ANNALES
POLONICI MATHEMATICI
120.1 (2017)

Weak solution and asymptotic behavior of magnetohydrodynamic flows of third grade fluids

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Abstract. The paper in concerned with the large time behavior of a perturbed MHD system, which is given by the coupling of a class of third grade fluid equation and the Maxwell equation, governing the fluid motion and the magnetic field respectively. We first derive the existence and uniqueness result for the system. Then we prove the existence of a finite-dimensional global attractor and an exponential attractor.

1. Introduction. We consider the large time behavior of solutions to the following system representing the motion of a third grade fluid in the presence of a magnetic field:

(1.1)
$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u - \alpha \operatorname{div}(A^2(u)) & \\ \qquad - \beta\operatorname{div}(|A(u)|^2A(u)) + \nabla\big(P+\frac{1}{2}|b|^2\big) - b\cdot\nabla b = f & \text{in } \Omega\times(0,\infty),\\ \partial_t b + u\cdot\nabla b - b\cdot\nabla u - \eta\Delta b = 0 & \text{in } \Omega\times(0,\infty),\\ \nabla\cdot b = \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty),\\ u(0)=u_0, \quad b(0)=b_0, & x\in\Omega, \end{cases}$$

where $\nu > 0$ is the constant kinematic viscosity, $\eta > 0$ is the constant magnetic diffusivity, α, β are material constants and $\beta > 0$. The unknown functions u, b, P are the fluid velocity, the magnetic field and the scalar pressure respectively; u_0, b_0, f are given functions representing the initial fluid velocity, the initial magnetic field and the forcing term respectively. Here A is the tensor defined as

$$A(u) = \nabla u + (\nabla u)^T$$

2010 *Mathematics Subject Classification*: Primary 35Q35, 37L30; Secondary 76D05.
Key words and phrases: magnetohydrodynamics, third grade fluid equation, attractor.
Received 13 December 2016; revised 25 July 2017 and 27 September 2017.
Published online 20 November 2017.

DOI: 10.4064/ap4161-9-2017

with $(\nabla u)^T$ being the transposition of the Jacobian matrix ∇u, and $|A(u)|^2$ denotes $\mathrm{tr}(A^2(u))$.

In the absence of a magnetic field, system (1.1) falls into the class of the following third grade fluid equations:

$$
\begin{cases}
\partial_t v - \nu \Delta u + (u \cdot \nabla) v + \sum_j v_j \nabla u_j \\
\qquad - (\alpha_1 + \alpha_2) \operatorname{div}(A^2(u)) - \beta \operatorname{div}(|A(u)|^2 A(u)) + \nabla P = f, \\
v = u - \alpha_1 \Delta u, \\
\operatorname{div} u = 0, \\
u(x,0) = u_0.
\end{cases} \tag{1.2}
$$

The third grade fluid is an important case of fluids of grade n introduced by Rivlin and Ericksen [33], for which the stress tensor is a polynomial of degree n in the first n Rivlin–Ericksen tensors defined recursively by

$$
A_1(u) = A(u) = \nabla u + (\nabla u)^T,
$$

$$
A_{k+1}(u) = \frac{\partial}{\partial t} A_k(u) + (u \cdot \nabla) A_k(u) + (\nabla u)^T A_k(u) + A_k(u) \nabla u.
$$

In the past years, third grade fluid equations have been widely studied: see for example [1, 8, 9, 13, 28, 33, 37, 43]. Some local existence and uniqueness results for initial data of arbitrary size, or global existence and uniqueness results for small initial data were obtained in the whole space $\mathbb{R}^n$, $n = 2, 3$, in [1, 37]. These results were then improved by Busuioc and Iftimie [8] and by Paicu [28]. Hamza and Paicu [17] studied a particular case of system (1.2). With the assumption that $\alpha_1 = 0$ (see [19] for its physical meaning), they proved the global existence and uniqueness of weak solutions to system (1.2) with $\mathbb{H}^1$ and $\mathbb{L}^2$ initial data. More recently, with the same assumption on α_1, Zhao and his co-authors [43] studied the time decay results for weak solutions to system (1.2) with zero forces, while in [10] we investigated the asymptotic behavior of solutions to system (1.2) with time dependent forces.

In recent years, the study of the motion of conducting non-Newtonian fluids in the presence of a magnetic field has attracted much attention due to their applications in various areas, such as the flow of plasma, the flow of mercury amalgams, and the handling of some biological fluids [2, 6, 18]. For example, in [35, 36, 14, 15], existence results and the control of some MHD equations arising from the coupling of a Ladyzhenskaya type model with the Maxwell equation were studied. In [31, 32] the large time behavior of solutions to some MHD equations for bipolar fluids were investigated. The existence of trajectory attractors and global attractors was obtained in different cases. More recently, in [16] (see also [18]), the existence and uniqueness results for solutions to the MHD equations for the second grade fluid were studied. Yet, as far as we know, few results have been obtained for the MHD equations of a third grade type fluid, although the third grade fluid equations have been extensively studied. Note that the MHD equations for a third grade fluid are quite different from the MHD equations for a bipolar fluid and the MHD equations for a second grade fluid. They are also different from the MHD equations composed of the Ladyzhenskaya type model and the Maxwell equation.

In this paper, we investigate the third grade fluid MHD system (1.1). We will explore the existence and uniqueness results and the large time behavior of solutions. Let $\Omega = [0, L]^3 \subseteq \mathbb{R}^3$, $L > 0$. We consider the problem with spatial periodic boundary conditions. We assume periodic boundary conditions on the initial data so that the corresponding solution is also space periodic. Furthermore, we make the assumption that the initial data and the forcing term are zero-spatial-mean functions ($\int_\Omega \varphi\, dx = 0$ for $\varphi = u_0, b_0, f$), so that the solutions have zero spatial mean, which allows us to use the Poincaré inequality. As we are mainly interested in the autonomous case, we assume that the forcing term f is time independent.

With proper assumptions on the material constants, we first prove the existence of a global solution for system (1.1) with $\mathbb{L}^2$ initial data. Then we show that if the initial fluid velocity and the initial magnetic field belong to $\mathbb{W}^{1,4}(\Omega)$ (indeed $\mathbb{H}^1(\Omega)$ is enough) and $\mathbb{H}^1(\Omega)$ respectively, the problem admits a regular solution, which is unique in the class of weak solutions. With these results, we then investigate the long time behavior of the solutions. By the method of short trajectories, we prove that system (1.1) possesses a finite-dimensional global attractor and an exponential attractor in $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega)$. The method of short trajectories was developed in [24, 26]. It has been used by many authors to study the large time behavior of solutions to various problems: see e.g. [7, 23, 25, 29, 30].

Note that system (1.1) could be viewed as a perturbation of the conventional MHD equations, which consist of the Navier–Stokes equation and the Maxwell equation. As a basic system in magnetohydrodynamics, the conventional MHD equations have been thoroughly studied in the past decades; we refer readers to [21, 38] for the classical results. It is well known that the conventional MHD equations have a unique global solution in dimension 2, while in dimension 3, the existence of a unique global solution remains open [38]. However, we will see that for the perturbed system (1.1), for small α there exists a unique global solution if the initial velocity and the magnetic field belong to $\mathbb{H}^1(\Omega)$ (see Remark 3.5). This is essentially due to the regularization effect brought by the term $-\beta \operatorname{div}(|A(u)|^2 A(u))$. It provides an $\mathbb{L}^4$ estimate for the gradient of the fluid velocity u, which helps us deal with the convective terms in the system.

This paper is organized as follows. In Section 2, we provide some preliminaries about function spaces. In Section 3, we prove the existence and uniqueness of a global solution, while in Section 4, we investigate the large time behavior of the solution.

2. Preliminaries. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. We denote their scalar product by $A \cdot B = \sum_{i,j=1}^{n} a_{ij} b_{ij}$, and in particular $|A|^2 = A \cdot A$. It is not difficult to show that $|A \cdot B| \le |A|\,|B|$. Moreover, for a matrix A, the vector $(\operatorname{div} A)_{n\times 1}$ is defined as $(\operatorname{div} A)_i = \sum_j \partial_j a_{ij}$, $i = 1, \ldots, n$.

For the domain $\Omega = [0, L]^3$, let $C_{\mathrm{p}}^{\infty}(\Omega)$ be the space of restrictions to Ω of infinitely differentiable functions that are L-periodic in each direction,

$$u(x + Le_j) = u(x), \qquad j = 1, 2, 3,$$

where e_j is the unit vector with 1 in the jth component. The Sobolev space $W_{\mathrm{p}}^{k,q}(\Omega)$ is the completion of $C_{\mathrm{p}}^{\infty}(\Omega)$ with respect to the norm of $W^{k,q}$,

$$\|u\|_{W^{k,q}} = \Big(\sum_{|l|\le k} \int_{\Omega} |D^l u|^q \, dx \Big)^{1/q},$$

in particular $W_{\mathrm{p}}^{0,q}(\Omega) = L_{\mathrm{p}}^{q}(\Omega)$ and $W_{\mathrm{p}}^{k,2}(\Omega) = H_{\mathrm{p}}^{k}(\Omega)$. Let

$$\dot{W}_{\mathrm{p}}^{k,q}(\Omega) = \Big\{ u \in W_{\mathrm{p}}^{k,q}(\Omega) : \int_{\Omega} u \, dx = 0 \Big\}.$$

For any function space X we denote by $\mathbb{X}$ the space X^3 endowed with the product structure, for example,

$$\mathbb{W}^{k,q}(\Omega) = [W^{k,q}(\Omega)]^3, \qquad \dot{\mathbb{W}}_{\mathrm{p}}^{k,q}(\Omega) = [\dot{W}_{\mathrm{p}}^{k,q}(\Omega)]^3.$$

In particular,

$$\mathbb{W}^{0,q}(\Omega) = \mathbb{L}^q(\Omega) = [L^q(\Omega)]^3, \qquad \dot{\mathbb{W}}_{\mathrm{p}}^{0,q}(\Omega) = \dot{\mathbb{L}}_{\mathrm{p}}^{q}(\Omega), \qquad \dot{\mathbb{W}}_{\mathrm{p}}^{2,q}(\Omega) = \dot{\mathbb{H}}_{\mathrm{p}}^{q}(\Omega).$$

We define

$$\begin{aligned}
\mathbb{H} &= \{u \in \dot{\mathbb{L}}_p^2(\Omega) : \operatorname{div} u = 0\}, &&\text{with } \|\cdot\| = \|\cdot\|_{\mathbb{L}^2} \text{ (the usual } \mathbb{L}^2 \text{ norm)},\\
\mathbb{V} &= \{u \in \dot{\mathbb{H}}_p^1(\Omega) : \operatorname{div} u = 0\}, &&\text{with } \|\cdot\|_{\mathbb{V}} = \|\nabla \cdot\|, \text{ and dual space } \mathbb{V}^*,\\
\mathbb{W} &= \{u \in \dot{\mathbb{W}}_p^{1,4}(\Omega) : \operatorname{div} u = 0\}, &&\text{with } \|\cdot\|_{\mathbb{W}} = \|\nabla \cdot\|_{\mathbb{L}^4}, \text{ and dual space } \mathbb{W}^*.
\end{aligned}$$

Define the linear "Stokes operator" $\mathcal{A}$ from $\mathbb{V}$ to $\mathbb{V}^*$ by

$$\langle \mathcal{A}u, v\rangle_{\mathbb{V}^*,\mathbb{V}} = \sum_{i,j=1}^{3} \int_{\Omega} \partial_j u_i \partial_j v_i \, dx, \qquad \forall u, v \in \mathbb{V},$$

and the bilinear operator $\mathcal{B}(u,v)$ from $\mathbb{V}\times\mathbb{V}$ into $\mathbb{V}^*$ as

$$\langle \mathcal{B}(u,v), w\rangle_{\mathbb{V}^*,\mathbb{V}} = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \qquad \forall u,v,w \in \mathbb{V}.$$

We also introduce the operators $\mathcal{K},\mathcal{J}$ from $\mathbb{W}$ into $\mathbb{W}^*$ as

$$\langle \mathcal{K}(u), \phi\rangle_{\mathbb{W}^*,\mathbb{W}} = \alpha \int_{\Omega} A^2(u)\cdot\nabla\phi\, dx \ (\le C\|A(u)\|^2_{\mathbb{L}^4}\|\nabla\phi\|_{\mathbb{L}^4}), \qquad \forall u,\phi\in\mathbb{W},$$

$$\langle \mathcal{J}(u), \phi\rangle_{\mathbb{W}^*,\mathbb{W}} = \beta \int_{\Omega} |A(u)|^2 A(u)\cdot\nabla\phi\, dx, \qquad \forall u,\phi\in\mathbb{W}.$$

Let $|\alpha| < \sqrt{2\nu\beta}$. We define $\mathcal{T}:\mathbb{W}\to\mathbb{W}^*$ as

$$\mathcal{T}(u) = (\nu - \nu\delta_0)\mathcal{A}u + (1-\delta_0)\mathcal{J}(u) + \mathcal{K}(u), \qquad \delta_0 = 1 - \sqrt{\alpha^2/2\nu\beta} \in (0,1).$$

Note that (see, e.g., [40, 41, 34])

$$\mathcal{A}u = -\Delta u \quad \text{ for } u \in D(\mathcal{A}) = \{u \in \dot{\mathbb{H}}^2_p(\Omega) : \operatorname{div} u = 0\}.$$

By the Hilbert–Schmidt theorem, one can deduce that $\mathcal{A}$ has a sequence of orthonormal eigenfunctions w_j belonging to $C^\infty_{\mathrm{p}}(\Omega)$ with zero mean in Ω. Since $\mathcal{A}$ is a self-adjoint positive operator with compact inverse, $\{w_j\}_{j=1}^\infty$ forms an orthonormal basis of the space $\mathbb{H}$. Moreover, $\{w_j\}_{j=1}^\infty$ also forms an orthogonal basis of the space $D(\mathcal{A}^{s/2}) = \{\dot{\mathbb{H}}^s_p(\Omega) : \operatorname{div} u = 0\}$ for any positive integer s (see [34, p. 198] for the details). Let P_m be the orthogonal projection form $\mathbb{H}$ onto the space spanned by $\{w_j\}_{j=1}^m$. Similar to [34, Lemma 7.5], (with minor modifications) we can show that for any $v \in D(\mathcal{A}^{s/2})$,

$$\|P_m v\|_{\dot{\mathbb{H}}^s_p} \le \|v\|_{\dot{\mathbb{H}}^s_p} \quad \text{and} \quad P_m v \to v \quad \text{in } \dot{\mathbb{H}}^s_p(\Omega) \text{ as } m\to\infty, \tag{2.1}$$

Furthermore, let $\mathbb{H}^{-s}$ be the dual space of $\dot{\mathbb{H}}^s_p(\Omega)$. We have

$$\begin{aligned} \|P_m f\|_{\mathbb{H}^{-s}} &= \sup_{\|v\|_{\dot{\mathbb{H}}^s_p}\le 1} |\langle P_m f, v\rangle| \\ &= \sup_{\|v\|_{\dot{\mathbb{H}}^s_p}\le 1} |\langle f, P_m v\rangle| \le \sup_{\|w\|_{\dot{\mathbb{H}}^s_p}\le 1} |\langle f, w\rangle| = \|f\|_{\mathbb{H}^{-s}}. \end{aligned} \tag{2.2}$$

Now, we recall some useful lemmas.

LEMMA 2.1 ([8, 17]). *The operator $\mathcal{J}$ is monotone: for any $u,v\in\mathbb{W}$,*

$$\langle \mathcal{J}(u) - \mathcal{J}(v), u-v\rangle_{\mathbb{W}^*,\mathbb{W}} \ge 0.$$

LEMMA 2.2 ([17]). *The operator $\mathcal{T}$ is monotone: for any $u,v\in\mathbb{W}$,*

$$\langle \mathcal{T}(u) - \mathcal{T}(v), u-v\rangle_{\mathbb{W}^*,\mathbb{W}} \ge 0.$$

Let $\mathcal{T}_1 : \mathbb{W}\to\mathbb{W}^*$ be defined as

$$\mathcal{T}_1(v) = \nu\mathcal{A}v + \mathcal{K}(v) + \mathcal{J}(v).$$

Thanks to the above lemmas, $\mathcal{T}_1$ is also a monotone operator.

The following Korn inequality plays an essential role in our analysis.

Lemma 2.3 ([4]). *Assume that* $1 < q < \infty$ *and* $\Omega = [0, L]^n$, $n = 2, 3$. *Let* $v \in \dot{\mathbb{W}}_p^{1,q}(\Omega)$. *Then there exists a positive constant* $c = c(q, \Omega)$ *such that*

$$\|\nabla v\|_{L^q} \le c\|A(v)\|_{L^q}.$$

3. Existence and uniqueness of global solutions

Definition 3.1. Let $u_0, b_0, f \in \mathbb{H}$. A *weak solution* of system (1.1) is a couple of functions (u, b) such that for any $T > 0$,

- $u \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^4(0, T; \mathbb{W})$ and $\partial_t u \in L^{4/3}(0, T; \mathbb{W}^*)$ with $u(0) = u_0$,
- $b \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V})$ and $\partial_t b \in L^2(0, T; \mathbb{V}^*)$ with $b(0) = b_0$,
- for any $\varphi \in \mathbb{W}$ and $\psi \in \mathbb{V}$, the couple (u, b) satisfies, for almost very $t \in [0, T]$,

$$\langle \partial_t u, \varphi\rangle_{\mathbb{W}^*,\mathbb{W}} + \int_\Omega (u\cdot\nabla)u\varphi\,dx - \int_\Omega (b\cdot\nabla)b\varphi\,dx + \int_\Omega \big(\nu\nabla u + \alpha A^2(u) + \beta|A(u)|^2 A(u)\big)\cdot\nabla\varphi\,dx = \int_\Omega f\varphi\,dx, \tag{3.1}$$

$$\langle \partial_t b, \psi\rangle_{\mathbb{V}^*,\mathbb{V}} + \int_\Omega \{(u\cdot\nabla)b - (b\cdot\nabla)u\}\psi\,dx + \eta\int_\Omega \nabla b\cdot\nabla\psi\,dx = 0. \tag{3.2}$$

Remark 3.2. Since $u \in L^4(0, T; \mathbb{W})$ with $\partial_t u \in L^{4/3}(0, T; \mathbb{W}^*)$ and $b \in L^2(0, T; \mathbb{V})$ with $\partial_t b \in L^2(0, T; \mathbb{V}^*)$, we may conclude that $u, b \in C([0, T]; \mathbb{H})$ (see [40, 41, 34]).

Theorem 3.3. *Let* $(u_0, b_0) \in \mathbb{H}\times\mathbb{H}$, $f \in \mathbb{H}$, $\beta > 0$, $|\alpha| < \sqrt{2\nu\beta}$. *Then problem* (1.1) *admits a global weak solution* (u, b) *such that, for any* $T > 0$ *and* $t \in (0, T]$,

$$\|u(t)\|^2 + \|b(t)\|^2 + \int_0^t (\|\nabla u\|^2 + \|A(u)\|_{\mathbb{L}^4}^4 + \|\nabla b\|^2)\,ds \le C(\|u_0\|^2 + \|b_0\|^2 + \|f\|^2), \tag{3.3}$$

where the positive constant C *depends on* α, β, ν, T. *Furthermore, if* $(u_0, b_0) \in \mathbb{W}\times\mathbb{V}$ *and* $|\alpha| < \sqrt{\nu\beta/2}$, *then* $(u, b) \in L^\infty(0, T; \mathbb{W}\times\mathbb{V}) \cap L^2(0, T; \dot{\mathbb{H}}_p^2(\Omega)\times\dot{\mathbb{H}}_p^2(\Omega))$ *with* $(u_t, b_t) \in L^2(0, T; \mathbb{H}\times\mathbb{H})$, *and*

$$\int_0^T (\|\Delta u\|^2 + \|\Delta b\|^2 + \|u_t\|^2 + \|b_t\|^2)\,ds + \|\nabla b(t)\|^2 + \|\nabla u(t)\|^2 + \|A(u)(t)\|_{\mathbb{L}^4}^4 \le C, \tag{3.4}$$

where the positive constant C *also depends on* $\alpha, \beta, \nu, \eta, T, u_0, b_0$.

Proof. We implement the Galerkin approximation method to prove the existence of weak solutions. Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of $\mathbb{H}$ consisting of eigenfunctions of the Stokes operator $\mathcal{A}$. Consider the following ordinary differential system:

$$\partial_t u_m = -P_m\mathcal{B}(u_m,u_m) + P_m\mathcal{B}(b_m,b_m) - \nu\mathcal{A}u_m - P_m[\mathcal{K}(u_m)] - P_m[\mathcal{J}(u_m))] + P_m f, \tag{3.5}$$

$$\partial_t b_m = P_m\mathcal{B}(b_m,u_m) - P_m\mathcal{B}(u_m,b_m) - \eta\mathcal{A}b_m, \tag{3.6}$$

$$u_m(0) = P_m u_0, b_m(0) = P_m b_0 \tag{3.7}$$

for the m-dimensional approximation $u_m(t), b_m(t)$ defined as

$$u_m(t) = \sum_{j=1}^{m} c_{jm}(t)w_j,$$

$$b_m(t) = \sum_{j=1}^{m} d_{jm}(t)w_j.$$

By the standard existence theorem for ordinary differential equations, for each m there exists a local solution (u_m, b_m) to system (3.5)–(3.7) in the interval $[0, T_m)$.

I: Estimates and compactness. Here and throughout the paper, C denotes a positive constant, which varies in different places. Multiplying (3.5) and (3.6) by $u_m(t)$ and $b_m(t)$ respectively, it is not difficult to obtain

$$\frac{d}{dt}(\|u_m\|^2 + \|b_m\|^2) + 2(\nu\|\nabla u_m\|^2 + \eta\|\nabla b_m\|^2) + 2\int_{\Omega}(\alpha A^2(u_m) + \beta|A(u_m)|^2A(u_m))\cdot\nabla u_m\,dx \le \frac{1}{\varepsilon}\|f\|^2 + \varepsilon\|\nabla u_m\|^2. \tag{3.8}$$

Thanks to [5], by the symmetry of $A(u_m)$ we have

$$\int_{\Omega}|A(u_m)|^2A(u_m)\cdot\nabla u_m\,dx = \tfrac{1}{2}\|A(u_m)\|_{\mathbb{L}^4}^4.$$

Similar to [17, p. 1103], from the Cauchy–Schwarz inequality and the fact that

$$|A^2(u_m)| = |A(u_m)A(u_m)| \le |A(u_m)|^2,$$

we deduce that

$$\left|\alpha\int_{\Omega}A^2(u_m)\cdot\nabla u_m\,dx\right| \le \frac{|\alpha|\delta}{2}\|A(u_m)\|_{\mathbb{L}^4}^4 + \frac{|\alpha|}{2\delta}\|\nabla u_m\|^2.$$

Since $|\alpha| < \sqrt{2\beta\nu}$, we can choose $\delta = \frac{2\beta\nu+\alpha^2}{4|\alpha|\nu}$ such that

$$\frac{|\alpha|\delta}{2} < \frac{\beta}{2} \quad\text{and}\quad \frac{|\alpha|}{2\delta} < \nu.$$

Taking ε small enough, we deduce from (3.8) that

$$\frac{d}{dt}(\|u_m\|^2 + \|b_m\|^2) + c_0\|\nabla u_m\|^2 + c_1\|A(u_m)\|_{\mathbb{L}^4}^4 + 2\eta\|\nabla b_m\|^2 \le C\|f\|^2 \tag{3.9}$$

for some positive constants c_0, c_1, which implies that for any fixed $T > 0$ and any $t \in (0, T]$,

$$\|u_m(t)\|^2 + \|b_m(t)\|^2 \le \|u_0\|^2 + \|b_0\|^2 + CT\|f\|^2, \tag{3.10}$$

$$\int_0^t \big(c_0\|\nabla u_m(s)\|^2 + c_1\|A(u_m)(s)\|_{\mathbb{L}^4}^4 + 2\eta\|\nabla b_m(s)\|^2\big)\, ds \le \|u_0\|^2 + \|b_0\|^2 + CT\|f\|^2. \tag{3.11}$$

Hence, from (3.10) and (3.11) we know that

u_m, b_m are uniformly bounded in $L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$,
∇u_m is uniformly bounded in $L^s(0,T;\mathbb{L}^s(\Omega))$ for all $2 \le s \le 4$.

To derive the compactness result for the sequence $\{(u_m, b_m)\}$, let us perform some estimates for $\partial_t u_m, \partial_t b_m$. Thanks to the estimates above for u_m, b_m and (2.1), it is easy to deduce that

$$\begin{aligned}
\|P_m\mathcal{B}(u_m,u_m)\|_{\mathbb{V}^*} &= \sup_{\|\phi\|_{\mathbb{V}}\le 1} |\langle P_m\mathcal{B}(u_m,u_m),\phi\rangle_{\mathbb{V}^*,\mathbb{V}}| \\
&= \sup_{\|\phi\|_{\mathbb{V}}\le 1}\Big|\int_\Omega (u_m\cdot\nabla)u_m P_m\phi\, dx\Big| = \sup_{\|\phi\|_{\mathbb{V}}\le 1}\Big|\int_\Omega (u_m\cdot\nabla)P_m\phi u_m\, dx\Big| \\
&\le \sup_{\|\phi\|_{\mathbb{V}}\le 1} C\|u_m\|^{1/2}\|u_m\|_{\mathbb{V}}^{3/2}\|P_m\phi\|_{\mathbb{V}} = C\|u_m\|^{1/2}\|u_m\|_{\mathbb{V}}^{3/2},
\end{aligned}$$

which then implies that

$$\|P_m\mathcal{B}(u_m,u_m)\|_{L^{4/3}(0,T;\mathbb{V}^*)} \le C\|u_m\|_{L^\infty(0,T;\mathbb{H})}^{1/2}\|u_m\|_{L^2(0,T;\mathbb{V})}^{3/2}. \tag{3.12}$$

Similarly,

$$\|P_m\mathcal{B}(b_m,b_m)\|_{L^{4/3}(0,T;\mathbb{V}^*)} \le C\|b_m\|_{L^\infty(0,T;\mathbb{H})}^{1/2}\|b_m\|_{L^2(0,T;\mathbb{V})}^{3/2}. \tag{3.13}$$

On the other hand,

$$\begin{aligned}
\|P_m\mathcal{B}(u_m,b_m)\|_{\mathbb{V}^*} &= \sup_{\|\phi\|_{\mathbb{V}}\le 1} |\langle P_m\mathcal{B}(u_m,b_m),\phi\rangle_{\mathbb{V}^*,\mathbb{V}}| \\
&= \sup_{\|\phi\|_{\mathbb{V}}\le 1}\Big|\int_\Omega (u_m\cdot\nabla)b_m P_m\phi\, dx\Big| = \sup_{\|\phi\|_{\mathbb{V}}\le 1}\Big|\int_\Omega (u_m\cdot\nabla)P_m\phi b_m\, dx\Big| \\
&\le \sup_{\|\phi\|_{\mathbb{V}}\le 1} \|P_m\phi\|_{\mathbb{V}}\|b_m\|_{\mathbb{L}^3}\|u_m\|_{\mathbb{L}^6} \le C\|b_m\|^{1/2}\|\nabla b_m\|^{1/2}\|u_m\|^{3/7}\|\nabla u_m\|_{\mathbb{L}^4}^{4/7},
\end{aligned}$$

and

$$\begin{aligned}\|P_m\mathcal{B}(b_m,u_m)\|_{\mathbb{V}^*} &= \sup_{\|\phi\|_{\mathbb{V}}\le 1} |\langle P_m\mathcal{B}(b_m,u_m),\phi\rangle_{\mathbb{V}^*,\mathbb{V}}| \\ &= \sup_{\|\phi\|_{\mathbb{V}}\le 1}\Big|\int_\Omega (b_m\cdot\nabla)u_m P_m\phi\,dx\Big| \le \sup_{\|\phi\|_{\mathbb{V}}\le 1}\|b_m\|\,\|\nabla u_m\|_{\mathbb{L}^3}\|P_m\phi\|_{\mathbb{V}} \\ &\le C\|b_m\|\,\|\nabla u_m\|_{\mathbb{L}^3}.\end{aligned}$$

Moreover, using (2.1) we deduce that

$$\begin{aligned}\|P_m\mathcal{J}(u_m)\|_{\mathbb{H}^{-2}} &= \sup_{\|\phi\|_{\dot{\mathbb{H}}^2_{\mathrm{p}}(\Omega)}\le 1} |\langle P_m\mathcal{J}(u_m),\phi\rangle_{\mathbb{H}^{-2},\dot{\mathbb{H}}^2_{\mathrm{p}}(\Omega)}| \\ &= \sup_{\|\phi\|_{\dot{\mathbb{H}}^2_{\mathrm{p}}(\Omega)}\le 1}\Big|\int_\Omega \mathcal{J}(u_m)P_m\phi\,dx\Big| \le \sup_{\|\phi\|_{\dot{\mathbb{H}}^2_{\mathrm{p}}(\Omega)}\le 1}\|A(u_m)\|^3_{\mathbb{L}^4}\|P_m\phi\|_{\mathbb{W}} \\ &\le C\sup_{\|\phi\|_{\dot{\mathbb{H}}^2_{\mathrm{p}}(\Omega)}\le 1}\|A(u_m)\|^3_{\mathbb{L}^4}\|P_m\phi\|_{\dot{\mathbb{H}}^2_{\mathrm{p}}(\Omega)} \le C\|A(u_m)\|^3_{\mathbb{L}^4}.\end{aligned}$$

We then obtain

$$(3.14)\quad \begin{aligned}&\|P_m\mathcal{B}(u_m,b_m)\|_{L^2(0,T;\mathbb{V}^*)} \\ &\quad\le C\|b_m\|^{1/2}_{L^\infty(0,T;\mathbb{H})}\|u_m\|^{3/7}_{L^\infty(0,T;\mathbb{H})}\|b_m\|^{1/2}_{L^2(0,T;\mathbb{V})}\|u_m\|^{4/7}_{L^4(0,T;\mathbb{W})},\end{aligned}$$

$$(3.15)\quad \|P_m\mathcal{B}(b_m,u_m)\|_{L^2(0,T;\mathbb{V}^*)} \le C\|b_m\|_{L^\infty(0,T;\mathbb{H})}\|\nabla u_m\|_{L^3(0,T;\mathbb{L}^3(\Omega))}.$$

In a similar way, we find that for large m ($m\ge m_0$),

$$(3.16)\quad \mathcal{A}u_m,\mathcal{A}b_m \in L^2(0,T;\mathbb{V}^*)\hookrightarrow L^{4/3}(0,T;\mathbb{W}^*),$$

$$(3.17)\quad \|P_m\mathcal{K}(u_m))\|_{L^2(0,T;\mathbb{V}^*)} \le C\|A(u_m)\|^2_{L^4(0,T;\mathbb{L}^4(\Omega))},$$

$$(3.18)\quad \|P_m\mathcal{J}(u_m)\|_{L^{4/3}(0,T;\mathbb{H}^{-2})} \le C\|A(u_m)\|^3_{L^4(0,T;\mathbb{L}^4(\Omega))}.$$

Combining (3.5), (3.6) and (3.12)–(3.18), we deduce that (for $m\ge m_0$)

$$(3.19)\quad \begin{aligned}&\partial_t u_m \text{ is uniformly bounded in } L^{4/3}(0,T;\mathbb{H}^{-2}),\\ &\partial_t b_m \text{ is uniformly bounded in } L^2(0,T;\mathbb{V}^*).\end{aligned}$$

Thanks to the Aubin–Simon type compactness results (see for example [41, 34, 39]), there exist vector-valued functions u,b such that, up to subsequences,

$$\begin{aligned} u_m,b_m\to u,b &\quad \text{weakly}^* \text{ in } L^\infty(0,T;\mathbb{H}),\\ u_m,b_m\to u,b &\quad \text{weakly in } L^2(0,T;\mathbb{V}),\\ u_m,b_m\to u,b &\quad \text{strongly in } L^2(0,T;\mathbb{H}),\\ u_m\to u &\quad \text{weakly in } L^4(0,T;\mathbb{W}),\\ \partial_t u_m\to\partial_t u &\quad \text{weakly}^* \text{ in } L^{4/3}(0,T;\mathbb{H}^{-2}),\\ \partial_t b_m\to\partial_t b &\quad \text{weakly}^* \text{ in } L^2(0,T;\mathbb{V}^*),\\ &u,b \text{ satisfy the estimates } (3.10),(3.11) \text{ and hence } (3.3).\end{aligned}$$

II: Passing to the limit. To prove that (u,b) is a weak solution, let us check that the equations

$$\partial_t u = -\mathcal{B}(u,u) + \mathcal{B}(b,b) - \nu\mathcal{A}u - \mathcal{K}(u) - \mathcal{J}(u) + f, \tag{3.20}$$

$$\partial_t b = \mathcal{B}(b,u) - \mathcal{B}(u,b) - \eta\mathcal{A}b \tag{3.21}$$

hold, respectively, in $L^{4/3}(0,T;\mathbb{W}^*)$ and $L^2(0,T;\mathbb{V}^*)$ by passing to the limit in the approximate problem (3.5)–(3.7). Let us consider the convergence of each term on the right hand sides of (3.5) and (3.6). We first prove that

$$P_m\mathcal{B}(u_m,u_m) \to \mathcal{B}(u,u) \quad \text{weakly}^* \text{ in } L^{4/3}(0,T;\mathbb{W}^*), \tag{3.22}$$

$$P_m\mathcal{B}(b_m,b_m) \to \mathcal{B}(b,b) \quad \text{weakly}^* \text{ in } L^{4/3}(0,T;\mathbb{W}^*), \tag{3.23}$$

$$P_m\mathcal{B}(u_m,b_m) \to \mathcal{B}(u,b) \quad \text{weakly}^* \text{ in } L^{2}(0,T;\mathbb{V}^*), \tag{3.24}$$

$$P_m\mathcal{B}(b_m,u_m) \to \mathcal{B}(b,u) \quad \text{weakly}^* \text{ in } L^{2}(0,T;\mathbb{V}^*). \tag{3.25}$$

Set

$$\mathbb{S} = \Big\{\sum_{j=1}^{k} \lambda_j(t)\xi_j : \lambda_j \in C^1([0,T]),\ \xi_j \in D(\mathcal{A}),\ k=1,2,\dots\Big\}.$$

It is not difficult to prove that $\mathbb{S}$ is dense in $L^4(0,T;\mathbb{W})$ and $L^2(0,T;\mathbb{V})$ (see [34, p. 211]). Hence, to prove (3.22), we only need to verify that for any fixed $\varPhi$ $(= \sum_{j=1}^{k} \lambda_j(t)\xi_j) \in \mathbb{S}$,

$$\int_0^T\int_\varOmega \big\{(u_m\cdot\nabla)u_m P_m\varPhi - (u\cdot\nabla)u\varPhi\big\}\,dx\,ds = I_1 + I_2 + I_3 \to 0, \tag{3.26}$$

where

$$I_1 = \int_0^T \sum_{j=1}^{k} \lambda_j(s) \int_\varOmega ((u_m-u)\cdot\nabla)u_m P_m\xi_j\,dx\,ds,$$

$$I_2 = \int_0^T \sum_{j=1}^{k} \lambda_j(s) \int_\varOmega (u\cdot\nabla)u(P_m\xi_j - \xi_j)\,dx\,ds,$$

$$I_3 = -\int_0^T \sum_{j=1}^{k} \lambda_j(s) \int_\varOmega (u\cdot\nabla)P_m\xi_j(u_m-u)\,dx\,ds.$$

Note that

$$I_1 \le C\|\varPhi\|_{L^4(0,T;\mathbb{L}^4(\varOmega))}\|\nabla u_m\|_{L^4(0,T;\mathbb{L}^4(\varOmega))}\|u-u_m\|_{L^2(0,T;\mathbb{L}^2(\varOmega))},$$

$$I_2 \le C\|u\|_{L^4(0,T;\mathbb{L}^4(\varOmega))}\|\nabla u\|_{L^4(0,T;\mathbb{L}^4(\varOmega))}\sum_{j=1}^{k}\|\xi_j - P_m\xi_j\|,$$

$$I_3 \le C\|\nabla\varPhi\|_{L^4(0,T;\mathbb{L}^4(\varOmega))}\|u\|_{L^4(0,T;\mathbb{L}^4(\varOmega))}\|u-u_m\|_{L^2(0,T;\mathbb{L}^2(\varOmega))}.$$

Thanks to the convergence of u_m (to u in $L^2(0,T;\mathbb{L}^2(\Omega))$) and of $P_m\xi_j$ (to ξ in $\mathbb{H}$), we conclude that

$$\int_0^T\int_\Omega \big\{(u_m\cdot\nabla)u_mP_m\Phi-(u\cdot\nabla)u\Phi\big\}\,dx\,ds\to 0\quad \text{as } m\to\infty. \tag{3.27}$$

In a similar way, we can obtain

$$\int_0^T\int_\Omega \big\{(b_m\cdot\nabla)b_mP_m\Phi-(b\cdot\nabla)b\Phi\big\}\,dx\,ds\to 0\quad \text{as } m\to\infty, \tag{3.28}$$

$$\int_0^T\int_\Omega \big\{(u_m\cdot\nabla)b_mP_m\Phi-(u\cdot\nabla)b\Phi\big\}\,dx\,ds\to 0\quad \text{as } m\to\infty, \tag{3.29}$$

$$\int_0^T\int_\Omega \big\{(b_m\cdot\nabla)u_mP_m\Phi-(b\cdot\nabla)u\Phi\big\}\,dx\,ds\to 0\quad \text{as } m\to\infty, \tag{3.30}$$

which implies respectively (3.23), (3.24) and (3.25).

On the other hand, note that $\mathcal{A}$ is a continuous linear mapping from $L^2(0,T;\mathbb{V})$ into $L^2(0,T;\mathbb{V}^*)$, and we conclude that

$$P_m\mathcal{A}b_m\to\mathcal{A}b\quad \text{weakly}^* \text{ in } L^2(0,T;\mathbb{V}^*). \tag{3.31}$$

Hence, combining (3.29)–(3.31), we get (3.21). Also, bearing in mind (3.27) and (3.28), to obtain (3.20) we only need to check that for any fixed Φ $(=\sum_{j=1}^k\lambda_j(t)\xi_j)\in\mathbb{S}$,

$$\langle P_m\mathcal{T}_1(u_m),\Phi\rangle_1\to\langle\mathcal{T}_1(u),\Phi\rangle_1, \tag{3.32}$$

where $\langle\,,\rangle_1$ is the duality product between $L^{4/3}(0,T;\mathbb{W}^*)$ and $L^4(0,T;\mathbb{W})$. Note that

$$\begin{aligned}\langle P_m\mathcal{T}_1(u_m),\Phi\rangle_1-\langle\mathcal{T}_1(u),\Phi\rangle_1&=\langle\mathcal{T}_1(u_m)-\mathcal{T}_1(u),\Phi\rangle_1+\langle\mathcal{T}_1(u_m),P_m\Phi-\Phi\rangle_1\\&\doteq J_1+J_2.\end{aligned}$$

For J_2, we have

$$\begin{aligned}J_2&=\int_0^T\sum_{j=1}^k\lambda_j\int_\Omega\big(\nu\nabla u_m+\alpha A^2(u_m)+\beta|A(u_m)|^2A(u_m)\big)\nabla(P_m\xi_j-\xi_j)\,dx\,ds\\&\le C\big(\|\nabla u_m\|_{L^2(0,T;\mathbb{L}^2(\Omega))}+\|A(u_m)\|^2_{L^4(0,T;\mathbb{L}^4(\Omega))}\big)\sum_{j=1}^k\|P_m\xi_j-\xi_j\|_{\mathbb{V}}\\&\quad+\|A(u_m)\|^3_{L^4(0,T;\mathbb{L}^4(\Omega))}\sum_{j=1}^k\|P_m\xi_j-\xi_j\|_{\mathbb{W}}.\end{aligned}$$

Since $\{w_j\}_{j=1}^{\infty}$ forms an orthonormal basis of the space $D(\mathcal{A}^{s/2})$ for any positive integer s, as we pointed in (2.2), taking s large enough such that $D(\mathcal{A}^{s/2}) \hookrightarrow \mathbb{W}$, we deduce that $P_m\xi_j$ converges to ξ_j in both the spaces $\mathbb{V}, \mathbb{W}$ for each j as m tends to infinity. Therefore, we conclude that J_2 converges to zero as m tends to infinity.

Now it remains to verify that J_1 converges to zero. This can be achieved by the monotone operator theory as in [17, 22, 20, 3]. Indeed since $\mathcal{T}_1(u_m)$ is uniformly (in m) bounded in $L^{4/3}(0,T;\mathbb{W}^*)$, we may assume that

$$\mathcal{T}_1(u_m) \to \varXi \quad \text{weakly}^* \text{ in } L^{4/3}(0,T;\mathbb{W}^*).$$

Hence passing to the limit in (3.5) we have

$$\partial_t u = -\mathcal{B}(u,u) + \mathcal{B}(b,b) - \varXi + f \quad \text{in } L^{4/3}(0,T;\mathbb{W}^*). \tag{3.33}$$

Taking u as a test function in (3.33), we obtain

$$\langle \varXi, u\rangle_1 = \tfrac{1}{2}\|u(0)\|^2 - \tfrac{1}{2}\|u(T)\|^2 + \langle \mathcal{B}(b,b), u\rangle_1 + \langle f, u\rangle_1. \tag{3.34}$$

Tanking u_m as a test function in (3.5) yields

$$\langle \mathcal{T}_1(u_m), u_m\rangle_1 = \tfrac{1}{2}\|u_m(0)\|^2 - \tfrac{1}{2}\|u_m(T)\|^2 + \langle \mathcal{B}(b_m,b_m) + f, u_m\rangle_1. \tag{3.35}$$

Due to the monotonicity of $\mathcal{T}_1$, for any $\varphi \in L^4(0,T;\mathbb{W})$ we have

$$\langle \mathcal{T}_1(u_m), u_m\rangle_1 \geq \langle \mathcal{T}_1(u_m), \varphi\rangle_1 + \langle \mathcal{T}_1(\varphi), u_m - \varphi\rangle_1. \tag{3.36}$$

Taking b as a test function in (3.21), we find that

$$\langle \mathcal{B}(b,b), u\rangle_1 = \tfrac{1}{2}\|b(0)\|^2 - \tfrac{1}{2}\|b(T)\|^2 - \int_0^T \|\nabla b\|^2\,dt. \tag{3.37}$$

On the other hand, taking b_m as a test function in (3.6), we get

$$\langle \mathcal{B}(b_m,b_m), u_m\rangle_1 = \tfrac{1}{2}\|b_m(0)\|^2 - \tfrac{1}{2}\|b_m(T)\|^2 - \int_0^T \|\nabla b_m\|^2\,dt. \tag{3.38}$$

Noting that $b_m(T), u_m(T)$ converge weakly in $\mathbb{H}$ to $b(T), u(T)$ respectively, and b_m converges to b weakly in $L^2(0,T;\mathbb{V})$, we have

$$\|b(T)\|^2 \leq \liminf_{m\to\infty} \|b_m(T)\|^2, \qquad \|u(T)\|^2 \leq \liminf_{m\to\infty} \|u_m(T)\|^2,$$
$$\int_0^T \|\nabla b\|^2\,dt \leq \liminf_{m\to\infty} \int_0^T \|\nabla b_m\|^2\,dt.$$

Hence inserting (3.37), (3.38) into (3.34), (3.35) and combining this with (3.36), we obtain

$$\langle \varXi, u\rangle_1 \geq \langle \mathcal{T}_1(u_m), u_m\rangle_1 \geq \langle \mathcal{T}_1(u_m), \varphi\rangle_1 + \langle \mathcal{T}_1(\varphi), u_m - \varphi\rangle_1.$$

Letting $m \to \infty$ yields

$$\langle \varXi - \mathcal{T}_1(\varphi), u - \varphi\rangle_1 \geq 0 \quad \text{for any } \varphi \in L^4(0,T;\mathbb{W}). \tag{3.39}$$

Take $\varphi = u + \delta\phi$ for $\delta > 0$ and $\phi \in L^4(0,T;\mathbb{W})$. Then after division by δ and letting $\delta \to 0$, we deduce that

$$\langle \varXi - \mathcal{T}_1(u), \phi\rangle_1 \geq 0 \quad \text{for any } \phi \in L^4(0,T;\mathbb{W}). \tag{3.40}$$

Hence $\varXi = \mathcal{T}_1(u)$ and of course J_1 converges to 0 as m tends to infinity.

At last, since we can easily check that $(u(0), b(0)) = (u_0, b_0)$ just as for the Navier–Stokes equations (see for example [34]), we omit the details and conclude that (u,b) is a weak solution to system (1.1).

III: Regular solutions. Now we prove the second part of Theorem 1.1. Multiplying the equation by $\Delta u_m, \Delta b_m$ and integrating, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \nu\|\Delta u_m\|^2 + \int_\Omega (u_m\cdot\nabla)u_m\Delta u_m\,dx \\ - \beta\int_\Omega \operatorname{div}(|A(u_m)|^2A(u_m))\Delta u_m\,dx - \alpha\int_\Omega \operatorname{div}[A^2(u_m)]\Delta u_m\,dx \\ \leq \frac{1}{4\varepsilon_1}\|f\|^2 + \varepsilon_1\|\Delta u_m\|^2 + \int_\Omega (b_m\cdot\nabla)b_m\Delta u_m\,dx,\end{aligned} \tag{3.41}$$

$$\frac{d}{dt}\|\nabla b_m\|^2 + \eta\|\Delta b_m\|^2 + \int_\Omega \{(u_m\cdot\nabla)b_m - (b_m\cdot\nabla)u_m\}\Delta b_m = 0. \tag{3.42}$$

Note that

$$\begin{aligned}\Big|\int_\Omega (u_m\cdot\nabla)u_m\Delta u_m\,dx\Big| &\leq \|\Delta u_m\|\,\|\nabla u_m\|_{\mathbb{L}^3}\|u_m\|_{\mathbb{L}^6} \\ &\leq \varepsilon_1\|\Delta u_m\|^2 + \frac{1}{4\varepsilon_1}\|\nabla u_m\|^2_{\mathbb{L}^3}\|\nabla u_m\|^2,\end{aligned} \tag{3.43}$$

$$\begin{aligned}\Big|\int_\Omega (b_m\cdot\nabla)u_m\Delta u_m\,dx\Big| &\leq \|\Delta u_m\|\,\|\nabla u_m\|_{\mathbb{L}^3}\|b_m\|_{\mathbb{L}^6} \\ &\leq \varepsilon_1\|\Delta u_m\|^2 + \frac{1}{4\varepsilon_1}\|\nabla u_m\|^2_{\mathbb{L}^3}\|\nabla b_m\|^2,\end{aligned} \tag{3.44}$$

$$\begin{aligned}\Big|\int_\Omega (u_m\cdot\nabla)b_m\Delta u_m\,dx\Big| &\leq \|\Delta u_m\|\,\|\nabla b_m\|_{\mathbb{L}^3}\|u_m\|_{\mathbb{L}^6} \\ &\leq C\|\Delta u_m\|\,\|\Delta b_m\|^{1/2}\|\nabla b_m\|^{1/2}\|\nabla u_m\|^{2/3}_{\mathbb{L}^3}\|u_m\|^{1/3} \\ &\leq \varepsilon_1(\|\Delta b_m\|^2 + \|\Delta u_m\|^2) + C_{\varepsilon_1}\|\nabla u_m\|^{8/3}_{\mathbb{L}^3}\|\nabla b_m\|^2,\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\Big|\int_\Omega (b_m\cdot\nabla)b_m\Delta u_m\,dx\Big| &= \Big|\sum_{i,j,k=1}^{3}\int_\Omega b_{mi}\partial_i b_{mj}\partial_k^2 u_{mj}\,dx\Big| \\ &= \Big|\sum_{i,j,k=1}^{3}\int_\Omega (\partial_k b_{mi}\partial_i b_{mj}\partial_k u_{mj} + b_{mi}\partial_i\partial_k b_{mj}\partial_k u_{mj})\,dx\Big| \\ &\leq \|\nabla b_m\|^2_{\mathbb{L}^3}\|\nabla u_m\|_{\mathbb{L}^3} + \|\Delta b_m\|\,\|\nabla u_m\|_{\mathbb{L}^3}\|b_m\|_{\mathbb{L}^6} \\ &\leq \varepsilon_1\|\Delta b_m\|^2 + C_{\varepsilon_1}\|\nabla u_m\|^2_{\mathbb{L}^3}\|\nabla b_m\|^2.\end{aligned} \tag{3.46}$$

On the other hand,

$$\beta \int_{\Omega} \operatorname{div}(|A(u_m)|^2 A(u_m)) \Delta u_m \, dx = \beta \sum_{i,j,k,l,\ell} \int_{\Omega} \partial_l (a_{ij}^2 a_{kl}) \partial_\ell^2 u_{mk} \, dx, \tag{3.47}$$

where $a_{ij} = a_{ij}(u_m) = \partial_i u_{mj} + \partial_j u_{mi}$, $i, j = 1, 2, 3$. Since $a_{ij} = a_{ji}$, we deduce from (3.47) by integration by parts that

$$\begin{aligned} &\beta \int_{\Omega} \operatorname{div}(|A(u_m)|^2 A(u_m)) \Delta u_m \, dx \\ &\quad = -\frac{\beta}{2} \sum_{i,j,k,l,\ell} \int_{\Omega} a_{ij}^2 a_{kl} \partial_\ell^2 a_{kl} \, dx \\ &\quad = \beta \sum_{i,j,k,l,\ell} \int_{\Omega} (a_{ij} \partial_\ell a_{ij})(a_{kl} \partial_\ell a_{kl}) \, dx + \frac{\beta}{2} \sum_{i,j,k,l,\ell} \int_{\Omega} a_{ij}^2 (\partial_\ell a_{kl})^2 \, dx \\ &\quad = \beta \sum_{\ell} \int_{\Omega} \{ (A(u_m) \cdot \partial_\ell A(u_m))^2 + \tfrac{1}{2} |A(u_m)|^2 |\partial_\ell A(u_m)|^2 \} \, dx. \end{aligned} \tag{3.48}$$

Moreover,

$$\begin{aligned} &\Big| \alpha \int_{\Omega} \operatorname{div}[A^2(u_m)] \Delta u_m \, dx \Big| \\ &\qquad \le \frac{|\alpha|}{2\epsilon} \int_{\Omega} |\operatorname{div}[A^2(u_m)]|^2 \, dx + \frac{\epsilon|\alpha|}{2} \int_{\Omega} |\Delta u_m|^2 \, dx \\ &\qquad \le \frac{|\alpha|}{2\epsilon} \sum_{i} \int_{\Omega} |\partial_i [A^2(u_m)]|^2 \, dx + \frac{\epsilon|\alpha|}{2} \int_{\Omega} |\Delta u_m|^2 \, dx \\ &\qquad \le \frac{2|\alpha|}{\epsilon} \sum_{s} \int_{\Omega} |\partial_i A(u_m)|^2 |A(u_m)|^2 \, dx + \frac{\epsilon|\alpha|}{2} \int_{\Omega} |\Delta u_m|^2 \, dx. \end{aligned} \tag{3.49}$$

Since $|\alpha| < \sqrt{\nu\beta/2}$, taking $\epsilon = \frac{\beta\nu + 2|\alpha|^2}{\beta|\alpha|}$ we obtain

$$2|\alpha|/\epsilon < \beta/2, \qquad \epsilon|\alpha|/2 < \nu. \tag{3.50}$$

Taking ε_1 small enough, we deduce from (3.41)–(3.46) and (3.48)–(3.50) that

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u_m\|^2 + \|\nabla b_m\|^2) + c_2 \|\Delta u_m\|^2 + c_3 \|\Delta b_m\|^2 \\ &\quad + c_4 \sum_{i} \int_{\Omega} |A(u_m)|^2 |\partial_i A(u_m)|^2 \, dx \\ &\le C_{\varepsilon_1} (\|\nabla u_m\|_{\mathbb{L}^3}^2 + \|\nabla u_m\|_{\mathbb{L}^3}^{8/3})(\|\nabla u_m\|^2 + \|\nabla b_m\|^2) + \frac{1}{2\varepsilon_1} \|f\|^2 \end{aligned} \tag{3.51}$$

for some positive constants c_2, c_3, c_4. Gronwall's inequality (see, e.g., [40, 34]) then implies that

$$\begin{aligned}(3.52)\qquad &\|\nabla u_m(t)\|^2 + \|\nabla b_m(t)\|^2\\ &\le C\Big(\|\nabla u_0\|^2 + \|\nabla b_0\|^2 + \frac{t}{2\varepsilon_1}\|f\|^2\Big)\\ &\quad \times \exp\Big\{\Big(\int_0^t \|\nabla u_m\|_{\mathbb{L}^3}^3\,dt\Big)^{2/3} t^{1/3} + \Big(\int_0^t \|\nabla u_m\|_{\mathbb{L}^3}^3\,dt\Big)^{8/9} t^{1/9}\Big\}.\end{aligned}$$

Note that (3.11) and the Korn inequality imply that

$$\int_0^t \|\nabla u_m\|_{\mathbb{L}^4}^4\,ds \le C\int_0^t \|A(u_m)\|_{\mathbb{L}^4}^4\,ds \le C, \qquad \forall 0 < t \le T.$$

Since

$$\int_0^t \|\nabla u_m\|_{\mathbb{L}^3}^3\,ds \le \int_0^t \|\nabla u_m\|^2\,ds + \int_0^t \|\nabla u_m\|_{\mathbb{L}^4}^4\,ds,$$

we deduce from (3.11) and (3.52) that

$$(3.53)\qquad \|\nabla u_m(t)\|^2 + \|\nabla b_m(t)\|^2 \le C\Big(\|\nabla u_0\|^2 + \|\nabla b_0\|^2 + \frac{T}{2\varepsilon_1}\|f\|^2\Big)$$

for all $0 < t \le T$, where the positive constant C depends on $\alpha, \beta, \nu, \eta, T$, u_0, b_0. Integrating (3.51), we have

$$\begin{aligned}(3.54)\qquad &\int_0^T \Big\{c_2\|\Delta u_m\|^2 + c_3\|\Delta b_m\|^2 + c_4\sum_i \int_\Omega |A(u_m)|^2|\partial_i A(u_m)|^2\,dx\Big\}\,ds\\ &\qquad\qquad \le C\Big(\|\nabla u_0\|^2 + \|\nabla b_0\|^2 + \frac{T}{2\varepsilon_1}\|f\|^2\Big),\end{aligned}$$

where the positive constant C depends on $\alpha, \beta, \nu, \eta, T, u_0, b_0$.

Now, let us multiply the equations by $\partial_t u_m, \partial_t b_m$ to deduce that

$$\begin{aligned}(3.55)\qquad &\|\partial_t u_m\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u_m\|^2 + \beta\int_\Omega |A(u_m)|^2 A(u_m)\cdot(\nabla u_m)_t\,dx\\ &= -\int_\Omega (u_m\cdot\nabla)u_m\partial_t u_m\,dx + \int_\Omega (b_m\cdot\nabla)b_m\partial_t u_m\,dx\\ &\quad + \alpha\int_\Omega \operatorname{div}(A^2(u_m))\partial_t u_m\,dx + \int_\Omega f_m\partial_t u_m\,dx,\end{aligned}$$

$$(3.56)\qquad \|\partial_t b_m\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla b_m\|^2 = -\int_\Omega \{(u_m\cdot\nabla)b_m + (b_m\cdot\nabla)u_m\}\partial_t b_m\,dx.$$

Similar to (3.48), due to the symmetry of $A(u_m)$, we have

$$(3.57)\qquad \beta\int_\Omega |A(u_m)|^2 A(u_m)\cdot(\nabla u_m)_t\,dx = \frac{\beta}{8}\frac{d}{dt}\int_\Omega |A(u_m)|^4\,dx.$$

On the other hand, note that

$$
\begin{aligned}
(3.58)\quad & \Big|\alpha \int_\Omega \operatorname{div}(A^2(u_m))\partial_t u_m\,dx\Big| \\
&\le \frac{|\alpha|}{2\epsilon}\sum_i \int_\Omega |\partial_i[A^2(u_m)]|^2\,dx + \frac{|\alpha|\epsilon}{2}\|\partial_t u_m\|^2 \\
&\le \frac{2|\alpha|}{\epsilon}\sum_i \int_\Omega |\partial_i A(u_m)|^2|A(u_m)|^2\,dx + \frac{|\alpha|\epsilon}{2}\|\partial_t u_m\|^2,
\end{aligned}
$$

$$
(3.59)\quad \Big|\int_\Omega (u_m\cdot\nabla)u_m\partial_t u_m\,dx\Big| \le C_\varepsilon\|\nabla u_m\|^2\|\nabla u_m\|^2_{\mathbb{L}^3} + \varepsilon\|\partial_t u_m\|^2,
$$

$$
(3.60)\quad \Big|\int_\Omega (b_m\cdot\nabla)b_m\partial_t u_m\,dx\Big| \le C_\varepsilon\|\nabla b_m\|^3\|\Delta b_m\| + \varepsilon\|\partial_t u_m\|^2,
$$

$$
(3.61)\quad \Big|\int_\Omega (b_m\cdot\nabla)u_m\partial_t b_m\,dx\Big| \le C_\varepsilon\|\nabla b_m\|^2\|\nabla u_m\|^2_{\mathbb{L}^3} + \varepsilon\|\partial_t b_m\|^2,
$$

$$
(3.62)\quad \Big|\int_\Omega (u_m\cdot\nabla)b_m\partial_t b_m\,dx\Big| \le C_\varepsilon\|\nabla u_m\|^2\|\nabla b_m\|\,\|\Delta b_m\| + \varepsilon\|\partial_t b_m\|^2,
$$

$$
(3.63)\quad \Big|\int_\Omega f_m\partial_t u_m\,dx\Big| \le C_\varepsilon\|f_m\|^2 + \varepsilon\|\partial_t u_m\|^2.
$$

Inserting (3.57)–(3.63) into (3.55) and (3.56), we obtain

$$
\begin{aligned}
(3.64)\quad & \|\partial_t u_m\|^2 + \frac{d}{dt}\Big(\|\nabla u_m\|^2 + \frac{\beta}{4}\|A(u_m)\|^4_{\mathbb{L}^4}\Big) \\
&\le C\sum_i \int_\Omega |\partial_i A(u_m)|^2|A(u_m)|^2\,dx + C\|\nabla b_m\|^3\|\Delta b_m\| \\
&\quad + C\{\|\nabla u_m\|^2\|\nabla u_m\|^2_{\mathbb{L}^3} + \|f_m\|^2\},
\end{aligned}
$$

$$
\begin{aligned}
(3.65)\quad & \|\partial_t b_m\|^2 + \frac{d}{dt}\|\nabla b_m\|^2 \\
&\le C\{\|\nabla u_m\|^2\|\nabla b_m\|\,\|\Delta b_m\| + \|\nabla b_m\|^2\|\nabla u_m\|^2_{\mathbb{L}^3}\}.
\end{aligned}
$$

Integrating (3.64) and (3.65) over $[0,t]$, $0 < t \le T$, and taking (3.52) and (3.54) into consideration, we deduce that

$$
\begin{aligned}
& \int_0^T (\|\partial_t u_m\|^2 + \|\partial_t b_m\|^2)\,dt + \|\nabla b_m(t)\|^2 + \|\nabla u_m(t)\|^2 + \frac{\beta}{4}\|A(u_m)(t)\|^4_{L^4} \\
&\quad \le \|\nabla u_0\|^2 + \|\nabla b_0\|^2 + \frac{\beta}{4}\|\nabla u_0\|^4_{L^4} + C\sum_i \int_{\Omega_T} |\partial_i A(u_m)|^2|A(u_m)|^2\,dx\,dt \\
&\qquad + \int_0^T (\|\nabla b_m\|^3 + \|\nabla u_m\|^2\|\nabla b_m\|)\|\Delta b_m\|\,dt \\
&\qquad + \int_0^T \{(\|\nabla b_m\|^2 + \|\nabla u_m\|^2)\|\nabla u_m\|^2_{\mathbb{L}^3} + \|f_m\|^2\}\,dt \\
&\quad \le C
\end{aligned}
$$

where $\Omega_T = \Omega \times (0,T)$, and the positive constant C depends on $\alpha, \beta, \nu, \eta, T$, u_0, b_0. Similar to [17, Section 4] (see also [34]), this combined with (3.54) gives (3.4) by letting $m \to \infty$. ∎

THEOREM 3.4. *Let (u_1, b_1) and (u_2, b_2) be weak solutions of system* (1.1) *corresponding to initial data (u_{10}, b_{10}) and (u_{20}, b_{20}) respectively. Assume that $|\alpha| < \sqrt{2\nu\beta}$ and $(u_2, b_2) \in L^\infty(0,T; \mathbb{W} \times \mathbb{V})$. Then, for $0 \le t \le T$, we have*

$$\begin{aligned} \|u_1(t) - u_2(t)\|^2 + \|b_1(t) - b_2(t)\|^2 \\ \le C(\|u_{10} - u_{20}\|^2 + \|b_{10} - b_{20}\|^2). \end{aligned} \tag{3.66}$$

Furthermore, for $0 < l < T/2$,

$$\begin{aligned} \|u_1 - u_2\|^2_{L^2(l,2l;\mathbb{V})} + \|b_1 - b_2\|^2_{L^2(l,2l;\mathbb{V})} \\ \le C(\|u_1 - u_2\|^2_{L^2(0,l;\mathbb{H})} + \|b_1 - b_2\|^2_{L^2(0,l;\mathbb{H})}). \end{aligned} \tag{3.67}$$

$$\begin{aligned} \|\partial_t u_1 - \partial_t u_2\|_{L^1(l,2l;(\dot{\mathbb{W}}^{3,2}_p(\Omega))^*)} + \|\partial_t b_1 - \partial_t b_2\|_{L^1(l,2l;(\dot{\mathbb{W}}^{3,2}_p(\Omega))^*)} \\ \le C(\|u_1 - u_2\|_{L^2(l,2l;\mathbb{V})} + \|b_1 - b_2\|_{L^2(l,2l;\mathbb{V})}). \end{aligned} \tag{3.68}$$

Proof. Set $\widetilde{u} = u_1 - u_2$ and $\widetilde{b} = b_1 - b_2$. It is obvious that $(\widetilde{u}, \widetilde{b})$ satisfies the system

$$\begin{cases} \partial_t \widetilde{u} + \nu \mathcal{A}\widetilde{u} + \mathcal{B}(u_1, \widetilde{u}) + B(\widetilde{u}, u_2) - \mathcal{B}(b_1, \widetilde{b}) - \mathcal{B}(\widetilde{b}, b_2) \\ \quad + (\mathcal{J}(u_1) - \mathcal{J}(u_2)) + \mathcal{K}(u_1) - \mathcal{K}(u_2) = 0, \\ \partial_t \widetilde{b} + \mathcal{B}(\widetilde{u}, b_1) + \mathcal{B}(u_2, \widetilde{b}) - \mathcal{B}(\widetilde{b}, u_2) - \mathcal{B}(b_1, \widetilde{u}) + \eta \mathcal{A}\widetilde{b} = 0, \\ \nabla \cdot \widetilde{b} = \nabla \cdot \widetilde{u} = 0, \\ \widetilde{u}_0 = u_{10} - u_{20}, \quad \widetilde{b}_0 = b_{10} - b_{20}. \end{cases} \tag{3.69}$$

Multiplying (3.69) by $\widetilde{u}, \widetilde{b}$ and integrating, bearing in mind that

$$\mathcal{T}(u) = (\nu - \nu\delta_0)\mathcal{A}u + (1 - \delta_0)\mathcal{J}(u) + \mathcal{K}(u), \qquad \delta_0 = 1 - \sqrt{\alpha^2/(2\nu\beta)} \in (0,1),$$

we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|^2 + \nu\delta_0\|\nabla\widetilde{u}\|^2 + \delta_0\langle \mathcal{J}(u_1) - \mathcal{J}(u_2), \widetilde{u}\rangle_{\mathbb{W}^*,\mathbb{W}} \\ = -\langle \mathcal{T}(u_1) - \mathcal{T}(u_2), \widetilde{u}\rangle_{\mathbb{W}^*,\mathbb{W}} - \int_\Omega (\widetilde{u} \cdot \nabla) u_2 \widetilde{u}\, dx \\ + \int_\Omega (b_1 \cdot \nabla)\widetilde{b}\widetilde{u}\, dx + \int_\Omega (\widetilde{b} \cdot \nabla) b_2 \widetilde{u}\, dx, \end{aligned} \tag{3.70}$$

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{b}\|^2 + \eta\|\nabla\widetilde{b}\|^2 = -\int_\Omega \{(\widetilde{u} \cdot \nabla) b_1 + (\widetilde{b} \cdot \nabla) u_2 + (b_1 \cdot \nabla)\widetilde{u}\}\widetilde{b}\, dx. \tag{3.71}$$

Since $\mathcal{J}$ and $\mathcal{T}$ are monotone operators (see Lemmas 2.1 and 2.2), and

$$\int_\Omega (b_1\cdot\nabla)\widetilde{b}\widetilde{u}\,dx = -\int_\Omega (b_1\cdot\nabla)\widetilde{u}\widetilde{b}\,dx,$$

we deduce from (3.70) and (3.71) that

$$\begin{aligned}(3.72)\qquad \frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|^2+\|\widetilde{b}\|^2)+\nu\delta_0\|\nabla\widetilde{u}\|^2+\eta\|\nabla\widetilde{b}\|^2 &\le \Big|\int_\Omega(\widetilde{u}\cdot\nabla)u_2\widetilde{u}\,dx\Big| + \Big|\int_\Omega(\widetilde{b}\cdot\nabla)b_2\widetilde{u}\,dx\Big| \\ &\quad + \Big|\int_\Omega(\widetilde{u}\cdot\nabla)b_1\widetilde{b}\,dx\Big| + \Big|\int_\Omega(\widetilde{b}\cdot\nabla)u_2\widetilde{b}\,dx\Big|.\end{aligned}$$

Note that, for $0\le t\le T$,

$$(3.73)\qquad \Big|\int_\Omega(\widetilde{u}\cdot\nabla)u_2\widetilde{u}\,dx\Big| \le C\|\widetilde{u}\|^{1/2}\|\nabla\widetilde{u}\|^{3/2} \le \varepsilon\|\nabla\widetilde{u}\|^2 + C_\varepsilon\|\widetilde{u}\|^2.$$

Similarly

$$(3.74)\qquad \Big|\int_\Omega(\widetilde{b}\cdot\nabla)u_2\widetilde{b}\,dx\Big| \le \varepsilon\|\nabla\widetilde{b}\|^2 + C_\varepsilon\|\widetilde{b}\|^2,$$

$$\begin{aligned}(3.75)\qquad \Big|\int_\Omega(\widetilde{b}\cdot\nabla)b_2\widetilde{u}\,dx\Big| &\le \|\nabla b_2\|\,\|\widetilde{u}\|_{\mathbb{L}^4}\|\widetilde{b}\|_{\mathbb{L}^4} \\ &\le C\|\widetilde{b}\|^{1/4}\|\nabla\widetilde{b}\|^{3/4}\|\widetilde{u}\|^{1/4}\|\nabla\widetilde{u}\|^{3/4} \\ &\le (\|\widetilde{b}\|^{1/2}+\|\widetilde{u}\|^{1/2})(\|\nabla\widetilde{u}\|^{3/2}+\|\nabla\widetilde{b}\|^{3/2}) \\ &\le C_\varepsilon(\|\widetilde{b}\|^2+\|\widetilde{u}\|^2)+\varepsilon(\|\nabla\widetilde{u}\|^2+\|\nabla\widetilde{b}\|^2),\end{aligned}$$

$$(3.76)\qquad \Big|\int_\Omega(\widetilde{u}\cdot\nabla)b_1\widetilde{b}\,dx\Big| \le C_\varepsilon(\|\widetilde{b}\|^2+\|\widetilde{u}\|^2)+\varepsilon(\|\nabla\widetilde{u}\|^2+\|\nabla\widetilde{b}\|^2).$$

Combining (3.72)–(3.76) and taking ε small enough, we get

$$(3.77)\qquad \frac{d}{dt}(\|\widetilde{u}\|^2+\|\widetilde{b}\|^2)+\nu\delta_0\|\nabla\widetilde{u}\|^2+\eta\|\nabla\widetilde{b}\|^2 \le C(\|\widetilde{u}\|^2+\|\widetilde{b}\|^2),$$

which implies that, for any $0\le s<t\le T$,

$$(3.78)\qquad \|\widetilde{u}(t)\|^2+\|\widetilde{b}(t)\|^2 \le e^{C(t-s)}(\|\widetilde{u}(s)\|^2+\|\widetilde{b}(s)\|^2).$$

This gives (3.66) by taking $s=0$. In particular, if $u_{10}=u_{20}$ and $b_{10}=b_{20}$, then $u_1(t)\equiv u_2(t)$ and $b_1(t)\equiv b_2(t)$.

Now for $l>0$, we take $s\in(0,l)$ and integrate (3.77) over $(s,2l)$ to obtain

$$\begin{aligned}(3.79)\qquad \|\widetilde{u}(2l)\|^2+\|\widetilde{b}(2l)\|^2+\int_s^{2l}(\nu\delta_0\|\nabla\widetilde{u}(t)\|^2+\eta\|\nabla\widetilde{b}(t)\|^2)\,dt \\ \le C\int_s^{2l}(\|\widetilde{u}(t)\|^2+\|\widetilde{b}(t)\|^2)\,dt+\|\widetilde{u}(s)\|^2+\|\widetilde{b}(s)\|^2.\end{aligned}$$

Using (3.78), we have

$$\int_s^{2l} (\|\nabla\widetilde{u}(t)\|^2 + \|\nabla\widetilde{b}(t)\|^2)\,dt \le C(\|\widetilde{u}(s)\|^2 + \|\widetilde{b}(s)\|^2). \tag{3.80}$$

Integrating (3.80) with respect to s over $(0, l)$, we get (3.67) immediately.

Finally, let us prove (3.68). Taking (φ, ψ) from the unit ball in $L^\infty(l, 2l; \dot{\mathbb{W}}_p^{3,2}(\Omega) \times \dot{\mathbb{W}}_p^{3,2}(\Omega))$ as test functions in (3.69), we deduce that

$$\begin{aligned} \Big|\int_l^{2l} \langle \partial_t\widetilde{u}, \varphi\rangle\,ds\Big| &\le \nu\int_l^{2l} |\langle \mathcal{A}\widetilde{u}, \varphi\rangle|\,ds + \int_l^{2l} |\langle \mathcal{J}(u_1) - \mathcal{J}(u_2), \varphi\rangle|\,ds \\ &\quad + \int_l^{2l} |\langle -\mathcal{B}(u_1, \widetilde{u}) - \mathcal{B}(\widetilde{u}, u_2) + \mathcal{B}(b_1, \widetilde{b}) + \mathcal{B}(\widetilde{b}, b_2), \varphi\rangle|\,ds \\ &\quad + \int_l^{2l} |\langle \mathcal{K}(u_1) - \mathcal{K}(u_2), \varphi\rangle|\,ds, \end{aligned} \tag{3.81}$$

$$\begin{aligned} \Big|\int_l^{2l} \langle \partial_t\widetilde{b}, \psi\rangle\,ds\Big| &\le \eta\int_l^{2l} |\langle \mathcal{A}b, \psi\rangle|\,ds \\ &\quad + \int_l^{2l} |\langle \mathcal{B}(\widetilde{u}, b_1) + \mathcal{B}(u_2, \widetilde{b}) - \mathcal{B}(\widetilde{b}, u_2) - \mathcal{B}(b_1, \widetilde{u}), \psi\rangle|\,ds. \end{aligned} \tag{3.82}$$

Let us estimate the terms on the right hand sides of (3.81), (3.82) one by one to derive (3.68). From the imbedding of Sobolev spaces we know that

$$\begin{aligned} &\|\varphi\|_{L^2(l,2l;\mathbb{V})}, \|\psi\|_{L^2(l,2l;\mathbb{V})} \le C, \\ &\|\psi\|_{L^\infty(l,2l;\mathbb{L}^\infty(\Omega))}, \|\varphi\|_{L^\infty(l,2l;\mathbb{L}^\infty(\Omega))} \le C, \\ &\|\nabla\psi\|_{L^\infty(l,2l;\mathbb{L}^\infty(\Omega))}, \|\nabla\varphi\|_{L^\infty(l,2l;\mathbb{L}^\infty(\Omega))} \le C. \end{aligned}$$

Thus it is easy to infer that

$$\nu\int_l^{2l} |\langle \mathcal{A}\widetilde{u}, \varphi\rangle|\,ds \le \nu\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})}\|\varphi\|_{L^2(l,2l;\mathbb{V})} \le C\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})}, \tag{3.83}$$

$$\eta\int_l^{2l} |\langle \mathcal{A}\widetilde{b}, \psi\rangle|\,ds \le \eta\|\widetilde{b}\|_{L^2(l,2l;\mathbb{V})}\|\psi\|_{L^2(l,2l;\mathbb{V})} \le C\|\widetilde{b}\|_{L^2(l,2l;\mathbb{V})}, \tag{3.84}$$

and also

$$\begin{aligned} &\int_l^{2l} |\langle \mathcal{B}(u_1, \widetilde{u}) + \mathcal{B}(\widetilde{u}, u_2) - B(b_1, \widetilde{b}) - \mathcal{B}(\widetilde{b}, b_2), \varphi\rangle|\,ds \\ &\qquad \le C(\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})} + \|\widetilde{b}\|_{L^2(l,2l;\mathbb{V})}), \end{aligned} \tag{3.85}$$

$$\int_l^{2l} |\langle \mathcal{B}(\widetilde{u}, b_1) + \mathcal{B}(u_2, \widetilde{b}) - \mathcal{B}(\widetilde{b}, u_2) - \mathcal{B}(b_1, \widetilde{u}), \psi\rangle|\, ds \leq C(\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})} + \|\widetilde{b}\|_{L^2(l,2l;\mathbb{V})}). \tag{3.86}$$

Furthermore, since $\nabla\varphi$ is bounded, we deduce that

$$\begin{aligned}
&\int_l^{2l} |\langle \mathcal{K}(u_1) - \mathcal{K}(u_2), \varphi\rangle|\, ds \\
&= \int_l^{2l} \Big| \int_\Omega (A^2(u_1) - A^2(u_2)) \cdot \nabla\varphi\, dx \Big|\, ds \\
&= \int_l^{2l} \Big| \int_\Omega \{A(\widetilde{u})A(u_1) + A(u_2)A(\widetilde{u})\} \cdot \nabla\varphi\, dx \Big|\, ds \\
&\leq C\|A(\widetilde{u})\|_{L^2(l,2l;\mathbb{L}^2(\Omega))} \big(\|A(u_1)\|_{L^2(l,2l;\mathbb{L}^2(\Omega))} + \|A(u_2)\|_{L^2(l,2l;\mathbb{L}^2(\Omega))}\big) \\
&\leq C\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})} (\|u_1\|_{L^2(l,2l;\mathbb{V})} + \|u_2\|_{L^2(l,2l;\mathbb{V})}) \leq C\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})},
\end{aligned} \tag{3.87}$$

where we have used the fact that

$$A^2(u_1) - A^2(u_2) = (A(u_1) - A(u_2))A(u_1) + A(u_2)(A(u_1) - A(u_2))$$

for the second equality. Similarly, we have

$$\begin{aligned}
&\int_l^{2l} |\langle \mathcal{J}(u_1) - \mathcal{J}(u_2), \varphi\rangle|\, ds \\
&= \int_l^{2l} \Big| \int_\Omega \big(|A(u_1)|^2 A(u_1) - |A(u_2)|^2 A(u_2)\big) \cdot \nabla\varphi\, dx \Big|\, ds \\
&\leq C \int_l^{2l} \int_\Omega \big\{ |A(u_1)|^2 |A(\widetilde{u})| + \big(|A(u_1)| + |A(u_2)|\big) |A(u_2)|\, |A(\widetilde{u})| \big\} \\
&\leq C\big(\|A(u_1)\|^2_{L^4(l,2l;\mathbb{L}^4(\Omega))} + \|A(u_2)\|^2_{L^4(l,2l;\mathbb{L}^4(\Omega))}\big) \|A(\widetilde{u})\|_{L^2(l,2l;\mathbb{L}^2(\Omega))} \\
&\leq C\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})},
\end{aligned} \tag{3.88}$$

where, for the first inequality, we have used the observation that

$$\begin{aligned}
&|A(u_1)|^2 A(u_1) - |A(u_2)|^2 A(u_2) \\
&\quad = |A(u_1)|^2 A(\widetilde{u}) + \big(|A(u_1)| - |A(u_2)|\big)\big(|A(u_1)| + |A(u_2)|\big) A(u_2).
\end{aligned}$$

Plugging (3.83)–(3.88) into (3.81) and (3.82), we conclude that

$$\Big| \int_l^{2l} \langle \partial_t \widetilde{u}, \varphi\rangle\, ds \Big| + \Big| \int_l^{2l} \langle \partial_t \widetilde{b}, \psi\rangle\, ds \Big| \leq C(\|\widetilde{u}\|_{L^2(l,2l;\mathbb{V})} + \|\widetilde{b}\|_{L^2(l,2l;\mathbb{V})}) \tag{3.89}$$

for any (φ, ψ) belonging to the unit ball of $L^\infty(l, 2l; \dot{\mathbb{W}}_p^{3,2}(\Omega) \times \dot{\mathbb{W}}_p^{3,2}(\Omega))$. Taking the supremum over all (φ, ψ) we obtain (3.68). ∎

Remark 3.5. Recall that, in the absence of a magnetic field, Hamza and Paicu [17] have proved the uniqueness of weak solutions for the reduced third grade fluid system. For the coupled system (1.1), we have not been able to prove the uniqueness of weak solutions due to the lack of regularity of the magnetic field b. However, under some additional assumptions, such as $b \in L^4(0, T; \mathbb{V})$, the uniqueness result could be recovered for system (1.1). However, we will not pursue this in detail, since the result is well known for Navier–Stokes equations and the proof is also not difficult.

4. Large time behavior. In this section, we use the l-trajectory method developed in [24, 26] to prove the existence of a finite-dimensional global attractor and an exponential attractor for system (1.1). For the convenience of the reader, we first give a brief review of the l-trajectory method.

4.1. The l-trajectory method

Definition 4.1 ([34, 3, 42, 27]). Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X. A subset $\mathscr{A} \subset X$ is called a *global attractor* for the semigroup if $\mathscr{A}$ enjoys the following properties:

(i) $\mathscr{A}$ is compact in X,
(ii) $\mathscr{A}$ is invariant, i.e., $S(t)\mathscr{A} = \mathscr{A}$ for any $t \geq 0$,
(iii) $\mathscr{A}$ attracts every bounded subset of X, i.e.,

$$\forall B \subset X \text{ bounded}, \qquad \lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathscr{A}) = 0,$$

where dist is the Hausdorff semidistance between sets in X, defined as

$$\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_X, \qquad \forall A, B \subset X.$$

Definition 4.2 ([34, 42, 27]). The *fractal dimension* of a compact set K in a Banach space X is defined as

$$d_f(K) = \limsup_{\epsilon\to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon}$$

where $N_\epsilon(K)$ is the minimal number of balls of radius ϵ in X needed to cover K.

Definition 4.3 ([12, 27, 11]). Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X. A subset $\mathscr{E} \subset X$ is called an *exponential attractor* for the semigroup if $\mathscr{E}$ is compact in X and enjoys the following properties:

(i) $\mathscr{E}$ is positively invariant, i.e., $S(t)\mathscr{E} \subseteq \mathscr{E}$ for any $t \geq 0$,
(ii) the fractal dimension $d_f(\mathscr{E})$ is finite,

(iii) $\mathscr{E}$ attracts bounded subsets of X exponentially, i.e.,

$$\forall B \subset X \text{ bounded}, \quad \operatorname{dist}(S(t)B, \mathscr{E}) \leq Q(\|B\|_X)e^{-ct},$$

where the positive constant c and the monotonic function Q are independent of B.

Let X, Y, Z be three Banach spaces, with X being reflexive and separable, such that $Y \hookrightarrow\hookrightarrow X$ (compact imbedding) and $X \hookrightarrow Z$. For $p_1 \in [2, \infty)$, $p_2 \in [1, \infty)$ and fixed $\tau > 0$, let

$$X_\tau = L^2(0, \tau; X), \qquad Y_\tau = \{u \in L^{p_1}(0, \tau; Y) : u' \in L^{p_2}(0, \tau; Z)\}.$$

Thanks to the well known Aubin–Simon type compactness results (see for example [41, 34, 39]), we have $Y_\tau \hookrightarrow\hookrightarrow X_\tau$.

Consider an abstract autonomous evolutionary problem

$$\begin{aligned} &\partial_t u = F(u(t)) \quad \text{in } X \ (t > 0), \\ &u(0) = u_0, \end{aligned} \tag{4.1}$$

where $F : X \to X$ is a nonlinear operator, $u_0 \in X$. Assume that, for any $T > 0$, system (4.1) admits at least one solution $u \in C(0, T; X_w)$, the space of weakly continuous functions from the interval $[0, T]$ to the Banach space X. Then for any fixed $l > 0$, the *l-trajectory* is any solution on the time interval $[0, l]$. The set of all l-trajectories is denoted by $\mathscr{X}_l$ and equipped with the topology of $X_l = L^2(0, l; X)$.

Now, we formulate the assumptions made in [26], which guarantee the existence of a global attractor and an exponential attractor for problem (4.1).

(A_1) For any $u_0 \in X$ and any $T > 0$, there exists a (not necessarily unique) solution $u \in C([0, T]; X_w) \cap Y_T$ to problem (4.1) on $[0, T]$ with $u(0) = u_0$. Moreover, for any solution the estimates of $\|u\|_{Y_T}$ are uniform with respect to $\|u_0\|_X$.

(A_2) There exists a bounded set $B^0 \subset X$ with the following properties: if u is an arbitrary solution with initial condition $u_0 \in X$ then (i) there exists $t_0 = t_0(\|u_0\|_X)$ such that $u(t) \in B^0$ for all $t \geq t_0$ and (ii) if $u_0 \in B^0$ then $u(t) \in B^0$ for all $t \geq 0$.

(A_3) Each l-trajectory has among all solutions a unique continuation, i.e., from an end point of an l-trajectory there starts at most one solution.

By (A_3), we can define a semigroup L_t on $\mathscr{X}_l$ as

$$L_t(\chi)(\tau) := u(t + \tau), \qquad \forall \tau \in [0, l],$$

where u is the unique solution on $[0, l + \tau]$ such that $u|_{[0,l]} = \chi$.

(A_4) For all $t > 0$, the operator $L_t : X_l \to X_l$ is continuous on $\mathscr{B}^0_l = \{\chi \in \mathscr{X}_l : \chi(0) \in B^0\}$.

(A_5) For some $\tau_0 > 0$, the closure of $L_{\tau_0}(\mathscr{B}_l^0)$ in X_l (denoted by $\mathscr{B}_l^1$) is included in $\mathscr{B}_l^0$, i.e., $\overline{L_{\tau_0}(\mathscr{B}_l^0)}^{X_l}$ $(\doteq \mathscr{B}_l^1) \subset \mathscr{B}_l^0$.

(A_6) There exist a space W_l and a constant $\tau > 0$ such that $W_l \hookrightarrow\hookrightarrow X_l$ and $L_\tau : X_l \to W_l$ is Lipschitz continuous on $\mathscr{B}_l^1$.

(A_7) The map $e : X_l \to X$, $e(\chi) = \chi(l)$, is continuous on $\mathscr{B}_l^1$.

(A_8) The map $e : X_l \to X$ is Hölder-continuous on $\mathscr{B}_l^1$.

(A_9) For all $\tau > 0$, the operators $L_t : X_l \to X_l$ are uniformly (with respect to $t \in [0, \tau]$) Lipschitz continuous on $\mathscr{B}_l^1$.

(A_{10}) For all $\tau > 0$, there exist $c > 0$ and $\gamma \in (0, 1]$ such that for all $\chi \in \mathscr{B}_l^1$ and $t_1, t_2 \in [0, \tau]$,

$$\|L_{t_1}(\chi) - L_{t_2}(\chi)\|_{X_l} \leq c|t_1 - t_2|^\gamma.$$

Note that the map $e : X_l \to X$ assigns to every l-trajectory its end point. It is meaningful since all trajectories are weakly continuous. Let $B^1 = e(\mathscr{B}_l^1)$. Thanks to (A_3), for every initial condition $u_0 \in B^1$, there exists a unique solution to (4.1), hence the solution operators S_t are well defined on B^1. Moreover, it is not difficult to find that B^1 is positively invariant with respect to S_t.

Now, under some of the assumptions above, we state the existence of a finite-dimensional global attractor and an exponential attractor for the dynamical system (S_t, B^1).

Theorem 4.4 ([26]). *Let assumptions* (A_1)–(A_5) *and* (A_7) *hold. Then there exists a global attractor* $\mathscr{A}$ *for the dynamical system* (S_t, B^1). *Moreover, if assumptions* (A_6), (A_8) *are satisfied then the fractal dimension of the attractor is finite.*

Theorem 4.5 ([26]). *Let* X *be a separable Hilbert space and let assumptions* (A_1)–(A_6) *and* (A_8)–(A_{10}) *hold. Then there exists an exponential attractor* $\mathscr{E}$ *for the dynamical system* (S_t, B^1).

4.2. Existence of global attractors and exponential attractors. The main results of this section can be stated as follows.

Theorem 4.6. *Assume that* $(u_0, b_0) \in \mathbb{H} \times \mathbb{H}$, $f \in \mathbb{H}$, $\beta > 0$ *and* $|\alpha| < \sqrt{\nu\beta/2}$. *Then system* (1.1) *has a finite-dimensional global attractor* $\mathscr{A}$ *in* $\mathbb{H} \times \mathbb{H}$.

Theorem 4.7. *Assume that* $(u_0, b_0) \in \mathbb{H} \times \mathbb{H}$, $f \in \mathbb{H}$, $\beta > 0$ *and* $|\alpha| < \sqrt{\nu\beta/2}$. *Then system* (1.1) *has an exponential attractor* $\mathscr{E}$ *in* $\mathbb{H} \times \mathbb{H}$.

Proof of Theorem 4.3. Thanks to Theorem 4.4, to prove the above theorem it is enough to check assumptions (A_1)–(A_6) and (A_8). Now let us check them one by one.

Assumption (A_1). For our problem, we set

$$X = \mathbb{H} \times \mathbb{H} \quad \text{and} \quad X_l = L^2(0, l; \mathbb{H} \times \mathbb{H}),$$

and

$$Y_l = \big\{(u, b) : u \in L^4(0, l; \mathbb{W}),\ \partial_t u \in L^{4/3}(0, l; \mathbb{W}^*) \\ \text{and } b \in L^2(0, l; \mathbb{V}),\ \partial_t b \in L^2(0, l; \mathbb{V}^*)\big\}.$$

Thanks to Theorem 3.3, for any $T > 0$ and any $(u_0, b_0) \in X$, there exists a weak solution $(u, b) \in C([0, T]; X)$ such that for any $\varphi \in L^4(0, T; \mathbb{W})$ and $\psi \in L^2(0, T; \mathbb{V})$,

$$\int_0^T \langle \partial_t u, \varphi\rangle_{\mathbb{W}^*,\mathbb{W}}\, dt = -\int_{\Omega_T} (u \cdot \nabla) u\varphi\, dx\, dt + \int_{\Omega_T} (b \cdot \nabla) b\varphi\, dx\, dt \\ - \int_{\Omega_T} \big\{(\nu\nabla u + \alpha A^2(u) + \beta|A(u)|^2 A(u)) \cdot \nabla\varphi + f\varphi\big\}\, dx\, dt, \tag{4.2}$$

$$\int_0^T \langle \partial_t b, \psi\rangle_{\mathbb{V}^*,\mathbb{V}}\, dt = -\int_{\Omega_T} \big\{[(u \cdot \nabla) b + (b \cdot \nabla) u]\psi - \eta\nabla b \cdot \nabla\psi\big\}\, dx\, dt. \tag{4.3}$$

Moreover,

$$\|u(t)\|^2_{L^2(0,T;\mathbb{V})} + \|A(u)(t)\|^4_{L^4(0,T;L^4(\Omega))} + \|b(t)\|^2_{L^2(0,T;\mathbb{V})} \\ \le C(\|u_0\|^2 + \|b_0\|^2 + \|f\|^2), \tag{4.4}$$

where the positive constant C depends on ν, α, β, T.

On the other hand, similar to (3.12)–(3.18) we can deduce from (4.2) and (4.4) that, for any weak solution (u, b),

$$|\langle \partial_t u, \varphi\rangle_{L^{4/3}(0,T;\mathbb{W}^*),L^4(0,T;\mathbb{W})}| \\ \le C\|u\|^{3/2}_{L^2(0,T;\mathbb{V})}\|\varphi\|_{L^4(0,T;\mathbb{V})} + C\|b\|^{3/2}_{L^2(0,T;\mathbb{V})}\|\varphi\|_{L^4(0,T;\mathbb{V})} \\ + C\|u\|_{L^2(0,T;\mathbb{V})}\|\varphi\|_{L^2(0,T;\mathbb{V})} + C\|A(u)\|^2_{L^4(0,T;\mathbb{L}^4(\Omega))}\|\varphi\|_{L^2(0,T;\mathbb{V})} \\ + C\|A(u)\|^3_{L^4(0,T;\mathbb{L}^4(\Omega))}\|\varphi\|_{L^4(0,T;\mathbb{W})} + CT\|f\|\,\|\varphi\|_{L^2(0,T;\mathbb{V})} \\ \le C(\|u\|^2_{L^2(0,T;\mathbb{V})} + \|b\|^2_{L^2(0,T;\mathbb{V})} + \|A(u)\|^4_{L^4(0,T;\mathbb{L}^4(\Omega))} + 1)\|\varphi\|_{L^4(0,T;\mathbb{W})} \\ \le C(\|u_0\|, \|b_0\|)\|\varphi\|_{L^4(0,T;\mathbb{W})}. \tag{4.5}$$

Also we can deduce from (4.3) and (4.4) that

$$|\langle \partial_t b, \psi\rangle_{L^2(0,T;\mathbb{V}^*),L^2(0,T;\mathbb{V})}| \le C\|u\|^{4/7}_{L^4(0,T;\mathbb{W})}\|b\|^{1/2}_{L^2(0,T;\mathbb{V})}\|\psi\|_{L^2(0,T;\mathbb{V})} \\ + C\|\nabla u\|_{L^3(0,T;\mathbb{L}^3(\Omega))}\|\psi\|_{L^2(0,T;\mathbb{V})} + C\|b\|_{L^2(0,T;\mathbb{V})}\|\psi\|_{L^2(0,T;\mathbb{V})} \\ \le C(\|u_0\|, \|b_0\|)\|\psi\|_{L^2(0,T;\mathbb{V})}. \tag{4.6}$$

From (4.5) and (4.6), we then conclude that

$$\|\partial_t u\|_{L^{4/3}(0,T;\mathbb{W}^*)} + \|\partial_t b\|_{L^2(0,T;\mathbb{V}^*)} \le C(\|u_0\|, \|b_0\|). \tag{4.7}$$

Now (A$_1$) follows from (4.4), (4.7) and the Korn inequality.

Assumption (A$_2$). Taking $\varphi = u$ and $\psi = v$ as test functions, similar to (3.9) in the proof of Theorem 3.3, we can deduce that

$$\frac{d}{dt}(\|u\|^2 + \|b\|^2) + \|\nabla u\|^2 + \|A(u)\|_{\mathbb{L}^4}^4 + \|\nabla b\|^2 \le C\|f\|^2,$$

where the positive constant C only depends on ν, α, β, η. Using the Poincaré inequality

$$\|u\| \le C_1\|\nabla u\|, \qquad \|b\| \le C_1\|\nabla b\|,$$

and Gronwall's inequality, we get

$$\|u(t)\|^2 + \|b(t)\|^2 \le (\|u_0\|^2 + \|b_0\|^2)e^{-C_1 t} + \frac{C\|f\|^2}{C_1}(1 - e^{-C_1 t}). \tag{4.8}$$

Taking B^0 as the ball centered at zero with radius $\sqrt{C/C_1}\,\|f\|$ in X, we obtain (A$_2$).

Assumption (A$_3$). For any trajectory $\chi \in \mathscr{X}_l$, there exists $0 < \tau < l$ such that $\chi(\tau) \in \mathbb{W} \times \mathbb{V}$, since $\chi \in L^4(0, l; \mathbb{W}) \times L^2(0, l; \mathbb{V})$. Thanks to Theorem 3.3, we know that there exists a solution $(u, b) \in L^\infty(0, T; \mathbb{W}\times\mathbb{V}) \cap L^2(0, T; \dot{\mathbb{H}}^2_p(\Omega)\times\dot{\mathbb{H}}^2_p(\Omega))$ starting from $\chi(\tau)$ with $(\partial_t u, \partial_t b) \in L^2(0, T; \mathbb{H}\times\mathbb{H})$ for any $T > 0$. Thanks to Theorem 3.4, such a solution is unique in the class of weak solutions. Thus, we obtain (A$_3$).

Assumption (A$_4$). Set $\mathscr{B}^0_l = \{\chi \in \mathscr{X}_l : \chi(0) \in B^0\}$, where B^0 is the set derived in (A$_2$). For any $\chi \in \mathscr{B}^0_l$, there exists a $s_0 \in (0, l/2)$ such that $\chi(s_0) \in \mathbb{W} \times \mathbb{V}$. Thanks to (A$_3$), there exists a unique solution $(u, b) \in L^\infty(s_0, T; \mathbb{W}\times\mathbb{V})$ starting from $\chi(s_0)$. The semigroup L_t is well defined on $\mathscr{X}_l$,

$$L_t(\chi)(\tau) := (u(t + \tau), b(t + \tau)), \qquad \forall\tau \in [0, l],$$

where (u, b) is the unique solution on $[0, l + \tau]$ such that $(u, b)|_{[0,l]} = \chi$.

For any $t > 0$, let (u_1, b_1) and (u_2, b_2) be solutions on $[0, t + l]$ whose restrictions to $[0, l]$ are χ_1, χ_2 respectively. Let $\tau \in [l/2, t + l/2]$. For any $s \in (0, l/2)$, thanks to Theorem 3.4 (see (3.78)), we have

$$\begin{aligned} &\|u_1(\tau + s) - u_2(\tau + s)\|^2 + \|b_1(\tau + s) - b_2(\tau + s)\|^2 \\ &\qquad\le e^{Ct}\|\chi_1(l/2 + s) - \chi_2(l/2 + s)\|^2. \end{aligned} \tag{4.9}$$

Integrating (4.9) with respect to s on $(0, l/2)$, we get

$$\begin{aligned} &\int_\tau^{\tau+l/2} \left(\|u_1(s) - u_2(s)\|^2 + \|b_1(s) - b_2(s)\|^2\right) ds \\ &\qquad\le e^{Ct}\|\chi_1 - \chi_2\|^2_{L^2(l/2,l;X)} \le e^{Ct}\|\chi_1 - \chi_2\|^2_{L^2(0,l;X)}. \end{aligned} \tag{4.10}$$

If $(k-1)l < t \le kl$ for some positive integer $k > 1$, then taking $\tau = (k-1)l, kl - l/2, kl, kl + l/2$ respectively, we deduce from (4.10) that

$$\int_{(k-1)l}^{kl-l/2} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)},$$

$$\int_{kl-l/2}^{kl} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)},$$

$$\int_{kl}^{kl+l/2} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)},$$

$$\int_{kl+l/2}^{kl+l} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)}.$$

Hence

$$\begin{aligned}\int_{t}^{t+l} &\left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \\ &\le \int_{(k-1)l}^{kl+l} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \\ &\le 4e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)}.\end{aligned} \tag{4.11}$$

If $0 < t \le l$, taking $\tau = l/2, l, 3l/2$ we have

$$\int_{l/2}^{l} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)},$$

$$\int_{l}^{3l/2} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)},$$

$$\int_{3l/2}^{2l} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \le e^{Ct}\|\chi_1-\chi_2\|^2_{L^2(0,l;X)},$$

which imply that

$$\begin{aligned}\int_{t}^{t+l} &\left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds \\ &\le \int_{l/2}^{2l} \left(\|u_1(s)-u_2(s)\|^2 + \|b_1(s)-b_2(s)\|^2\right) ds + \int_0^{l/2} \|\chi_1-\chi_2\|^2_X\, ds \\ &\le (3e^{Ct}+1)\|\chi_1-\chi_2\|^2_{L^2(0,l;X)}.\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain (A_4).

Assumption (A_5). Note that (A_2) implies that $\mathscr{B}_l^0$ is positively invariant with respect to L_t. Thus to verify (A_5), it is enough to check that $\mathscr{B}_l^0$ is closed in X_l, that is, if $\{\chi_n\}(=\{(u_n,b_n)|_{[0,l]}\})$ is a sequence in $\mathscr{B}_l^0$ converging to some $\chi(=(u,b)|_{[0,l]})$ in X_l then χ is also a trajectory with $\chi(0)\in B^0$.

From Assumption (A_1), it follows that $\{\chi_n\}$ is bounded in Y_l, that is,

$$\|u_n\|^2_{L^2(0,l;\mathbb{V})}+\|A(u_n)\|^4_{L^4(0,l;L^4(\Omega))}+\|b_n\|^2_{L^2(0,l;\mathbb{V})}\le C,$$
$$\|\partial_t u_n\|_{L^{4/3}(0,T;\mathbb{W}^*)}+\|\partial_t b_n\|_{L^2(0,T;\mathbb{V}^*)}\le C.$$

Thus up to subsequences,

$$\begin{aligned}
u_n,b_n &\to u,b && \text{weakly in } L^2(0,l;\mathbb{V}),\\
u_n,b_n &\to u,b && \text{strongly in } L^2(0,l;\mathbb{H}),\\
u_n &\to u && \text{weakly in } L^4(0,l;\mathbb{W}),\\
\partial_t u_n &\to \partial_t u && \text{weakly}^* \text{ in } L^{4/3}(0,l;\mathbb{W}^*),\\
\partial_t b_n &\to \partial_t b && \text{weakly}^* \text{ in } L^2(0,l;\mathbb{V}^*).
\end{aligned}$$

Thus similar to the proof of Theorem 3.3, using the monotone operator theory we can verify that $\chi=(u,b)|_{[0,l]}$ is a weak solution of problem (1.1) in $[0,l]$. It remains to show that $\chi(0)=(u(0),b(0))\in B^0$. Indeed, since $\chi_n\to\chi$ in $L^2(0,l;\mathbb{H}\times\mathbb{H})$, we infer that $\chi_n(t)\to\chi(t)$ in $\mathbb{H}\times\mathbb{H}$ for almost every t. Note that B^0 is a closed ball in $\mathbb{H}\times\mathbb{H}$. Thus $\chi(t)\in B^0$ for almost every $t\in[0,l]$. Finally, thanks to the continuity of $\chi:[0,l]\to\mathbb{H}\times\mathbb{H}$, we conclude that $\chi(0)\in B^0$, proving (A_5).

Assumption (A_6). Set
$W_l=\{(u,b)\in L^2(0,l;\mathbb{V}\times\mathbb{V}):(\partial_t u,\partial_t b)\in L^1(0,l;(\dot{\mathbb{W}}_p^{3,2}(\Omega))^*\times(\dot{\mathbb{W}}_p^{3,2}(\Omega))^*)\}$.
Note that

$$\mathbb{V}\times\mathbb{V}\hookrightarrow\hookrightarrow\mathbb{H}\times\mathbb{H}\hookrightarrow(\dot{\mathbb{W}}_p^{3,2}(\Omega))^*\times(\dot{\mathbb{W}}_p^{3,2}(\Omega))^*.$$

By the Aubin–Simon type compactness results, we have $W_l\hookrightarrow\hookrightarrow X_l$. It remains to verify that for some $\tau>0$, $L_\tau:X_l\to W_l$ is Lipschitz continuous on $\mathscr{B}_l^1$. From the definition of $\mathscr{B}_l^1$ and the regularity results in Theorem 3.3 (see (3.4)), we know that L_τ is bounded in $L^\infty(0,l;\mathbb{W}\times\mathbb{V})\cap L^2(0,l;\dot{\mathbb{H}}_p^2(\Omega)\times\dot{\mathbb{H}}_p^2(\Omega))$. Thus thanks to Theorem 3.4 (see (3.67), (3.68)), for any χ_1,χ_2 in $\mathscr{B}_l^1$,

$$\|L_l(\chi_1)-L_l(\chi_2)\|_{W_l}\le C\|\chi_1-\chi_2\|_{X_l}.$$

Hence (A_6) holds.

Assumption (A_7). Let $\chi_1,\chi_2\in\mathscr{B}_l^1$. From the regularity of $\mathscr{B}_l^1$, we know that χ_1,χ_2 are bounded in $L^\infty(0,l;\mathbb{W}\times\mathbb{V})\cap L^2(0,l;\dot{\mathbb{H}}_p^2(\Omega)\times\dot{\mathbb{H}}_p^2(\Omega))$. Thus thanks to Theorem 3.4 (see (3.78)), for any $0<s<l$,

$$\|\chi_1(l)-\chi_2(l)\|^2_X\le C\|\chi_1(s)-\chi_2(s)\|^2_X.$$

Integrating with respect to s over $(0,l)$, we obtain

$$\|\chi_1(l)-\chi_2(l)\|_X \le C\|\chi_1-\chi_2\|_{X_l},$$

which is exactly (A_7).

Assumption (A_8). It follows directly from (A_7).

This completes the proof of Theorem 4.6.

Proof of Theorem 4.4. Thanks to Theorem 4.5, it is enough to prove (A_9) and (A_{10}). Note that (A_9) is a simple consequence of (4.11) and (4.12), so we only need to verify (A_{10}). Thanks to Theorem 3.1, the set $\mathscr{B}_l^1$ is bounded in $L^\infty(0,l;\mathbb{W}\times\mathbb{V})\cap L^2(0,l;\dot{\mathbb{H}}_p^2(\Omega)\times\dot{\mathbb{H}}_p^2(\Omega))$ and $\{\partial_t\chi:\chi\in\mathscr{B}_l^1\}$ is bounded in $L^2(0,l;\mathbb{H}\times\mathbb{H})$. Here $\partial_t\chi$ is the partial derivative of χ with respect to t (i.e., $(\partial_t u,\partial_t b)|_{[0,l]}$). Thus for any $\chi\in\mathscr{B}_l^1$ and any $t_1<t_2\in[0,\tau]$,

$$\begin{aligned}
(4.13)\quad &\int_0^l \|L_{t_1}\chi(s)-L_{t_2}\chi(s)\|_X^2\,ds\\
&=\int_0^l\Big\{\|u(t_2+s)-u(t_1+s)\|^2+\|b(t_2+s)-b(t_1+s)\|^2\Big\}\,ds\\
&=\int_0^l\Big\{\Big\|\int_{t_1+s}^{t_2+s}\partial_\varsigma u(\varsigma)\,d\varsigma\Big\|^2+\Big\|\int_{t_1+s}^{t_2+s}\partial_\varsigma b(\varsigma)\,d\varsigma\Big\|^2\Big\}\,ds\\
&=\int_0^l\Big\{\int_\Omega\Big|\int_{t_1+s}^{t_2+s}\partial_\varsigma u(\varsigma)\,d\varsigma\Big|^2\,dx+\int_\Omega\Big|\int_{t_1+s}^{t_2+s}\partial_\varsigma b(\varsigma)\,d\varsigma\Big|^2\,dx\Big\}\,ds\\
&\le\int_0^l\Big\{(t_2-t_1)\int_\Omega\int_{t_1+s}^{t_2+s}[|\partial_\varsigma u(\varsigma)|^2+|\partial_\varsigma b(\varsigma)|^2]\,d\varsigma\,dx\Big\}\,ds\\
&=\int_0^l\Big\{(t_2-t_1)\int_{t_1+s}^{t_2+s}\|\partial_\varsigma u(\varsigma)\|^2\,d\varsigma+(t_2-t_1)\int_{t_1+s}^{t_2+s}\|\partial_\varsigma b(\varsigma)\|^2\,d\varsigma\Big\}\,ds\\
&\le\int_0^l(t_2-t_1)\big(\|\partial_\varsigma u(\varsigma)\|_{L^2(t_1+s,t_2+s;\mathbb{H})}+\|\partial_\varsigma b(\varsigma)\|_{L^2(t_1+s,t_2+s;\mathbb{H})}\big)\,ds.
\end{aligned}$$

Since $\mathscr{B}_l^1$ is positively invariant, we find that

$$\|\partial_\varsigma u(\varsigma)\|_{L^2(t_1+s,t_2+s;\mathbb{H})}+\|\partial_\varsigma b(\varsigma)\|_{L^2(t_1+s,t_2+s;\mathbb{H})}$$

is bounded by some constant depending on τ,l. Then it is easy to deduce from (4.13) that

$$\|L_{t_1}(\chi)-L_{t_2}(\chi)\|_{X_l}\le C|t_1-t_2|^{1/2}.$$

Thus we obtain (A_{10}) and complete the proof of Theorem 4.7.

Acknowledgments. The authors would like to thank the referees for the careful reading of the manuscripts, and especially for some invaluable suggestions, which helped us improve the paper greatly. This work was partially supported by the NSFC (No. 11701002) and NSF of Anhui Province (No. 1708085MA02).

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