

Local geodesics between toric plurisubharmonic functions with infinite energy

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Abstract. We study the behavior of weak geodesics in the space of plurisubharmonic functions on pseudoconvex domains. We investigate the continuity of geodesics between toric plurisubharmonic functions using the convex analytic method. As a result, we characterize the continuity of geodesics by means of the Lelong numbers.

1. Introduction. We study geodesics in the space of toric plurisubharmonic functions on a bounded hyperconvex Reinhardt domain.

In Kähler geometry, Mabuchi [Mab] defined a metric on the space of Kähler potentials on a compact complex manifold. Then the geodesics with respect to the Mabuchi metric have been studied. Donaldson [Don] and Semmes [Sem] proved that the geodesics correspond to solutions of a complex Monge–Ampère equation. To study them, a concept of weak geodesics is introduced as weak solutions. A weak geodesic is defined as an envelope of quasi-plurisubharmonic functions with boundary conditions (for more details, see [Dar]). As an analogue in the case of pseudoconvex domains, weak geodesics in the space of plurisubharmonic functions are studied in [Ras16]. Since weak geodesics are defined in a pluripotential-theoretic manner, it is natural to consider their analogs in pseudoconvex domains.

Let us give some definitions. Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and $S := \{1 < |\zeta| < e\} \subset \mathbb{C}$ be an annulus. Let u_0 and u_1 be negative plurisubharmonic functions on Ω . We define the function \hat{u} on $\Omega \times S$ by

$$\hat{u} := \text{usc} \left(\sup \left\{ \hat{v} \in \text{PSH}(\Omega \times S) : \begin{array}{l} \hat{v} \leq 0, \\ \limsup_{\log|\zeta| \rightarrow j} \hat{v}(z, \zeta) \leq u_j(z) \text{ for } j = 0, 1 \end{array} \right\} \right),$$

where usc means the upper semicontinuous regularization. Then \hat{u} is a

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plurisubharmonic function on $\Omega \times S$. We define

$$u_t := \hat{u}(\cdot, e^t)$$

for $0 < t < 1$ and we call the family $\{u_t\}_{t \in [0,1]}$ of plurisubharmonic functions the (weak) geodesic between u_0 and u_1 . In this paper, we will simply call it the geodesic.

When the endpoints u_0 and u_1 are in the finite energy class $\mathcal{F}_1(\Omega)$ [Ceg], the properties of such geodesics are studied in [Ras16]. In particular, geodesics are continuous at $t = 0$ and $t = 1$ in this case. We note that functions in $\mathcal{F}_1(\Omega)$ have zero Lelong numbers. The Lelong number of a plurisubharmonic function u at z is defined as $\nu(u, z) = \liminf_{\zeta \rightarrow z} \frac{u(\zeta)}{\log|\zeta - z|}$; it measures how singular u is near z . From this point of view, functions in $\mathcal{F}_1(\Omega)$ can be regarded as having weak singularities. On the contrary, in [Ras16], it is also shown that geodesics may be discontinuous at $t = 0$ or 1 when u_0 or u_1 have positive Lelong numbers. A simple example is when $\Omega = \Delta \subset \mathbb{C}$ is the unit disc, $u_0 = 0$ and $u_1 = \log|z|$. Then the geodesic between u_0 and u_1 is $u_t = \log|z|$ independent of $t \in (0, 1)$, thus it is discontinuous at $t = 0$. When u_0 and u_1 have zero Lelong numbers and may have infinite energy, the behavior was not clear from the results in [Ras16].

Our point of interest is to analyze the behavior of geodesics between plurisubharmonic functions more closely. For this purpose, we restrict our attention to the toric case, which is easy to treat using techniques of convex analysis. We say that a plurisubharmonic function is toric if it depends only on $|z_1|, \dots, |z_n|$.

Our main theorem is as follows.

THEOREM 1.1. *Let Ω be a bounded hyperconvex Reinhardt domain. Let u_0 and u_1 be plurisubharmonic functions on Ω . Assume that:*

- (i) u_0 and u_1 are toric;
- (ii) $\lim_{z \rightarrow \zeta} u_j(z) = 0$ for every $\zeta \in \partial\Omega$ ($j = 0, 1$).

Let $\{u_t\}_{0 < t < 1}$ be the geodesic between u_0 and u_1 . Then $u_t \rightarrow u_0$ in capacity as $t \rightarrow 0$ if and only if $\nu(u_0 \circ \phi, 0) \geq \nu(u_1 \circ \phi, 0)$ for every curve of the form $\phi : \zeta \mapsto (a_1 \zeta^{b_1}, \dots, a_n \zeta^{b_n})$, where $a_i \in \mathbb{C}^$ and $b_i \in \mathbb{Z}_{>0}$.*

Here, a family $\{u_t\}_{t > 0}$ of plurisubharmonic functions is said to converge to u_0 in capacity as $t \rightarrow 0$ if $\lim_{t \rightarrow 0} \text{Cap}(K \cap \{|u_t - u_0| > \epsilon\}) = 0$ for every $\epsilon > 0$ and every compact subset $K \subset \Omega$. The capacity of a Borel subset $E \subset \Omega$ is defined as

$$\text{Cap}(E) := \sup \left\{ \int_E (dd^c u)^n : u \in \text{PSH}(\Omega), -1 \leq u \leq 0 \right\}.$$

Note that the polar sets of u_0 and u_1 in Theorem 1.1 satisfy $u_j^{-1}(-\infty) \subset \{0\}$ by the assumptions.

In [Dar], a similar result is proved in the compact Kähler case. Our result can be seen as an analog in the pseudoconvex case.

As a special case, we can describe the behavior of geodesics between toric subharmonic functions on the unit disc. In this case we only have to check the Lelong numbers of u_0 and u_1 at the origin.

COROLLARY 1.2. *Let u_0 and u_1 be toric subharmonic functions on the unit disc in \mathbb{C} . Assume that u_0 and u_1 have zero boundary values. Then $u_t \rightarrow u_0$ in capacity as $t \rightarrow 0$ if and only if $\nu(u_0, 0) \geq \nu(u_1, 0)$.*

By using this corollary, we can easily construct an example of a geodesic $\{u_t\}$ such that $u_0 = 0$, u_1 has infinite energy, and $u_t \rightarrow u_0$ in capacity (see Example 3.5).

The proof of Theorem 1.1 is essentially convex-analytic. It is well-known that toric plurisubharmonic functions in n complex variables correspond to convex functions in n real variables which are increasing in each variable. In the proof, we use this fact and techniques of convex analysis. We also use a theorem proved in [Dar]. In [Dar], the behavior of geodesics has been studied in Kähler settings. A similar proof works in our situation as well.

Here we remark that the condition on Lelong numbers in Theorem 1.1 seems to have some connections with the concept of Greenifications [Ras13]. For a negative plurisubharmonic function u on a bounded pseudoconvex domain Ω , we define the *Greenification* g_u of u by

$$g_u := \text{usc}(\sup\{v \in \text{PSH}(\Omega) : v \leq 0 \text{ and } v \leq u + C \text{ for some } C\}).$$

Then the condition on Lelong numbers in Theorem 1.1 can be expressed as an inclusion of some sets called indicators in [Ras13], which implies that $g_{u_0} \leq g_{u_1}$. We expect that this result can also be generalized to general plurisubharmonic functions via this terminology.

The organization of the paper is as follows. In Section 2, the properties of toric plurisubharmonic functions, the associated convex functions and their Legendre transforms are treated. In Section 3, we prove Theorem 1.1 and give an example.

2. Toric plurisubharmonic functions, associated convex functions, and Legendre transforms. Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex Reinhardt domain with $0 \in \Omega$. A plurisubharmonic function on Ω is said to be *toric* if it depends only on $|z_1|, \dots, |z_n|$. Define

$$C_0 := \{(\log |z_1|, \dots, \log |z_n|) : (z_1, \dots, z_n) \in \Omega, z_1, \dots, z_n \neq 0\} \subset \mathbb{R}^n.$$

Note that C_0 is a convex domain in \mathbb{R}^n by assumption. By the standard argument [Dem, Chapter I, Section 5], there is a one-to-one correspondence between toric plurisubharmonic functions u on Ω and convex functions f

on C_0 which are increasing in each variable, namely

$$u(z_1, \dots, z_n) = f(\log |z_1|, \dots, \log |z_n|).$$

When $u^{-1}(-\infty) \subset \{0\}$, for any $x \in C_0$ and $1 \leq j \leq n$ we have

$$(2.1) \quad \lim_{t \rightarrow -\infty} f(x_1 + t, \dots, x_{j-1} + t, x_j, x_{j+1} + t, \dots, x_n + t) \neq -\infty.$$

As in [Gue], it is useful to consider the *Legendre transform* f^* of a function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined as

$$f^*(x^*) := \sup\{\langle x^*, x \rangle - f(x) : x \in \mathbb{R}^n\}.$$

We regard a function f on C_0 as a function on \mathbb{R}^n by defining $f(x) = \infty$ when $x \notin C_0$. For properties of the Legendre transform, see for example [Roc, Chapter 12]. Note that f^{**} is the greatest lower semicontinuous function with $f^{**} \leq f$. Note also that the Legendre transform is continuous for decreasing sequences, i.e. if f_j decreases to f pointwise, then f_j^* increases to f^* pointwise.

Here we introduce the following notation for convex envelopes. Let C be a convex subset of \mathbb{R}^n and let f and g be upper semicontinuous functions on C . We define the *convex envelope* of f as

$$P_C(f) := \sup\{u : u \text{ is a convex function on } C \text{ such that } u \leq f\}.$$

We also define $P_C(f, g) := P_C(\min(f, g))$. If the set C is clear from the context, we write $P(f)$. These functions can be written as $P(f) = f^{**}$ and $P(f, g) = (\min(f, g))^{**} = (\max(f^*, g^*))^*$.

Following [Gue], for a convex function f , we introduce a function \hat{f} representing the slope of f at infinity and a subset $\Gamma(f)$ of \mathbb{R}^n called the Newton convex body (we note that we use different signature conventions).

DEFINITION 2.1. Let f be a convex function on C_0 increasing in each variable. Fix $a \in C_0$.

(1) We define a function \hat{f} on $\mathbb{R}_{\leq 0}^n \setminus \{0\}$ by

$$\hat{f}(w) := \lim_{t \rightarrow \infty} \frac{f(a + tw)}{t}.$$

(2) We define the *Newton convex body* $\Gamma(f)$ associated to f by

$$\Gamma(f) := \{\lambda \in \mathbb{R}^n : \langle \lambda, \cdot \rangle \leq f + O(1)\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^n .

By the definition, $\Gamma(f)$ equals the domain of the Legendre transform f^* , i.e. the set $\{x^* \in \mathbb{R}^n : f^*(x^*) \in \mathbb{R}\}$. We have the following relation between \hat{f} and $\Gamma(f)$.

LEMMA 2.2 ([Gue, Lemma 1.19]). *Let f and a be as in Definition 2.1. Then*

$$\overline{\Gamma(f)} = \Gamma(\hat{f}) = \{\lambda \in \mathbb{R}^n : \langle \lambda, \cdot \rangle \leq \hat{f}\}.$$

In the next section, we use these facts to prove the main theorem.

3. Proof of the main theorem. To prove the main theorem, we use the following theorem, which is proved in [Dar] in the compact Kähler setting. The same proof works for our situation.

THEOREM 3.1 ([Dar, Theorem 5.2]). *Let Ω be a bounded hyperconvex domain. Let $u_0, u_1 \in \text{PSH}(\Omega)$ be such that $u_0, u_1 \not\equiv -\infty$. Assume that $u_0, u_1 = 0$ on $\partial\Omega$. Let $\{u_t\}_t$ be the geodesic between u_0 and u_1 . Then $\lim_{t \rightarrow 0} u_t = u_0$ in capacity if and only if $P_{[u_1]}(u_0) = u_0$. Here, $P_{[u_1]}(u_0) := (\lim_{c \rightarrow \infty} P_\Omega(u_0, u_1 + C))^*$ where $P_\Omega(u, v)$ denotes the greatest plurisubharmonic function w satisfying $w \leq \min(u, v)$.*

By the above theorem, we can rewrite Theorem 1.1 in terms of convex analysis as follows:

PROPOSITION 3.2. *Let f, g be convex functions increasing in each variable. Assume that $f = g = 0$ on ∂C_0 and f, g satisfy condition (2.1). Then the following two conditions are equivalent:*

- (1) $\lim_{c \rightarrow \infty} P_{C_0}(f, g + c) = f$ a.e. on C_0 ;
- (2) for each $a = (a_1, \dots, a_n) \in C_0$ and $b = (b_1, \dots, b_n) \in \mathbb{N}^n$, we have

$$\lim_{t \rightarrow -\infty} \frac{f(a + tb)}{t} \geq \lim_{t \rightarrow -\infty} \frac{g(a + tb)}{t}.$$

Here, ‘‘a.e.’’ in (1) is not necessary. Indeed, the left-hand side is an increasing sequence of convex functions, thus the limit is also convex. Since two convex functions on C_0 coincide almost everywhere, they are equal on all of C_0 .

3.1. 1-dimensional case. First we prove the theorem in the 1-dimensional case. In this case, we have an elementary proof.

PROPOSITION 3.3. *Let f and g be convex increasing functions on $\mathbb{R}_{\leq 0}$ bounded above. Then*

$$\lim_{c \rightarrow \infty} P_{\mathbb{R}_{\leq 0}}(f, g + c) = f \text{ on } \mathbb{R}$$

if and only if $\lim_{t \rightarrow -\infty} f(t)/t \geq \lim_{t \rightarrow -\infty} g(t)/t.$

Proof. First we assume that $\lim_{c \rightarrow \infty} P(f, g + c) = f$. Adding a constant to f and g , we can also assume that $f(0) = 0$. We note that, under this assumption, $P(f, g + c)(0) = 0$ for sufficiently large c (since g is

bounded below by a linear function). Fix $\epsilon > 0$ and $M_0 > 0$. By assumption, $P(f, g + c)(-M_0) \rightarrow f(-M_0)$ as $c \rightarrow \infty$. This is an increasing limit, thus we can choose $c_0 > 0$ satisfying $P(f, g + c_0)(-M_0) \geq f(-M_0) - \epsilon$. Then, for $M_1 > M_0$, we have

$$P(f, g + c_0)(-M_1) \geq \frac{M_1}{M_0} P(f, g + c_0)(-M_0)$$

from $P(f, g + c)(0) = 0$ and the convexity of P . Using these inequalities and $P(f, g + c_0) \leq g + c_0$, we find that

$$\begin{aligned} \frac{g(-M_1)}{-M_1} + \frac{c_0}{-M_1} &\leq \frac{P(f, g + c_0)(-M_1)}{-M_1} \leq \frac{P(f, g + c_0)(-M_0)}{-M_0} \\ &\leq \frac{f(-M_0)}{-M_0} - \frac{\epsilon}{-M_0}. \end{aligned}$$

Thus, taking $M_1 \rightarrow \infty$, we obtain

$$\lim_{M \rightarrow \infty} \frac{g(-M)}{-M} \leq \frac{f(-M_0)}{-M_0} - \frac{\epsilon}{-M_0}$$

for each ϵ and M_0 . We get the desired inequality when we take $\epsilon \rightarrow 0$ and $M_0 \rightarrow \infty$.

Conversely, assume $\lim_{t \rightarrow -\infty} f(t)/t \geq \lim_{t \rightarrow -\infty} g(t)/t$. The inequality $\lim P(f, g + c) \leq f$ is trivial. To show the opposite inequality, fix $\epsilon > 0$. By the assumption, $f(-M)/(-M) + \epsilon > g(-M)/(-M)$ for sufficiently large M , say for $M > M_0$. Equivalently, $f(-M) - \epsilon M < g(-M)$ for $M > M_0$. It follows that $f(t) + \epsilon t < g(t) + C$ for every $t < 0$ and sufficiently large $C > 0$. We also have $f(t) + \epsilon t < f(t)$ trivially, thus by the definition of $P(f, g + C)$,

$$f(t) + \epsilon t \leq P(f, g + C) \leq \lim_{c \rightarrow \infty} P(f, g + c).$$

Since this holds for every $\epsilon > 0$, we have the conclusion. ■

3.2. Higher-dimensional case. First we prove (1) \Rightarrow (2) in Proposition 3.2.

Proof of (1) \Rightarrow (2). Take $a = (a_1, \dots, a_n) \in C_0$ and $b = (b_1, \dots, b_n) \in \mathbb{N}^n$. We denote by L the half-line $\{a + tb : t < 0\} \subset C_0$. Then

$$P_{C_0}(f, g + c) \leq P_L(f, g + c)$$

on L , because the left-hand side is a convex function on L not greater than f and $g + c$. Since (1) holds everywhere in C_0 , we have $\lim_{c \rightarrow \infty} P_L(f, g + c) = f$ on L . Thus we can apply the 1-dimensional case to $f|_L$ and $g|_L$ and get the conclusion. ■

Next we prove the opposite implication. We shall prove that condition (2) implies a relation of Newton convex bodies.

LEMMA 3.4. *Under condition (2) of Proposition 3.2, we have*

$$\overline{\Gamma(f)} \subset \overline{\Gamma(g)}.$$

Proof. Condition (2) means that $\hat{f}(w) \geq \hat{g}(w)$ for every $w \in \mathbb{Q}_{<0}^n$ (for a fixed $a \in C_0$). By convexity of \hat{f} and \hat{g} , this inequality holds for every $w \in \mathbb{R}_{<0}^n$. Thus the conclusion follows from Lemma 2.2. ■

Proof of (2)⇒(1). We have

$$\lim_{c \rightarrow \infty} P(f, g + c) = \left(\lim_{c \rightarrow \infty} \max(f^*, g^* - c) \right)^*$$

and $f = f^{**}$ (on C_0). Define $Q := \lim_{c \rightarrow \infty} \max(f^*, g^* - c)$. We want to prove that $Q^* = f^{**}$. Since $Q^{***} = Q^*$, it suffices to show that $Q^{**} = f^*$.

Because $\max(f^*, g^* - c) \geq f^*$, we have $Q = \lim_{c \rightarrow \infty} \max(f^*, g^* - c) \geq f^*$. We claim that $Q = f^*$ on the interior $\Gamma(f)^\circ$ of $\Gamma(f)$. By Lemma 3.4, we have $\overline{\Gamma(f)} \subset \overline{\Gamma(g)}$. Thus by convexity of $\Gamma(f)$ and $\Gamma(g)$ it follows that $\Gamma(f)^\circ \subset \Gamma(g)$. This means that g^* is finite on $\Gamma(f)^\circ$. Thus

$$Q = \lim_{c \rightarrow \infty} \max(f^*, g^* - c) = f^*$$

on $\Gamma(f)^\circ$, and we have proved the claim. On $\mathbb{R}^n \setminus \Gamma(f)$, we have $Q = f^* = \infty$.

Since $f^* \leq Q$ and $f^* = Q$ on $\Gamma(f)^\circ \cup (\mathbb{R}^n \setminus \Gamma(f))$, and Q^{**} is the greatest convex lower semicontinuous function with $Q^{**} \leq Q$, we deduce that $f^* = Q^{**}$ on $\Gamma(f)^\circ \cap (\mathbb{R}^n \setminus \Gamma(f))$. Because f^* and Q^{**} are convex and lower semicontinuous, we conclude that $f^* = Q^{**}$. ■

3.3. An example

EXAMPLE 3.5. We consider the subharmonic function

$$u^\alpha(z) := -(-\log |z|)^\alpha + C_{\alpha, \epsilon}$$

on a disc $\Delta_{1-\epsilon} := \{|z| < 1 - \epsilon\}$ in \mathbb{C} for $0 < \alpha \leq 1$, where $C_{\alpha, \epsilon}$ is chosen so that $u^\alpha = 0$ on $\partial\Delta_{1-\epsilon}$. The Lelong number is $\nu(u^\alpha, 0) = 0$ when $\alpha < 1$ and $\nu(u^\alpha, 0) = 1$ when $\alpha = 1$. We consider the geodesic between $u_0 = 0$ and $u_1 = u^\alpha$. Then, by Theorem 1.1, $u_t \rightarrow 0$ in capacity if and only if $\alpha < 1$. By a straightforward computation, the L^1 -energy $E_1(u^\alpha) := \int_{\Delta_{1-\epsilon}} (-u^\alpha) dd^c u^\alpha$ of u^α is finite when $\alpha < 1/2$ and infinite when $\alpha \geq 1/2$. Thus we have an example of a continuous geodesic with an endpoint having infinite energy.

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