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Automorphisms of Toeplitz $\mathscr{B}$-free systems

by

Aurelia DYMEK

Presented by Feliks PRZYTYCKI

Summary. For each $\mathscr{B}$-free subshift given by $\mathscr{B} = \{2^i b_i\}_{i\in\mathbb{N}}$, where $\{b_i\}_{i\in\mathbb{N}}$ is a set of pairwise coprime odd numbers greater than one, it is shown that the automorphism group of the subshift consists solely of powers of the shift.

1. Introduction. Consider the compact space $\{0,1\}^{\mathbb{Z}}$ of two-sided sequences provided with the product topology. On this space we have the natural $\mathbb{Z}$-action by the *left shift* S, i.e.

$$S((x_m)_{m\in\mathbb{Z}}) = (y_m)_{m\in\mathbb{Z}},$$

where $y_m = x_{m+1}$ for each $m \in \mathbb{Z}$.

We say that a set $Z \subset \{0,1\}^{\mathbb{Z}}$ is a *subshift* if it is closed and invariant under the shift, i.e. $SZ = Z$. For any subshift system (Z,S), by the *automorphism group* $C(S)$ we mean the set of all homeomorphisms $U\colon Z \to Z$ which commute with S. The set $C(S)$ is a group.

A natural source of subshifts is to take $x \in \{0,1\}^{\mathbb{Z}}$ and (X_x,S), where $X_x := \overline{\mathcal{O}_S(x)}$ with $\mathcal{O}_S(x) := \{S^n x : n \in \mathbb{Z}\}$ (equivalently, $X_x = \{y \in \{0,1\}^{\mathbb{Z}} :$ each block appearing in y appears in $x\}$). Each $x = (x_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ can be identified with $\mathbb{1}_{\operatorname{supp} x}$, where $\operatorname{supp} x := \{n \in \mathbb{Z} : x_n = 1\}$ is the *support* of x.

In the present paper, we will study the case where the support is the set $\mathcal{F}_{\mathscr{B}} := \mathbb{Z} \setminus \mathcal{M}_{\mathscr{B}}$ of *$\mathscr{B}$-free numbers* for some $\mathscr{B} \subset \mathbb{N}$, where $\mathcal{M}_{\mathscr{B}}$ is the set of $\mathscr{B}$-multiples, i.e. $\mathcal{M}_{\mathscr{B}} = \bigcup_{b\in\mathscr{B}} b\mathbb{Z}$. Let $\eta = \mathbb{1}_{\mathcal{F}_{\mathscr{B}}}$. By a *$\mathscr{B}$-free subshift* we mean (X_η, S). We will constantly assume that $\mathscr{B}$ is *primitive*, that is, for any $b, b' \in \mathscr{B}$ if $b \mid b'$ then $b = b'$.

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When (X, S) is a subshift, its automorphism group is countable because each of its members is a coding [14]. It is of interest how complicated $C(S)$ can be; see e.g. [6] and [9, 10] for some recent results and the references therein. In the present paper, we consider the $\mathscr{B}$-free systems whose dynamical properties are under intensive study; see e.g. [1], [2], [5], [17], [19], [20], and especially [3] for more historical references. When $\mathscr{B}$ is infinite, and its elements are pairwise coprime and $\sum_{b\in\mathscr{B}} 1/b < \infty$ (the *Erdős case*), it has been proved that the automorphism group is *trivial*, i.e. it consists solely of the powers of the shift [18]. Recall that the Erdős case implies that (X_η, S) is proximal and non-minimal [1]. On the other hand, if a $\mathscr{B}$-free subshift (X_η, S) is minimal then it must be Toeplitz [3]; as a matter of fact, η itself has to be Toeplitz [16].

The main result of this paper is the following.

THEOREM 1.1. *Let $\{b_i\}_{i\in\mathbb{N}}$ be a set of pairwise coprime odd integers greater than one and let $\mathscr{B} = \{2^i b_i\}_{i\in\mathbb{N}}$. Then the automorphism group of the $\mathscr{B}$-free subshift (X_η, S) is trivial, where $\eta = \mathbb{1}_{\mathcal{F}_\mathscr{B}}$.*

Here we consider a minimal case, in contrast to [18], where a proximal case has been studied. Moreover, our examples are *coalescent*, i.e. all continuous maps commuting with S are homeomorphisms (see Corollary 3.9). We recall that the automorphism group of a general Toeplitz subshift need not be trivial (see, e.g., [4]).

As shown in [3], the $\mathscr{B}$-free systems that are taut have interesting dynamical properties. We show that $\mathscr{B}$-free systems which are minimal (equivalently, Toeplitz) are taut (see last section). Moreover, we formulate some open questions.

2. Preliminaries

2.1. Toeplitz subshifts. In this subsection, we recall the definition and properties of Toeplitz subshifts on the space of 0-1 sequences indexed by $\mathbb{Z}$.

We say that $x \in \{0,1\}^\mathbb{Z}$ is a *Toeplitz sequence* whenever for any $n \in \mathbb{Z}$ there exists $s_n \in \mathbb{N}$ such that $x(n + k \cdot s_n) = x(n)$ for any $k \in \mathbb{Z}$. A subshift (Z, S), $Z \subset \{0,1\}^\mathbb{Z}$, is said to be *Toeplitz* if $Z = \overline{\mathcal{O}_S(y)}$ for some Toeplitz sequence $y \in \{0,1\}^\mathbb{Z}$.

DEFINITION 2.1 (due to S. Williams [21]). Let $s \in \mathbb{N}$. Let $\mathrm{Per}_s(x)$ we denote $\{n \in \mathbb{Z} : \forall_{k\in\mathbb{Z}}\ x(n) = x(n + ks)\}$. (If $s' \mid s$ then $\mathrm{Per}_{s'}(x) \subset \mathrm{Per}_s(x)$.) By an *essential period* of x we mean s for which $\mathrm{Per}_{s'}(x) \neq \mathrm{Per}_s(x) \neq \emptyset$ for any positive integer $s' < s$. Finally, a *periodic structure* of x is any sequence $s = (s_m)_{m\in\mathbb{N}}$ of essential periods such that $s_m \mid s_{m+1}$ for each $m \in \mathbb{N}$ and

$$\bigcup_{m\in\mathbb{N}} \mathrm{Per}_{s_m}(x) = \mathbb{Z}. \tag{2.1}$$

Every Toeplitz sequence has a periodic structure, which is not unique because any subsequence of a periodic structure is a periodic structure too. As in [11], we can take s_m to be the least common multiple of the minimal periods of the positions of zeros and ones in $[-m, m]$.

DEFINITION 2.2 (see [15]). A Toeplitz sequence $x \in \{0,1\}^{\mathbb{Z}}$ is *regular* if there exists a periodic structure $(s_m)_{m\in\mathbb{N}}$ of x such that

$$\lim_{m\to\infty} \frac{|\mathrm{Per}_{s_m}(x) \cap [0, s_m)|}{s_m} = 1.$$

Let $(p_t)_{t=1}^{\infty}$ be a periodic structure of a Toeplitz sequence $x \in \{0,1\}^{\mathbb{Z}}$, and let G be the inverse limit group $\varprojlim \mathbb{Z}/p_t\mathbb{Z}$, that is,

$$G = \{(n_t)_{t=1}^{\infty} : n_t \in \mathbb{Z}/p_t\mathbb{Z} \text{ and } n_{t+1} \equiv n_t \pmod{p_t} \text{ for any } t \geq 1\}.$$

Notice that G is metrizable by the metric

$$|(n_t)_{t=1}^{\infty}, (n_t')_{t=1}^{\infty})| = \max\left\{\frac{1}{i+1} : n_i \neq n_i'\right\}$$

for any $(n_t)_{t=1}^{\infty}, (n_t')_{t=1}^{\infty} \in G$. We denote $(n, n, \ldots)$ by $\overline{n}$ for any $n \in \mathbb{Z}$. Let T be the translation of G by the unit element $\overline{1}$. Then G is a compact monothetic group with generator $\overline{1}$. In [21], it is proved that the system (G, T) is the *maximal equicontinuous factor* of $(\overline{\mathcal{O}_S(x)}, S)$, i.e. (G, T) is the largest system such that the family $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous and there exists a continuous surjective map $\pi\colon \overline{\mathcal{O}_S(x)} \to G$ such that $\pi \circ S = T \circ \pi$ (π is called a *factor map*). In other words, (G, T) is the largest equicontinuous factor of $(\overline{\mathcal{O}_S(x)}, S)$ (any other equicontinuous factor of $(\overline{\mathcal{O}_S(x)}, S)$ is a factor of (G, T)). Every topological dynamical system has a maximal equicontinuous factor, which is unique up to conjugacy (see, e.g., [12]).

Let $A_t \in \{0, 1, \_\}^{p_t}$ be a block such that

$$A_t(n) = \begin{cases} x(n) & \text{if } x(n) = x(n + kp_t) \text{ for each } k \in \mathbb{Z}, \\ \_ & \text{otherwise,} \end{cases}$$

for any $n \in \{0, 1, \ldots, p_t - 1\}$. By a *filled place* in A_t we mean each $i \in \{0, 1, \ldots, p_t - 1\}$ such that $A_t(i) \in \{0, 1\}$. We call the symbol _ a *hole*. The p_t*-skeleton* of x is the sequence obtained from x by replacing $x(n)$ by a hole for all $n \notin \mathrm{Per}_{p_t}(x)$. If $x \in \{0,1\}^{\mathbb{Z}}$ is a regular Toeplitz sequence then the sequence $(A_t)_{t=1}^{\infty}$ of blocks satisfies the following conditions:

(A) the block A_{t+1} is obtained as a concatenation $A_t \ldots A_t$, where some holes are filled by symbols 0 or 1,
(B) $\lim_{t\to\infty} r_t/p_t = 1$, where r_t is the number of filled places in A_t,
(C) for every $i \in \mathbb{N}$ there exists t such that $A_t(i) \in \{0, 1\}$.

For any given $t \geq 1$ let $\{n : A_t(n) = \_\} = \{I_1^{(t)} < I_2^{(t)} < \cdots < I_{s_t}^{(t)}\}$ be the set of all positions of holes in A_t.

DEFINITION 2.3 (see [4]). We say that a Toeplitz sequence $x \in \{0,1\}^{\mathbb{Z}}$ has *property* (Sh) (separated holes) if

$$k_t := \min(\{I_{j+1}^{(t)} - I_j^{(t)} : j = 1, \ldots, s_t - 1\} \cup \{p_t - I_{s_t}^{(t)} + I_1^{(t)}\}) \tag{2.2}$$

tends to ∞ as $t \to \infty$.

As mentioned in [4], each sequence $(A_t)_{t=1}^{\infty}$ of blocks satisfying (A)–(C) determines a Toeplitz sequence x, which may be periodic.

REMARK 2.4. For Toeplitz sequences satisfying (Sh), each continuous $U : \overline{\mathcal{O}_S(x)} \to \overline{\mathcal{O}_S(x)}$ which commutes with S is a homeomorphism. In other words, such Toeplitz subshifts are topologically coalescent [4, Proposition 3].

By a *t-symbol* of x we mean any block A of length p_t such that $A = x[\ell p_t, \ell p_t + p_t - 1]$ for some $\ell \in \mathbb{Z}$. For each t-symbol of x we have $\{n \in \{0,1,\ldots,p_t - 1\} : A(n) = A_t(n)\} = \{0,1,\ldots,p_t - 1\} \setminus \{I_1^{(t)}, \ldots, I_{s_t}^{(t)}\}$.

Let $g = (n_t)_{t=1}^{\infty} \in G$. We denote

$$A_t(g) = A_t A_t[n_t, n_t + p_t - 1],$$

where $A_t A_t$ is the concatenation of blocks. Notice that $A_t(g)$ has length p_t, and $(\{I_1^{(t)}, \ldots, I_{s_t}^{(t)}\} - n_t) \bmod p_t$ are all positions of holes in $A_t(g)$. Denote this set by $\{J_1^{(t)}(g) < \cdots < J_{s_t}^{(t)}(g)\}$. The sequence $(A_t(g))_{t=1}^{\infty}$ satisfies conditions (A) and (B), so it determines a two-sided sequence $x(g) \in \{0,1,\_\}^{\mathbb{Z}}$ such that for any $t \geq 1$ and any $0 \leq i < p_t$ satisfying $A_t(g)(i) \in \{0,1\}$ we have

$$x(g)(i + \ell p_t) = A_t(g)(i) \quad \text{for all } \ell \in \mathbb{Z}.$$

Each of the t-symbols of $x(g)$ coincides with $A_t A_t[n_t, n_t + p_t - 1]$ except at the places $J_1^{(t)}(g), \ldots, J_{s_t}^{(t)}(g)$. By the definition of $A_t(g)$, we obtain $\{J_k^{(t)}(g) : k = 1, \ldots, s_t\} = \{I_k^{(t)} - n_t \pmod{p_t} : k = 1, \ldots, s_t\}$.

Let $G_0 = \{g \in G : x(g)$ is a 0-1 Toeplitz sequence$\}$ and $G_2 = \{g = (n_t)_{t=1}^{\infty} \in G : A_t(n_t) = \_$ for each $t \geq 0\}$. Then $G_0 = G \setminus G_1$, where $G_1 = G_2 + \mathbb{Z}\overline{1}$. Moreover, G_0 is of Haar measure 1 [4, p. 48].

Let π be a factor map from $(\overline{\mathcal{O}_S(x)}, S)$ to (G, T), defined as in [21]. Then $\pi^{-1}(\{g\}) = \{y \in \overline{\mathcal{O}_S(x)} : y$ has the same p_t-skeleton as $S^{n_t}x$ for any $t \geq 1\}$ for any $g = (n_t)_{t=1}^{\infty} \in G$. Assume that a Toeplitz sequence $x \in \{0,1\}^{\mathbb{Z}}$ has property (Sh). Then $\pi^{-1}(\{g\})$ contains at most two elements for any $g \in G$ [4, Remark 4] and $\pi^{-1}(\{g\}) = \{x(g)\}$ for any $g \in G_0$ [4, Remark 2].

If $U \in C(S)$ then U induces a continuous map U' on G commuting with T. Indeed, notice that $\overline{0} \in G_0$ because $x(\overline{0}) = x$. Set

$$U' T^n \overline{0} = \pi U S^n x \quad \text{for any } n \in \mathbb{Z}. \tag{2.3}$$

Because π is a factor map and $U \in C(S)$, we have $\pi U S^n x = \pi S^n U x = T^n \pi U x = \overline{n} + \pi U x = \overline{n} + U'\overline{0}$. Hence $U' : \{T^n \overline{0} : n \in \mathbb{Z}\} \to G$ is the restriction of translation by $U'\overline{0}$. Because G is a monothetic group, U' is translation by $g_0 := U'\overline{0}$. In this case it is natural to say that g_0 can be *lifted* to U.

Notice that if $g_0 \in G$ can be lifted to U, then U is unique. Indeed, suppose $U_1 \neq U_2 \in C(S)$ induce a map $U' \in C(T)$. Then because $(\overline{\mathcal{O}_S(x)}, S)$ is minimal, $U_1 y \neq U_2 y$ for any $y \in \overline{\mathcal{O}_S(x)}$. Notice that $U'\pi = \pi U_1 = \pi U_2$. So $U'g \in G_1$ for any $g \in G$; but G_1 is of Haar measure zero, so this contradicts U' being a translation.

The question arises which elements $g_0 \in G$ can be lifted to elements of $C(S)$.

2.2. General lemmas. In this subsection, we include the general well-known facts about arithmetic progressions and a fact about $\mathscr{B}$-free systems. These facts will be used later.

LEMMA 2.5. *Let $b \in \mathbb{Z}$ and $a, m \in \mathbb{Z} \setminus \{0\}$. Then the congruence*

$$ax \equiv b \pmod{m} \tag{2.4}$$

has a solution $x \in \mathbb{Z}$ if and only if $\gcd(a, m) \mid b$.

In [3], a characterization of periodicity of the characteristic function of $\mathscr{B}$-free numbers is given:

LEMMA 2.6 ([3, Proposition 4.25]). *Let $\mathscr{B} \subset \mathbb{N}$ be primitive, i.e. for any $b, b' \in \mathscr{B}$, if $b \mid b'$ then $b = b'$. Then $\mathscr{B}$ is finite if and only if η is periodic with minimal period* $\operatorname{lcm}(\mathscr{B})$.

LEMMA 2.7. *Let $a, b, r \in \mathbb{N}$. If* $\gcd(a, b) \mid r$ *then*

$$(a\mathbb{Z} + r) \cap b\mathbb{Z} = \operatorname{lcm}(a, b)\mathbb{Z} + bs \tag{2.5}$$

for some $s \in \mathbb{Z}$ such that $bs \equiv r \pmod{a}$. Moreover, if $\gcd(a, b) \nmid r$ then $(a\mathbb{Z} + r) \cap b\mathbb{Z} = \emptyset$.

LEMMA 2.8. *Let $a, r, m \in \mathbb{N}$. Then $|(a\mathbb{Z} + r) \cap [0, ma)| = m$.*

3. Subfamily of Toeplitz $\mathscr{B}$-free systems. Let $\mathscr{B} = \{2^i b_i\}_{i \in \mathbb{N}}$, where $\{b_i\}_{i \in \mathbb{N}}$ is a set of pairwise coprime odd integers greater than 1. We say that $n \in \mathbb{Z}$ is *$\mathscr{B}$-free* if $n \notin \bigcup_{i \in \mathbb{N}} 2^i b_i \mathbb{Z}$. We denote by $\mathcal{F}_{\mathscr{B}}$ the set of all $\mathscr{B}$-free numbers and by η its characteristic function. This means that

$$\eta(n) = \begin{cases} 1 & \text{if } n \text{ is } \mathscr{B}\text{-free}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

for any $n \in \mathbb{Z}$.

Let $p_t = 2^t b_1 \ldots b_t$ for any $t \in \mathbb{N}$. We will prove that η is a Toeplitz sequence with periodic structure $(p_t)_{t \in \mathbb{N}}$.

Lemma 3.1 ([3, Example 3.1]). *The sequence η is a Toeplitz sequence.*

Lemma 3.2. *For any $t \in \mathbb{N}$ the number p_t is an essential period of η.*

Proof. Let $s < p_t$ be a positive integer, m be odd and a be a non-negative integer such that $s = m2^a$. Three cases can occur:

(I) there exists $1 \le i \le t-1$ such that $\gcd(b_i, s) < b_i$,
(II) $\gcd(b_j, s) = b_j$ for all $1 \le j \le t-1$ and $\gcd(b_t, s) < b_t$,
(III) $\gcd(b_j, s) = b_j$ for all $1 \le j \le t$.

Assume that condition (I) holds. Let $b_i' = \gcd(b_i, s)$ and $\ell \in \mathbb{Z}$.

We claim that

$$\eta(2^{t-1}b_i' + p_t\ell) = 1. \tag{3.2}$$

Suppose not, so $2^j b_j \mid 2^{t-1}b_i' + p_t\ell$ for some $j \in \mathbb{N}$. Because b_i' is odd, we have $2^t \nmid 2^{t-1}b_i' + p_t\ell = 2^{t-1}(b_i' + 2b_1 \dots b_t\ell)$. Hence $j < t$ and because b_j is odd and greater than 1, we obtain $b_j \mid b_i' + 2b_1 \dots b_t\ell$, which implies $b_j \mid b_i' \mid b_i$. Because the b_r are pairwise coprime, we have $j = i$; but then $b_i = b_i'$, which contradicts condition (I). Hence (3.2) holds.

Moreover,

$$2^{\min(a,i)}b_i' = \gcd(2^i b_i, s) \mid 2^{t-1}b_i'. \tag{3.3}$$

Indeed, 2^i and b_i are coprime, so $\gcd(2^i b_i, s) = \gcd(2^i, s)\gcd(b_i, s) = 2^{\min(a,i)}b_i'$. Because $i \le t-1$, we have $\min(a, i) \le t-1$, proving (3.3).

By Lemma 2.5 and (3.3), there exists $n \in \mathbb{Z}$ such that $2^{t-1}b_i' + sn \equiv 0 \pmod{2^i b_i}$, so $\eta(2^{t-1}b_i' + sn) = 0$. Hence $2^{t-1}b_i' \in \mathrm{Per}_{p_t}(\eta)$ but $2^{t-1}b_i' \notin \mathrm{Per}_s(\eta)$.

Assume that condition (II) holds. Then

$$\eta(2^t b_t + 2^{t+1}s) = 1. \tag{3.4}$$

Indeed, suppose not, so there exists $j \in \mathbb{N}$ such that $2^j b_j \mid 2^t b_t + 2^{t+1}s$. Because b_t is odd, we have $2^{t+1} \nmid 2^t b_t + 2^{t+1}s = 2^t(b_t + 2s)$. Hence $j \le t$ and because b_j is odd and greater than 1, we obtain $b_j \mid b_t + 2s$. If $j < t$ then by condition (II), we have $b_j \mid s$, which implies $b_j \mid b_t$, which is impossible. So $j = t$ and $b_t \mid 2s$. But this contradicts condition (II). Hence (3.4) holds.

Moreover, $2^t b_t \mid 2^t b_t + p_t\ell = 2^t b_t(1 + b_1 \dots b_{t-1}\ell)$ for any $\ell \in \mathbb{Z}$. So $2^t b_t \in \mathrm{Per}_{p_t}(\eta)$ but $2^t b_t \notin \mathrm{Per}_s(\eta)$.

Assume that condition (III) holds. Then $a < t$ because $b_1, \dots, b_t$ are pairwise coprime and $s < p_t = 2^t b_1 \dots b_t$.

We claim that

$$\eta(2^t b_t + s) = 1. \tag{3.5}$$

Suppose not, so there exists $j \in \mathbb{N}$ such that $2^j b_j \mid 2^t b_t + s$. Notice that $2^{a+1} \nmid 2^t b_t + s = 2^a(2^{t-a}b_t + m)$. Hence $j \le a < t$. Because b_j is odd, we obtain

$b_j \mid 2^{t-a}b_t + m$, which together with (III) implies $b_j \mid b_t$; this is impossible. So (3.5) holds. Hence $2^t b_t \notin \mathrm{Per}_s(\eta)$ but $2^t b_t \in \mathrm{Per}_{p_t}(\eta)$.

By the above, p_t is an essential period of η. ■

LEMMA 3.3. *The sequence η is a Toeplitz sequence with periodic structure $(p_t)_{t\geq 1}$.*

Proof. Let $t \in \mathbb{N}$. Then by Lemma 3.2, p_t is an essential period. Notice that $p_{t+1} = 2p_t b_{t+1}$, so $p_t \mid p_{t+1}$. In [3] it is proved that $\bigcup_{t\geq 1} \mathrm{Per}_{2^{t+1}b_1\ldots b_t}(\eta) = \mathbb{Z}$. Because $2^{t+1}b_1 \ldots b_t \mid p_{t+1}$, we get $\bigcup_{t\geq 1} \mathrm{Per}_{p_t}(\eta) = \mathbb{Z}$ as desired. ■

LEMMA 3.4. *The Toeplitz sequence η is not periodic.*

Proof. Because $\{b_i\}_{i\in\mathbb{N}}$ is a set of pairwise coprime odd numbers, $\mathscr{B}$ is primitive and infinite. By Lemma 2.6, η is not periodic. ■

Now we will give a characterization of places where A_t has holes, and compute the number of holes.

LEMMA 3.5. *Let $0 \leq s < p_t$. Then s is a hole in A_t if and only if*

$$2^t \mid s \quad \text{and} \quad b_i \nmid s \text{ for any } 1 \leq i \leq t. \tag{3.6}$$

The number of holes in A_t equals

$$\prod_{i=1}^{t}(b_i - 1) =: s_t. \tag{3.7}$$

Proof. Let $0 \leq s < p_t$. Let m be odd and a be a non-negative integer such that $s = m2^a$. Three cases can occur:

(i) $2^j b_j \mid s$ for some $1 \leq j \leq t$,
(ii) $2^t \nmid s$ and $2^i b_i \nmid s$ for any $1 \leq i \leq t$,
(iii) $2^t \mid s$ and $2^i b_i \nmid s$ for any $1 \leq i \leq t$.

Assume that condition (i) holds. Then

$$2^j b_j \mid s + p_t \ell = 2^j b_j \left(\frac{s}{2^j b_j} + 2^{t-j} \frac{b_1 \ldots b_t}{b_j} \ell \right)$$

for any $\ell \in \mathbb{Z}$. So $s \in \mathrm{Per}_{p_t}(\eta)$.

Assume that condition (ii) holds. Then we claim that

$$2^i b_i \nmid s + p_t \ell \quad \text{for any } i \in \mathbb{N} \text{ and } \ell \in \mathbb{Z}. \tag{3.8}$$

Indeed, suppose that $2^i b_i \mid s + p_t \ell$ for some $\ell \in \mathbb{Z}$ and $i \in \mathbb{N}$. Notice that $2^{a+1} \nmid s + p_t \ell = 2^a(m + 2^{t-a} b_1 \ldots b_t \ell)$. So $i \leq a < t$, which implies $2^i b_i \mid p_t$. Hence $2^i b_i \mid s$, which contradicts (ii).

So (3.8) holds. This implies $s \in \mathrm{Per}_{p_t}(\eta)$.

Assume that condition (iii) holds. Notice that if (iii) holds then $2^i \mid s$ for any $1 \leq i \leq t$. So (iii) is equivalent to $2^t \mid s$ and $b_i \nmid s$ for any $1 \leq i \leq t$. We claim that $s \notin \mathrm{Per}_{p_t}(\eta)$. Indeed, we have two possibilities:

- $\eta(s) = 1$. Then $2^t = \gcd(2^{t+1}b_{t+1}, p_t) \mid s$. So by Lemma 2.5, there exists $n \in \mathbb{Z}$ such that $p_t n + s \equiv 0 \pmod{2^{t+1}b_{t+1}}$. Hence $\eta(s + p_t n) = 0$.
- $\eta(s) = 0$. If $a = t$ then $2^{t+1} \nmid s + 2p_t = 2^t(m + 2b_1 \dots b_t)$. Suppose that

$$2^j b_j \mid s + 2p_t \quad \text{for some } j \in \mathbb{N}. \tag{3.9}$$

Then $j \le t$. Because b_j is odd, we have $b_j \mid m + 2b_1 \dots b_t$, which implies $b_j \mid m$. But this contradicts (iii). So (3.9) does not hold. If $a > t$ then because $b_1, \dots, b_t$ are odd, we obtain $2^{t+1} \nmid s + p_t = 2^t(2^{a-t}m + b_1 \dots b_t)$. Suppose that

$$2^j b_j \mid s + p_t \quad \text{for some } j \in \mathbb{N}. \tag{3.10}$$

Then $j \le t$. Because $2^j b_j \mid p_t$, we have $2^j b_j \mid s$. But this contradicts (iii). So (3.10) does not hold. Hence $\eta(s + p_t) = 1$.

There are $b_1 \dots b_t$ non-negative multiples of 2^t smaller than p_t. We need to know how many of them are not multiples of any b_i for $1 \le i \le t$. We will compute how many of the non-negative multiples of 2^t smaller than p_t are multiples of b_i for some $1 \le i \le t$. This is equivalent to computing the cardinality of $\bigcup_{i=1}^t B_i$, where $B_i = \{0 \le w < b_1 \dots b_t : b_i \mid w\}$ for any $1 \le i \le t$. Because $\{b_i\}_{i \in \mathbb{N}}$ are pairwise coprime, for any $\emptyset \ne J \subset \{1, \dots, t\}$ we have

$$\bigcap_{j \in J} B_j = \bigcap_{j \in J} b_j \mathbb{Z} \cap [0, b_1 \dots b_t) = \Big(\prod_{j \in J} b_j\Big)\mathbb{Z} \cap [0, b_1 \dots b_t).$$

By Lemma 2.8, we obtain

$$\Big|\bigcap_{j \in J} B_j\Big| = \frac{b_1 \dots b_t}{\prod_{j \in J} b_j}. \tag{3.11}$$

By the Inclusion-Exclusion Principle and (3.11), we have

$$\begin{aligned}\Big|\bigcup_{i=1}^t B_i\Big| &= \sum_{\emptyset \ne J \subset \{1,\dots,t\}} (-1)^{|J|-1} \Big|\bigcap_{j \in J} B_j\Big| = \sum_{\emptyset \ne J \subset \{1,\dots,t\}} (-1)^{|J|-1} \frac{b_1 \dots b_t}{\prod_{j \in J} b_j} \\ &= b_1 \dots b_t \sum_{\emptyset \ne J \subset \{1,\dots,t\}} (-1)^{|J|-1} \frac{1}{\prod_{j \in J} b_j}.\end{aligned}$$

Hence the number of $0 \le s < p_t$ satisfying (iii) equals

$$\begin{aligned}b_1 \dots b_t \Big(1 - \sum_{\emptyset \ne J \subset \{1,\dots,t\}} (-1)^{|J|-1} \frac{1}{\prod_{j \in J} b_j}\Big) &= b_1 \dots b_t \prod_{i=1}^t \Big(1 - \frac{1}{b_i}\Big) \\ &= \prod_{i=1}^t (b_i - 1). \ \blacksquare\end{aligned}$$

Corollary 3.6 ([3, Remark 3.2]). *The Toeplitz sequence η is regular.*

Proof. By Lemma 3.5, we have

$$\lim_{t\to\infty}\frac{|(\mathbb{Z}\setminus \mathrm{Per}_{p_t}(\eta))\cap[0,p_t)|}{p_t}=\lim_{t\to\infty}\frac{\prod_{i=1}^{t}(b_i-1)}{2^t b_1\dots b_t}\le\lim_{t\to\infty}\frac{1}{2^t}=0.$$

The assertion follows. ∎

COROLLARY 3.7. *We have* $k_t=2^t$ (*see* (2.2)).

Proof. By (3.6) and $2^t\,|\,p_t$, we have $2^t\,|\,k_t$. Notice that for any $t\ge 1$ the two smallest numbers satisfying (3.6) are 2^t and 2^{t+1}, so $I_1^{(t)}=2^t$ and $I_2^{(t)}=2^{t+1}$. Hence $k_t=2^t$. ∎

LEMMA 3.8. *The point* η *has property* (Sh).

Proof. Let $t\ge 1$. As proved above, the block A_t has holes at some non-zero multiples of 2^t (see (3.6)). Moreover, $2^t\,|\,p_t$. Hence the distance between consecutive holes in A_t is at least 2^t, so η has separated holes. ∎

COROLLARY 3.9. *The system* (X_η,S) *is coalescent, i.e. each continuous map* $U\colon X_\eta\to X_\eta$ *commuting with* S *is a homeomorphism.*

Proof. By [4, Proposition 3], any Toeplitz system $\overline{\mathcal{O}_S(x)}$ such that x has property (Sh) is coalescent. Hence by Lemma 3.8, the system (X_η,S) is coalescent. ∎

In the following lemmas, we will give specific properties of holes in A_t.

LEMMA 3.10. *Let* $t\ge 1$. *Then for any* $1\le i\le t$ *we have*

$$\left\{\frac{I_1^{(t)}}{2^t},\dots,\frac{I_{s_t}^{(t)}}{2^t}\right\}\bmod b_i=\{1,\dots,b_i-1\}.$$

Proof. Let $1\le i\le t$ and $1\le j\le b_i-1$. Set

$$A:=(b_i\mathbb{Z}+j)\cap[0,b_1\dots b_t)\cap\bigcup_{m=1}^{t}b_m\mathbb{Z},$$
$$A_F:=(b_i\mathbb{Z}+j)\cap\bigcap_{n\in F}b_n\mathbb{Z}\quad\text{ for any } F\subset\{1,\dots,t\}\setminus\{i\}.$$

Because $b_1,\dots,b_t$ are pairwise coprime, we have

$$\bigcap_{n\in F}b_n\mathbb{Z}=\Big(\prod_{n\in F}b_n\Big)\mathbb{Z}.$$

By Lemma 2.7, for any $F\subset\{1,\dots,t\}\setminus\{i\}$, we obtain

$$A_F=\Big(\prod_{n\in F\cup\{i\}}b_n\Big)\mathbb{Z}+\Big(\prod_{n\in F}b_n\Big)s, \tag{3.12}$$

where $s \in \mathbb{Z}$ satisfies $(\prod_{n\in F} b_n)s \equiv j \pmod{b_i}$. By Lemma 2.8, the intersection $A_F \cap [0, b_1 \dots b_t)$ has $b_1 \dots b_t / \prod_{n\in F\cup\{i\}} b_n$ elements. By the Inclusion-Exclusion Principle (using the same arguments as in the proof of (3.7)), we find that the number of elements of A equals

$$\sum_{1\le n\ne i\le t} \frac{b_1\dots b_t}{b_i b_n} - \sum_{\substack{1\le n\ne i\ne n'\le t\\ n\ne n'}} \frac{b_1\dots b_t}{b_i b_n b_{n'}} + \dots + (-1)^t \frac{b_1\dots b_t}{\prod_{n=1}^t b_n}$$
$$= \frac{b_1\dots b_t}{b_i}\Big(1 - \prod_{1\le n\ne i\le t}\Big(1-\frac{1}{b_n}\Big)\Big).$$

Hence the number of elements of A is smaller than the number of elements of $(b_i\mathbb{Z}+j)\cap[0,b_1\dots b_t)$, which, by Lemma 2.8, equals $b_1\dots b_t/b_i$. Hence there exists $m \in (b_i\mathbb{Z}+j)\cap[0,b_1\dots b_t)$ such that $b_n \nmid m$ for any $1\le n\le t$. So, by (3.6), we have $m \in \{I_1^{(t)}/2^t, \dots, I_{s_t}^{(t)}/2^t\}$. The assertion follows. ∎

Lemma 3.11. *Suppose $h = (n_t)_{t=1}^\infty \in G$ can be lifted. Then there exists $t_0 \ge 1$ such that for any $t \ge t_0$,*

$$I_1^{(t)} - J_1^{(t)}(h) = \dots = I_{s_t}^{(t)} - J_{s_t}^{(t)}(h) =: k',$$

where k' depends on t_0.

Proof. Suppose $h = (n_t)_{t=1}^\infty \in G$ can be lifted to $U \in C(S)$. By the Curtis–Hedlund–Lyndon Theorem [14, Theorem 3.4], there exist $I \in \mathbb{Z}$ and $k \in \mathbb{N}$ and a function $f\colon \{0,1\}^k \to \{0,1\}$ such that

$$U(y)(m) = f(y[m+I, m+I+k-1]) \tag{3.13}$$

for any $m \in \mathbb{Z}$ and any $y \in \overline{\mathcal{O}_S(\eta)}$. Without loss of generality we can assume that $I = 0$.

Let $t_0 \ge 1$ be so large that the distance between consecutive holes in A_{t_0} is greater than k, i.e. $2^{t_0} > k$ (see Corollary 3.7). Consider $t \ge t_0$.

As in [4, proof of Proposition 3], by coding arguments, we deduce that each hole in $A_t(g)$ is less than k distant from some hole in A_t, more precisely $I_i^{(t)} - J_i^{(t)}(h) < k$ for any $1 \le i \le s_t$.

Let $k' = I_1^{(t_0)} - J_1^{(t_0)}(h)$. There exists $m \in \mathbb{N}$ such that $J_2^{(t_0)}(h) = J_1^{(t_0)}(h) + 2^{t_0}m$. Notice that

$$0 \le I_2^{(t_0)} - J_2^{(t_0)}(h) = I_1^{(t_0)} - J_1^{(t_0)}(h) + 2^{t_0}\frac{I_2^{(t_0)} - I_1^{(t_0)}}{2^{t_0}} - 2^{t_0}m$$
$$= k' + 2^{t_0}\Big(\frac{I_2^{(t_0)} - I_1^{(t_0)}}{2^{t_0}} - m\Big) < 2^{t_0}.$$

So

$$\frac{I_2^{(t_0)} - I_1^{(t_0)}}{2^{t_0}} - m = 0,$$

which implies $I_2^{(t_0)} - J_2^{(t_0)}(h) = k'$. We can repeat the same argument for any $j = 3, 4, \ldots, s_{t_0}$ and obtain $I_j^{(t_0)} - J_j^{(t_0)}(h) = k'$ for any such j.

Let $t = t_0 + 1$. Then by the definitions of A_t and $A_t(h)$ (see (A)), we have the inclusions

$$\{I_1^{(t)}, \ldots, I_{s_t}^{(t)}\} \subset \bigcup_{j=0}^{2b_t-1} (\{I_1^{(t_0)}, \ldots, I_{s_{t_0}}^{(t_0)}\} + jp_{t_0})$$

and

$$\{J_1^{(t)}(h), \ldots, J_{s_t}^{(t)}(h)\} \subset \bigcup_{j=0}^{2b_t-1} (\{J_1^{(t_0)}(h), \ldots, J_{s_{t_0}}^{(t_0)}(h)\} + jp_{t_0}).$$

Hence for any $1 \le i \le s_t$ there exist $1 \le i_0 \le s_{t_0}$ and $0 \le j < 2b_t$ such that $I_i^{(t)} = I_{i_0}^{(t_0)} + jp_{t_0}$ and $J_i^{(t)}(h) = J_{i_0}^{(t_0)}(h) + jp_{t_0}$. Indeed, $I_i^{(t)} - J_i^{(t)}(h) < k < 2^{t_0}$. So $I_i^{(t)} - J_i^{(t)}(h) = I_{i_0}^{(t_0)} - J_{i_0}^{(t_0)}(h) = k'$. ∎

COROLLARY 3.12. *Suppose $h = (n_t)_{t=1}^{\infty} \in G$ can be lifted. Then there exists $t_0 \ge 1$ such that for any $t \ge t_0$ we have*

$$(\{I_1^{(t)}, \ldots, I_{s_t}^{(t)}\} - n_t + k') \bmod p_t = \{I_1^{(t)}, \ldots, I_{s_t}^{(t)}\}.$$

Moreover, there exists $t_0 \ge 1$ such that $2^t \mid n_t - k'$ for any $t \ge t_0$.

4. Proof of Theorem 1.1. Assume that $h = (n_t)_{t=1}^{\infty} \in G \setminus \mathbb{Z}\overline{1}$ can be lifted to $U \in C(S)$. Let $t \ge t_0$ be large enough to satisfy the conclusion of Lemma 3.11 and $0 < n_t - k'$. Such a t exists because the coordinates of h are unbounded.

We claim

$$b_i \mid n_t - k' \quad \text{for each } 1 \le i \le t. \tag{4.1}$$

Indeed, by Corollary 3.12, we have $2^t \mid n_t - k'$. Assume that $(n_t - k')/2^t \equiv j \pmod{b_i}$ for some $1 \le i \le t$ and some $1 \le j \le b_i - 1$. Then by Lemma 3.10, there exists $1 \le m \le s_t$ such that $I_m^{(t)}/2^t \equiv j \pmod{b_i}$. So $b_i \mid (I_m^{(t)} - n_t + k')/2^t$, which implies $b_i \mid I_m^{(t)} - n_t + k'$. Because $b_i \mid p_t$, we obtain $b_i \mid (I_m^{(t)} - n_t + k') \bmod p_t$. By Corollary 3.12, we have $(I_m^{(t)} - n_t + k') \bmod p_t \in \{I_1^{(t)}, \ldots, I_{s_t}^{(t)}\}$. But this contradicts (3.6). So (4.1) holds.

Hence $n_t - k'$ is a multiple of $b_1, \ldots, b_t$ and 2^t, which are coprime. Because $0 \le n_t, k' < p_t$, we have $n_t - k' = 0$. But this contradicts $n_t - k' > 0$.

Hence the set of elements in G which can be lifted is $\mathbb{Z}\overline{1}$. This set corresponds to the powers of the shift. The assertion follows.

5. Final remarks

REMARK 5.1. There exist regular Toeplitz sequences satisfying condition (Sh) whose automorphism groups are non-trivial. For more information see [4].

REMARK 5.2 ([1]). We will give an example of a Toeplitz sequence $x \in \{0,1\}^{\mathbb{Z}}$ for which the block A_t has two holes for each $t \geq 1$ and the Boolean complement $\neg$ is an element of the automorphism group; recall that $(\neg y)(n) = 0$ if and only if $y(n) = 1$ for any $y \in \overline{\mathcal{O}_S(x)}$. Notice that $\neg$ is generated by the code $f\colon \{0,1\} \to \{0,1\}$ of length 1 defined by $f(0) = 1$ and $f(1) = 0$. Because $\neg$ is continuous and invertible for the full shift, to prove $\neg$ is an element of the automorphism group of the Toeplitz subshift, it is sufficient to show that $\neg(\overline{\mathcal{O}_S(x)}) \subset \overline{\mathcal{O}_S(x)}$.

Let $n \in \mathbb{N}$ and $B_1 \in \{0,1\}^n$. For any $B \in \{0,1\}^n$ we denote by $\widetilde{B} \in \{0,1\}^n$ the Boolean complement of B, i.e. $\widetilde{B}(i) = f(B(i))$ for any $0 \leq i < n$. Let $A_1 = B_1\_\widetilde{B}_1\_$ and $(c_t)_{t\in\mathbb{N}} \subset \{0,1\}$. Then set $B_2 := B_1c_1\widetilde{B}_1c_1B_1$ and $A_2 := B_2\_\widetilde{B}_2\_$. Notice that A_2 is the concatenation $A_1A_1A_1$, where all but two holes are filled by 0 or 1. Let $B_{t+1} = B_tc_t\widetilde{B}_tc_tB_t$ and $A_{t+1} = B_{t+1}\_\widetilde{B}_{t+1}\_$ for any $t \in \mathbb{N}$. Notice that the sequence $(A_t)_{t=1}^{\infty}$ of blocks satisfies (A)–(C), so it determines a Toeplitz sequence $x \in \{0,1\}^{\mathbb{Z}}$. By the definition of x, it is non-periodic and for any block B appearing in x its Boolean complement $\widetilde{B}$ appears in x. So $\neg(\overline{\mathcal{O}_S(x)}) \subset \overline{\mathcal{O}_S(x)}$.

For $A \subset \mathbb{N}$, we recall several notions of asymptotic density:

$$\underline{d}(A) := \liminf_{N\to\infty} \frac{1}{N}|A \cap [1,N]| \qquad (\textit{lower density} \text{ of } A),$$
$$\overline{d}(A) := \limsup_{N\to\infty} \frac{1}{N}|A \cap [1,N]| \qquad (\textit{upper density} \text{ of } A).$$

If the lower and the upper density of A coincide, their common value $d(A) := \underline{d}(A) = \overline{d}(A)$ is called the *density* of A.

The *logarithmic density* of A is

$$\boldsymbol{\delta}(A) := \lim_{N\to\infty} \frac{1}{\log N} \sum_{a\in A,\, 1\leq a\leq N} \frac{1}{a},$$

whenever the limit exists. A set $\mathscr{B} \subset \mathbb{N}\setminus\{1\}$ is called *Behrend* if $d(\mathcal{M}_{\mathscr{B}}) = 1$ (see [7, 8]). A set $\mathscr{B}$ is *taut* when it is primitive, i.e. for any $b, b' \in \mathscr{B}$, if $b \mid b'$ then $b = b'$, and $\mathscr{B}$ does not contain $c\mathscr{A}$ with $c \in \mathbb{N}$ and $\mathscr{A} \subset \mathbb{N}\setminus\{1\}$ that is Behrend. In [13], it is proved that for any $\mathscr{B} \subset \mathbb{N}$ there exists a primitive $\mathscr{B}' \subset \mathscr{B}$ such that $\mathcal{M}_{\mathscr{B}} = \mathcal{M}_{\mathscr{B}'}$. As shown in [3], we have a "good" theory of $\mathscr{B}$-free subshifts, for both topological dynamics and ergodic theory points

([1]) This example was pointed out to me by W. Bułatek.

of view whenever $\mathscr{B}$ is taut. On the other hand, Toeplitz $\mathscr{B}$-free systems (X_η, S) play a special role in the theory of $\mathscr{B}$-free systems. The Toeplitz case can be characterized by the minimality of (X_η, S) [3], more precisely by the fact that η itself is a Toeplitz sequence [16].

We prove the following.

PROPOSITION 5.3. *For any primitive $\mathscr{B} \subset \mathbb{N} \setminus \{1\}$, if $\eta = \mathbb{1}_{\mathcal{F}_\mathscr{B}}$ is a Toeplitz sequence, then $\mathscr{B}$ is taut.*

Proof. Assume that $\mathbb{1}_{\mathcal{F}_\mathscr{B}}$ is a Toeplitz sequence and $\mathscr{B}$ is not taut. Then for some $c \in \mathbb{N}$ and $\mathscr{A} \subset \mathbb{N} \setminus \{1\}$ such that $d(\mathcal{M}_\mathscr{A}) = 1$ we have $c\mathscr{A} \subset \mathscr{B}$. So

$$\frac{1}{c} = d(c\mathcal{M}_\mathscr{A}) \le \underline{d}(\mathcal{M}_\mathscr{B} \cap c\mathbb{Z}) \le \underline{d}(c\mathbb{Z}) = \frac{1}{c}. \tag{5.1}$$

Notice that $c' \notin \mathscr{B}$ for any $c' \mid c$ because $\mathscr{B}$ is primitive. So $\eta(c) = 1$. By the assumption that η is a Toeplitz sequence, there exists $m \in \mathbb{N}$ such that

$$\eta(c + m\ell) = 1 \quad \text{for any } \ell \in \mathbb{Z}.$$

Hence $c + m\mathbb{Z} \subset \mathcal{F}_\mathscr{B}$, which implies $(c + m\mathbb{Z}) \cap c\mathbb{Z} \subset \mathcal{F}_\mathscr{B}$. By Lemma 2.7, we have

$$(c + m\mathbb{Z}) \cap c\mathbb{Z} = \operatorname{lcm}(c, m)\mathbb{Z} + c.$$

So $\underline{d}(\mathcal{F}_\mathscr{B} \cap c\mathbb{Z}) \ge 1/\operatorname{lcm}(c, m)$, which contradicts (5.1). ■

REMARK 5.4. The sequences we have considered here are regular Toeplitz sequences, hence the systems they determine are strictly ergodic [15] and of zero entropy. There exist non-regular Toeplitz sequences of $\mathscr{B}$-free origin—see [16, Example 4.2].

The following questions seem to be natural: How to characterize those Toeplitz sequences which are of $\mathscr{B}$-free origin? Which non-regular Toeplitz sequences are of $\mathscr{B}$-free origin? Can entropy be positive in the Toeplitz $\mathscr{B}$-free case? Is the automorphism group always trivial (question by Lemańczyk for all $\mathscr{B}$-free systems)?

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Aurelia Dymek
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: aurbart@mat.umk.pl