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ϕ-minimal rotational surfaces in pseudo-Galilean space with density

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Abstract. Rotational surfaces in the pseudo-Galilean 3-space G_3^1 have some types according to pseudo-Euclidean rotations and isotropic rotations. We study rotational surfaces in G_3^1 with a log-linear density and investigate ϕ-minimal rotational surfaces.

1. Introduction. Minimal surfaces are one of the main objects which have drawn geometers' interest for a long time. In particular, Euler found that the only minimal rotational surfaces are the planes and the catenoids, and Catalan proved that the planes and the helicoids are the only minimal ruled surfaces in the Euclidean 3-space $\mathcal{E}^3$. In 1983, Kobayashi [9] classified space-like minimal ruled surfaces and rotational surfaces in the Minkowski 3-space $\mathcal{L}^3$, and Van de Woestyne [16] extended it to the Lorentz version in 1988.

As a new category in geometry, a manifold with density (also called a weighted manifold) appears in many areas of mathematics, and the study of and interest in such manifolds has increased due to their applications in probability and statistics. It was instrumental in Perelman's proof of the Poincaré conjecture [13].

A manifold with density is a Riemannian manifold M with a positive density function, which is used to weigh the volume and area. Consider a surface in the Euclidean 3-space $\mathcal{E}^3$ with density e^ϕ. Then the weighted mean curvature H_ϕ (also called ϕ-mean curvature) of a surface M in $\mathcal{E}^3$ with density e^ϕ is defined by

$$H_\phi = H - \tfrac{1}{2}\langle \nabla\phi, N\rangle, \tag{1.1}$$

where H is the mean curvature, N is the unit normal vector of M and $\nabla\phi$ is

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the gradient of ϕ. The ϕ-mean curvature H_ϕ was introduced by Gromov [6] and it is a natural generalization of the mean curvature H of a surface. As one can see in [15], the mean curvature can be expressed as an isotropic surface free energy and the weighted mean curvature is the same as the anisotropic surface free energy. A surface with $H_\phi = 0$ is called a ϕ-minimal surface or a weighted minimal surface in $\mathcal{E}^3$. A weighted minimal surface is the same as a self-similar surface, which is a solution of the mean curvature flow. Some results on manifolds with density and related topics can be found in [1], [2], [7], [8], [10], [12] and [14].

In particular, Hieu and Hoang [7] studied ruled surfaces and translation surfaces in $\mathcal{E}^3$ with density e^z, and they classified the ϕ-minimal ruled surfaces and the ϕ-minimal translation surfaces. Also, López [10] considered a log-linear density $e^{\alpha x+\beta y+\gamma z}$, $\alpha, \beta, \gamma \in \mathcal{R}$, and he classified the ϕ-minimal translation surfaces and the ϕ-minimal cyclic surfaces in the Euclidean 3-space $\mathcal{E}^3$. Belarbi and Belkhelfa [1] investigated properties of ϕ-minimal graphs in $\mathcal{E}^3$ with a linear density. Recently, the first author [18] studied translation surfaces in the Galilean 3-space G_3 with a log-linear density, and completely classified the ϕ-minimal translation surfaces in G_3. The Galilean space is one of the Cayley–Klein geometries defined by a Lie group.

In this article, we focus on a class of rotational surfaces in the pseudo-Galilean 3-space G_3^1. We investigate some types of ϕ-minimal rotational surfaces according to pseudo-Euclidean rotations and isotropic rotations in G_3^1 with a log-linear density $e^{\alpha x+\beta y+\gamma z}$.

2. Preliminaries. In 1872, F. Klein in his Erlangen program proposed how to classify and characterize geometries on the basis of projective geometry and group theory. He showed that the Euclidean and non-Euclidean geometries could be considered as spaces that are invariant under a given group of transformations. Any geometry motivated by this approach is called a Cayley–Klein geometry. Actually, the formal definition of a Cayley–Klein geometry is a pair (G, H), where G is a Lie group and H is a closed Lie subgroup of G such that the (left) coset space G/H is connected. G/H is called the space of the geometry or simply the *Cayley–Klein geometry.*

The pseudo-Galilean geometry is one of the real Cayley–Klein geometries with projective signature $(0, 0, +, -)$. The *absolute figure* of the pseudo-Galilean geometry is an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f a line in ω and I a fixed hyperbolic involution of f.

Homogeneous coordinates in G_3^1 are introduced in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the hyperbolic involution by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : x_3 : x_2)$. With respect to the absolute figure, metric relations are introduced (see [4]). In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ is defined by (see [11])

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1| & \text{if } x_1 \neq x_2, \\ \sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2|} & \text{if } x_1 = x_2. \end{cases}$$

The group of motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned} \bar{x} &= a + x, \\ \bar{y} &= b + cx + y\cosh\varphi + z\sinh\varphi, \\ \bar{z} &= d + ex + y\sinh\varphi + z\cosh\varphi; \end{aligned}$$

more details can be found in [4, 5].

The pseudo-Galilean scalar product of two vectors $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ in G_3^1 is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} x_1 x_2 & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\ y_1 y_2 - z_1 z_2 & \text{if } x_1 = 0 \text{ and } x_2 = 0, \end{cases} \tag{2.1}$$

and the pseudo-Galilean norm of $\mathbf{x}$ is given by

$$\|\mathbf{x}\| = \begin{cases} |x_1| & \text{if } x_1 \neq 0, \\ \sqrt{|y_1^2 - z_1^2|} & \text{if } x_1 = 0. \end{cases}$$

A vector $\mathbf{x}$ is called *isotropic* if $x_1 = 0$, otherwise it is called *non-isotropic*. All unit non-isotropic vectors are the form $(1, y_1, z_1)$. An isotropic vector $\mathbf{x} = (0, y_1, z_1)$ of G_3^1 is said to be *spacelike* if $y_1^2 - z_1^2 > 0$, *timelike* if $y_1^2 - z_1^2 < 0$ and *lightlike* if $y_1^2 - z_1^2 = 0$. A non-lightlike isotropic vector is a *unit vector* if $y_1^2 - z_1^2 = \pm 1$.

The pseudo-Galilean cross product of $\mathbf{x}$ and $\mathbf{y}$ on G_3^1 is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}, \tag{2.2}$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a C^r-regular surface Σ, $r \geq 1$, in G_3^1 parametrized by

$$\mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

We denote by x_{u_i}, y_{u_i} and z_{u_i} the partial derivatives of the functions x, y and z with respect to u_i $(i = 1, 2)$, respectively. The parametrized surface is *admissible* if it does not have pseudo-Euclidean tangent planes, that is, if and only if $x_{u_i} \neq 0$ for some $i = 1, 2$.

On the other hand, the matrix of the first fundamental form ds^2 of Σ is given by

$$ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_1 du_1 + g_2 du_2)^2$ and $ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$. Here $g_i = x_{u_i}$ and $h_{ij} = \langle \tilde{\mathbf{x}}_{u_i}, \tilde{\mathbf{x}}_{u_j} \rangle$ $(i, j = 1, 2)$ means the Euclidean scalar product of the projections $\tilde{\mathbf{x}}_{u_k}$ of $\mathbf{x}_{u_k}$ onto the yz-plane.

The unit normal vector field N of Σ is defined by

$$N = \frac{1}{\omega}(0, x_{u_1} z_{u_2} - x_{u_2} z_{u_1}, x_{u_1} y_{u_2} - x_{u_2} y_{u_1}),$$

where the positive function ω is given by

$$\omega = \sqrt{|(x_{u_1} z_{u_2} - x_{u_2} z_{u_1})^2 - (x_{u_1} y_{u_2} - x_{u_2} y_{u_1})^2|}.$$

Since the surface is admissible, the normal vector is isotropic. From this, the coefficients L_{ij} $(i, j = 1, 2)$ of the second fundamental form of Σ are given by

$$L_{ij} = \frac{\epsilon}{g_1}\langle g_1 \tilde{\mathbf{x}}_{u_i u_j} - g_{i,j} \tilde{\mathbf{x}}_{u_1}, N \rangle = \frac{\epsilon}{g_2}\langle g_2 \tilde{\mathbf{x}}_{u_i u_j} - g_{i,j} \tilde{\mathbf{x}}_{u_2}, N \rangle,$$

where ϵ denotes the sign of the unit normal vector N and $g_{i,j} = \partial g_i / \partial u_j$. The Gaussian curvature K and the mean curvature H of Σ are defined by [11]

$$K = -\epsilon \frac{L_{11} L_{22} - L_{12}^2}{\omega^2}, \tag{2.3}$$

$$H = -\frac{\epsilon}{2\omega^2}(g_2^2 L_{11} - 2 g_1 g_2 L_{12} + g_1^2 L_{22}). \tag{2.4}$$

Now, we define rotational surfaces in the pseudo-Galilean 3-space G_3^1.

A rotational surface in the Euclidean space is generated by revolving an arbitrary curve about an arbitrary axis. In the pseudo-Galilean space, however, there are different cases of curves (non-isotropic or isotropic) as well as different cases of rotations (pseudo-Euclidean or isotropic). Some cases of rotational surfaces in G_3^1 were studied in [11] and [17].

Case 1: The rotation is pseudo-Euclidean.

Suppose that a non-isotropic curve C lies in the xy-plane or xz-plane. Then C can be represented by

$$C(u) = (f(u), g(u), 0) \quad \text{or} \quad C(u) = (f(u), 0, g(u)),$$

where g is a positive function and f is a smooth function on an open interval I.

On the other hand, a pseudo-Euclidean rotation in G_3^1 is given by the normal form

$$\begin{aligned} \bar{x} &= x, \\ \bar{y} &= y \cosh t + z \sinh t, \\ \bar{z} &= y \sinh t + z \cosh t. \end{aligned} \tag{2.5}$$

Thus the corresponding rotational surface can be written as

$$\mathbf{x}(u,v) = (f(u), g(u)\cosh v, g(u)\sinh v) \tag{2.6}$$

or

$$\mathbf{x}(u,v) = (f(u), g(u)\sinh v, g(u)\cosh v), \tag{2.7}$$

for any $v \in \mathcal{R}$.

Case 2: The rotation is isotropic.

First of all, without loss of generality, we may assume that an isotropic curve C lies in the yz-plane, and so it is given by

$$C(u) = (0, f(u), g(u))$$

for some smooth functions f and g. In this case, an isotropic rotation in G_3^1 is given by the normal form

$$\begin{aligned} \bar{x} &= x + bt, \\ \bar{y} &= y + xt + bt^2/2, \\ \bar{z} &= z \end{aligned} \tag{2.8}$$

and the rotational surface can be parametrized by

$$\mathbf{x}(u,v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right), \tag{2.9}$$

where $t \in \mathcal{R}$ and b is a positive constant.

Next, we assume, again without loss of generality, that a non-isotropic curve C lies in the isotropic xy-plane and is parametrized by

$$C(u) = (f(u), g(u), 0)$$

for some smooth functions f and g. In this case, an isotropic rotation in G_3^1 is given by the normal form [3]

$$\begin{aligned} \bar{x} &= x + bt, \\ \bar{y} &= y, \\ \bar{z} &= z + xt + bt^2/2. \end{aligned} \tag{2.10}$$

Thus, the rotational surface is defined by

$$\mathbf{x}(u,v) = \left(f(u) + v, g(u), vf(u) + \frac{v^2}{2b}\right), \tag{2.11}$$

where $t \in \mathcal{R}$ and b is a positive constant.

3. Minimal rotational surfaces with a linear density. First of all, we classify the rotational surfaces generated by a non-isotropic curve with zero ϕ-mean curvature.

3.1. The rotation is pseudo-Euclidean. Let M be a rotational surface generated by a unit speed non-isotropic curve $C(u) = (u, g(u), 0)$ in G_3^1. Then M is parametrized by

$$\mathbf{x}(u,v) = (u, g(u)\cosh v, g(u)\sinh v), \tag{3.1}$$

where g is a positive function.

The unit normal vector field N of the surface M is given by

$$N = (0, \cosh v, \sinh v). \tag{3.2}$$

By a straightforward computation, the mean curvature H of M is

$$H = \frac{1}{2g(u)}. \tag{3.3}$$

If M is a rotational surface in G_3^1 with a log-linear density e^{ϕ}, where $\phi = \alpha x + \beta y + \gamma z$, α, β, γ not all zero, then the ϕ-mean curvature H_ϕ of M is

$$H_\phi = H - \tfrac{1}{2}\langle N, \nabla\phi\rangle \tag{3.4}$$

and the ϕ-minimality condition $H_\phi = 0$ of M becomes

$$\frac{1}{g(u)} - \langle (0, \cosh v, \sinh v), (\alpha, \beta, \gamma)\rangle = 0, \tag{3.5}$$

where $\nabla\phi = (\alpha, \beta, \gamma)$ is the gradient of ϕ.

If $\alpha \neq 0$, the vector (α, β, γ) is non-isotropic. From this $1/g(u) = 0$, a contradiction.

If $\alpha = 0$, the vector $\nabla\phi$ is isotropic, which implies that $1/g(u) = \beta\cosh v - \gamma\sinh v$. Thus, $g(u)$ is a non-zero positive constant function and the surface has a constant mean curvature. According to [11], the surface M is a hyperbolic sphere as a timelike surface.

Now, we consider a rotational surface M defined by (2.7). Then, similarly to the above, we also see that the surface M with an isotropic gradient vector is a hyperbolic sphere as a spacelike surface.

THEOREM 3.1. *Let M be a ϕ-minimal rotational surface given by* (2.6) *or* (2.7) *in G_3^1 with a log-linear density $e^{\alpha x + \beta y + \gamma z}$.*

(1) *If $\alpha \neq 0$, then there are no ϕ-minimal rotational surfaces.*
(2) *If $\alpha = 0$, then M is an open part of a timelike or spacelike hyperbolic sphere.*

3.2. The rotation is isotropic. Let M be a rotational surface generated by an isotropic curve $C(u) = (0, f(u), g(u))$ in G_3^1. Then the parametrization of M is given by

$$\mathbf{x}(u,v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right), \tag{3.6}$$

where f and g are smooth functions and $b \neq 0$.

We may assume $C(u)$ is parametrized by arc length, i.e., $g'(u)^2 - f'(u)^2 = \epsilon$. In this case, the unit normal vector field N and the mean curvature H of M are given by

$$N = (0, -g'(u), -f'(u)) \quad \text{and} \quad H = -\frac{\epsilon f''(u)}{2g'(u)}.$$

Suppose that M is a ϕ-minimal surface in G_3^1 with a log-linear density $e^{\alpha x+\beta y+\gamma z}$. Then from (3.4) we get

$$\epsilon \frac{f''(u)}{g'(u)} + \langle (0, -g'(u), -f'(u)), (\alpha, \beta, \gamma) \rangle = 0. \tag{3.7}$$

Let us distinguish two cases according to the value of α:

CASE 1: $\alpha \neq 0$. In this case, M is a minimal surface and the functions $f(u)$ and $g(u)$ are linear. Thus, M is a parabolic sphere according to [11].

CASE 2: $\alpha = 0$. From (3.7), we have the equation

$$\epsilon f''(u) - \beta(f'(u)^2 + \epsilon) + \gamma f'(u)\sqrt{f'(u)^2 + \epsilon} = 0. \tag{3.8}$$

CASE 2-1: $\gamma = 0$. We set $p(u) = f'(u)$. Then (3.8) becomes $\epsilon p'(u) = \beta(p^2(u) + \epsilon)$. If $\epsilon = 1$, then $p(u) = \tan(\beta u + c_1)$, and it follows that

$$f(u) = \frac{1}{\beta} \ln |\sec(\beta u + c_1)| + c_2 \tag{3.9}$$

for some constants c_1 and c_2. By the relation between $f(u)$ and $g(u)$, we get

$$g(u) = \frac{1}{\beta} \ln |\sec(\beta u + c_1) + \tan(\beta u + c_1)| + c_2. \tag{3.10}$$

If $\epsilon = -1$, a straightforward computation as above shows that

$$\begin{aligned} f(u) &= \frac{1}{\beta} \ln |\sinh(\beta u + c_1)| + c_2, \\ g(u) &= \frac{1}{\beta} \ln \left| \tanh\left(\frac{\beta u + c_1}{2} \right) \right| + c_2, \end{aligned} \tag{3.11}$$

for some constants c_1 and c_2.

CASE 2-2: $\beta = 0$. Take $p(u) = f'(u)$. Then (3.8) reads

$$\epsilon p'(u) + \gamma p(u)\sqrt{p^2(u) + \epsilon} = 0. \tag{3.12}$$

Since $p(u) = 0$ is a trivial solution of (3.12), $f(u)$ is constant and $g(u) = u + c_1$ with $c_1 \in \mathcal{R}$. Now, assume that $p(u) \neq 0$. If $\epsilon = 1$, the general solution is

$$p(u) = \frac{1}{\sinh(\gamma u + c_1)}.$$

It follows that

$$f(u) = -\frac{2}{\gamma}\tanh^{-1}(e^{\gamma u+c_1}) + c_2, \tag{3.13}$$

$$g(u) = \frac{1}{\gamma}\ln|\sinh(\gamma u + c_1)| + c_2, \tag{3.14}$$

for some constants c_1 and c_2.

If $\epsilon = -1$, by computing as above we get

$$\begin{aligned} f(u) &= \frac{1}{\gamma}\ln|\sec(\gamma u + c_1) + \tan(\gamma u + c_1)| + c_2, \\ g(u) &= \frac{1}{\gamma}\ln|\sec(\gamma u + c_1)| + c_2, \end{aligned} \tag{3.15}$$

for some constants c_1 and c_2.

THEOREM 3.2. *Let M be a ϕ-minimal rotational surface parametrized by*

$$\mathbf{x}(u,v) = \left(bv, f(u) + \frac{v^2}{2b}, g(u)\right)$$

in G_3^1 with a log-linear density $e^{\alpha x+\beta y+\gamma z}$.

(1) *If $\alpha \neq 0$, then M is an open part of a parabolic sphere given by $f(u) = c_1 u + c_2$ and $g(u) = c_3 u + c_4$.*
(2) *If $\alpha = 0$ and $\gamma = 0$, then for a timelike surface we have*

$$\begin{aligned} f(u) &= \frac{1}{\beta}\ln|\sec(\beta u + c_1)| + c_2, \\ g(u) &= \frac{1}{\beta}\ln|\sec(\beta u + c_1) + \tan(\beta u + c_1)| + c_2, \end{aligned}$$

and for a spacelike surface we have

$$\begin{aligned} f(u) &= \frac{1}{\beta}\ln|\sinh(\beta u + c_1)| + c_2, \\ g(u) &= \frac{1}{\beta}\ln\left|\tanh\left(\frac{\beta u + c_1}{2}\right)\right| + c_2. \end{aligned}$$

(3) *If $\alpha = 0$ and $\beta = 0$, then M is an open part of a parabolic cylinder given by $f(u) = c_1$, $g(u) = u + c_2$ or is parametrized by*

$$\begin{aligned} f(u) &= -\frac{2}{\gamma}\tanh^{-1}(e^{\gamma u+c_1}) + c_2, \\ g(u) &= \frac{1}{\gamma}\ln|\sinh(\gamma u + c_1)| + c_2 \end{aligned}$$

for a timelike surface, and

$$f(u) = \frac{1}{\gamma}\ln|\sec(\gamma u + c_1) + \tan(\gamma u + c_1)| + c_2,$$
$$g(u) = \frac{1}{\gamma}\ln|\sec(\gamma u + c_1)| + c_2$$

for a spacelike surface, where c_1 and c_2 are constant.

Remark 3.3. In general, the normal vector of a surface in G_3^1 is always isotropic, therefore ϕ-minimal rotational surfaces in G_3^1 with a log-linear density $e^{\alpha x+\beta y+\gamma z}$ for $\alpha \neq 0$ are just those with density 1. Thus, such surfaces are minimal in G_3^1 and classified by Šipuš and Divjak [11].

Now, we have to solve (3.8),

$$\epsilon f''(u) - \beta(f'(u)^2 + \epsilon) + \gamma f'(u)\sqrt{f'(u)^2 + \epsilon} = 0,$$

for $\alpha = 0$ and $\beta\gamma \neq 0$. It could be hard to get a general solution, but not an analytical solution. To get a special solution, we take $\epsilon = -1$ and $\beta = -\gamma$. Then (3.8) can be written as

$$-\tfrac{1}{2}p'(u) + \gamma p(u)\sqrt{p(u)+1} + \gamma(p(u)+1)\sqrt{p(u)} = 0, \tag{3.16}$$

where we set $p(u) = f'(u)^2 - 1 \geq 0$. By a long calculation, the solution of (3.16) is given by

$$\sqrt{p(u)} - \sqrt{p(u)+1} = \gamma u + c_1$$

with $c_1 \in \mathcal{R}$, or equivalently

$$p(u) = \frac{((\gamma u + c_1)^2 - 1)^2}{4(\gamma u + c_1)^2}.$$

Since $p(u) = f'(u)^2 - 1$ and $f'(u)^2 - g'(u)^2 = 1$, we have

$$g(u) = \frac{\gamma}{4}u^2 + \frac{c_1}{2}u - \frac{1}{2\gamma}\ln|\gamma u + c_1| + c_2 \tag{3.17}$$

with $c_2 \in \mathcal{R}$, and it follows that

$$f(u) = \frac{\gamma}{4}u^2 + \frac{c_1}{2}u + \frac{1}{2\gamma}\ln|\gamma u + c_1| + c_2. \tag{3.18}$$

Theorem 3.4. *Let M be a spacelike rotational surface given by* (2.9) *in G_3^1 with a log-linear density $e^{-\gamma y+\gamma z}$. If M is a ϕ-minimal surface, then M is generated by an isotropic curve $C(u) = (0, f(u), g(u))$, where $f(u)$ and $g(u)$ are given by* (3.18) *and* (3.17)*, respectively.*

Finally, we consider isotropic rotational surfaces generated by a non-isotropic curve.

Suppose that a non-isotropic curve C is parametrized by arc length, that is, $C(u) = (u, g(u), 0)$. Then the surface M is given by

$$\mathbf{x}(u,v) = \left(u+v, g(u), uv + \frac{v^2}{2b}\right), \tag{3.19}$$

where b is a positive constant.

In this case, the unit normal vector field N and the mean curvature H of M are given by

$$N = \frac{1}{\omega}\left(0, u - v + \frac{1}{b}v, -g'(u)\right),$$
$$H = -\frac{1}{2\omega^3}\left(g''(u)\left(u + \frac{v}{b} - v\right) - 2g'(u) + \frac{1}{b}g'(u)\right),$$

where $\omega = \sqrt{|(u - v + (1/b)v)^2 - g'(u)^2|}$.

Suppose that M is a ϕ-minimal surface in G_3^1 with a log-linear density $e^{\alpha x+\beta y+\gamma z}$. Then from (3.4) we get

$$\left[g''(u)\left(u + \frac{v}{b} - v\right) - 2g'(u) + \frac{1}{b}g'(u)\right] + \omega^2\left\langle \left(0, u - v + \frac{1}{b}v, -g'(u)\right), (\alpha, \beta, \gamma)\right\rangle = 0. \tag{3.20}$$

Let us distinguish two cases according to the value of α:

Case 1: $\alpha \neq 0$. In this case, we have

$$g''(u)\left(u + \frac{v}{b} - v\right) - 2g'(u) + \frac{1}{b}g'(u) = 0.$$

It follows that

$$\left(\frac{1}{b} - 1\right)g''(u) = 0, \qquad ug''(u) + \left(\frac{1}{b} - 2\right)g'(u) = 0.$$

A solution of the ODE system is either $b = 1$ and $g(u) = c_1u^2 + c_2$, or $b = 1/2$ and $g(u) = d_1u + d_2$, for some constants $c_1 \neq 0$, c_2, d_1 and d_2. Thus, M is an open part of a parabolic sphere, an isotropic plane or a parabolic cylinder as a minimal surface (see [3]).

Case 2: $\alpha = 0$. In this case, (3.20) implies

$$g''(u)\left(u + \frac{v}{b} - v\right) - 2g'(u) + \frac{1}{b}g'(u) + \varepsilon\left((u - v + \frac{1}{b}v)^2 - g'(u)^2\right)\left(\beta(u - v + \frac{1}{b}v) + \gamma g'(u)\right) = 0, \tag{3.21}$$

where ε is the sign of the unit normal vector of the surface.

A direct computation of the left-hand side of (3.21) gives a polynomial in v with functions of u as coefficients, and thus they must be zero. From the coefficients of v in (3.21), we have

$$
\begin{aligned}
v^3 &: \quad \varepsilon\left(\frac{1}{b}-1\right)^3\beta = 0,\\
v^2 &: \quad \left(\frac{1}{b}-1\right)^2(\beta u+\gamma g'(u)) + 2\beta\left(\frac{1}{b}-1\right)^2 u = 0,\\
v &: \quad \left(\frac{1}{b}-1\right)g''(u) + 2\varepsilon\left(\frac{1}{b}-1\right)u(\beta u+\gamma g'(u))\\
&\qquad + \varepsilon\beta\left(\frac{1}{b}-1\right)(u^2-g'(u)^2) = 0,\\
v^0 &: \quad ug''(u) + \left(\frac{1}{b}-2\right)g'(u) + \varepsilon(u^2-g'(u)^2)(\beta u+\gamma g'(u)) = 0.
\end{aligned}
\tag{3.22}
$$

Case 2-1: $\gamma = 0$. From the first equation of (3.22), we have $b = 1$. It follows that the last equation of (3.22) implies

$$
ug''(u) - g'(u) + \beta u^3 - \beta u {g'}^2(u) = 0 \tag{3.23}
$$

when the surface is timelike, that is, $\varepsilon = 1$.

To solve the ODE, we take $g'(u)/u = 1 + 1/p(u)$. Then (3.23) reads

$$
p'(u) + 2\beta u p(u) + \beta u = 0,
$$

and its general solution is given by

$$
p(u) = -1/2 + c_1 e^{-\beta u^2}
$$

for some constant c_1. Thus, we get

$$
g'(u) = u + \frac{u e^{\beta u^2}}{c_1 - \frac{1}{2}e^{\beta u^2}},
$$

which implies that we can obtain the general solution of the ODE (3.23):

$$
g(u) = \frac{1}{2}u^2 - \frac{1}{\beta}\ln\left|c_1 - \frac{1}{2}e^{\beta u^2}\right| + c_2
$$

for some constants c_1 and c_2.

Next, we assume that the surface is spacelike, that is, $\varepsilon = -1$. By using a similar method, we have

$$
g(u) = \frac{1}{2\beta}\ln\frac{(c_1 e^{\beta u^2} - 1/2)^2}{|c_1 e^{\beta u^2}|} + c_2
$$

for some constants c_1 and c_2.

Case 2-2: $\beta = 0$. From the second equation of (3.22) we get $b = 1$ or $g'(u) = 0$.

If $b = 1$ and the surface is timelike, the last equation of (3.22) gives

$$ug''(u) - g'(u) + \gamma u^2 g'(u) - \gamma g'^3(u) = 0. \tag{3.24}$$

Let $p(u) = g'(u)/u$ in (3.24). Then we can obtain a Riccati differential equation in terms of u:

$$p'(u) + \gamma u p(u) = \gamma u p^3(u), \tag{3.25}$$

and its solution is given by

$$p^2(u) = \frac{1}{1 + c_1 e^{\gamma u^2}}.$$

It follows that the general solution of (3.24) becomes

$$g(u) = \frac{1}{2\gamma} \ln \frac{\sqrt{1 + c_1 e^{\gamma u^2}} + 1}{\sqrt{1 + c_1 e^{\gamma u^2}} - 1} + c_2,$$

where c_1, c_2 are constant. The surface with $g'(u) = 0$ is an isotropic plane.

Next, we assume that the surface is spacelike. By using a similar method, we get

$$g(u) = \frac{1}{2\gamma} \ln \frac{\sqrt{1 + c_1 e^{-\gamma u^2}} + 1}{\sqrt{1 + c_1 e^{-\gamma u^2}} - 1} + c_2,$$

where c_1 and c_2 are constant.

THEOREM 3.5. *Let M be a ϕ-minimal rotational surface parametrized by*

$$\mathbf{x}(u, v) = \mathbf{x}(u, v) = \left(u + bv, g(u), uv + b\frac{v^2}{2}\right)$$

in G_3^1 with a log-linear density $e^{\alpha x + \beta y + \gamma z}$.

(1) *If $\alpha \neq 0$, then M is an open part of an isotropic plane, a parabolic cylinder or a parabolic sphere.*

(2) *If $\alpha = 0$ and $\gamma = 0$, then for a timelike surface we have*

$$g(u) = \frac{1}{2}u^2 - \frac{1}{\beta} \ln \left| c_1 - \frac{1}{2} e^{\beta u^2} \right| + c_2 \quad \text{and} \quad b = 1,$$

and for a spacelike surface we have

$$g(u) = \frac{1}{2\beta} \ln \frac{(c_1 e^{\beta u^2} - 1/2)^2}{|c_1 e^{\beta u^2}|} + c_2 \quad \text{and} \quad b = 1,$$

where c_1 and c_2 are constant.

(3) *If $\alpha = 0$ and $\beta = 0$, then M is an open part of an isotropic plane or is parametrized by*

$$g(u) = \frac{1}{2\gamma} \ln \frac{\sqrt{1 + c_1 e^{\varepsilon \gamma u^2}} + 1}{\sqrt{1 + c_1 e^{\varepsilon \gamma u^2}} - 1} + c_2 \quad \text{and} \quad b = 1,$$

where $c_1, c_2 \in \mathcal{R}$ and ε is the sign of the unit normal vector of the surface.

Remark 3.6. In the case $\beta\gamma \neq 0$ in (3.22), it could be hard to get a general solution but not a numerical solution.

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