

Reciprocity theorems for Bettin–Conrey sums

by

JUAN S. AULI (Hanover, NH), ABDELMEJID BAYAD (Évry) and
MATTHIAS BECK (San Francisco, CA)

Dedicated to the memory of Tom M. Apostol

1. Introduction and statement of results. Our point of departure is recent work of Bettin and Conrey [9, 8] on the period functions of Eisenstein series. Their initial motivation was the derivation of an exact formula for the second moments of the Riemann zeta function, but their work naturally gave rise to a family of finite arithmetic sums of the form

$$c_a\left(\frac{h}{k}\right) = k^a \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \zeta\left(-a, \frac{m}{k}\right),$$

where $a \in \mathbb{C}$, h and k are positive coprime integers, and $\zeta(a, x)$ denotes the *Hurwitz zeta function*

$$\zeta(a, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^a},$$

initially defined for $\Re(a) > 1$ and meromorphically continued to the a -plane. We call $c_a(h/k)$ and its natural generalizations appearing below *Bettin–Conrey sums*.

There are two major motivations to study these sums. The first is that $c_0(h/k)$ is essentially a *Vasyunin sum*, which in turn makes a critical appearance in the Nyman–Beurling–Báez-Duarte approach to the Riemann hypothesis through the twisted mean-square of the Riemann zeta function on the critical line (see, e.g., [5, 15]). Bettin–Conrey’s work, for $a = 0$, implies that there is a hidden symmetry of this mean-square.

2010 *Mathematics Subject Classification*: Primary 11F20; Secondary 11L03, 11M35.

Key words and phrases: Dedekind sum, cotangent sum, Bettin–Conrey sum, reciprocity theorem, Hurwitz zeta function, period function, quantum modular form, Estermann zeta function.

Received 3 August 2016; revised 13 June 2017.

Published online 24 November 2017.

The second motivation, and the central theme of our paper, is that the Bettin–Conrey sums satisfy a *reciprocity theorem*:

$$c_a\left(\frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a} c_a\left(\frac{-k}{h}\right) + \frac{k^a a \zeta(1-a)}{\pi h}$$

extends from its initiation domain \mathbb{Q} to an (explicit) analytic function on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, making c_a nearly an example of a *quantum modular form* in the sense of Zagier [17]. In fact, Zagier’s “Example 0” is the *Dedekind sum*

$$s(h, k) = \frac{1}{4k} \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \cot\left(\frac{\pi m}{k}\right),$$

which is, up to a trivial factor, $c_{-1}(h/k)$. Dedekind sums first appeared in the transformation properties of the Dedekind eta function and satisfy the reciprocity theorem [10, 14]

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12} \left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right).$$

We now recall the precise form of Bettin–Conrey’s reciprocity theorem.

THEOREM 1.1 (Bettin–Conrey [8]). *If h and k are positive coprime integers then*

$$c_a\left(\frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a} c_a\left(\frac{-k}{h}\right) + \frac{k^a a \zeta(1-a)}{\pi h} = -i \zeta(-a) \psi_a\left(\frac{h}{k}\right),$$

where

$$\begin{aligned} \psi_a(z) &= \frac{i}{\pi z} \frac{\zeta(1-a)}{\zeta(-a)} - \frac{i}{z^{1+a}} \cot\left(\frac{\pi a}{2}\right) + i \frac{g_a(z)}{\zeta(-a)}, \\ g_a(z) &= -2 \sum_{1 \leq n \leq M} (-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a) (2\pi z)^{2n-1} \\ &\quad + \frac{1}{\pi i} \int_{(-1/2-2M)} \zeta(s) \zeta(s-a) \Gamma(s) \frac{\cos(\frac{\pi a}{2})}{\sin(\pi \frac{s-a}{2})} (2\pi z)^{-s} ds. \end{aligned}$$

Here B_k denotes the k th Bernoulli number, M is any integer with $M \geq -\frac{1}{2} \min\{0, \Re(a)\}$, and the integration is over the vertical line $\Re(s) = -1/2 - 2M$.

We note that Bettin and Conrey initially defined $\psi_a(z)$ through

$$\psi_a(z) = E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right),$$

in other words, $\psi_a(z)$ is the *period function* of the *Eisenstein series* of weight $a + 1$,

$$E_{a+1}(z) = 1 + \frac{2}{\zeta(-a)} \sum_{n \geq 1} \sigma_a(n) e^{2\pi i n z},$$

where $\sigma_a(n) = \sum_{d|n} d^a$, and then showed that $\psi_a(z)$ satisfies the properties of Theorem 1.1.

We have several goals. We start by showing that the right-hand side of Theorem 1.1 can be simplified by employing an integration technique for Dedekind-like sums that goes back to Rademacher [14]. This yields our first main result:

THEOREM 1.2. *Let $\Re(a) > 1$ and suppose h and k are positive coprime integers. Then for any $0 < \varepsilon < \min\{1/h, 1/k\}$,*

$$h^{1-a} c_{-a}\left(\frac{h}{k}\right) + k^{1-a} c_{-a}\left(\frac{k}{h}\right) = \frac{a\zeta(a+1)}{\pi(hk)^a} - \frac{(hk)^{1-a}}{2i} \int_{(\varepsilon)} \frac{\cot(\pi h z) \cot(\pi k z)}{z^a} dz.$$

Theorem 1.2 implies that the function

$$F(a) = \int_{(\varepsilon)} \frac{\cot(\pi h z) \cot(\pi k z)}{z^a} dz$$

has a holomorphic continuation to the whole complex plane. In particular, in this sense Theorem 1.2 can be extended to all complex a .

Second, we employ Theorem 1.2 to show that when a is an odd negative integer, the right-hand side of the reciprocity theorem can be explicitly given in terms of Bernoulli numbers.

THEOREM 1.3. *Let $n > 1$ be an odd integer and suppose h and k are positive coprime integers. Then*

$$\begin{aligned} h^{1-n} c_{-n}\left(\frac{h}{k}\right) + k^{1-n} c_{-n}\left(\frac{k}{h}\right) \\ = \left(\frac{2\pi i}{hk}\right)^n \frac{1}{i(n+1)!} \left(n B_{n+1} + \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m} \right). \end{aligned}$$

Our third main result is a consequence of Theorem 1.3: in conjunction with Theorem 1.1, it implies the following explicit formulas for $\psi_a(z)$ and $g_a(z)$ when a is an odd negative integer.

THEOREM 1.4. *If $n > 1$ is an odd integer then for all $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$,*

$$\begin{aligned} \psi_{-n}(z) &= \frac{(2\pi i)^n}{\zeta(n)(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} z^{m-1}, \\ g_{-n}(z) &= \frac{(2\pi i)^n}{i(n+1)!} \sum_{m=0}^n \binom{n+1}{m+1} B_{m+1} B_{n-m} z^m. \end{aligned}$$

In [8, Theorem 2], Bettin and Conrey computed the Taylor series of $g_a(z)$ and remarked that, if a is a negative integer, $\pi g_a^{(m)}(1)$ is a rational polynomial in π^2 . Theorem 1.4 generalizes this remark. We will prove Theorems 1.2–1.4 in Section 2. We note that both Theorem 1.3 and 1.4 can also be derived directly from Theorem 1.1.

Our next goal is to study natural generalizations of $c_a(h/k)$. Taking a leaf from Zagier’s generalization of $s(h, k)$ to *higher-dimensional Dedekind sums* [16] and its variation involving cotangent derivatives [7], let k_0, k_1, \dots, k_n be positive integers such that $(k_0, k_j) = 1$ for $j = 1, \dots, n$, let m_0, m_1, \dots, m_n be nonnegative integers, $a \neq -1$ a complex number, and define the *generalized Bettin–Conrey sum*

$$c_a \left(\begin{matrix} k_0 & k_1 & \cdots & k_n \\ m_0 & m_1 & \cdots & m_n \end{matrix} \right) = k_0^a \sum_{l=1}^{k_0-1} \zeta^{(m_0)} \left(-a, \frac{l}{k_0} \right) \prod_{j=1}^n \cot^{(m_j)} \left(\frac{\pi k_j l}{k_0} \right).$$

Here $\zeta^{(m_0)}(a, z)$ denotes the m_0 th derivative of the Hurwitz zeta function with respect to z .

This notation mimics that of Dedekind cotangent sums; note that

$$c_s \left(\frac{h}{k} \right) = c_s \left(\begin{matrix} k & h \\ 0 & 0 \end{matrix} \right).$$

In Section 3, we will prove reciprocity theorems for generalized Bettin–Conrey sums, paralleling Theorems 1.2 and 1.3, as well as more special cases that give, we think, interesting identities.

Our final goal is to relate the particular generalized Bettin–Conrey sum

$$\sum_{m=1}^{q-1} \cot^{(k)}(\pi m x) \zeta \left(-a, \frac{m}{q} \right)$$

with evaluations of the *Estermann zeta function* $\sum_{n \geq 1} \sigma_a(n) e^{2\pi i n x} / n^s$ at integers s ; see Section 4.

2. Proofs of main results. In order to prove Theorem 1.2, we need two lemmas.

LEMMA 2.1. *Let m be a nonnegative integer. Then*

$$\lim_{y \rightarrow \infty} \cot^{(m)}(\pi(x \pm iy)) = \begin{cases} \mp i & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Furthermore, this convergence is uniform with respect to x in a fixed bounded interval.

Proof. Since $\cot z = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$, we may estimate

$$|i + \cot(\pi(x + iy))| = \frac{2}{|e^{i(2\pi x)} - e^{2\pi y}|} \leq \frac{2}{||e^{i(2\pi x)}| - |e^{2\pi y}||} = \frac{2}{|1 - e^{2\pi y}|}.$$

As the rightmost term vanishes as $y \rightarrow \infty$, we see that

$$\lim_{y \rightarrow \infty} \cot(\pi(x + iy)) = -i.$$

Similarly, the inequality

$$|-i + \cot(\pi(x - iy))| = \frac{2}{|e^{i(2\pi x)}e^{2\pi y} - 1|} \leq \frac{2}{||e^{i(2\pi x)}e^{2\pi y}| - 1|} = \frac{2}{|e^{2\pi y} - 1|}$$

implies that $\lim_{y \rightarrow \infty} \cot(\pi(x - iy)) = i$. Since

$$|\csc(\pi(x + iy))| = \frac{2e^{\pi y}}{|e^{i\pi x} - e^{-i\pi x}e^{2\pi y}|} \leq \frac{2e^{\pi y}}{||e^{i\pi x}| - |e^{-i\pi x}e^{2\pi y}||} = \frac{2e^{\pi y}}{|1 - e^{2\pi y}|},$$

it follows that $\lim_{y \rightarrow \infty} \csc(\pi(x + iy)) = 0$. Similarly,

$$|\csc(\pi(x - iy))| = \frac{2e^{\pi y}}{|e^{i\pi x}e^{2\pi y} - e^{-i\pi x}|} \leq \frac{2e^{\pi y}}{||e^{i\pi x}e^{2\pi y}| - |e^{-i\pi x}||} = \frac{2e^{\pi y}}{|e^{2\pi y} - 1|}$$

implies that $\lim_{y \rightarrow \infty} \csc(\pi(x - iy)) = 0$. We remark that $\frac{d}{dz}(\cot z) = -\csc^2 z$ and

$$\frac{d}{dz}(\csc z) = -\csc z \cot z,$$

so all the derivatives of $\cot z$ have a $\csc z$ factor, and therefore

$$\lim_{y \rightarrow \infty} \cot^{(m)}(\pi(x \pm iy)) = \begin{cases} \mp i & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Since the convergence above is independent of x , the limit is uniform with respect to x in a fixed bounded interval. ■

Lemma 2.1 implies that

$$\lim_{y \rightarrow \infty} \cot^{(m)}(\pi h(x \pm iy)) = \lim_{y \rightarrow \infty} \cot^{(m)}(\pi k(x \pm iy)) = \begin{cases} \mp i & \text{if } m = 0, \\ 0 & \text{if } m > 0, \end{cases}$$

uniformly with respect to x in a fixed bounded interval.

The proof of the following lemma is hinted at by Apostol [3].

LEMMA 2.2. *If $\Re(a) > 1$ and $R > 0$, then $\zeta(a, x + iy)$ vanishes uniformly with respect to $x \in [0, R]$ as $y \rightarrow \pm\infty$.*

Proof. We begin by showing that $\zeta(a, z)$ vanishes as $\Im(z) \rightarrow \pm\infty$ if $\Re(z) > 0$. Since $\Re(a) > 1$ and $\Re(z) > 0$, we have the integral representation [13, eq. 25.11.25]

$$\zeta(a, z) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{t^{a-1}e^{-zt}}{1 - e^{-t}} dt,$$

which may be written as

$$(2.1) \quad \zeta(a, z) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{t^{a-1} e^{-t\Re(z)}}{1 - e^{-t}} e^{-it\Im(z)} dt.$$

Note that for fixed $\Re(z)$,

$$\int_0^\infty \frac{t^{a-1} e^{-t\Re(z)}}{1 - e^{-t}} dt = \zeta(a, \Re(z))\Gamma(a)$$

and

$$\int_0^\infty \left| \frac{t^{a-1} e^{-t\Re(z)}}{1 - e^{-t}} \right| dt = \int_0^\infty \frac{t^{\Re(a)-1} e^{-t\Re(z)}}{1 - e^{-t}} dt = \zeta(\Re(a), \Re(z))\Gamma(\Re(a)),$$

so the Riemann–Lebesgue lemma (see, for example, [12, Theorem 16]) implies that

$$\int_0^\infty \frac{t^{a-1} e^{-t\Re(z)}}{1 - e^{-t}} e^{-it\Im(z)} dt$$

vanishes as $\Im(z) \rightarrow \pm\infty$. By (2.1), this means that for $\Re(z)$ fixed, $\zeta(a, z)$ vanishes as $\Im(z) \rightarrow \pm\infty$. In other words, $\zeta(a, x + iy) \rightarrow 0$ pointwise with respect to $x > 0$ as $y \rightarrow \pm\infty$.

Moreover, the vanishing of $\zeta(a, x + iy)$ as $y \rightarrow \pm\infty$ is uniform with respect to $x \in [0, R]$. Indeed, denote $g(t) = \frac{t^{a-1} e^{-tR}}{1 - e^{-t}}$; then (2.1) implies that

$$\int_0^\infty g(t) dt = \Gamma(a)\zeta(a, R) \quad \text{and} \quad \int_0^\infty |g(t)| dt = \Gamma(\Re(a))\zeta(\Re(a), R).$$

The Riemann–Lebesgue lemma shows that $\lim_{|z| \rightarrow \infty} \int_0^\infty g(t) e^{-itz} dt = 0$. If $x \in (0, R]$, we may write

$$\Gamma(a)\zeta(a, x \pm iy) = \int_0^\infty \frac{t^{a-1} e^{-tx}}{1 - e^{-t}} e^{\mp ity} dt = \int_0^\infty g(t) e^{-it(\pm y - i(x - R))} dt.$$

Since $g(t)$ does not depend on x , the speed at which $\zeta(a, x \pm iy)$ vanishes depends on R and $y^2 + (x - R)^2$. However, we know that $0 \leq |x - R| < R$, so the speed of the vanishing depends only on R .

Finally, note that

$$\zeta(a, iy) = \sum_{n=0}^\infty \frac{1}{(iy + n)^a} = \sum_{n=0}^\infty \frac{1}{(1 + iy + n)^a} + \frac{1}{(iy)^a} = \zeta(a, 1 + iy) + \frac{1}{(iy)^a},$$

so $\zeta(a, iy) \rightarrow 0$ as $y \rightarrow \pm\infty$, and the speed at which $\zeta(s, iy)$ vanishes depends on that of $\zeta(s, 1 + iy)$. Thus, $\zeta(s, x + iy) \rightarrow 0$ uniformly as $y \rightarrow \pm\infty$, as long as $x \in [0, R]$. ■

Proof of Theorem 1.2. The idea is to use Cauchy’s residue theorem to integrate the function

$$f(z) = \cot(\pi h z) \cot(\pi k z) \zeta(a, z)$$

along $C(M, \varepsilon)$ as $M \rightarrow \infty$, where $C(M, \varepsilon)$ denotes the positively oriented rectangle with vertices $1 + \varepsilon + iM$, $\varepsilon + iM$, $\varepsilon - iM$ and $1 + \varepsilon - iM$, for $M > 0$ and $0 < \varepsilon < \min\{1/h, 1/k\}$ (see Figure 1).

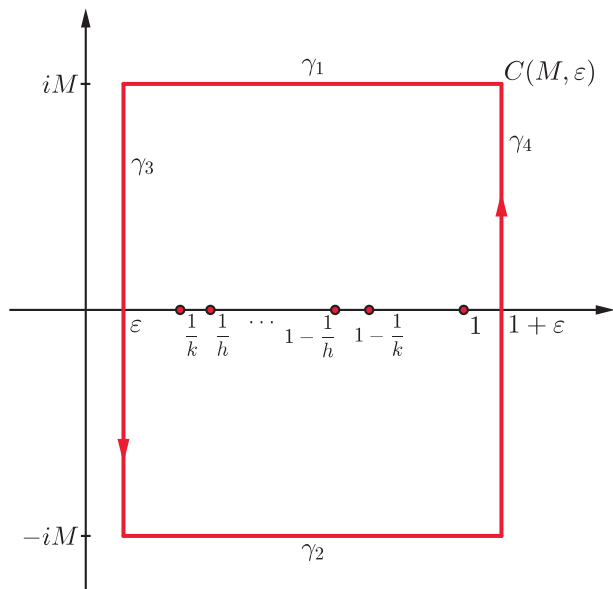


Fig. 1. The closed contour $C(M, \varepsilon)$

Henceforth, $a \in \mathbb{C}$ is such that $\Re(a) > 1$, (h, k) is a pair of coprime positive integers, and $f(z)$ and $C(M, \varepsilon)$ are as above, unless otherwise stated. Since $\zeta(a, z)$ is analytic inside $C(M, \varepsilon)$, the only poles of $f(z)$ are those of the cotangent factors. Thus, the fact that h and k are coprime implies that a complete list of the possible poles of $f(z)$ inside $C(M, \varepsilon)$ is

$$E = \left\{ \frac{1}{h}, \dots, \frac{h-1}{h}, \frac{1}{k}, \dots, \frac{k-1}{k}, 1 \right\},$$

and each of these poles is (at most) simple, with the exception of 1, which is (at most) double. For $m \in \{1, \dots, h-1\}$,

$$\begin{aligned} \operatorname{Res}_{z=m/h} f(z) &= \cot\left(\frac{\pi k m}{h}\right) \cos(\pi m) \zeta\left(a, \frac{m}{h}\right) \operatorname{Res}_{z=m/h} \frac{1}{\sin(\pi h z)} \\ &= \frac{1}{\pi h} \cot\left(\frac{\pi k m}{h}\right) \zeta\left(a, \frac{m}{h}\right). \end{aligned}$$

Of course, an analogous result is true for $\text{Res}_{z=m/k} f(z)$ for all $m \in \{1, \dots, k - 1\}$, and therefore

$$\sum_{z_0 \in E} \text{Res}_{z=z_0} f(z) = \text{Res}_{z=1} f(z) + \frac{1}{\pi h} \sum_{m=1}^{h-1} \cot\left(\frac{\pi km}{h}\right) \zeta\left(a, \frac{m}{h}\right) + \frac{1}{\pi k} \sum_{m=1}^{k-1} \cot\left(\frac{\pi hm}{k}\right) \zeta\left(a, \frac{m}{k}\right),$$

or equivalently

$$(2.2) \quad h^{1-a} c_{-a}\left(\frac{h}{k}\right) + k^{1-a} c_{-a}\left(\frac{k}{h}\right) = \pi(hk)^{1-a} \left(\left(\sum_{z_0 \in E} \text{Res}_{z=z_0} f(z) \right) - \text{Res}_{z=1} f(z) \right).$$

We now determine $\text{Res}_{z=1} f(z)$. The Laurent series of the cotangent function about 0 is given by

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \frac{2}{945}z^5 + \dots,$$

so, by the periodicity of $\cot z$, for $z \neq 1$ in a small neighborhood of $z = 1$,

$$\cot(\pi kz) = \left(\frac{1}{\pi k}\right) \frac{1}{z-1} - \frac{\pi k}{3}(z-1) - \frac{(\pi k)^3}{45}(z-1)^3 - \frac{2(\pi k)^5}{945}(z-1)^5 + \dots,$$

and similarly

$$\cot(\pi hz) = \left(\frac{1}{\pi h}\right) \frac{1}{z-1} - \frac{\pi h}{3}(z-1) - \frac{(\pi h)^3}{45}(z-1)^3 - \frac{2(\pi h)^5}{945}(z-1)^5 + \dots.$$

Since $\zeta(a, z)$ is analytic in a small neighborhood of 1, Taylor's theorem implies that

$$\zeta(a, z) = \sum_{n=0}^{\infty} b_n(z-1)^n,$$

where $b_n = \zeta^{(n)}(a, 1)/n!$ for $n = 0, 1, 2, \dots$ (derivatives relative to z). Thus, the expansion of $f(z)$ about 1 is of the form

$$\frac{b_0}{\pi^2 hk} \left(\frac{1}{z-1}\right)^2 + \left(\frac{b_1}{\pi^2 hk}\right) \frac{1}{z-1} + (\text{analytic part}).$$

Since $a \neq 0, 1$, we know that $\frac{\partial}{\partial z} \zeta(a, z) = -a\zeta(a+1, z)$ [13, eq. 25.11.17], so $b_1 = -a\zeta(a+1, 1) = -a\zeta(a+1)$. Hence $\text{Res}_{z=1} f(z) = -a\zeta(a+1)/(\pi^2 hk)$ and it then follows from (2.2) that

$$(2.3) \quad h^{1-a} c_{-a}\left(\frac{h}{k}\right) + k^{1-a} c_{-a}\left(\frac{k}{h}\right) = \frac{a\zeta(a+1)}{\pi(hk)^a} + \frac{\pi}{(hk)^{a-1}} \sum_{z_0 \in E} \text{Res}_{z=z_0} f(z).$$

We now turn to the computation of $\sum_{z_0 \in E} \operatorname{Res}_{z=z_0} f(z)$ via Cauchy’s residue theorem, which together with (2.3) will provide the reciprocity we are after. Note that the function $f(z)$ is analytic on any two closed contours $C(M_1, \varepsilon)$ and $C(M_2, \varepsilon)$, and since the poles inside these two contours are the same, we may apply Cauchy’s residue theorem to both contours and deduce that

$$\int_{C(M_1, \varepsilon)} f(z) dz = \int_{C(M_2, \varepsilon)} f(z) dz.$$

In particular, this implies that

$$(2.4) \quad \lim_{M \rightarrow \infty} \int_{C(M, \varepsilon)} f(z) dz = 2\pi i \sum_{z_0 \in E} \operatorname{Res}_{z=z_0} f(z).$$

Let γ_1 be the path along $C(M, \varepsilon)$ from $1 + \varepsilon + iM$ to $\varepsilon + iM$. Similarly, define γ_2 from $\varepsilon - iM$ to $1 + \varepsilon - iM$, γ_3 from $\varepsilon + iM$ to $\varepsilon - iM$, and γ_4 from $1 + \varepsilon - iM$ to $1 + \varepsilon + iM$ (see Figure 1). Since $\Re(a) > 1$,

$$\zeta(a, z + 1) = \sum_{n=0}^{\infty} \frac{1}{(n + z + 1)^a} = \sum_{n=1}^{\infty} \frac{1}{(n + z)^a} = \zeta(a, z) - \frac{1}{z^a},$$

and so the periodicity of $\cot z$ implies that

$$\int_{\gamma_4} f(z) dz = - \int_{\gamma_3} f(z) dz + \int_{\gamma_3} \frac{\cot(\pi hz) \cot(\pi kz)}{z^a} dz.$$

Lemmas 2.1 and 2.2 show that $f(z)$ vanishes uniformly as $M \rightarrow \infty$ (uniformly with respect to $\Re(z) \in [\varepsilon, 1 + \varepsilon]$), so

$$\lim_{M \rightarrow \infty} \int_{\gamma_1} f(z) dz = 0 = \lim_{M \rightarrow \infty} \int_{\gamma_2} f(z) dz.$$

This means that

$$\lim_{M \rightarrow \infty} \int_{C(M, \varepsilon)} f(z) dz = \lim_{M \rightarrow \infty} \left(\int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz \right),$$

and it follows from (2.3) and (2.4) that

$$\begin{aligned} h^{1-a} c_{-a} \left(\frac{h}{k} \right) + k^{1-a} c_{-a} \left(\frac{k}{h} \right) &= \frac{a\zeta(a+1)}{\pi(hk)^a} + \frac{(hk)^{1-a}}{2i} \int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{\cot(\pi hz) \cot(\pi kz)}{z^a} dz. \end{aligned}$$

This completes the proof of Theorem 1.2. ■

To prove Theorem 1.3, we now turn to the particular case in which $a = n > 1$ is an odd integer and study Bettin–Conrey sums of the form c_{-n} .

Let $\Psi^{(n)}(z)$ denote the $(n + 2)$ th polygamma function (see, for example, [13, Sec. 5.15]). It is well known that for n a positive integer,

$$\zeta(n + 1, z) = \frac{(-1)^{n+1}\Psi^{(n)}(z)}{n!}$$

whenever $\Re(z) > 0$ (see, for instance, [13, eq. 25.11.12]), so for $n > 1$ we may write

$$c_{-n}\left(\frac{h}{k}\right) = \frac{(-1)^n}{k^n(n-1)!} \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \Psi^{(n-1)}\left(\frac{m}{k}\right).$$

By the reflection formula for the polygamma functions [13, eq. 5.15.6],

$$\Psi^{(n)}(1 - z) + (-1)^{n+1}\Psi^{(n)}(z) = (-1)^n \pi \cot^{(n)}(\pi z),$$

we know that if n is odd, then

$$\Psi^{(n-1)}\left(1 - \frac{m}{k}\right) - \Psi^{(n-1)}\left(\frac{m}{k}\right) = \pi \cot^{(n-1)}\left(\frac{\pi m}{k}\right)$$

for each $m \in \{1, \dots, k - 1\}$. Therefore,

$$\begin{aligned} 2 \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \Psi^{(n-1)}\left(\frac{m}{k}\right) &= \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \Psi^{(n-1)}\left(\frac{m}{k}\right) \\ &\quad + \sum_{m=1}^{k-1} \cot\left(\frac{\pi(k-m)h}{k}\right) \Psi^{(n-1)}\left(1 - \frac{m}{k}\right), \end{aligned}$$

which implies that

$$\begin{aligned} 2 \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \Psi^{(n-1)}\left(\frac{m}{k}\right) &= \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \left(\Psi^{(n-1)}\left(\frac{m}{k}\right) - \Psi^{(n-1)}\left(1 - \frac{m}{k}\right)\right) \\ &= -\pi \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \cot^{(n-1)}\left(\frac{\pi m}{k}\right). \end{aligned}$$

This means that for $n > 1$ odd, c_{-n} is essentially a Dedekind cotangent sum. Indeed, using the notation in [7], we have

$$\begin{aligned} c_{-n}\left(\frac{h}{k}\right) &= \frac{\pi}{2k^n(n-1)!} \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \cot^{(n-1)}\left(\frac{\pi m}{k}\right) \\ &= \frac{\pi}{2(n-1)!} \mathbf{c} \left(\begin{array}{c|cc} k & h & 1 \\ n-1 & 0 & n-1 \\ 0 & 0 & 0 \end{array} \right). \end{aligned}$$

Thus Theorem 1.3 is an instance of Theorem 1.2. Its significance is a reciprocity instance for Bettin–Conrey sums of the form c_{-n} in terms of Bernoulli numbers. For this reason we give the details of its proof.

Proof of Theorem 1.3. We consider the closed contour $\tilde{C}(M, \varepsilon)$ defined as the positively oriented rectangle with vertices $1 + iM$, iM , $-iM$ and $1 - iM$, with indentations (to the right) of radius $0 < \varepsilon < \min\{1/h, 1/k\}$ around 0 and 1 (see Figure 2).

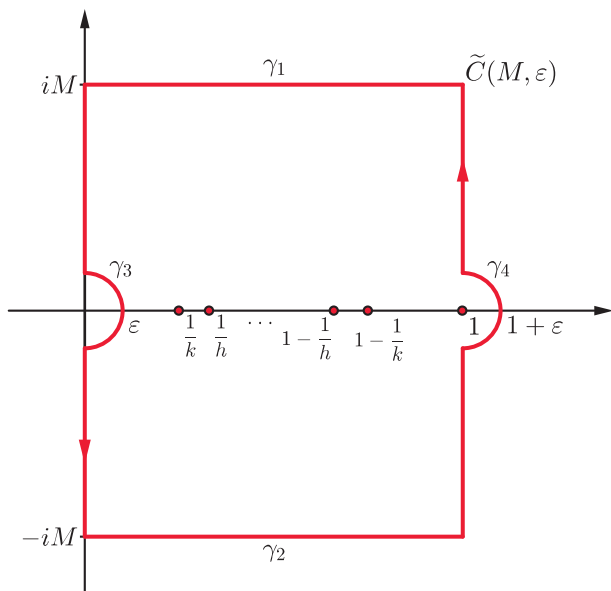


Fig. 2. The closed contour $\tilde{C}(M, \varepsilon)$

Since $\tilde{C}(M, \varepsilon)$ contains the same poles of $f(z) = \cot(\pi hz) \cot(\pi kz) \zeta(n, z)$ as the closed contour $C(M, \varepsilon)$ in Figure 1 used to prove Theorem 1.2, we may apply Cauchy’s residue theorem, letting $M \rightarrow \infty$, and we only need to determine $\lim_{M \rightarrow \infty} \int_{\tilde{C}(M, \varepsilon)} f(z) dz$ in order to deduce a reciprocity law for the sums c_{-n} .

As in the case of $C(M, \varepsilon)$, the integrals along the horizontal paths vanish, so using the periodicity of the cotangent to add integrals along parallel paths, as we did when considering $C(M, \varepsilon)$, we obtain

$$(2.5) \quad \lim_{M \rightarrow \infty} \int_{\tilde{C}(M, \varepsilon)} f(z) dz = \lim_{M \rightarrow \infty} \left(\int_{iM}^{i\varepsilon} g(z) dz + \int_{-i\varepsilon}^{-iM} g(z) dz \right) + \int_{\gamma_3} g(z) dz,$$

where γ_3 denotes the indented path around 0 and

$$g(z) = \frac{\cot(\pi kz) \cot(\pi hz)}{z^n}.$$

Given that $g(z)$ is an odd function, the vertical integrals cancel, and we may apply Cauchy’s residue theorem to integrate $g(z)$ along the positively oriented circle of radius ε and centered at 0 to deduce that

$$\lim_{M \rightarrow \infty} \int_{\tilde{C}(M, \varepsilon)} f(z) dz = -\pi i \operatorname{Res}_{z=0} g(z).$$

This is the main reason to use the contour $\tilde{C}(M, \varepsilon)$ instead of $C(M, \varepsilon)$. Indeed, integration along $\tilde{C}(M, \varepsilon)$ exploits the parity of the function $g(z)$, allowing us to cancel the vertical integrals in (2.5).

The expansion of the cotangent function is

$$\pi z \cot(\pi z) = \sum_{m=0}^{\infty} \frac{(2\pi i)^m B_m}{m!} z^m,$$

with the convention that B_1 must be redefined to be zero. Thus,

$$\cot(\pi k z) = \sum_{m=-1}^{\infty} \frac{(2i)(2\pi i k)^m B_{m+1}}{(m+1)!} z^m,$$

and of course an analogous result holds for h . Hence,

$$(2.6) \quad \operatorname{Res}_{z=0} g(z) = \frac{(2i)(2\pi i)^n}{\pi h k (n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m},$$

and given that $\zeta(n+1) = -\frac{(2\pi i)^{n+1}}{2(n+1)!} B_{n+1}$ [13, eq. 25.6.2], the Cauchy residue theorem and (2.3) yield

$$(2.7) \quad \left(\frac{2\pi i}{hk}\right)^n \frac{1}{i(n+1)!} \left(n B_{n+1} + \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m} \right) = h^{1-n} c_{-n} \left(\frac{h}{k}\right) + k^{1-n} c_{-n} \left(\frac{k}{h}\right).$$

Finally, note that the convention $B_1 := 0$ is irrelevant in (2.7), since B_1 in this sum is always multiplied by a Bernoulli number with odd index larger than 1. ■

Note that Theorem 1.3 is essentially the same as the reciprocity deduced by Apostol for Dedekind–Apostol sums [2]. This is a consequence of the fact that for $n > 1$ an odd integer, $c_{-n}(h/k)$ is a multiple of the Dedekind–Apostol sum $s_n(h, k)$. Indeed, for such n [3, Theorem 1],

$$s_n(h, k) = in!(2\pi i)^{-n} c_{-n}(h/k).$$

It is worth mentioning that although the Dedekind–Apostol sum $s_n(h, k)$ is trivial for n even [2, eq. (4.13)], in the sense that $s_n(h, k)$ is independent of h , the Bettin–Conrey sum $c_{-n}(h/k)$ is not.

The following is an immediate consequence of Theorems 1.2 and 1.3.

COROLLARY 2.3. *Let $n > 1$ be an odd integer and suppose h and k are positive coprime integers. Then for any $0 < \varepsilon < \min\{1/h, 1/k\}$,*

$$\int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{\cot(\pi hz) \cot(\pi kz)}{z^n} dz = \frac{2(2\pi i)^n}{hk(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} h^m k^{n+1-m}.$$

Proof of Theorem 1.4. Since $c_{-n}(-k/h) = -c_{-n}(k/h)$, it follows from Theorems 1.1 and 1.3 that

$$(2.8) \quad \psi_{-n}\left(\frac{h}{k}\right) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} \left(\frac{h}{k}\right)^{m-1}.$$

The function

$$\phi_{-n}(z) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} z^{m-1}$$

is analytic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and, by [8, Theorem 1], so is ψ_{-n} . Let

$$S_n = \{z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \mid \psi_{-n}(z) = \phi_{-n}(z)\}.$$

Since all positive rationals can be written in reduced form, it follows from (2.8) that $\mathbb{Q}_{>0} \subseteq S_n$. Thus, S_n is not a discrete set and, as both ψ_{-n} and ϕ_{-n} are analytic on the connected open set $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, [11, Theorem 1.2(ii), p. 90] implies that $\psi_{-n} = \phi_{-n}$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. That is,

$$\psi_{-n}(z) = \frac{(2\pi i)^n}{\zeta(n)(n+1)!} \sum_{m=0}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} z^{m-1}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Now

$$\psi_{-n}(z) = \frac{i}{\pi z} \frac{\zeta(1+n)}{\zeta(n)} - iz^{n-1} \cot\left(\frac{-\pi n}{2}\right) + i \frac{g_{-n}(z)}{\zeta(n)},$$

and since n is odd, $\cot(-\pi n/2) = 0$ and $\zeta(n+1) = -\frac{(2\pi i)^{n+1}}{2(n+1)!} B_{n+1}$ [13, eq. 25.6.2], so

$$\begin{aligned} g_{-n}(z) &= \frac{i(2\pi i)^n B_{n+1}}{(n+1)!} \left(\frac{1}{z}\right) - i\zeta(n)\psi_{-n}(z) \\ &= \frac{-i(2\pi i)^n}{(n+1)!} \sum_{m=1}^{n+1} \binom{n+1}{m} B_m B_{n+1-m} z^{m-1} \\ &= \frac{-i(2\pi i)^n}{(n+1)!} \sum_{m=0}^n \binom{n+1}{m+1} B_{m+1} B_{n-m} z^m. \blacksquare \end{aligned}$$

Clearly, Theorem 1.4 is a particular case of Bettin–Conrey’s [8, Theorem 3]. However, the proofs are independent, so Theorem 1.4 is stronger (in the particular case $a = -n$, with $n > 1$ an odd integer), because it completely

determines g_{-n} and shows that g_{-n} is a polynomial. In particular, it becomes obvious that if $a \in \mathbb{Z}_{\leq 1}$ is odd and $(a, m) \neq (0, 0)$, then $\pi g_a^{(m)}(1)$ is a rational polynomial in π^2 .

3. Generalizations of Bettin–Conrey sums. Now we study the generalized Bettin–Conrey sum

$$c_a \left(\begin{matrix} k_0 & k_1 & \cdots & k_n \\ m_0 & m_1 & \cdots & m_n \end{matrix} \right) = k_0^a \sum_{l=1}^{k_0-1} \zeta^{(m_0)} \left(-a, \frac{l}{k_0} \right) \prod_{j=1}^n \cot^{(m_j)} \left(\frac{\pi k_j l}{k_0} \right).$$

Henceforth, B_n denotes the n th Bernoulli number with the convention $B_1 := 0$.

The following reciprocity theorem generalizes Theorem 1.2.

THEOREM 3.1. *Let $d \geq 2$ and suppose that k_1, \dots, k_d is a list of pairwise coprime positive integers and m_0, m_1, \dots, m_d are nonnegative integers. If $\Re(a) > 1$ and $0 < \varepsilon < \min_{1 \leq j \leq d} \{1/k_j\}$ then*

$$\begin{aligned} \sum_{j=1}^d \frac{(-1)^{m_j}}{\pi} \sum_{\substack{l_0 + \cdots + \widehat{l_j} + \cdots + l_d = m_j \\ l_0, \dots, \widehat{l_j}, \dots, l_d \geq 0}} \binom{m_j}{l_0, \dots, \widehat{l_j}, \dots, l_d} \left(\prod_{\substack{t=1 \\ t \neq j}}^d (\pi k_t)^{l_t} \right) k_j^{a-1} c_{-a}(j) \\ = - \sum_{l_0=0}^{m_1 + \cdots + m_d + d - 1} \sum_{l_1 + \cdots + l_d = -l_0 - 1} \prod_{j=0}^d a_{l_j} \\ + \frac{(-1)^{m_0} a^{(m_0)}}{2\pi i} \int_{\varepsilon + i\infty}^{\varepsilon - i\infty} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{a+m_0}} dz, \end{aligned}$$

where $x^{(n)} = \prod_{l=0}^{n-1} (x + l)$ is the rising factorial,

$$c_{-a}(j) = c_{-a} \left(\begin{matrix} k_j & k_1 & \cdots & \widehat{k_j} & \cdots & k_d \\ m_0 + l_0 & m_1 + l_1 & \cdots & m_j + l_j & \cdots & m_d + l_d \end{matrix} \right),$$

and for $j = 0, 1, \dots, d$ we define

$$a_{l_j} = \begin{cases} \frac{(-1)^{m_0+l_0} a^{(m_0+l_0)} \zeta(a + m_0 + l_0)}{l_0!} & \text{if } j=0 \text{ and } l_0 \geq 0, \\ \frac{(2i)^{l_j+m_j+1} B_{l_j+m_j+1} (\pi k_j)^{l_j+m_j} (l_j + 1)^{(m_j)}}{(l_j + m_j + 1)!} & \text{if } j \neq 0 \text{ and } l_j \geq 0, \\ \frac{(-1)^{m_j} m_j!}{\pi k_j} & \text{if } j \neq 0 \text{ and } l_j = -(m_j + 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The argument is analogous to that for Theorem 1.2. Again, $\Re(a) > 1$, k_1, \dots, k_d is a list of pairwise coprime positive integers and $0 < \varepsilon < \min_{1 \leq j \leq d} \{1/k_j\}$. In addition, m_0, m_1, \dots, m_d is a list of nonnegative integers,

$$f(z) = \zeta^{(m_0)}(a, z) \prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)$$

and, as before, $C(M, \varepsilon)$ denotes the positively oriented rectangle with vertices $1 + \varepsilon + iM$, $\varepsilon + iM$, $\varepsilon - iM$ and $1 + \varepsilon - iM$, where $M > 0$ (see Figure 1).

For each j , we know that $\cot(\pi k_j z)$ is analytic on and inside $C(M, \varepsilon)$, with the exception of the poles $1/k_j, \dots, (k_j - 1)/k_j$. This means that except for these poles, $\cot^{(m_j)}(\pi k_j z)$ is analytic on and inside $C(M, \varepsilon)$, so the analyticity of $\zeta^{(m_0)}(a, z)$ on and inside $C(M, \varepsilon)$ implies that a complete list of (possible) poles of f is

$$E = \left\{ \frac{1}{k_1}, \dots, \frac{k_1 - 1}{k_1}, \dots, \frac{1}{k_d}, \dots, \frac{k_d - 1}{k_d}, 1 \right\}.$$

Let $j \in \{1, \dots, d\}$ and $q \in \{1, \dots, k_j - 1\}$. Then the Laurent series of $\cot(\pi k_j z)$ about q/k_j is of the form $\left(\frac{1}{\pi k_j}\right) \frac{1}{z - q/k_j} + (\text{analytic part})$, so near q/k_j ,

$$\cot^{(m_j)}(\pi k_j z) = \frac{(-1)^{m_j} m_j!}{\pi k_j} \left(z - \frac{q}{k_j}\right)^{-(m_j+1)} + (\text{analytic part}).$$

Since $(k_j, k_t) = 1$ for $t \neq j$, it follows from Taylor’s theorem that for $t \neq j$ the expansion

$$\cot^{(m_t)}(\pi k_t z) = \sum_{l_t=0}^{\infty} \frac{(\pi k_t)^{l_t}}{l_t!} \cot^{(m_t+l_t)}\left(\frac{\pi k_t q}{k_j}\right) \left(z - \frac{q}{k_j}\right)^{l_t}$$

is valid near q/k_j . Taylor’s theorem also yields

$$\zeta^{(m_0)}(a, z) = \sum_{l_0=0}^{\infty} \frac{\zeta^{(m_0+l_0)}(a, q/k_j)}{l_0!} \left(z - \frac{q}{k_j}\right)^{l_0}$$

near q/k_j . Hence, we may write $\text{Res}_{z=q/k_j} f(z)$ as

$$\frac{(-1)^{m_j} m_j!}{\pi k_j} \sum_{\substack{l_0 + \dots + \widehat{l_j} + \dots + l_d = m_j \\ l_0, \dots, \widehat{l_j}, \dots, l_d \geq 0}} \frac{\zeta^{(m_0+l_0)}(a, q/k_j)}{l_0!} \prod_{\substack{t=1 \\ t \neq j}}^d \frac{(\pi k_t)^{l_t}}{l_t!} \cot^{(m_t+l_t)}\left(\frac{\pi k_t q}{k_j}\right).$$

Therefore $\sum_{q=1}^{k_j-1} \text{Res}_{z=q/k_j} f(z)$ is given by

$$\begin{aligned} & \frac{(-1)^{m_j}}{\pi} \sum_{\substack{l_0+\dots+\widehat{l}_j+\dots+l_d=m_j \\ l_0,\dots,\widehat{l}_j,\dots,l_d \geq 0}} \binom{m_j}{l_0, \dots, \widehat{l}_j, \dots, l_d} \left(\prod_{\substack{t=1 \\ t \neq j}}^d (\pi k_t)^{l_t} \right) \\ & \quad \times \frac{1}{k_j} \sum_{q=1}^{k_j-1} \zeta^{(m_0+l_0)} \left(a, \frac{q}{k_j} \right) \prod_{\substack{t=1 \\ t \neq j}}^d \cot^{(m_t+l_t)} \left(\frac{\pi k_t q}{k_j} \right) \\ & = \frac{(-1)^{m_j}}{\pi} \sum_{\substack{l_0+\dots+\widehat{l}_j+\dots+l_d=m_j \\ l_0,\dots,\widehat{l}_j,\dots,l_d \geq 0}} \binom{m_j}{l_0, \dots, \widehat{l}_j, \dots, l_d} \left(\prod_{\substack{t=1 \\ t \neq j}}^d (\pi k_t)^{l_t} \right) k_j^{a-1} c_{-a}(j). \end{aligned}$$

As this holds for all j , we conclude that

$$\begin{aligned} (3.1) \quad & \sum_{z_0 \in E \setminus \{1\}} \text{Res}_{z=z_0} f(z) \\ & = \sum_{j=1}^d \frac{(-1)^{m_j}}{\pi} \sum_{\substack{l_0+\dots+\widehat{l}_j+\dots+l_d=m_j \\ l_0,\dots,\widehat{l}_j,\dots,l_d \geq 0}} \binom{m_j}{l_0, \dots, \widehat{l}_j, \dots, l_d} \left(\prod_{\substack{t=1 \\ t \neq j}}^d (\pi k_t)^{l_t} \right) k_j^{a-1} c_{-a}(j). \end{aligned}$$

We now compute $\text{Res}_{z=1} f(z)$. For each $j \in \{1, \dots, d\}$, we know that $\cot(\pi k_j z)$ has an expansion about 1 of the form

$$\cot(\pi k_j z) = \frac{1}{\pi k_j(z-1)} + \sum_{n=0}^{\infty} \frac{(2i)^{n+1} B_{n+1} (\pi k_j)^n}{(n+1)!} (z-1)^n,$$

so the Laurent expansion of $\cot^{(m_j)}(\pi k_j z)$ about 1 is given by

$$\begin{aligned} \cot^{(m_j)}(\pi k_j z) & = \frac{(-1)^{m_j} m_j!}{\pi k_j (z-1)^{m_j+1}} \\ & \quad + \sum_{l_j=0}^{\infty} \frac{(2i)^{l_j+m_j+1} B_{l_j+m_j+1} (\pi k_j)^{l_j+m_j} (l_j+1)^{(m_j)}}{(l_j+m_j+1)!} (z-1)^{l_j} \\ & = \sum_{l_j=-\infty}^{\infty} a_{l_j} (z-1)^{l_j}, \end{aligned}$$

where

$$a_{l_j} = \begin{cases} \frac{(2i)^{l_j+m_j+1} B_{l_j+m_j+1} (\pi k_j)^{l_j+m_j} (l_j+1)^{(m_j)}}{(l_j+m_j+1)!} & \text{if } l_j \geq 0, \\ \frac{(-1)^{m_j} m_j!}{\pi k_j} & \text{if } l_j = -(m_j+1), \\ 0 & \text{otherwise.} \end{cases}$$

Now, Taylor’s theorem implies that the expansion of $\zeta^{(m_0)}(a, z)$ about 1 is of the form

$$\zeta^{(m_0)}(a, z) = \sum_{l_0=0}^{\infty} a_{l_0} (z - 1)^{l_0},$$

where

$$a_{l_0} = \frac{\zeta^{(m_0+l_0)}(a, 1)}{l_0!} = \frac{(-1)^{m_0+l_0} a^{(m_0+l_0)} \zeta(a + m_0 + l_0)}{l_0!}.$$

Therefore,

$$\operatorname{Res}_{z=1} f(z) = \sum_{l_0=0}^{m_1+\dots+m_d+d-1} \sum_{l_1+\dots+l_d=-l_0-1} \prod_{j=0}^d a_{l_j}.$$

Since $\frac{\partial}{\partial z} \zeta(a, z) = -a\zeta(a + 1, z)$, we have

$$\begin{aligned} \zeta^{(m_0)}(a, z + 1) &= (-1)^{m_0} \zeta(a + m_0, z + 1) a^{(m_0)} \\ &= (-1)^{m_0} a^{(m_0)} \sum_{n=0}^{\infty} \frac{1}{(n + z + 1)^{a+m_0}} \\ &= (-1)^{m_0} a^{(m_0)} \left(\zeta(a + m_0, z) - \frac{1}{z^{a+m_0}} \right) \\ &= \zeta^{(m_0)}(a, z) - \frac{(-1)^{m_0} a^{(m_0)}}{z^{a+m_0}}. \end{aligned}$$

This means that

$$\int_{1+\varepsilon-iM}^{1+\varepsilon+iM} f(z) dz + \int_{\varepsilon+iM}^{\varepsilon-iM} f(z) dz = (-1)^{m_0} a^{(m_0)} \int_{\varepsilon+iM}^{\varepsilon-iM} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{a+m_0}} dz.$$

As in the proof of Theorem 1.2, it follows from Lemmas 2.1 and 2.2 that the integrals along the horizontal segments of $C(M, \varepsilon)$ vanish, so the Cauchy residue theorem implies that

$$\sum_{z_0 \in E} \operatorname{Res}_{z=z_0} f(z) = \frac{(-1)^{m_0} a^{(m_0)}}{2\pi i} \int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{a+m_0}} dz.$$

The result then follows from (3.1) and the computation of $\operatorname{Res}_{z=1} f(z)$. ■

An analogue of Theorem 1.3 is valid for generalized Bettin–Conrey sums.

THEOREM 3.2. *Let $d \geq 2$ and suppose that k_1, \dots, k_d is a list of pairwise coprime positive integers and m_0, m_1, \dots, m_d are nonnegative integers. If $n > 1$ is an integer and $m_0 + n + d + \sum_{j=1}^d m_j$ is odd, then*

$$\begin{aligned} & \sum_{j=1}^d \frac{(-1)^{m_j}}{\pi} \sum_{\substack{l_0+\dots+\widehat{l}_j+\dots+l_d=m_j \\ l_0,\dots,\widehat{l}_j,\dots,l_d \geq 0}} \binom{m_j}{l_0, \dots, \widehat{l}_j, \dots, l_d} \left(\prod_{\substack{t=1 \\ t \neq j}}^d (\pi k_t)^{l_t} \right) k_j^{n-1} c_{-n}(j) \\ &= - \sum_{l_0=0}^{m_1+\dots+m_d+d-1} \sum_{l_1+\dots+l_d=-l_0-1} \prod_{j=0}^d a_{l_j} \\ & \quad + \frac{(-1)^{m_0+1} n^{(m_0)}}{2} \sum_{l_1+\dots+l_d=n+m_0-1} \prod_{j=1}^d a_{l_j} \end{aligned}$$

where

$$c_{-n}(j) = c_{-n} \left(\begin{array}{c|cccc} k_j & k_1 & \dots & \widehat{k}_j & \dots & k_d \\ m_0 + l_0 & m_1 + l_1 & \dots & m_j + l_j & \dots & m_d + l_d \end{array} \right)$$

and

$$a_{l_j} = \begin{cases} \frac{(-1)^{m_0+l_0} n^{(m_0+l_0)} \zeta(n+m_0+l_0)}{l_0!} & \text{if } j = 0 \text{ and } l_0 \geq 0, \\ \frac{(2i)^{l_j+m_j+1} B_{l_j+m_j+1} (\pi k_j)^{l_j+m_j} (l_j+1)^{(m_j)}}{(l_j+m_j+1)!} & \text{if } j \neq 0 \text{ and } l_j \geq 0, \\ \frac{(-1)^{m_j} m_j!}{\pi k_j} & \text{if } j \neq 0 \text{ and } l_j = -(m_j+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Theorem 1.3, the contour $\widetilde{C}(M, \varepsilon)$ is defined as the positively oriented rectangle with vertices $1+iM$, iM , $-iM$ and $1-iM$, with indentations (to the right) of radius $0 < \varepsilon < \min_{1 \leq j \leq d} \{1/k_j\}$ around 0 and 1 (see Figure 2). Since this closed contour contains the same poles of f as $C(M, \varepsilon)$, we may apply Cauchy’s residue theorem, letting $M \rightarrow \infty$, and we only need to determine $\lim_{M \rightarrow \infty} \int_{\widetilde{C}(M, \varepsilon)} f(z) dz$ in order to deduce a reciprocity law for the generalized Bettin–Conrey sums of the form c_{-n} .

Since $m_0 + n + d + \sum_{j=1}^d m_j$ is odd, the function

$$g(z) = \frac{(-1)^{m_0} n^{(m_0)} \prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{n+m_0}}$$

is odd:

$$\begin{aligned} g(-z) &= \frac{(-1)^{m_0+d+\sum_{j=1}^d m_j} n^{(m_0)}}{(-1)^{n+m_0} z^{n+m_0}} \prod_{j=1}^d \cot^{(m_j)}(\pi k_j z) \\ &= \frac{(-1)^{d+\sum_{j=1}^d m_j}}{(-1)^{n+m_0}} g(z) = -g(z). \end{aligned}$$

Let $\gamma(M, \varepsilon)$ be the indentation around zero along $\tilde{C}(M, \varepsilon)$. Then

$$\begin{aligned} \lim_{M \rightarrow \infty} \int_{\tilde{C}(M, \varepsilon)} f(z) dz &= \lim_{M \rightarrow \infty} \left(\int_{iM}^{\varepsilon i} g(z) dz + \int_{-\varepsilon i}^{-iM} g(z) dz \right) + \int_{\gamma(M, \varepsilon)} g(z) dz \\ &= \int_{\gamma(M, \varepsilon)} g(z) dz. \end{aligned}$$

Since g is odd, the Cauchy residue theorem implies that

$$\int_{\gamma(M, \varepsilon)} g(z) dz = -\pi i \operatorname{Res}_{z=0} g(z).$$

Consequently,

$$\sum_{z_0 \in E} \operatorname{Res}_{z=z_0} f(z) = -\frac{1}{2} \operatorname{Res}_{z=0} g(z).$$

For each $j \in \{1, \dots, d\}$ we have an expansion of the form

$$\cot^{(m_j)}(\pi k_j z) = \sum_{l_j=-\infty}^{\infty} a_{l_j} z^{l_j},$$

where

$$a_{l_j} = \begin{cases} \frac{(2i)^{l_j+m_j+1} B_{l_j+m_j+1} (\pi k_j)^{l_j+m_j} (l_j+1)^{(m_j)}}{(l_j+m_j+1)!} & \text{if } l_j \geq 0, \\ \frac{(-1)^{m_j} m_j!}{\pi k_j} & \text{if } l_j = -(m_j+1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{Res}_{z=0} g(z)$ is given by

$$(-1)^{m_0} n^{(m_0)} \sum_{l_1+\dots+l_d=n+m_0-1} \prod_{j=1}^d a_{l_j}.$$

Therefore,

$$\sum_{z_0 \in E} \operatorname{Res}_{z=z_0} f(z) = \frac{(-1)^{m_0+1} n^{(m_0)}}{2} \sum_{l_1+\dots+l_d=n+m_0-1} \prod_{j=1}^d a_{l_j}. \blacksquare$$

From Theorems 3.1 and 3.2, we deduce a computation of the integral

$$\int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{n+m_0}} dz$$

in terms of the sequences $B(m_j)_{l_j}$, whenever $n \in \mathbb{Z}_{>1}$ and $m_0 + n + d + \sum_{j=1}^d m_j$ is odd, which generalizes Corollary 2.3:

COROLLARY 3.3. *Let $d \geq 2$ and suppose that k_1, \dots, k_d is a list of pairwise coprime positive integers and m_0, m_1, \dots, m_d are nonnegative integers. If $n > 1$ is an integer and $m_0 + n + d + \sum_{j=1}^d m_j$ is odd, then for all $0 < \varepsilon < \min_{1 \leq j \leq d} \{1/k_j\}$,*

$$\int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{\prod_{j=1}^d \cot^{(m_j)}(\pi k_j z)}{z^{n+m_0}} dz = -\pi i \sum_{l_1+\dots+l_d=n+m_0-1} \prod_{j=1}^d a_{l_j},$$

where the sequences $\{a_{l_j}\}$ are as in Theorem 3.2, for $j = 1, \dots, d$.

The consideration of the case $m_0 = m_1 = \dots = m_n = 0$ leads to the definition of higher-dimensional Bettin–Conrey sums,

$$\begin{aligned} c_a(k_0; k_1, \dots, k_n) &= c_a \left(\begin{array}{c|ccc} k_0 & k_1 & \cdots & k_n \\ \hline 0 & 0 & \cdots & 0 \end{array} \right) \\ &= k_0^a \sum_{m=1}^{k_0-1} \zeta \left(-a, \frac{m}{k_0} \right) \prod_{l=1}^n \cot \left(\frac{\pi k_l m}{k_0} \right) \end{aligned}$$

for $a \neq -1$ complex and k_0, k_1, \dots, k_n a list of positive numbers such that $(k_0, k_j) = 1$ for each $j \neq 0$.

Of course, higher-dimensional Bettin–Conrey sums satisfy Theorems 3.1 and 3.2. In particular, if $0 < \varepsilon < \min_{1 \leq l \leq d} \{1/k_l\}$, $\Re(a) > 1$ and k_1, \dots, k_d is a list of pairwise coprime positive integers, then

$$\begin{aligned} &\sum_{j=1}^d k_j^{a-1} c_{-a}(k_j; k_1, \dots, \widehat{k_j}, \dots, k_d) \\ &= -\pi \sum_{l_0=0}^{d-1} \sum_{l_1+\dots+l_d=-l_0-1} a_{l_0} a_{l_1} \cdots a_{l_d} + \frac{1}{2i} \int_{\varepsilon+i\infty}^{\varepsilon-i\infty} \frac{\prod_{j=1}^d \cot(\pi k_j z)}{z^a} dz, \end{aligned}$$

where

$$a_{l_j} = \begin{cases} \frac{(-1)^{l_0} a^{(l_0)} \zeta(a + l_0)}{l_0!} & \text{if } j = 0 \text{ and } l_0 \geq 0, \\ \frac{(2i)^{l_j+1} B_{l_j+1}(\pi k_j)^{l_j}}{(l_j + 1)!} & \text{if } j \neq 0 \text{ and } l_j \geq 0, \\ \frac{1}{\pi k_j} & \text{if } j \neq 0 \text{ and } l_j = -1, \\ 0 & \text{otherwise.} \end{cases}$$

4. Derivative cotangent sums and critical values of Estermann zeta. As usual, for $a, x \in \mathbb{C}$, let $\sigma_a(n) = \sum_{d|n} d^a$ and $e(x) = e^{2\pi i x}$. For a given rational number x , the Estermann zeta function is defined through

the Dirichlet series:

$$(4.1) \quad E(s, x, a) = \sum_{n \geq 1} \sigma_a(n) e(nx) n^{-s},$$

initially defined for $\Re(s) > \max\{1, \Re(a) + 1\}$ and analytically continued to the whole s -plane with possible poles at $s = 1, a + 1$. For $x = p/q$ with $(p, q) = 1$ and $q > 1$,

$$E(s, x, a) - q^{1+a-2s} \zeta(s - a) \zeta(s)$$

is an entire function of s . By use of the Hurwitz zeta function we observe that

$$(4.2) \quad E(s, x, a) = q^{a-2s} \sum_{m, n=1}^q e(mnx) \zeta(s - a, m/q) \zeta(s, n/q).$$

We consider the sums

$$(4.3) \quad C(a, s, x) = q^a \sum_{m=1}^{q-1} e(mx) \Phi(-s, 1, e(mx)) \zeta(-a, m/q),$$

where $\Phi(s, z, \lambda) = \sum_{n \geq 0} \lambda^n / (z + n)^s$ is *Lerch’s transcendent function*, defined for $z \neq 0, -1, -2, \dots$ if $|\lambda| < 1$, and $\Re(s) > 1$ if $|\lambda| = 1$, and analytically continued in λ .

The purpose of this section is to establish relationships between $C(a, s, x)$ and values of the Estermann zeta function at integers s . We start with some preliminary results.

LEMMA 4.1. *Let k be a nonnegative integer. Then*

$$(4.4) \quad \lambda \Phi(s, z + 1, \lambda) = \Phi(s, z, \lambda) - z^{-s},$$

$$(4.5) \quad \Phi(-k, z, \lambda) = -\frac{B_{k+1}(z; \lambda)}{k + 1},$$

$$(4.6) \quad B_k(0; e(x)) = \begin{cases} \frac{1}{2i} \cot(\pi x) - \frac{1}{2} & \text{if } k = 1, \\ \frac{k}{(2i)^k} \cot^{(k-1)}(\pi x) & \text{if } k > 1. \end{cases}$$

Proof. Equation (4.4) follows from the special case $m = 1$ in [13, eq. 25.14.4]. Equation (4.5) can be found in [4, p. 164]. Equation (4.6) follows from [1, Lemma 2.1] and [6, Theorem 4]. ■

Lemma 4.1 implies that for a positive integer $s = k$, the sum $C(a, k, x)$ defined in (4.3) is, up to a constant factor, the k th-derivative cotangent sum

$$C(a, k, x) = -\frac{1}{(2i)^{k+1}} q^a \sum_{m=1}^{q-1} \cot^{(k)}(\pi mx) \zeta(-a, m/q).$$

LEMMA 4.2. *Let p, q be coprime positive integers and $x = p/q$. For any $n \in \mathbb{Z}$ and $z \in \mathbb{C}$ with $\Re(z) > 0$,*

$$(4.7) \quad \sum_{m=0}^{q-1} e(mnx)\zeta(s, z + m/q) = q^s \Phi(s, qz, e(nx)).$$

Proof. Writing $m = kq + j$ with $j = 0, 1, \dots, q - 1$, we have

$$q^s \Phi(s, qz, e(nx)) = q^s \sum_{m=0}^{\infty} \frac{e(nmx)}{(m + qz)^s} = \sum_{j=0}^{q-1} e(njx) \sum_{k=0}^{\infty} \frac{1}{(k + z + j/q)^s}. \blacksquare$$

PROPOSITION 4.3. *Let p, q be coprime positive integers and $x = p/q$. Then*

$$E(-s, x, a - s) = q^a \sum_{m=1}^{q-1} e(mx)\zeta(-a, m/q)\Phi(-s, 1, e(mx)) + q^a \zeta(-s)\zeta(-a),$$

$$E(-s, x, a - s) = q^s \sum_{n=1}^{q-1} e(nx)\zeta(-s, n/q)\Phi(-a, 1, e(nx)) + q^s \zeta(-a)\zeta(-s).$$

THEOREM 4.4. *Let a, k be nonnegative integers. Then*

$$E(-k, x, a - k) = C(a, k, x) + q^a \zeta(-k)\zeta(-a) \quad \text{if } k \geq 1,$$

$$E(-k, x, a - k) = C(k, a, x) + q^k \zeta(-k)\zeta(-a) \quad \text{if } a \geq 1,$$

and

$$E(0, x, a) = C(a, 0, x) - \frac{1}{2}\zeta(-a),$$

$$E(0, x, a) = C(0, a, x) - \frac{1}{2}\zeta(-a) \quad \text{with } a \geq 1.$$

COROLLARY 4.5. *Let a, k be nonnegative integers. For any rational number $x \neq 0$,*

$$C(a, k, x) - C(k, a, x) = \begin{cases} 0 & \text{if } k = 0 \text{ or } a = 0, \\ (q^a - q^k)\zeta(-k)\zeta(-a) & \text{otherwise.} \end{cases}$$

Acknowledgements. We thank Sandro Bettin and an anonymous referee for valuable comments. Abdelmejid Bayad was partially supported by the FDIR of the Université d'Évry Val d'Essonne; Matthias Beck was partially supported by the US National Science Foundation (DMS-1162638).

References

[1] V. S. Adamchik, *On the Hurwitz function for rational arguments*, Appl. Math. Comput. 187 (2007), 3–12.
 [2] T. M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Math. J. 17 (1950), 147–157.

- [3] T. M. Apostol, *Theorems on generalized Dedekind sums*, Pacific J. Math. 2 (1952), 1–9.
- [4] T. M. Apostol, *On the Lerch zeta function*, Pacific J. Math. 1 (1951), 161–167.
- [5] L. Báez-Duarte, M. Balazard, B. Landreau, et E. Saias, *Étude de l'autocorrélation multiplicative de la fonction 'partie fractionnaire'*, Ramanujan J. 9 (2005), 215–240.
- [6] A. Bayad and J. Chikhi, *Möbius inversion formulae for Apostol–Bernoulli type polynomials and numbers*, Math. Comp. 82 (2013), 2327–2332.
- [7] M. Beck, *Dedekind cotangent sums*, Acta Arith. 109 (2003), 109–130.
- [8] S. Bettin and J. B. Conrey, *Period functions and cotangent sums*, Algebra Number Theory 7 (2013), 215–242.
- [9] S. Bettin and J. B. Conrey, *A reciprocity formula for a cotangent sum*, Int. Math. Res. Notices 2013, no. 24, 5709–5726.
- [10] R. Dedekind, *Erläuterungen zu den Fragmenten XXVIII*, in: Collected Works of Bernhard Riemann, Dover Publ., New York, 1953, 466–478.
- [11] S. Lang, *Complex Analysis*, 4th ed., Grad. Texts in Math. 103, Springer, New York, 1999.
- [12] M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ. Press, Cambridge, 1958.
- [13] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, US Dept. of Commerce, Nat. Inst. Standards and Technology, Washington, DC, and Cambridge Univ. Press, Cambridge, 2010.
- [14] H. Rademacher and E. Grosswald, *Dedekind Sums*, Math. Assoc. America, Washington, DC, 1972.
- [15] V. I. Vasyunin, *On a biorthogonal system associated with the Riemann hypothesis*, Algebra i Analiz 7 (1995), no. 3, 118–135 (in Russian).
- [16] D. Zagier, *Higher dimensional Dedekind sums*, Math. Ann. 202 (1973), 149–172.
- [17] D. Zagier, *Quantum modular forms*, in: Quanta of Maths, Clay Math. Proc. 11, Amer. Math. Soc., Providence, RI, 2010, 659–675.

Juan S. Auli
Department of Mathematics
Dartmouth College
Hanover, NH 03755, U.S.A.
E-mail: juan.s.auli.gr@dartmouth.edu

Matthias Beck
Department of Mathematics
San Francisco State University
San Francisco, CA 94132, U.S.A.
E-mail: mattbeck@sfsu.edu

Abdelmejid Bayad
Univ. Évry
Université Paris-Saclay
I.B.G.B.I.
23 Bd. de France
91025 Évry Cedex, France
E-mail: abayad@maths.univ-evry.fr

