

The Zorn property for holomorphic functions

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Abstract. The aim of this paper is to investigate Zorn's property for the space (E_B, τ_E) , where E_B the linear hull of some compact, absolutely convex subset B of a Fréchet space E and the topology τ_E on E_B is induced by the topology of E . Furthermore, holomorphic extensions from (E_B, τ_E) are considered. Based on these results, we establish some results on extension of a Fréchet-valued continuous function f to an entire function from a non-polar balanced convex compact subset B of a Fréchet space whenever f is approximated fast enough on B by a sequence of polynomials.

1. Introduction. In 1945 Max Zorn [Zo] showed that the set of points of continuity of a Gâteaux holomorphic mapping between Banach spaces is open and closed. This means that a Gâteaux holomorphic mapping on a domain D in a Banach space is holomorphic whenever it is continuous at a single point in D . This result was not developed further until the 1960's when a number of mathematicians extended it to some other locally convex spaces and provided counter-examples in general locally convex spaces.

A domain D in a locally convex space E has the *Zorn property* if for every locally convex space F and every Gâteaux holomorphic function $f : D \rightarrow F$ the set of points of continuity of f is open and closed.

We recall that a *domain* in a locally convex space E is a connected open subset of E . Thus a domain D has the Zorn property if and only if every Gâteaux holomorphic function which is continuous at a single point is everywhere continuous. A domain D with the Zorn property is also called a *Zorn space*. To verify that D has the Zorn property it suffices to consider Banach-valued Gâteaux holomorphic functions. Complete metrizable spaces, Baire spaces, products of complete metrizable spaces, DFS-spaces (the duals

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of Fréchet–Schwartz spaces), products of DFS-spaces have Zorn’s property (see [No1, Theoreme 1.3.1]). Hirschowitz [Hi] showed that Zorn’s theorem no longer holds for a strict inductive limit of Banach spaces, and the Cartesian product of Zorn spaces is not necessarily a Zorn space. For example, the Cartesian product of a reflexive, separable Banach space with the dual of the space ω of all complex numbers does not have Zorn’s property. However, if $E = \prod_{i \in I} E_i$ and $\prod_{i \in I_1} E_i$ is a Zorn space for each finite subset I_1 of I then E is a Zorn space. Hirschowitz also gave an example of a locally convex nuclear space which is not Zorn (see also [Di1, Example 3.12]). Since every nuclear space is the surjective limit of inner product spaces, we conclude that there exist inner product spaces which are not Zorn. Nachbin [Na1] gave an example of a holomorphic function on $H(\mathbb{C})$ which is not uniformly holomorphic. If $\{p_n\}_{n \in \mathbb{N}}$ is an increasing family of seminorms on $H(\mathbb{C})$ which defines its topology then $(H(\mathbb{C}), p_n)$ is not a Zorn space for any n sufficiently large. Examples of spaces which are Zorn and which are not Zorn are given in [Di2]. In the same work, Dineen introduced the concept of surjective limit and used it to generate collections of locally convex spaces which are of interest as regards holomorphy. He extended Zorn’s theorem to different collections of spaces and to different definitions of holomorphy.

Wilansky [Wi, Exercise 3-1-4] and Borwein, Lucet and Mordukhovich [BLM, Example 2.4] give examples of non-complete Baire normed linear spaces. These are non-complete spaces with the Zorn property. We can also find more examples of non-complete Zorn spaces in [No1, pp. 32, Examples 2, 3].

Dineen and Lourenço [DL] proved that a DFM-space (the dual of a Montel–Fréchet space) with an absolute basis has the Zorn property. Dineen [Di3] made the following conjecture:

Silva (or Mackey) continuous Gâteaux holomorphic functions on DFM-spaces are continuous.

This conjecture, which is still open, is known to hold for DFS-spaces and for DFM-spaces with an absolute basis [DL]. According to Dineen [Di3], this conjecture requires a deep study of convergent sequences which are not Mackey convergent.

In this paper we give further examples of dense subspaces of Fréchet spaces which are Zorn spaces. Namely, in Section 3 we give conditions in the form of linear topological invariants for a Fréchet (nuclear or Schwartz with an absolute Schauder basis) space E to have a convex balanced compact subset B such that the space (E_B, τ_E) has the Zorn property, where E_B is the linear hull of B and τ_E is the topology on E_B induced by the topology of E (Example 3.3, Theorem 3.4). Holomorphic extensions from (E_B, τ_E) are also considered. This is natural because there is a connection between polar

subsets of a locally convex space and extension of holomorphic functions, as shown by an important result of Noverraz [No2] in 1974.

Some applications of the results of Section 3 are given in Section 4. Two results on convergence of Tauberian type for a sequence of Banach-valued polynomials on a locally convex space are proved (Proposition 4.3 and Theorem 4.4). We also establish convergence theorems of Tauberian type for a sequence of polynomials p_m between Fréchet spaces. Theorems 4.7 and 4.8 give conditions under which a Fréchet-valued function f which is defined, continuous, and approximated fast enough by a sequence $\{p_m\}_{m \geq 1}$ of Fréchet-valued polynomials on a non-polar balanced convex compact subset B of E can be extended to an entire function.

Before proving the main results of this paper, we review in Section 2 some linear topological invariants of Fréchet spaces introduced and investigated by Vogt, and some notions of holomorphic functions on locally convex spaces pertaining to our work.

2. Preliminaries and some auxiliary results

2.1. General notation. We shall use the standard notation of the theory of locally convex spaces as presented in the book of Jarchow [Ja]. A locally convex space is always a complex vector space with a locally convex Hausdorff topology.

Given a Fréchet space E , we always assume that its locally convex structure is generated by an increasing system $(\|\cdot\|_k)_{k \geq 1}$ of seminorms. Then we let E_k be the completion of the canonically normed space $E/\ker \|\cdot\|_k$ and $\omega_k : E \rightarrow E_k$ be the canonical map. We also denote by U_k the set $\{x \in E : \|x\|_k < 1\}$.

For an absolutely convex subset B of E we define a norm $\|\cdot\|_B^*$ with values in $[0, +\infty]$ on E' , the strong dual space of E , by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}.$$

Obviously $\|\cdot\|_B^*$ is the gauge functional of B° . Instead of $\|\cdot\|_{U_k}^*$ we write $\|\cdot\|_k^*$. We denote by E_B the linear hull of B , which becomes a normed space in a canonical way if B is bounded.

We let ω denote the countable product of copies of \mathbb{C} . This is also the space of all sequences of complex numbers. It is a Fréchet nuclear space which does not admit a continuous norm.

For a locally convex space E we use E'_{bor} to denote E' equipped with the bornological topology associated with the strong topology β . When E is Fréchet, E'_{bor} is the inductive dual defined by Berezanskii [Be], and moreover, by [Di1, Example 1.24 and Propositions 3.19(c) and 3.22(b)], E'_{bor} is the space of continuous 1-homogeneous polynomials on E endowed with either the τ_ω or τ_δ topology.

The following result is necessary for the proofs in the last section.

LEMMA 2.1. *Let E be a Fréchet space. Then E is closed in $(E'_{\text{bor}})'_{\beta}$.*

Proof. Since E is Fréchet, it is a closed subspace of any locally convex space in which it lies. Since metrizable spaces are infrabarrelled (or quasi-barrelled), [Ja, Proposition 11.2.2] implies that E is a subspace of the Fréchet space E'' . Hence E is a closed subspace of E'' . By [Ja, Proposition 13.3.1(b)] the spaces E'_{β} and E'_{bor} have the same bounded sets. Since the topologies on E'' and $(E'_{\text{bor}})'_{\beta}$ are the topologies of uniform convergence over all bounded subsets of E'_{β} and E'_{bor} respectively, it follows that E'' and $(E'_{\text{bor}})'_{\beta}$ induce the same topology on E . Hence E is closed in $(E'_{\text{bor}})'_{\beta}$. ■

2.2. Some linear topological invariants. Let E be a Fréchet space with a fundamental system $(\|\cdot\|_k)_{k \geq 1}$ of seminorms. We say that E has *property* $(\tilde{\Omega})$ if

$$(\tilde{\Omega}) \quad \forall p \exists q, d > 0 \forall k \exists C > 0 : \quad \|\cdot\|_q^{*1+d} \leq C \|\cdot\|_k^* \|\cdot\|_p^{*d}.$$

Let B be a closed, bounded, absolutely convex set in E (written $B \in \mathcal{B}(E)$). We say that E has *property* $(\tilde{\Omega}_B)$ if

$$(\tilde{\Omega}_B) \quad \forall p \exists q, d > 0, C > 0 : \quad \|\cdot\|_q^{*1+d} \leq C \|\cdot\|_B^* \|\cdot\|_p^{*d}.$$

The above properties have been introduced and investigated by Vogt [V1, V2, V3]. Hereafter, whenever E has property $(\tilde{\Omega})$ (resp. $(\tilde{\Omega}_B)$) we write $E \in (\tilde{\Omega})$ (resp. $E \in (\tilde{\Omega}_B)$).

Note that if E is Fréchet–Schwartz, then $E \in (\tilde{\Omega})$ if and only if there exists $B \in \mathcal{K}(E)$ such that $E \in (\tilde{\Omega}_B)$, where $\mathcal{K}(E)$ is the family of compact absolutely convex subsets of E [MV, Lemma 3.6].

2.3. Holomorphic functions. Let E and F be locally convex spaces and D be a domain in E . A function $f : D \rightarrow F$ is *Gâteaux holomorphic* if for every $a \in D$, $b \in E$ and $\varphi \in F'$ (the dual space of F), the \mathbb{C} -valued function of one complex variable

$$\lambda \mapsto (\varphi \circ f)(a + \lambda b)$$

is holomorphic on a neighbourhood of $0 \in \mathbb{C}$.

The function f is said to be *holomorphic* if it is Gâteaux holomorphic and continuous.

We denote by $H(D, F)$ the vector space of all holomorphic functions on D with values in F . This space is equipped with the compact-open topology τ_0 . Instead of $H(D, \mathbb{C})$ we write $H(D)$. We let $H_b(D)$ denote the space of holomorphic functions on D which are bounded on every bounded set in D , and $H_{ub}(E)$ will indicate the set of holomorphic functions which are

bounded on rU for some neighbourhood U of $0 \in E$ and for all $r > 0$. Note that $H_{ub}(E) \subseteq H_b(E)$.

For details concerning holomorphic functions on locally convex spaces we refer to the book of Dineen [Di1].

2.4. Polar sets. Let D be a domain in a locally convex space E . An upper-semicontinuous function $\varphi : D \rightarrow [-\infty, +\infty)$ is said to be *plurisubharmonic* (written $\varphi \in \text{PSH}(D)$) if φ is subharmonic on every one-dimensional section of D .

A subset $B \subset E$ is said to be *polar* in D if there exists $\varphi \in \text{PSH}(D)$ such that $\varphi \not\equiv -\infty$ and $\varphi|_B = -\infty$.

We say that B is *non-polar* if B is not polar.

A subset X of a domain D in a locally convex space E is *locally polar* at $a \in X$ if for every open neighbourhood U of a in D , $X \cap U$ is a polar subset of U . If X is a polar subset of D then X is locally polar at each point of X . In particular, if $X \subset D$ contains a point a and X is not locally polar at a then X is not a polar subset of D . By Josefson [Jo], the converse is true for domains in \mathbb{C}^n , $n \in \mathbb{N}$. This result appears to be unknown in infinite-dimensional locally convex spaces.

Polarity of subsets in infinite-dimensional spaces has been dealt with in many places, and we refer the reader to [DMV1, DMV2, DN] for further details.

The following is a property of non-polar sets in a Fréchet space.

LEMMA 2.2. *If B is a non-polar subset of a Fréchet space E then B spans a dense subspace E_B of E .*

Proof. If the closure of E_B is not equal to E then, by the Hahn–Banach theorem, there exists $\varphi \in E'$, $\varphi \neq 0$, such that $\varphi(E_B) = 0$. Then $v := \log |\varphi|$ is a plurisubharmonic function, $v \not\equiv 0$, and $B \subset E_B \subset \{z : v(z) = -\infty\}$. This contradicts the fact that B is non-polar. ■

3. The main results. In this section we will investigate the Zorn property of the linear hull of some convex balanced compact subset B of a Fréchet space. First, we discuss holomorphic extensions from a dense subset of a domain in a Fréchet space. We omit the easy proof of the following lemma.

LEMMA 3.1. *Let D_0 be a dense subset of a domain D in a Fréchet space E . Then for every bounded sequence $\{x_n\}_{n \geq 1} \subset D$ there exists a bounded sequence $\{y_n\}_{n \geq 1} \subset D_0$ such that $\{x_n : n \geq 1\} \subset \overline{\{y_n : n \geq 1\}}$.*

THEOREM 3.2. *Let D_0 be a dense subset of a domain D in a Fréchet space E . Then*

$$H_b(D_0) = H_b(D).$$

Proof. Without loss of generality we may assume that D is convex and $0 \in D_0 \subset D$. Set

$$\mathcal{B} = \{B \subset D : \text{there exists a bounded set } B_0 \subset D_0 \text{ with } B \subset \overline{B_0} \text{ in } D\}.$$

Let $f \in H_b(D_0)$ and write

$$f(z) = \sum_{n=0}^{\infty} P_n f(z)$$

where

$$P_n f(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{f(tz)}{t^{n+1}} dt.$$

Since f is bounded on every bounded set in D_0 , it follows that the series is uniformly convergent on every bounded set in D_0 . Then the series of the extended n -homogeneous polynomials

$$\sum_{n=0}^{\infty} \widehat{P_n f}$$

converges uniformly on all sets in \mathcal{B} .

Because $\{x\} \in \mathcal{B}$ for any $x \in D$, it follows that $\widehat{f} := \sum_{n=0}^{\infty} \widehat{P_n f}$ is defined on D .

By Lemma 3.1, $\{x_n : n \in \mathbb{N}\} \in \mathcal{B}$ for any bounded sequence $\{x_n\}$ in D . Hence \widehat{f} is continuous and bounded on all bounded sets in D . Indeed, otherwise there exists a bounded sequence $\{x_n\}_{n \geq 1} \subset D$ such that

$$\sup_{n \geq 1} |\widehat{f}(x_n)| = \infty.$$

By Lemma 3.1 there exists a bounded sequence $\{y_n\}_{n \geq 1} \subset D_0$ such that $\{x_n : n \geq 1\} \subset \overline{\{y_n : n \geq 1\}}$. Then \widehat{f} is not bounded on $\overline{\{y_n : n \geq 1\}}$, which is impossible.

Finally, it is easy to see that $\{za + wb : |z| + |w| \leq 1\} \in \mathcal{B}$ for all $a, b \in D$. Hence \widehat{f} is Gâteaux holomorphic. ■

The following example is due to Meise–Vogt [MV, Theorem 3.9 and Corollary 3.10] (see also [Di1, Proposition 6.23]).

EXAMPLE 3.3. Let E be a nuclear Fréchet space with the topology τ_E and $E \in (\widetilde{\Omega})$. Then there exists a non-polar set $B \in \mathcal{K}(E)$ such that (E_B, τ_E) has the Zorn property and

$$H_{ub}(E_B, \tau_E) = H_b(E_B, \tau_E) = H(E_B, \tau_E).$$

Indeed, since $E \in (\widetilde{\Omega})$ it follows from [MV, Lemma 3.6] that there exists $B \in \mathcal{K}(E)$ such that $E \in (\widetilde{\Omega}_B)$. Note that, by [DMV1, Theorem 9], B is non-polar. Then, by Lemma 2.2, E_B is dense in E .

Now, let f be a Gâteaux holomorphic on (E_B, τ_E) which is holomorphic at $0 \in (E_B, \tau_E)$. Then f is holomorphic on some τ_E -neighbourhood of $0 \in (E_B, \tau_E)$. By the denseness of E_B in E and Theorem 3.2, f has a holomorphic extension to some zero-neighbourhood in E .

On the other hand, since the canonical mapping $i : (E_B, \|\cdot\|_B) \rightarrow (E_B, \tau_E)$ is continuous it follows that $f \circ i$ is a Gâteaux holomorphic function on the Banach space $(E_B, \|\cdot\|_B)$ which is continuous at 0. By the classical theorem of Zorn [No1, Théorème 1.3.1], $f \circ i \in H(E_B, \|\cdot\|_B)$. Then, by [MV, Theorem 3.9(iv)], $f \in H(E_B, \tau_E)$.

Finally, the equalities follow from [MV, Corollary 3.10]. ■

In the following theorem we suppose that E is a Fréchet–Schwartz space with absolute basis $(e_j)_{j \geq 1}$. In this case, the coordinate functionals $e_j^* : E \rightarrow \mathbb{C}$ form a dual basis for E' , the strong dual of E . This implies that each $x \in E$ has an expansion $x = \sum_{j \geq 1} e_j^*(x)e_j$ and E contains a fundamental system $(\|\cdot\|_n)_{n \geq 1}$ of seminorms of the form

$$\left\| \sum_{j \geq 1} e_j^*(x)e_j \right\|_n = \sum_{j \geq 1} |e_j^*(x)| \|e_j\|_n.$$

In particular, for all j and n satisfying $\|e_j\|_n \neq 0$, we have

$$\begin{aligned} (3.1) \quad \|e_j^*\|_n^* &= \sup\{|e_n^*(x)| : \|x\|_n \leq 1\} \\ &= \sup\left\{|e_n^*(x)| : \sum_{j \geq 1} |e_j^*(x)| \|e_j\|_n \leq 1\right\} = \frac{1}{\|e_j\|_n}. \end{aligned}$$

THEOREM 3.4. *Let E be a Fréchet–Schwartz space with an absolute Schauder basis and topology τ_E such that $E \in (\tilde{\Omega})$. Then there exists a non-polar set $B \in \mathcal{K}(E)$ such that (E_B, τ_E) has the Zorn property. Moreover,*

$$H(E_B, \tau_E) = H_{ub}(E).$$

Proof. Let $(\|\cdot\|_\gamma)_{\gamma \geq 1}$ be an increasing fundamental system of seminorms for E and let $(e_j)_{j \geq 1}$ be an absolute Schauder basis of E .

Since $E \in (\tilde{\Omega})$, by [MV, Lemma 3.6] there exists $K \in \mathcal{K}(E)$ such that $E \in (\tilde{\Omega}_K)$. Then we have

$$(3.2) \quad \forall \alpha \exists \beta \geq \alpha, d > 0, C > 0 : \quad \|\cdot\|_\beta^{*1+d} \leq C \|\cdot\|_K^* \|\cdot\|_\alpha^{*d}.$$

Since $(e_j)_{j \geq 1}$ is an absolute Schauder basis of E , for every continuous seminorm p on E the series

$$\sum_{j \geq 1} |\langle e_j^*, x \rangle| p(e_j)$$

is convergent for every $x \in E$. This means that the sequence $\{p(e_j)e_j^*\}_{j \geq 1}$ is $\sigma(E', E)$ -convergent to 0 in E' . Then, since E is Fréchet–Montel, it follows

from [Ja, Theorem 11.6.2, p. 231] that $\{p(e_j)e_j^*\}_{j \geq 1}$ is $\beta(E', E)$ -convergent to 0 in E' for every continuous seminorm p on E . Hence $\|e_j^*\|_K^* e_j \rightarrow 0 \in E$.

Let

$$A = \overline{\text{acx}} \left(K \cup \bigcup_{j \geq 1} \|e_j^*\|_K^* e_j \right).$$

Clearly A is a compact convex balanced subset of E and by [HK, proof of Theorem 3.1] it is non-polar. As $\|x\|_A = \inf\{|\lambda| : x \in \lambda A\}$ and $\|e_j^*\|_K^* e_j \in A$ it follows that

$$(3.3) \quad \|e_j^*\|_K^* \|e_j\|_A = \| \|e_j^*\|_K^* e_j \|_A \leq 1 \quad \forall j \geq 1.$$

Choose a convex balanced compact subset B of E containing A such that the canonical map $j : (E_A, \|\cdot\|_A) \rightarrow (E_B, \|\cdot\|_B)$ is compact. Since A is non-polar, B is also non-polar. Let f be Gâteaux holomorphic on (E_B, τ_E) which is holomorphic at $0 \in (E_B, \tau_E)$. Since the canonical mapping $i : (E_B, \|\cdot\|_B) \rightarrow (E_B, \tau_E)$ is continuous it follows that $f \circ i$ is a Gâteaux holomorphic function on the Banach space $(E_B, \|\cdot\|_B)$ which is continuous at 0. By the classical theorem of Zorn, $f \circ i$ is a holomorphic function on $(E_B, \|\cdot\|_B)$, and hence is bounded on compact subsets of $(E_B, \|\cdot\|_B)$. In particular, $f(\lambda A)$ is bounded for every complex number λ . This implies that $f \circ i \circ j$ is a holomorphic function of bounded type on $(E_A, \|\cdot\|_A)$. We have thus proved the following estimates. For every $s > 0$,

$$N(s) := \sup\{|f(x)| : \|x\|_A \leq s\} < \infty.$$

On the other hand, since f is holomorphic at $0 \in (E_B, \tau_E)$ we can find $\alpha \in \mathbb{N}$ such that

$$\sup\{|f(x)| : x \in E_B \cap U_\alpha\} = M_\alpha < +\infty.$$

For all $x \in E_B \cap U_\alpha$ we write

$$f(x) = \sum_{n \geq 0} P_n f(x)$$

where

$$P_n f(x) = \frac{1}{2\pi i} \int_{|t|=1} \frac{f(tx)}{t^{n+1}} dt.$$

Since B is non-polar, by Lemma 2.2, E_B is dense in E . Thus the extended n -homogeneous polynomials (still denoted by $P_n f$) satisfy

$$\sup\{|P_n f(x)| : x \in U_\alpha\} = M_\alpha < \infty.$$

Let $\widehat{P_n f}$ be the symmetric n -linear form associated with $P_n f$. Take $\beta \geq \alpha$ and $d, C > 0$ such that (3.2) holds.

(i) First, we consider the case where E admits a continuous norm.

Without loss of generality we may assume that $\|\cdot\|_1$ is a continuous norm. Then by (3.1)–(3.3) we have

$$\left(\frac{1}{\|e_j\|_\beta}\right)^{1+d} \leq C \frac{1}{\|e_j\|_A} \left(\frac{1}{\|e_j\|_\alpha}\right)^d \quad \text{for all } j \geq 1.$$

In our next set of estimates we use the following generalized arithmetic-geometric inequality: If $0 \leq \theta \leq 1$ and a and b are positive real numbers then

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b.$$

Now, for every $r > 0$ and for $x = \sum_{j \geq 1} e_j^*(x) e_j \in E$ with

$$\|x\|_\beta = \sum_{j \geq 1} |e_j^*(x)| \|e_j\|_\beta \leq r$$

we have

$$\begin{aligned} \sum_{n \geq 0} |P_n f(x)| &= \sum_{n \geq 0} \left| P_n f \left(\sum_{j \geq 1} e_j^*(x) e_j \right) \right| \\ &\leq \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} |e_{j_1}^*(x)| \cdots |e_{j_n}^*(x)| |\widehat{P_n f}(e_{j_1}, \dots, e_{j_n})| \\ &= \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} |e_{j_1}^*(x)| \|e_{j_1}\|_\beta \cdots |e_{j_n}^*(x)| \|e_{j_n}\|_\beta \frac{|\widehat{P_n f}(e_{j_1}, \dots, e_{j_n})|}{\|e_{j_1}\|_\beta \cdots \|e_{j_n}\|_\beta} \\ &\leq \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} C^{n/(1+d)} |e_{j_1}^*(x)| \|e_{j_1}\|_\beta \cdots |e_{j_n}^*(x)| \|e_{j_n}\|_\beta \\ &\quad \times \left[\frac{|\widehat{P_n f}(e_{j_1}, \dots, e_{j_n})|}{\|e_{j_1}\|_A \cdots \|e_{j_n}\|_A} \right]^{1/(1+d)} \left[\frac{|\widehat{P_n f}(e_{j_1}, \dots, e_{j_n})|}{\|e_{j_1}\|_\alpha \cdots \|e_{j_n}\|_\alpha} \right]^{d/(1+d)} \\ &\leq \sum_{n \geq 0} \frac{1}{\varrho^{n/(1+d)}} \sum_{j_1, \dots, j_n \geq 1} |e_{j_1}^*(x)| \|e_{j_1}\|_\beta \cdots |e_{j_n}^*(x)| \|e_{j_n}\|_\beta \\ &\quad \times \left[\frac{|\widehat{P_n f}(C \varrho e_{j_1}, \dots, C \varrho e_{j_n})|}{\|e_{j_1}\|_A \cdots \|e_{j_n}\|_A} \right]^{1/(1+d)} \left[\frac{|\widehat{P_n f}(e_{j_1}, \dots, e_{j_n})|}{\|e_{j_1}\|_\alpha \cdots \|e_{j_n}\|_\alpha} \right]^{d/(1+d)} \\ &\leq \sum_{n \geq 0} \frac{1}{\varrho^{n/(1+d)}} \sum_{j_1, \dots, j_n \geq 1} |e_{j_1}^*(x)| \|e_{j_1}\|_\beta \cdots |e_{j_n}^*(x)| \|e_{j_n}\|_\beta \\ &\quad \times \left[\frac{1}{1+d} \frac{|\widehat{P_n f}(C \varrho e_{j_1}, \dots, C \varrho e_{j_n})|}{\|e_{j_1}\|_A \cdots \|e_{j_n}\|_A} + \frac{d}{1+d} \frac{|\widehat{P_n f}(e_{j_1}, \dots, e_{j_n})|}{\|e_{j_1}\|_\alpha \cdots \|e_{j_n}\|_\alpha} \right] \\ &\leq \sum_{n \geq 0} \frac{1}{\varrho^{n/(1+d)}} \sum_{j_1, \dots, j_n \geq 1} |e_{j_1}^*(x)| \|e_{j_1}\|_\beta \cdots |e_{j_n}^*(x)| \|e_{j_n}\|_\beta \frac{n^n}{n!} \left[\frac{N(C \varrho)}{1+d} + \frac{dM_\alpha}{1+d} \right] \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{N(C\varrho)}{1+d} + \frac{dM_\alpha}{1+d} \right] \sum_{n \geq 0} \frac{n^n}{n! \varrho^{n/(1+d)}} \left(\sum_{j \geq 1} |e_j^*(x)| \|e_j\|_\beta \right)^n \\
 &\leq \left[\frac{N(C\varrho)}{1+d} + \frac{dM_\alpha}{1+d} \right] \sum_{n \geq 0} \frac{n^n r^n}{n! \varrho^{n/(1+d)}} < \infty
 \end{aligned}$$

for $\varrho \geq (2er)^{1+d}$. Hence $f \in H_{ub}(E)$.

(ii) Now we consider the general case.

Choose α (we can assume that $\alpha = 1$) such that

$$\sup\{|f(z)| : z \in U_1\} < \infty.$$

Set

$$J = \{j \in \mathbb{N} : \|e_j\|_1 \neq 0\}$$

and write

$$E = E^1 \oplus E^2$$

where E^1 is the subspace of E with the absolute Schauder basis $\{e_j : j \in J\}$ and the continuous norm $\|\cdot\|_1|_{E^1}$, and $E^2 = \ker \|\cdot\|_1$. Let $\pi_1 : E \rightarrow E^1$ be the canonical projection. Note that $U_1 = U_1^1 \times E^2$ where $U_1^1 = \pi_1(U_1) = \{x_1 \in E^1 : \|x_1\|_1 < 1\}$. Moreover, if $E \in (\tilde{\Omega}_K)$ then $E^1 \in (\tilde{\Omega}_{K_1})$, where $K_1 = \pi_1(K)$. On the other hand, since f is Gâteaux holomorphic on E_K , Liouville’s theorem implies that f can be factorized through $\pi_1 : E \rightarrow E^1$ by $f_1 : E^1_{K_1} \rightarrow \mathbb{C}$ given by $f_1(x_1) = f(x_1, 0)$, $x_1 \in E^1_{K_1}$. Applying (i) to f_1 we get $f_1 \in H_{ub}(E^1)$. Hence $f = f_1 \circ \pi_1 \in H_{ub}(E)$.

The theorem is proved. ■

4. Application to the problem of Tauberian convergence. Let E, F be locally convex spaces over \mathbb{C} , and D be a domain in E . The problem of Tauberian convergence is to look for additional properties to ensure that every sequence of F -valued holomorphic functions defined and converging on a subset of D is convergent everywhere on D . An important example is Vitali’s theorem, where subsets admitting a limit point in D are considered and local uniform boundedness is a possible additional property. In the case where E, F are finite-dimensional, it seems that the easiest proof of Vitali’s theorem is given with the help of Montel’s theorem. However, Montel’s theorem no longer holds in the vector-valued case if the underlying Banach space is infinite-dimensional. It is surprising that Vitali’s theorem is still valid (see [AN] and [QLD]).

Using the results in the previous section, we investigate the holomorphic extension of a Fréchet-valued continuous function f to an entire function from a non-polar balanced convex compact subset B of a Fréchet space whenever f is approximated fast enough on B by a sequence of polynomials.

Let us begin with the case where f is a Banach-valued continuous function defined on a compact non-polar subset of a finite-dimensional space.

First, we recall the following result about negligible sets due to Bedford and Taylor [Kl, Theorem 4.7.6] and prove a slight modification of the classical inequality on estimating the sup-norm of polynomials with values in a seminormed space.

PROPOSITION 4.1. *Let $\{u_m\}_{m \geq 1}$ be a sequence of plurisubharmonic functions defined on an open subset D of \mathbb{C}^N . If the sequence is locally uniformly bounded from above then the set*

$$\left\{ z \in D : \limsup_{m \rightarrow \infty} u_m(z) < \left(\limsup_{m \rightarrow \infty} u_m \right)^*(z) \right\}$$

is polar.

LEMMA 4.2. *Let K, L be compact sets in \mathbb{C}^N , with K non-polar. Let F be a vector space equipped with a seminorm $\|\cdot\|$. Then there exists $C_{K,L} > 0$ depending only on K and L such that for every continuous polynomial $p : \mathbb{C}^N \rightarrow F$ of degree m we have*

$$\frac{1}{m} \log \|p\|_L \leq \frac{1}{m} \log \|p\|_K + C_{K,L}.$$

Proof. Consider the Siciak extremal function

$$V_K(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^N), u|_K \leq 0\},$$

where $\mathcal{L}(\mathbb{C}^N)$ is the Lelong class of plurisubharmonic functions φ with logarithmic growth on \mathbb{C}^N , i.e.,

$$\varphi(z) \leq \log(1 + |z|) + C \quad \forall z \in \mathbb{C}^N$$

where C is a constant independent of z . Since K is compact and non-polar, by a well known result of Siciak (see [Kl, Corollary 5.2.2 and Theorem 5.2.4]), the function V_K is bounded from above on compact subsets of \mathbb{C}^N .

On the other hand, since $\deg p = m$, it is easy to see that

$$\varphi(z) := \frac{1}{m} \log \|p(z)\| \quad \forall z \in \mathbb{C}^N$$

belongs to the class $\mathcal{L}(\mathbb{C}^N)$. Then, since the function

$$\psi : z \mapsto \frac{1}{m} \log \|p(z)\| - \frac{1}{m} \log \|p\|_K$$

belongs to $\mathcal{L}(\mathbb{C}^N)$ and $\psi|_K \leq 0$, we have $\psi(z) \leq V_K(z)$ for all $z \in \mathbb{C}^N$, that is,

$$\frac{1}{m} \log \|p(z)\| - \frac{1}{m} \log \|p\|_K \leq V_K(z)$$

for all $z \in \mathbb{C}^N$. Hence, letting $C_{K,L} := \sup_{z \in L} V_K(z)$, we see that

$$\begin{aligned} \sup_{z \in L} \frac{1}{m} \log \|p(z)\| &= \frac{1}{m} \log \|p\|_L \leq \frac{1}{m} \log \|p\|_K + \sup_{z \in L} V_K(z) \\ &\leq \frac{1}{m} \log \|p\|_K + C_{K,L}. \quad \blacksquare \end{aligned}$$

PROPOSITION 4.3. *Let F be a Banach space and f be an F -valued continuous function defined on a compact non-polar subset X of \mathbb{C}^N . Let $\{p_m\}_{m \geq 1}$ be a sequence of F -valued continuous polynomials with $\deg p_m \leq m$ such that*

$$\lim_{m \rightarrow \infty} \|f(z) - p_m(z)\|^{1/m} = 0 \quad \forall z \in X.$$

Then:

- (i) f can be extended to a holomorphic function (still denoted by f) on \mathbb{C}^N ;
- (ii) for every $\varepsilon > 0$ and every compact set $K \subset \mathbb{C}^N$ there exists m_0 such that for all $m \geq m_0$ we have

$$\|f - p_m\|_K^{1/m} := \sup_{z \in K} \|f(z) - p_m(z)\|^{1/m} < \varepsilon.$$

Proof. (i) Since $p_m \rightarrow f$ pointwise on X , we deduce that the sequence $\{p_m\}_{m \geq 1}$ is pointwise bounded on X . Thus, since X is non-polar, there exists a non-polar set $X' \subset X$ such that

$$\sup_{m \geq 1} \|p_m\|_{X'} < \infty.$$

Indeed, for every $k \geq 1$ we set

$$X_k := \{z \in X : |p_m(z)| \leq k \ \forall m \geq 1\}.$$

Since the sequence $\{p_m\}_{m \geq 1}$ is pointwise bounded on X , we deduce that $X = \bigcup_{k \geq 1} X_k$. It follows that there exists k_0 such that X_{k_0} is non-polar. Since X is compact and each p_m is continuous it follows that

$$X_{k_0} = \bigcap_{m=1}^{\infty} \{z \in X : |p_m(z)| \leq k_0\}$$

is compact.

Next, since X_{k_0} is compact and non-polar, using Lemma 4.2 we deduce that the sequence $\{m^{-1} \log \|p_m\|\}_{m \geq 1}$ is locally uniformly bounded from above on \mathbb{C}^N . Now for $m \geq 1$ we define the plurisubharmonic function

$$u_m(z) := \frac{1}{2^m} \log \|p_{2^m}(z) - p_{2^{m-1}}(z)\|, \quad z \in \mathbb{C}^N.$$

It is easy to check that the sequence $\{u_m\}_{m \geq 1}$ is also locally uniformly bounded from above. Thus the function u defined by

$$u(z) := \limsup_{m \rightarrow \infty} u_m(z) \quad \forall z \in \mathbb{C}^N$$

is locally bounded from above as well. Fix $x \in X$. We claim that $u(x) = -\infty$. Indeed, given $\varepsilon \in (0, 1)$, we choose m_0 large enough such that

$$\|f(x) - p_m(x)\| < \varepsilon^m \quad \forall m \geq m_0.$$

By the triangle inequality, for every $m \geq m_0$ we obtain

$$\|p_{2^m}(x) - p_{2^{m-1}}(x)\| \leq \|f(x) - p_{2^m}(x)\| + \|f(x) - p_{2^{m-1}}(x)\| < \varepsilon^{2^m} + \varepsilon^{2^{m-1}}.$$

This implies that

$$u(x) = \limsup_{m \rightarrow \infty} u_m(x) < \frac{\log \varepsilon}{2}.$$

By letting $\varepsilon \downarrow 0$ we get $u(x) = -\infty$, as claimed. This means that $u \equiv -\infty$ on X_{k_0} .

On the other hand, by Proposition 4.1, the set $\{z \in \mathbb{C}^N : u(z) \neq u^*(z)\}$ is polar. Thus $u^* = -\infty$ on a non-polar subset of \mathbb{C}^N . This forces $u^* \equiv -\infty$ on \mathbb{C}^N . Hence $u \equiv -\infty$ on \mathbb{C}^N as well. By Hartogs' lemma for sequences of subharmonic functions, we deduce that $\{u_m\}_{m \geq 1}$ converges uniformly to $-\infty$ on compact sets of \mathbb{C}^N . Now fix a compact subset K of \mathbb{C}^N and $\varepsilon > 0$. Then there exists $m_0 \geq 1$ such that

$$u_m(z) < \log \varepsilon \quad \forall m \geq m_0, \forall z \in K.$$

Therefore

$$\|p_{2^m} - p_{2^{m-1}}\|_K < \varepsilon^{2^m} \quad \forall m \geq m_0.$$

Hence the sequence $\{p_{2^m}\}_{m \geq 1}$ is uniformly convergent on compact sets of \mathbb{C}^N to a holomorphic function \tilde{f} . Since the convergence occurs in particular on X , we infer that \tilde{f} is the desired holomorphic extension.

(ii) We set

$$v(z) = \limsup_{m \rightarrow \infty} \frac{1}{m} \log \|f(z) - p_m(z)\| \quad \forall z \in D.$$

By our assumption, $v(a) = -\infty$ for every $a \in X$. Moreover, as the sequence $(m^{-1} \log \|f - p_m\|)_m$ consists of plurisubharmonic functions which are locally bounded from above, we infer that the upper regularization

$$v^*(z) = \limsup_{\xi \rightarrow z} v(\xi)$$

is also plurisubharmonic on D . Thus, using Bedford–Taylor's theorem and non-polarity of X , we infer that $v^*(z) = -\infty$ on a non-polar portion of X . This forces $v^* \equiv -\infty$ on D , and hence $v \equiv -\infty$ on D .

So for each $\varepsilon \in (0, 1)$ we have

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log \|f(z) - p_m(z)\| < \log \varepsilon \quad \forall z \in D.$$

Now, it suffices to apply Hartogs' lemma to reach the conclusion that for every compact subset K of D , there exists m_0 such that if $m \geq m_0$ then

$$\|f - p_m\|_K^{1/m} < \varepsilon. \blacksquare$$

Next, we give a result on convergence of Tauberian type for a sequence of Banach-valued polynomials on a locally convex space.

THEOREM 4.4. *Let E be an locally convex space, F be a Banach space, f be a F -valued holomorphic function defined on a domain $D \subset E$, and $\{p_m\}_{m \geq 1}$ be a sequence of F -valued polynomials with $\deg p_m \leq m$. Assume that there exists a subset $X \subset D$ satisfying the following conditions:*

- (i) $\lim_{m \rightarrow \infty} \|f(z) - p_m(z)\|^{1/m} = 0$ for all $z \in X$;
- (ii) for every point $a \in D \setminus X$, there exists a finite-dimensional complex affine subspace L passing through a such that $D_a \cap X$ is compact and non-polar as a subset of L , where D_a is the connected component of $D \cap L$ that contains a .

Then for every $x \in D$ we have

$$\lim_{m \rightarrow \infty} \|f(x) - p_m(x)\|^{1/m} = 0.$$

Proof. It suffices to prove the result for $x \in D \setminus X$. Fix $x \in D \setminus X$. By the assumption, we can find an affine complex subspace L of finite dimension such that $D_x \cap X$ is non-polar, where D_x is the connected component of $D \cap L$ that contains x . It now suffices to apply Proposition 4.3 to $D := D_x$ and $p'_m := p_m|_L$. The application is possible since $\deg p'_m \leq m$ for every m . \blacksquare

The following example shows that there exists a locally convex space satisfying condition (ii) of Theorem 4.4.

EXAMPLE 4.5. Let $D := \bigoplus_{n \in \mathbb{N}} D_n \subset \mathbb{C}^{(\mathbb{N})}$ be an open set and $X := \bigoplus_{n \in \mathbb{N}} X_n \subset D$ where $X_n = \{|z| \leq 1\} \subset D_n = \{|z| < 2\} \subset \mathbb{C}_n = \mathbb{C}$ for all $n \in \mathbb{N}$.

For each $a = (a_n)_{n \in \mathbb{N}} = (a_1, \dots, a_{n_0}, 0, 0, \dots) \in D$ we consider the complex subspace $L = \mathbb{C}^{n_0} = \prod_{k=1}^{n_0} \mathbb{C}_k$ of $\mathbb{C}^{(\mathbb{N})}$. Then $a \in L$ and $L \cap X = \prod_{k=1}^{n_0} X_k$ is compact and non-polar in L .

Now, we give an example of a non-polar set in ω .

EXAMPLE 4.6. Let $X := \prod_{j=1}^{\infty} X_j \subset \omega$ where $X_j \subset \mathbb{C}$, $j \in \mathbb{N}$. Then X is non-polar if and only if each X_j is non-polar for all $j \in \mathbb{N}$.

Proof. It is obvious that if there exists $j \in \mathbb{N}$ such that X_j is polar then X is also polar.

Now, suppose that X_j is non-polar for all $j \in \mathbb{N}$. Then, for each $j \in \mathbb{N}$, choose a point $z_j^0 \in X_j$ such that X_j is non-polar at z_j^0 .

If $z^0 := (z_j^0)$ then $z^0 \in X$. Now suppose $\varphi \in \text{PSH}(\omega)$ and $\varphi|_X \equiv -\infty$. Take an open neighbourhood $U := \prod_{j=1}^n U_j \times \prod_{j>n} \mathbb{C}_j$ of z^0 , where $\mathbb{C}_j = \mathbb{C}$ for all $j > n$, such that φ is bounded from above on U . Since every bounded above subharmonic function on \mathbb{C} is constant, this implies that $\varphi(x, y) = \varphi(x, 0)$ for all $x \in \prod_{j=1}^n U_j$ and all $y \in \prod_{j>n} \mathbb{C}_j$. If $x \in \prod_{j=1}^n X_j$ and $y \in \prod_{j>n} X_j$, we have $\varphi(x, y) = -\infty$, and hence $\varphi(x, y) = -\infty$ for all $x \in \prod_{j=1}^n X_j$ and $y \in \prod_{j>n} \mathbb{C}_j$. Since $\prod_{j=1}^n X_j$ is clearly non-polar at $z^n := (z_j^0)_{j=1}^n \in \prod_{j=1}^n X_j$, this implies $\varphi(x, 0) = -\infty$ for all $x \in \prod_{j=1}^n U_j$. Hence $\varphi \equiv -\infty$ on U , and as U has non-empty interior, on ω . ■

We proceed now to establish the result for a sequence of polynomials between Fréchet spaces.

THEOREM 4.7. *Let E be an infinite-dimensional nuclear Fréchet space with topology τ_E such that $E \in (\widehat{\Omega})$ and let F be a Fréchet space with topology defined by an increasing sequence $(\|\cdot\|_{n \geq 1})$ of seminorms. Then there exists a non-polar set $B \in \mathcal{K}(E)$ such that every F -valued continuous function f defined on B satisfying the following conditions:*

- (i) f is holomorphic at $0 \in (E_B, \tau_E)$;
- (ii) there exists a sequence $\{p_m\}_{m \geq 1}$ of F -valued polynomials on E with $\deg p_m \leq m$ such that

$$\lim_{m \rightarrow \infty} \|f(z) - p_m(z)\|_n^{1/m} = 0 \quad \forall z \in B, \forall n \geq 1$$

can be extended to a holomorphic function on E .

Proof. As in Example 3.3, we can find a non-polar set $B \in \mathcal{K}(E)$ such that E_B is a dense subspace of E .

We consider two cases.

(a) Suppose $F = \mathbb{C}$. We denote by \mathcal{F} the family of all finite-dimensional subspaces $P \neq 0$ of E_B . Since B is a closed balanced convex set, $B \cap P$ is a closed balanced convex neighbourhood of 0 in P for every $P \in \mathcal{F}$. Since P is finite-dimensional, $B \cap P$ is not polar in P . By Proposition 4.3 there exists $\widehat{f}_P \in H(E_B \cap P, F)$ such that $\widehat{f}_P = f$ on $B \cap P$. On the other hand, if P and Q are finite-dimensional subspaces of E_B then the identity principle for holomorphic functions implies that

$$\widehat{f}_P|_{P \cap Q} = \widehat{f}_{P \cap Q} = \widehat{f}_Q|_{P \cap Q}.$$

Hence the family $\{\widehat{f}_P\}_{P \in \mathcal{F}}$ defines a Gâteaux holomorphic function, which we still denote by f , on E_B . By the hypothesis and Example 3.3 we have $f \in H(E)$.

(b) Suppose now that F is a general Fréchet space. Obviously, for all $z \in B$ we have

$$\lim_{m \rightarrow \infty} |(u \circ f)(z) - (u \circ p_m)(z)|^{1/m} = 0 \quad \text{for all } u \in F'.$$

It follows from (a) that $u \circ f$ has an extension $\widehat{u \circ f} \in H(E)$ for all $u \in F'$.

We denote by $H(E)_{\text{bor}}$ the space $H(E)$ endowed with the bornological topology associated with the compact-open topology. Since E is Fréchet, [Di1, Proposition 3.19(c)] implies that $H(E)_{\text{bor}} = (H(E), \tau_\delta)$.

By the identity principle we define a linear mapping

$$T : F'_{\text{bor}} \rightarrow H(E)_{\text{bor}}$$

by letting

$$T(u)(z) = \widehat{(u \circ f)}(z), \quad z \in E, u \in F'_{\text{bor}}.$$

Suppose that $u_\alpha \rightarrow u$ in F'_{bor} and $T(u_\alpha) \rightarrow v$ in $H(E)_{\text{bor}}$ as $\alpha \rightarrow \infty$. This implies, in particular, that $[T(u_\alpha)](z) \rightarrow v(z)$ for all $z \in E$. However, if $z \in B$ then

$$[T(u_\alpha)](z) = \widehat{(u_\alpha \circ f)}(z) = u_\alpha(f(z)) \rightarrow u(f(z)),$$

and so $v(z) = u(f(z))$ for all $z \in B$. By the principle of analytic continuation we have $v(z) = u(f(z))$ for all z in the closed subspace of E spanned by B . Since B spans a dense subspace of E (see Lemma 2.2), and both v and $u \circ f$ are continuous on E , this implies that $v = u \circ f$. Hence T has a closed graph. Moreover, since F is Fréchet, by [Ja, Theorem 13.4.2] we have $\beta(F', F)_{\text{bor}} = \eta(F', F)$ on F' . This implies that F'_{bor} is ultrabornological, that is, an inductive limit of Banach spaces. On the other hand, because E is nuclear Fréchet and $E \in (\tilde{\Omega})$, [MV, Corollary 3.10] implies that $E \in (H_{ub})$, and hence, by [MV, Proposition 4.1], $H(E)_{\text{bor}}$ is an (LF)-space. Consequently, by the closed graph theorem of Grothendieck [Gr, Introduction, Theorem B], T is continuous.

Now we define a map $\hat{f} : E \rightarrow F''_{\text{bor}}$ by

$$[\hat{f}(z)](u) = [T(u)](z), \quad z \in E, u \in F'_{\text{bor}}.$$

Since T is continuous and point evaluations on $H(E)_{\text{bor}}$ are continuous, it follows that $\hat{f}(z) \in F''_{\text{bor}}$ for all $z \in E$. Moreover, for each fixed $u \in F'_{\text{bor}}$ the mapping

$$z \in E \mapsto [T(u)](z)$$

is holomorphic, that is,

$$\hat{f} : E \rightarrow (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))$$

is a continuous mapping. For all $a, b \in E$ and all $u \in F'_{\text{bor}}$ the mapping

$$\lambda \in \mathbb{C} \mapsto u \circ \hat{f}(a + \lambda b) = \widehat{(u \circ f)}(a + \lambda b)$$

is a Gâteaux holomorphic mapping, and hence

$$\hat{f} : E \rightarrow (F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))$$

is holomorphic.

By [Ja, 8.13.2 and 8.13.3], F'_{bor} is a complete locally convex space. Hence by [Ho, Theorem 4, p. 210] applied to F'_{bor} we see that $(F''_{\text{bor}}, \sigma(F''_{\text{bor}}, F'_{\text{bor}}))$

and $(F'_{\text{bor}})'_{\beta}$ have the same bounded sets. An application of [Na2, Proposition 13] shows that

$$\widehat{f} : E \rightarrow (F'_{\text{bor}})'_{\beta}$$

is holomorphic.

Let j denote the canonical injection from F into F'' . If $z \in B$ and $\widehat{f}(z) \neq j(f(z))$ then there exists $u \in F'$ such that

$$\widehat{f}(z)(u) \neq j(f(z))(u) = u(f(z)).$$

This, however, contradicts the fact that for all $z \in B$ we have

$$\widehat{f}(z)(u) = (\widehat{u \circ f})(z) = (u \circ f)(z) = u(f(z)).$$

We now fix a non-zero $z \in B$. Then there exists a unique sequence $(a_{n,z})_{n=1}^{\infty}$ in F'' such that for all $\lambda \in \mathbb{C}$,

$$\widehat{f}(\lambda z) = \sum_{n=0}^{\infty} a_{n,z} \lambda^n.$$

Since $\widehat{f}(0) = f(0) = a_{0,z}$ it follows that $a_{0,z} \in F$. Now suppose that $(a_{j,z})_{j=0}^n \subset F$. When $|\lambda| \leq 1$, we have $\widehat{f}(\lambda z) = f(\lambda z) \in F$. Hence, if $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, then

$$\frac{\widehat{f}(\lambda z) - \sum_{j=0}^n a_{j,z} \lambda^j}{\lambda^{n+1}} = \sum_{j=n+1}^{\infty} a_{j,z} \lambda^{j-n-1} \in F.$$

Since F is complete, we see, on letting λ tend to 0, that $a_{n+1,z} \in F$. By induction $a_{n,z} \in F$ for all n , and hence $\widehat{f}(\lambda z) \in F$ for all $\lambda \in \mathbb{C}$ and all $z \in B$. Since \widehat{f} is continuous and F is a closed subspace of $(F'_{\text{bor}})'_{\beta}$ (see Lemma 2.1), we have shown that $\widehat{f} : E \rightarrow F$ is holomorphic. ■

An argument similar to that used in the proof of Theorem 4.7 but using Theorem 3.4 proves the following

THEOREM 4.8. *Let E be a Fréchet-Schwartz space with an absolute Schauder basis and topology τ_E such that $E \in (\widetilde{\Omega})$, and let F be a Fréchet space with topology defined by an increasing sequence $(\|\cdot\|)_{n \geq 1}$ of seminorms. Then there exists a non-polar set $B \in \mathcal{K}(E)$ such that every F -valued continuous function f defined on B satisfying the following conditions:*

- (i) f is holomorphic at $0 \in (E_B, \tau_E)$;
- (ii) there exists a sequence $\{p_m\}_{m \geq 1}$ of F -valued polynomials on E with $\text{deg } p_m \leq m$ such that

$$\lim_{m \rightarrow \infty} \|f(z) - p_m(z)\|_n^{1/m} = 0 \quad \forall z \in B, \forall n \geq 1$$

can be extended to a holomorphic function on E .

Note that Theorems 4.7 and 4.8 immediately yield the following:

For every compact set $L \subset E_B$ and for any finite-dimensional complex subspace $P \subset E$,

$$\lim_{m \rightarrow \infty} \|f - p_m\|_{n, L \cap P}^{1/m} := \lim_{m \rightarrow \infty} \sup_{z \in L \cap P} \|f(z) - p_m(z)\|_n^{1/m} = 0 \quad \forall n \geq 1.$$

In particular, for every $x \in E$ we have

$$\lim_{m \rightarrow \infty} \|f(x) - p_m(x)\|_n^{1/m} = 0 \quad \forall n \geq 1.$$

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