ s\ \text{square-free}}} Q_c(s;N).$$

We remark that only the case $S \le N^c$ is meaningful, hence we always assume this. Having nontrivial upper bounds on $\mathfrak{Q}_c(S,N)$ immediately implies a

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lower bound on the number of distinct square-free parts of the integers $\lfloor n^c \rfloor$, $1 \le n \le N$. In turn, this can be reformulated as a lower bound on the number of distinct quadratic fields in the sequence of fields $\mathbb{Q}(\sqrt{\lfloor n^c \rfloor})$, $1 \le n \le N$.

1.2. Notation. Among other methods, our results are also based on the square sieve of Heath-Brown [14] coupled with a bound of character sums with PS-sequences due to Baker and Banks [3]. We also employ the method of exponential sums.

Throughout the paper, as usual $U \ll V$ and $U = O(V)$ are both equivalent to the inequality $|U| \le BV$ with some constant $B > 0$, which may depend on the parameter c (and sometimes, where obvious, on some other auxiliary parameters), but it is always uniform with respect to our main parameters N, s and S.

For two quantities U and V which among other parameters also depend on N, we use $U \preccurlyeq V$ to denote that $U \le V N^{o(1)}$ as $N \to \infty$.

We also write $u \sim U$ to denote that $U < u \le 2U$.

The letters ℓ and p, with or without subscripts, always denote prime numbers.

As usual (k/q) denotes the *Jacobi symbol* modulo q, which we use only for prime q, when it is called the *Legendre symbol*, or for products of two primes.

We also use $\square$ to denote a nonspecified integer square, that is, $n = \square$ is equivalent to the statement that n is a perfect square and thus we can write

$$Q_c(s;N) = \sum_{\substack{n \le N \\ \lfloor n^c \rfloor = s\square}} 1.$$

2. Main results

2.1. Results for single s. We start with an asymptotic formula for $Q_c(s;N)$ for the values of c close to 1, which is obtained by using the method of exponential sums.

THEOREM 2.1. *For any $c > 1$, $c \notin \mathbb{N}$, let $\gamma = 1/c$. Then*

$$\begin{aligned} Q_c(s;N) - \gamma(2\gamma-1)^{-1} s^{-1/2} N^{1-c/2} \\ \ll s^{-5/19} N^{(3+5c)/19} + s^{-3/8} N^{3c/8} + s^{-1/2} N^{(3c-2)/6} \\ + s^{-1/4} N^{(2+3c)/12} + s^{-1/5} N^{(1+c)/5} \end{aligned}$$

as $N \to \infty$.

For the special case $s = 1$, this gives an asymptotic formula for $1 < c < 32/29 \approx 1.10345$ and nontrivial upper bounds of $Q_c(1;N)$ for $1 < c < 8/3 \approx 2.66667$.

For larger values of c we have a less explicit bound, which is nontrivial for any $c > 2$. This bound depends on an absolute constant $\beta(c) > 0$, depending only on c, such that for any positive integers N and q, for character sums

$$T_{c,\chi}(q;N) = \sum_{N/2<n\le N} \chi(\lfloor n^c \rfloor) \tag{2.1}$$

with a primitive Dirichlet character χ modulo q (see [16, Chapter 3] for a background on characters), we have

$$T_{c,\chi}(q;N) \ll q^{1/2} N^{1-\beta(c)}. \tag{2.2}$$

The existence of such $\beta(c)$ for any $c > 2$ of the form

$$\beta(c) = \beta/c^2 \tag{2.3}$$

with an absolute constant $\beta > 0$ is essentially a result of Baker and Banks [3, Theorem 1.6], which we also present as Lemma 3.7 below.

Theorem 2.2. *For any $s \ge 1$, $c > 2$, $c \notin \mathbb{N}$ and $\beta(c)$ satisfying* (2.2), *let $\gamma = 1/c$. Then*

$$Q_c(s;N) \ll N^{1-\beta(c)/2+o(1)}.$$

Remark 2.1. The proof of Theorem 2.2 is based on the square sieve method of Heath-Brown [14]. This seems to be the first application of this method in the context of PS-sequences for large c, where usually the method of exponential sums is valid only for small c. This has become possible because of the recent results of Baker and Banks [3].

2.2. Results on average over s. Here we show that using a result of Fouvry and Iwaniec [7, Theorem 3], when c is near to 1, we can take advantage of averaging over s and estimate the sum $\mathfrak{Q}_c(S,N)$ better than via a direct application of Theorem 2.1.

Our result, as is natural to expect, depends on the function

$$\Phi(S) = \sum_{\substack{s\le S\\ s\ \text{square-free}}} s^{-1/2}.$$

Using the well known result [11, Theorem 333]

$$\sum_{\substack{s\le t\\ s\ \text{square-free}}} 1 = \frac{6}{\pi^2} t + O(\sqrt{t})$$

and partial summation, we easily derive

$$\Phi(S) = \frac{12}{\pi^2} S^{1/2} + O(\log S). \tag{2.4}$$

Theorem 2.3. *For any $c > 1$, $c \notin \mathbb{N}$, we have*

$$\mathfrak{Q}_c(S,N) - \frac{12\gamma}{\pi^2(2\gamma-1)} S^{1/2}N^{1-c/2} \lll S^{1/5}N^{(1+2c)/5} + S^{5/8}N^{3c/8} + S^{1/8}N^{(2+3c)/8} + SN^{1-c}$$

with $\gamma = 1/c$ as $N \to \infty$.

In particular, we have:

Corollary 2.4. *For any $c > 1$, $c \notin \mathbb{N}$, any $\varepsilon > 0$ and $S \le N^{\tau(c)-\varepsilon}$, where*

$$\tau(c) = \begin{cases} (8-3c)/5 & \text{for } 1 < c \le 12/7, \\ 2(2-c) & \text{for } c > 12/7, \end{cases}$$

we have

$$\mathfrak{Q}_c(S,N) = o(N) \quad \text{as } N \to \infty.$$

Clearly Corollary 2.4 is nontrivial only for $1 < c < 2$.

Remark 2.2. To get a nontrivial upper bound on $\mathfrak{Q}_c(S,N)$ for large c, we use the square sieve again as in the proof of Theorem 2.2. This is supplemented by another argument which allows us to take advantage of the additional averaging over s.

Theorem 2.5. *For any $c > 2$, $c \notin \mathbb{N}$ and $\beta(c)$ satisfying (2.2), we have*

$$\mathfrak{Q}_c(S,N) \lll SN^{1-\beta(c)} + S^{3/4}N^{1-\beta(c)/2} \quad \text{as } N \to \infty.$$

In particular:

Corollary 2.6. *For any $c > 2$, $c \notin \mathbb{N}$, $\beta(c)$ satisfying (2.2), and any $\varepsilon > 0$, for $S \le N^{2\beta(c)/3-\varepsilon}$, we have*

$$\mathfrak{Q}_c(S,N) = o(N) \quad \text{as } N \to \infty.$$

Corollaries 2.4 and 2.6 cover the full range $c > 1$, $c \notin \mathbb{N}$, provided (2.2) holds. Hence, combining this with (2.3), which we have by Lemma 3.7 below, we obtain:

Corollary 2.7. *For any $c > 1$, $c \notin \mathbb{N}$, there exists a constant $\vartheta(c) > 0$ such that the square-free parts of almost all integers of the type $\lfloor n^c \rfloor$, $n \le N$, are larger than $N^{\vartheta(c)}$.*

3. Preparations

3.1. Some general statements. As usual, we define the function $\psi(u) = u - \lfloor u \rfloor - 1/2$. We use the following result of Vaaler [21] (see also [9, Theorem A.6]).

Lemma 3.1. *Let $H \geq 1$. There are functions $a(h)$ and $b(h)$ such that for $1 \leq |h| \leq H$ we have*

$$a(h) \ll 1/|h|, \qquad b(h) \ll 1/H,$$

and

$$\Big|\psi(t) - \sum_{1\leq |h|\leq H} a(h)e(ht)\Big| \leq \sum_{|h|\leq H} b(h)e(ht).$$

Note that we can take explicitly

$$a(h) = (2\pi i h)^{-1} F\Big(\frac{h}{H+1}\Big) \quad \text{and} \quad b(h) = \frac{1}{2H+2}\Big(1 - \frac{|h|}{H+1}\Big),$$

with $F(u) = \pi u(1-|u|)\cot(\pi u)$. We also remark that the right hand side of the inequality in Lemma 3.1 is a real nonnegative number, so no absolute value symbol is necessary. It is important to notice that the summation on the right hand side also includes $h = 0$.

We need the following technical result [9, Lemma 2.4].

Lemma 3.2. *Let*

$$L(Z) = \sum_{i=1}^{u} A_i Z^{a_i} + \sum_{j=1}^{v} B_j Z^{-b_j},$$

where A_i, B_j, a_i and b_j are positive. Let $0 \leq Z_1 \leq Z_2$. Then there is some $Z \in (Z_1, Z_2]$ with

$$L(Z) \ll \sum_{i=1}^{u}\sum_{j=1}^{v} (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{u} A_i Z_1^{a_i} + \sum_{j=1}^{v} B_j Z_2^{-b_j},$$

where the implied constant depends only on u and v.

We also need the following third derivative test for mean values of exponential sums [18, Theorem 1].

Lemma 3.3. *Let $M \geq 1$ and λ be positive real numbers and let H be a positive integer. If $f : [1, M] \to \mathbb{R}$ is a real valued function with three continuous derivatives, which satisfies*

$$\lambda \leq |f^{(3)}(x)| \ll \lambda \quad \textit{for } 1 \leq x \leq M,$$

then for the sum

$$S = \frac{1}{H}\sum_{h=H+1}^{2H}\Big|\sum_{m=1}^{M_h} e\Big(\frac{h}{H} f(m)\Big)\Big|,$$

where the integer M_h satisfies $1 \leq M_h \leq M$ for each $h \in [H+1, 2H]$, we have

$$S \lll M\lambda^{1/6}H^{-1/9} + M\lambda^{1/5} + M^{3/4} + \lambda^{-1/3} \quad \textit{as } M \to \infty.$$

The following is a form of the square sieve of Heath-Brown [14] which is given by Friedlander and Iwaniec [8, Proposition 3.1] (combined with the trivial observation that if for some integers r and s we have $r = s\square$ then $rs = \square$).

Lemma 3.4. *Let a_r, $r = 1, \ldots, R$, be an arbitrary finite sequence of nonnegative real numbers and let $P \geq 2$. Then*

$$\sum_{\substack{r=1\\ r=s\square}}^{R} a_r \leq 10P^{-2} \sum_{r=1}^{R} a_r \bigg(\bigg(\sum_{P<p\leq 2P} \left(\frac{sr}{p}\right) \log p \bigg)^2 + (\log sr)^2 \bigg).$$

One can apply Lemma 3.4 directly to $Q_c(s;N)$ but it is technically easier to work with dyadic intervals, so we define

$$Q_c^*(s;N) = \sum_{\substack{N/2<n\leq N\\ \lfloor n^c \rfloor = s\square}} 1. \tag{3.1}$$

Taking a_r to be the characteristic function of the event $r = \lfloor n^c \rfloor$ for some positive integer $N/2 < n \leq N$ we obtain:

Corollary 3.5. *For any positive integers N, $P \geq 2$ and s, we have*

$$Q_c^*(s;N) \ll P^{-2} \sum_{N/2<n\leq N} \bigg(\sum_{P<p\leq 2P} \left(\frac{s\lfloor n^c \rfloor}{p}\right) \log p \bigg)^2 + P^{-2} N \log^2 N.$$

We need the following mean value estimate for real character sums, which is Theorem 1 of [15] (see also [16, Theorem 7.20]).

Lemma 3.6. *For any integers $M, N \geq 1$ and complex numbers a_n, $n = 1, \ldots, N$, we have*

$$\sum_{\substack{m\leq M\\ m \text{ square-free}}} \bigg| \sum_{\substack{n\leq N\\ n \text{ square-free}}} a_n \left(\frac{n}{m}\right) \bigg|^2 \leq (MN)^{o(1)} (M+N) \sum_{n\leq N} |a_n|^2$$

as $MN \to \infty$.

3.2. Character sums with PS-sequences. We now recall the following bound on the sums $T_{c,\chi}(q;x)$ defined by (2.1), given by Baker and Banks [3, Theorem 1.6] (used with $y = x = N/2$), which is nontrivial for any $c > 2$ (provided that N is sufficiently large compared to q).

Lemma 3.7. *Let $N \geq 2$ and $q \geq 3$. Then for $c > 2$, $c \notin \mathbb{N}$, there exists an absolute constant $\beta > 0$ such that*

$$T_{c,\chi}(q;N) \ll q^{1/2} N^{1-\beta/c^2}.$$

In particular, Lemma 3.7 shows that (2.3) is satisfied for some $\beta > 0$, and thus the assumption (2.2) is not void.

There is no doubt that the value of β in Lemma 3.7 can be explicitly evaluated.

3.3. Exponential sums with monomials. We need the following bound due to Fouvry and Iwaniec [7, Theorem 3]. We remark that the more recent bound of Robert and Sargos [19, Theorem 1] does not bring any improvement to our results (as the bounds of [7, Theorem 3] and [19, Theorem 1] have some common terms, and these are exactly the terms that dominate in our applications).

LEMMA 3.8. *Let $\alpha, \alpha_1, \alpha_2$ be real constants such that*

$$\alpha \neq 1 \quad \text{and} \quad \alpha\alpha_1\alpha_2 \neq 0.$$

Let $M, M_1, M_2, x \geq 1$ and let

$$\varPhi = (\varphi_m)_{m\sim M} \quad \text{and} \quad \varPsi = (\psi_{m_1,m_2})_{m_1\sim M_1, m_2\sim M_2}$$

be two sequences of complex numbers supported on $m \sim M$, $m_1 \sim M_1$ and $m_2 \sim M_2$ with $|\varphi_m| \leq 1$ and $|\psi_{m_1,m_2}| \leq 1$. Then for the sum

$$S_{\varPhi,\varPsi}(x; M, M_1, M_2) = \sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} \varphi_m \psi_{m_1,m_2} e\left(x\frac{m^{\alpha}m_1^{\alpha_1}m_2^{\alpha_2}}{M^{\alpha}M_1^{\alpha_1}M_2^{\alpha_2}}\right),$$

we have

$$\begin{aligned} S_{\varPhi,\varPsi}&(x; M, M_1, M_2) \\ &\ll \big(x^{1/4}M^{1/2}(M_1M_2)^{3/4} + M^{7/10}M_1M_2 + M(M_1M_2)^{3/4} \\ &\qquad + x^{-1/4}M^{11/10}M_1M_2\big)\log^2(2MM_1M_2). \end{aligned}$$

4. Proofs of main results

4.1. Proof of Theorem 2.1. For $c > 1$, $c \notin \mathbb{N}$, let $\gamma = 1/c$. It is easy to see that $\lfloor n^c \rfloor = s\square$ if and only if

$$(s\square)^{\gamma} \leq n < (s\square + 1)^{\gamma}.$$

We also set $M = s^{-1/2}N^{c/2}$. Using a similar argument to that in Heath-Brown [13] (and in many other works on PS-sequences), we have

$$Q_c(s; N) = \sum_{m\leq M} \big(\lfloor -s^{\gamma}m^{2\gamma}\rfloor - \lfloor -(sm^2+1)^{\gamma}\rfloor\big) + O(1).$$

Let

$$\psi(x) = x - \lfloor x \rfloor - 1/2.$$

Then we obtain

$$(4.1)\qquad Q_c(s;N) = \sum_{m\le M}((sm^2+1)^\gamma - s^\gamma m^{2\gamma}) - \sum_{m\le M}\psi(-s^\gamma m^{2\gamma}) + \sum_{m\le M}\psi(-(sm^2+1)^\gamma) + O(1).$$

The first term on the right side is

$$(4.2)\qquad \sum_{m\le M}((sm^2+1)^\gamma - s^\gamma m^{2\gamma}) = \sum_{m\le M} s^\gamma m^{2\gamma}(\gamma m^{-2}s^{-1} + O(m^{-4}s^{-2})) = \gamma(2\gamma-1)^{-1}s^{\gamma-1}(N^{c/2}s^{-1/2})^{2\gamma-1} + O\left(1\right) = \gamma(2\gamma-1)^{-1}N^{1-c/2}s^{-1/2} + O\left(1\right),$$

which gives the desired main term.

Now we need to estimate the other two sums with the ψ-functions. We only estimate the first sum with $\psi(-s^\gamma m^{2\gamma})$; the other with $\psi(-(sm^2+1)^\gamma)$ can be treated similarly and admits the same upper bound.

We now fix some parameter $H \ge 2$ and using Lemma 3.1, we obtain

$$(4.3)\qquad \sum_{m\le M}\psi(-s^\gamma m^{2\gamma}) \ll |E_1(N,H,c)| + |E_2(N,H,c)| + H^{-1}s^{-1/2}N^{c/2},$$

where

$$E_1(N,H,c) = \sum_{m\le M}\sum_{0<|h|\le H} a(h)e(-hs^\gamma m^{2\gamma}),$$
$$E_2(N,H,c) = \sum_{m\le M}\sum_{0<|h|\le H} b(h)e(-hs^\gamma m^{2\gamma})$$

(the term $H^{-1}s^{-1/2}N^{c/2}$ corresponds to the choice $h=0$ in the summation on the right hand side in Lemma 3.1). We deal with $E_1(N,H,c)$ first. Switching the summations, we have

$$(4.4)\qquad E_1(N,H,c) \ll \log^2(HM) \max_{\substack{1\ll T\ll H\\ 1\ll L\ll M}} S(T,L),$$

where

$$S(T,L) = \frac{1}{T}\sum_{h\sim T}\Big|\sum_{m\sim L} e(hs^\gamma m^{2\gamma})\Big|.$$

By Lemma 3.3 with $f(m) = Ts^\gamma(m+L)^{2\gamma}$ and thus

$$\lambda = 4\gamma(\gamma-1)(2\gamma-1)Ts^\gamma(2L)^{2\gamma-3},$$

we obtain

$$\begin{aligned}S(T,L) \lll\, & T^{1/18}s^{\gamma/6}L^{(3+2\gamma)/6} + T^{1/5}s^{\gamma/5}L^{(2+2\gamma)/5}\\ & + L^{3/4} + T^{-1/3}s^{-\gamma/3}L^{(3-2\gamma)/3}.\end{aligned}$$

Since

$$\begin{aligned}&L^{(3+2\gamma)/6} \ll s^{-(3+2\gamma)/12}N^{(2+3c)/12}, && L^{(2+2\gamma)/5} \ll s^{-(1+\gamma)/5}N^{(1+c)/5},\\ &L^{3/4} \ll s^{-3/8}N^{3c/8}, && L^{1-2\gamma/3} \ll s^{-(3-2\gamma)/6}N^{(3c-2)/6},\end{aligned}$$

this implies

$$\begin{aligned}E_1(N,H,c) \lll\, & H^{1/18}s^{-1/4}N^{(2+3c)/12} + H^{1/5}s^{-1/5}N^{(1+c)/5}\\ & + s^{-3/8}N^{3c/8} + s^{-1/2}N^{(3c-2)/6}.\end{aligned}$$

Similarly,

$$\begin{aligned}E_2(N,H,c) \lll\, & H^{1/18}s^{-1/4}N^{(2+3c)/12} + H^{1/5}s^{-1/5}N^{(1+c)/5}\\ & + s^{-3/8}N^{3c/8} + s^{-1/2}N^{(3c-2)/6}.\end{aligned}$$

Applying Lemma 3.2 with $(u,v) = (2,1)$, $(Z_1,Z_2) = (1,N^{c/2})$,

$$(A_1,A_2,B_1) = (s^{-1/4}N^{(2+3c)/12}, s^{-1/5}N^{(1+c)/5}, s^{-1/2}N^{c/2})$$

and

$$(a_1,a_2,b_1) = (1/18, 1/5, 1)$$

to the bounds on terms in (4.3), we obtain the following terms:

$$\begin{aligned}(A_1^{b_1}B_1^{a_1})^{1/(a_1+b_1)} &= \big(s^{-1/4}N^{(2+3c)/12}(s^{-1/2}N^{c/2})^{1/18}\big)^{18/19}\\ &= s^{-5/19}N^{(3+5c)/19},\\ (A_2^{b_1}B_1^{a_2})^{1/(a_2+b_1)} &= \big(s^{-1/5}N^{(1+c)/5}(s^{-1/2}N^{c/2})^{1/5}\big)^{5/6}\\ &= s^{-1/4}N^{(2+3c)/12}.\end{aligned}$$

Furthermore,

$$A_1Z_1^{a_1} = s^{-1/4}N^{(2+3c)/12} \quad\text{and}\quad A_2Z_1^{a_2} = s^{-1/5}N^{(1+c)/5}$$

(clearly, the term involving $Z_2 = N^{c/2}$ never dominates). Therefore

$$\begin{aligned}&\sum_{m\le s^{-1/2}N^{c/2}} \psi(-s^{\gamma}m^{2\gamma})\\ &\qquad \lll s^{-5/19}N^{(3+5c)/19} + s^{-1/4}N^{(2+3c)/12} + s^{-1/5}N^{(1+c)/5}\\ &\qquad\quad + s^{-3/8}N^{3c/8} + s^{-1/2}N^{(3c-2)/6}.\end{aligned}$$

Now the result follows from (4.1) and (4.2).

4.2. Proofs of Theorems 2.2 and 2.5. We fix some integer P with $2 \le P \le N$ (to be optimised later). It is also clear that it is enough to obtain the desired bounds for $Q_c^*(s;N)$ defined by (3.1).

Using Corollary 3.5 and then opening the square, changing the order of summation and separating the diagonal terms (with the total contribution at most $NP^{1+o(1)}$), we obtain

$$\begin{aligned} Q_c^*(s;N) &\ll P^{-2} \sum_{\substack{P<\ell,p\le 2P \\ \ell\ne p}} \log \ell \log p \left| \sum_{N/2<n\le N} \left(\frac{s\lfloor n^c \rfloor}{\ell p}\right) \right| + P^{-1}N^{1+o(1)} \\ &\lll P^{-2} \sum_{\substack{P<\ell,p\le 2P \\ \ell\ne p}} \left| \sum_{N/2<n\le N} \left(\frac{\lfloor n^c \rfloor}{\ell p}\right) \right| + P^{-1}N. \end{aligned}$$

We remark that s is not present anymore in the expression on the right hand side, and thus the estimates below are uniform in s.

Note that the Jacobi symbols here are primitive characters, thus (2.2) applies and yields

$$Q_c^*(s;N) \lll N^{1-\beta(c)}P + NP^{-1}.$$

Taking $P = N^{\beta(c)/2}$, we get

$$Q_c^*(s;N) \lll N^{1-\beta(c)/2},$$

which concludes the proof of Theorem 2.2.

To prove Theorem 2.5 we only need to consider

$$\mathfrak{Q}_c^*(S,N) = \sum_{\substack{s\le S \\ s \text{ square-free}}} Q_c^*(s;N).$$

By Corollary 3.5 again, we have

$$\mathfrak{Q}_c^*(S,N) \lll P^{-2} \sum_{\substack{P<\ell,p\le 2P \\ \ell\ne p}} \left| \sum_{\substack{s\le S \\ s \text{ square-free}}} \left(\frac{s}{\ell p}\right) \sum_{N/2<n\le N} \left(\frac{\lfloor n^c \rfloor}{\ell p}\right) \right| + P^{-1}SN.$$

Applying (2.2), we see that

$$\mathfrak{Q}_c^*(S,N) \lll P^{-1}N^{1-\beta(c)} \sum_{\substack{r\le 4P^2 \\ r \text{ square-free}}} \left| \sum_{\substack{s\le S \\ s \text{ square-free}}} \left(\frac{s}{r}\right) \right| + P^{-1}SN.$$

Now by the Cauchy inequality, Lemma 3.6 and choosing an optimal P, we get

$$\mathfrak{Q}_c^*(S,N) \lll (PS^{1/2} + S)N^{1-\beta(c)} + P^{-1}SN \ll SN^{1-\beta(c)} + S^{3/4}N^{1-\beta(c)/2},$$

which yields Theorem 2.5.

4.3. Proof of Theorem 2.3

4.3.1. *Preliminaries.* We proceed as in the proof of Theorem 2.1, so

$$\mathfrak{Q}_c(S,N) = S_0 - E_1 + E_2 + O(1), \tag{4.5}$$

where

$$S_0 = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} ((sm^2+1)^\gamma - s^\gamma m^{2\gamma})$$

contributes to the main term, and

$$E_1 = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} \psi(-s^\gamma m^{2\gamma}) \quad \text{and} \quad E_2 = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} \psi(-(sm^2+1)^\gamma)$$

contribute to the error term.

4.3.2. *Evaluation of the main term S_0.* Using (4.2), we compute S_0 directly as follows:

$$S_0 = \gamma(2\gamma-1)^{-1} N^{1-c/2} \Phi(S) + O(SN^{1-c}). \tag{4.6}$$

4.3.3. *Reductions in the error terms E_1 and E_2.* By Lemma 3.1, we obtain the following analogue of (4.3):

$$E_1 \ll |E_{11}| + |E_{12}| + H^{-1} N^{c/2} S^{1/2}, \tag{4.7}$$

where

$$E_{11} = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} \sum_{0<|h|\le H} a(h) e(-hs^\gamma m^{2\gamma}),$$

$$E_{12} = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} \sum_{0<|h|\le H} b(h) e(-hs^\gamma m^{2\gamma})$$

for some $H \ge 2$. Using the same H, we also have

$$E_2 \ll |E_{21}| + |E_{22}| + H^{-1} N^{c/2} S^{1/2}, \tag{4.8}$$

where

$$E_{21} = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} \sum_{0<|h|\le H} a(h) e(-h(sm^2+1)^\gamma),$$

$$E_{22} = \sum_{\substack{sm^2 \le N^c \\ s \le S \\ s \text{ square-free}}} \sum_{0<|h|\le H} b(h) e(-h(sm^2+1)^\gamma)$$

As usual, the sums E_{12} and E_{22} can be estimated similarly to E_{11} and E_{21}, respectively, and by partial summation, E_{21} can be converted to an exponential sum which is similar to E_{11} (see [13, Section 2] for details). In particular, we obtain the same upper bounds for E_{11}, E_{12}, E_{21} and E_{22}. Hence we only concentrate on E_{11}.

4.3.4. *Estimating* E_{11}. Using

$$\mu^2(s) = \sum_{s=rd^2} \mu(d),$$

we can write

$$E_{11} = \sum_{0<|h|\le H} a(h) \sum_{\substack{rd^2m^2\le N^c\\ rd^2\le S}} \mu(d)e(-hr^\gamma d^{2\gamma}m^{2\gamma}).$$

Then, splitting the ranges of variables into dyadic ranges, for some real positive parameters R, D and M satisfying

$$RD^2M^2 \ll N^c \quad \text{and} \quad RD^2 \ll S, \tag{4.9}$$

we obtain

$$E_{11} \lll \sum_{0<|h|\le H} h^{-1}|S(R,D,M;h)|, \tag{4.10}$$

where

$$S(R,D,M;h) = \sum_{\substack{r\sim R,\, d\sim D,\, m\sim M,\\ rd^2m^2\le N^c,\, rd^2\le S}} \mu(d)e(-hr^\gamma d^{2\gamma}m^{2\gamma}).$$

Now we estimate $S(R,D,M;h)$. Clearly we can assume that $1 < c < 2$ as for $c > 2$ the result is trivial (due to the presence of the term $S^{1/5}N^{(1+2c)/5} > N^{(1+2c)/5} \ge N$ for $c > 2$). We can remove the restrictive conditions

$$rd^2m^2 \le N^c \quad \text{and} \quad rd^2 \le S$$

at the cost of a small factor $(SN)^{o(1)}$ in a standard way (see, for example, [12, Sections 2.3 and 3.2]), which yields

$$S(R,D,M;h) \lll \Big| \sum_{r\sim R,\, d\sim D,\, m\sim M} \alpha_1(r)\alpha_2(d)\alpha_3(m)e(hr^\gamma d^{2\gamma}m^{2\gamma})\Big| + M$$

with some coefficients $|\alpha_i(n)| \le 1$ for $n \in \mathbb{N}$ and $i = 1,2,3$. Applying Lemma 3.8 to the right hand side of the above formula, we get

$$\begin{aligned} S(R,D,M;h) \lll\ & (hR^\gamma D^{2\gamma}M^{2\gamma})^{1/4}R^{1/2}(DM)^{3/4} + R^{7/10}DM \\ & + R(DM)^{3/4} + (hR^\gamma D^{2\gamma}M^{2\gamma})^{-1/4}R^{11/10}DM. \end{aligned}$$

Noting $\gamma > 1/2$, it is easy to check that the fourth term can be absorbed by the third term on the right side. Thus by conditions (4.9), we have

$$E_{11} \lll H^{1/4}S^{1/8}N^{1/4+3c/8} + S^{1/5}N^{c/2} + S^{5/8}N^{3c/8}. \tag{4.11}$$

4.3.5. *Concluding the proof.* Bound (4.11) together with (4.7) and (4.8) yields

$$|E_1| + |E_2| \lll H^{1/4}S^{1/8}N^{1/4+3c/8} + H^{-1}N^{c/2}S^{1/2} + S^{1/5}N^{c/2} + S^{5/8}N^{3c/8}.$$

Now Lemma 3.2 gives

$$|E_1| + |E_2| \lll S^{1/5}N^{(1+2c)/5} + S^{5/8}N^{3c/8} + S^{1/8}N^{1/4+3c/8}, \tag{4.12}$$

where the term $S^{1/5}N^{c/2}$ is absorbed by $S^{1/5}N^{(1+2c)/5}$, since we suppose $1 < c < 2$. Using the bound (4.12) together with (2.4), (4.5) and (4.6), and noting that the contribution of $O(\log S)$ in (2.4) can also be absorbed by $S^{1/5}N^{(1+2c)/5}$, we obtain the desired result.

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Kui Liu
School of Mathematics and Statistics
Qingdao University
No. 308, Ningxia Road, Shinan
Qingdao, Shandong 266071, P.R. China
E-mail: liukui@qdu.edu.cn

Igor E. Shparlinski
Department of Pure Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
E-mail: igor.shparlinski@unsw.edu.au

Tianping Zhang (corresponding author)
School of Mathematics and Information Science
Shaanxi Normal University
Xi'an, Shaanxi 710119, P.R. China
E-mail: tpzhang@snnu.edu.cn