ACTA ARITHMETICA
181.4 (2017)

Galois action on $\bar{\mathbb{Q}}$-isogeny classes of abelian L-surfaces with quaternionic multiplication

by

Santiago Molina (Barcelona)

1. Introduction. Let L be a number field. An abelian variety $A/\bar{\mathbb{Q}}$ is called an *abelian L-variety* if for each $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$ there exists an isogeny $\mu_\sigma : {}^\sigma A \to A$ with the following compatibility condition: $\psi \circ \mu_\sigma = \mu_\sigma \circ {}^\sigma\psi$ for all $\psi \in \mathrm{End}(A)$.

In this note, we deal with the two-dimensional situation and the so-called *fake elliptic curves* or *abelian surfaces with quaternionic multiplication (QM)*, that is, pairs $(A, \imath)$ where A is an abelian surface and $\imath$ is an embedding of a quaternion order $\mathcal{O}$ into the endomorphism ring of A. We remark that, by setting $A = E \times E$, where E is an elliptic curve, and $\imath$ the obvious embedding of $\mathcal{O} = \mathrm{M}_2(\mathbb{Z})$ into $\mathrm{End}(E \times E)$, the theory of elliptic curves can be seen as a special case. In this scenario, we say that $(A, \imath)$ is an *abelian L-surface with QM* if, for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$, there exists an isogeny $\mu_\sigma : {}^\sigma A \to A$ as well, but with the above compatibility condition only satisfied for the image of the quaternionic multiplication $\imath$.

The moduli problem that classifies abelian surfaces with QM is solved by the classical *Shimura curves* X_Γ. If $P \in X_\Gamma$ corresponds to an abelian L-surface with QM $(A, \imath)$, we can interpret the $\bar{\mathbb{Q}}$-isogeny class of $(A, \imath)$ as a set $[P]$ of points in $X_\Gamma(\bar{\mathbb{Q}})$. Directly from the definition, we can deduce that $[P]$ is stable under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$.

The main aim of this paper is to construct a projective Galois representation ρ attached to $(A, \imath)$ of the form

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \to (\mathcal{O} \otimes \mathbb{A}_{\mathrm{f}})^\times / (\mathrm{End}(A, \imath) \otimes \mathbb{Q})^\times,$$

where $\mathbb{A}_{\mathrm{f}}$ is the ring of finite adeles and $\mathrm{End}(A, \imath)$ is the set of endomorphisms

---

2010 *Mathematics Subject Classification*: Primary 11G18; Secondary 11G10.
Key words and phrases: Galois action on points in Shimura curves, isogeny classes of abelian surfaces with QM.
Received 4 May 2017.
Published online 8 December 2017.

DOI: 10.4064/aa170504-20-9

commuting with every element of the image of $\imath$, and to give a concise description of the Galois action on $[P]$ through this representation ρ.

With this in mind, we have to first give a description of the $\bar{\mathbb{Q}}$-isogeny class $[P]$. In §3 we show that there is a bijective correspondence between the set $[P] \subset X_\Gamma$ and the double coset space

$$\Gamma\backslash(\mathcal{O}\otimes\mathbb{A}_{\mathrm{f}})^\times/(\mathrm{End}(A,\imath)\otimes\mathbb{Q})^\times$$

for some compact open subgroup Γ. This provides a purely algebraic description of the $\bar{\mathbb{Q}}$-isogeny class of $(A,\imath)$. Finally, in §4, we introduce the main result of this note (Theorem 4.2): The Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$ on the $\bar{\mathbb{Q}}$-isogeny class $[P] \subset X_\Gamma$ is given by the map

$$\begin{aligned}\mathrm{Gal}(\bar{\mathbb{Q}}/L) \times \Gamma\backslash(\mathcal{O}\otimes\mathbb{A}_{\mathrm{f}})^\times/(\mathrm{End}(A,\imath)\otimes\mathbb{Q})^\times \\ \to \Gamma\backslash(\mathcal{O}\otimes\mathbb{A}_{\mathrm{f}})^\times/(\mathrm{End}(A,\imath)\otimes\mathbb{Q})^\times, \quad (\sigma,[b]) \mapsto [b\rho(\sigma)],\end{aligned}$$

where $[b]$ denotes the class of $b \in (\mathcal{O}\otimes\mathbb{A}_{\mathrm{f}})^\times$ in the double coset space $\Gamma\backslash(\mathcal{O}\otimes\mathbb{A}_{\mathrm{f}})^\times/(\mathrm{End}(A,\imath)\otimes\mathbb{Q})^\times$.

The current interest in abelian L-varieties began, for $L=\mathbb{Q}$, when K. Ribet observed that non-CM absolutely simple factors of the modular Jacobians $J_1(N)$ are in fact abelian $\mathbb{Q}$-varieties [Rib]. Actually, by the proof of Serre's conjecture on representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ [Ser, 3.2.4], every so-called *building block* (a $\mathbb{Q}$-variety whose endomorphism algebra is a central division algebra over a totally real number field F with Schur index $t=1$ or $t=2$ and $t[F:\mathbb{Q}] = \dim A$) is an absolutely simple factor up to isogeny of a modular Jacobian $J_1(N)$.

In order to explore the relation of the representation ρ to modularity, one realizes that the construction of ρ given in §2 imitates the classical construction of the ℓ-adic Galois representation of an elliptic curve defined over L or the projective ℓ-adic Galois representation attached to an elliptic $\mathbb{Q}$-curve. In fact, in the trivial case of $A = E\times E$ and E one of such objects, ρ is the projectivization of the product over all ℓ of the corresponding classical ℓ-adic representations. Something analogous happens when $(A,\imath)$ is a building block, namely, a non-CM abelian $\mathbb{Q}$-surface with QM by an order in a division algebra. We know that A is an absolutely simple factor of an abelian variety A_{GL_2} defined over $\mathbb{Q}$ of GL_2-type. In §8 we show that ρ is the projectivization of the product over all ℓ of the ℓ-adic Galois representations attached to the abelian variety $A_{\mathrm{GL}_2}/\mathbb{Q}$.

Since for $L=\mathbb{Q}$ the projective representation ρ is related to well known classical ℓ-adic representations, we expect the norm of ρ to be characterized by the cyclotomic character. In §6 we introduce the dual of an abelian surface with QM and we describe the Weil pairing attached to it. Our explicit description of the Weil pairing allows us to compute the norm of ρ and to prove that it is indeed provided by the cyclotomic character (Theorem 6.5).

Apart from the interesting relations between ρ and certain abelian varieties of GL_2-type in the non-CM case, the CM case is of interest as well. One can prove that any CM abelian surface with QM (which we know is modular) is in fact an abelian $\mathbb{Q}$-surface with QM (Proposition 7.1). Moreover, the corresponding points in X_Γ classifying the $\bar{\mathbb{Q}}$-isogeny class of a CM abelian ($\mathbb{Q}$-)surface with QM are classical *Heegner points*. This implies, by Shimura's reciprocity law, that the Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set $[P] \subset X_\Gamma(\bar{\mathbb{Q}})$ of Heegner points is described via class field theory. Using this fact, we prove in Proposition 7.1 (as a direct consequence of Theorem 4.2) that, in the CM case, the projective representation ρ factors through the inverse of the Artin map. This provides a complete description of ρ in this case.

Notation. Let $\hat{\mathbb{Z}}$ denote the completion of $\mathbb{Z}$, hence $\hat{\mathbb{Z}} = \varprojlim(\mathbb{Z}/N\mathbb{Z})$. Let $\mathbb{A}_{\mathrm{f}}$ denote the ring of finite adeles, $\mathbb{A}_{\mathrm{f}} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$. Note that $\mathbb{Q}/\mathbb{Z} = \varinjlim \mathbb{Z}/N\mathbb{Z}$, therefore

$$\begin{aligned}\mathrm{End}(\mathbb{Q}/\mathbb{Z}) &= \mathrm{Hom}(\varinjlim(\mathbb{Z}/N\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathrm{Hom}(\mathbb{Z}/N\mathbb{Z}, \mathbb{Q}/\mathbb{Z})\\ &= \varprojlim \mathrm{Hom}(\mathbb{Z}/N\mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) = \varprojlim(\mathbb{Z}/N\mathbb{Z}) = \hat{\mathbb{Z}}.\end{aligned}$$

Write $\mathbb{A}_{\mathbb{Q}}$ for the ring of adeles of $\mathbb{Q}$.

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D, and let $\mathcal{O}$ be an Eichler order in B. We denote by Tr and Norm the reduced trace and the reduced norm in B respectively: for any $b \in B$, $\mathrm{Tr}(b) = b + \bar{b}$ and $\mathrm{Norm}(b) = b\bar{b}$, where $\bar{b}$ stands for the conjugate of the quaternion b.

Let G be the group scheme over $\mathbb{Z}$ such that $G(R) = (\mathcal{O}^{\mathrm{opp}} \otimes R)^\times$ for all rings R, where $\mathcal{O}^{\mathrm{opp}}$ is the opposite algebra to $\mathcal{O}$. Note that the group $G(\mathbb{A}_{\mathrm{f}})$ does not depend on the Eichler order $\mathcal{O}$ chosen since it is maximal locally for all but finitely many places. We will denote by $b \mapsto \bar{b}$ the conjugate of any element of $G(\mathbb{A}_{\mathrm{f}})$.

Write $\hat{\mathcal{O}} = \mathcal{O} \otimes \hat{\mathbb{Z}}$; then we have the isomorphism $\hat{\mathcal{O}} = \varprojlim(\mathcal{O}/N\mathcal{O})$. Moreover, $\varinjlim(\mathcal{O}/N\mathcal{O}) = B/\mathcal{O}$ as left $\mathcal{O}$-modules. Applying the above argument, we have $\mathrm{End}_{\mathcal{O}}(B/\mathcal{O}) = \hat{\mathcal{O}}^{\mathrm{opp}}$, where $\hat{\mathcal{O}}^{\mathrm{opp}}$ acts on $B/\mathcal{O}$ via right translation. Hence we can identify $G(\mathbb{A}_{\mathrm{f}}) = (\mathrm{End}_{\mathcal{O}}(B/\mathcal{O}) \otimes \mathbb{Q})^\times$.

Given an abelian variety A defined over $\mathbb{C}$, we will denote by $\mathrm{End}(A)$ the algebra of endomorphisms of A defined over $\mathbb{C}$. Note that if A admits a model over $\bar{\mathbb{Q}}$, then $\mathrm{End}(A) = \mathrm{End}_{\bar{\mathbb{Q}}}(A)$. Throughout this paper, we will denote $\mathrm{End}^0 := \mathrm{End} \otimes \mathbb{Q}$.

2. Abelian L-surfaces with quaternionic multiplication. Let F be a field. An *abelian surface with QM by $\mathcal{O}$* over F is a pair $(A, \imath)$ where A/F is an abelian surface and $\imath$ is an embedding $\mathcal{O} \hookrightarrow \mathrm{End}_F(A)$ optimal in the sense that $\imath(B) \cap \mathrm{End}_F(A) = \imath(\mathcal{O})$ (here we denote also by $\imath$ the scalar extension $\imath : B \hookrightarrow \mathrm{End}^0_F(A)$), and such that for every $\alpha \in \mathcal{O}$, the endomorphism $\imath(\alpha)$

is defined over F. If the order $\mathcal{O}$ is clear from the context, we will call $(A,\imath)$ just a *QM-abelian surface*. Let us consider the subring

$$\mathrm{End}(A,\imath)=\{\lambda\in\mathrm{End}_F(A):\lambda\circ\imath(o)=\imath(o)\circ\lambda \text{ for all } o\in\mathcal{O}\}.$$

If $(A,\imath)$ is defined over $\mathbb{C}$, then $\mathrm{End}^0(A,\imath)$ can be either $\mathbb{Q}$ or an imaginary quadratic field K; in this last situation we say that $(A,\imath)$ has *complex multiplication* (CM).

Definition 2.1. Two abelian surfaces $(A,\imath)$ and $(A',\imath')$ with QM by $\mathcal{O}$ defined over F are *isogenous* or $\mathcal{O}$*-isogenous* if there exist an isogeny $\mu: A'\to A$ defined over F satisfying $\mu\circ\imath'(\alpha)=\imath(\alpha)\circ\mu$ for all $\alpha\in\mathcal{O}$. We will then call $\mu:(A',\imath')\to(A,\imath)$ an $\mathcal{O}$*-isogeny.*

Assume that $(A,\imath)$ is defined over $\mathbb{C}$ and let $\hat{T}(A)=\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},A_{\mathrm{tor}})$ be its Tate module. Since $\hat{T}(A)$ is a $\hat{\mathbb{Z}}$-module of rank 4 with $\mathrm{End}(\hat{T}(A))=\mathrm{End}(A)\otimes\hat{\mathbb{Z}}$ and $\imath(\mathcal{O})=\imath(B)\cap\mathrm{End}(A)$, we conclude that $\hat{T}(A)\simeq\mathcal{O}\otimes\hat{\mathbb{Z}}=\hat{\mathcal{O}}$ and $A_{\mathrm{tor}}\simeq B/\mathcal{O}$ as $\mathcal{O}$-modules. This implies that, for any $\mathcal{O}$-isogeny $\mu:(A,\imath)\to(A',\imath')$, we have an isomorphism $\ker(\mu)\simeq I_\mu/\mathcal{O}$ as $\mathcal{O}$-modules, for some left fractional $\mathcal{O}$-ideal I_μ. We define $\deg(\mu)$ to be $\mathrm{Norm}(I_\mu)^{-1}$. With this definition, the multiplication-by-n $\mathcal{O}$-isogeny has degree $\deg(n)=n^2$, instead of n^4. Moreover, we have an inclusion $\ker(\mu)\subseteq\ker(\deg(\mu))$ provided by

$$\ker(\mu)=I_\mu/\mathcal{O}\subseteq\mathrm{Norm}(I_\mu)\mathcal{O}/\mathcal{O}=\ker(\deg(\mu)).$$

This implies that there exists an $\mathcal{O}$-isogeny $\hat{\mu}:(A',\imath')\to(A,\imath)$ such that $\mu\circ\hat{\mu}=\hat{\mu}\circ\mu=\deg(\mu)$ and $\deg(\mu)=\deg(\hat{\mu})$. We call that isogeny the *dual* $\mathcal{O}$*-isogeny* of μ.

Definition 2.2. Let $L/\mathbb{Q}$ be a number field. An *abelian L-surface* $(A,\imath)$ *with QM by* $\mathcal{O}$ is an abelian surface with QM by $\mathcal{O}$ over $\bar{\mathbb{Q}}$ such that, for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/L)$, there exists an isogeny $\mu_\sigma:{}^\sigma A\to A$ (defined over $\bar{\mathbb{Q}}$) such that $\mu_\sigma\circ{}^\sigma\imath(o)=\imath(o)\circ\mu_\sigma$ for all $o\in\mathcal{O}$. Equivalently, $({}^\sigma A,{}^\sigma\imath)$ and $(A,\imath)$ are $\mathcal{O}$-isogenous for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/L)$, where ${}^\sigma\imath$ is defined by ${}^\sigma\imath(o):={}^\sigma(\imath(o))$ for $o\in\mathcal{O}$.

Given an abelian L-surface $(A,\imath)$ with QM we shall construct a map

$$\rho_{(A,\imath,\varphi)}:\mathrm{Gal}(\bar{L}/L)\to G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)^\times$$

that describes the Galois action on the Tate module. In order to do this, we will fix an $\mathcal{O}$-module isomorphism $\varphi:A_{\mathrm{tor}}\to B/\mathcal{O}$. The following result shows that to choose such an isomorphism φ is equivalent to choosing an $\mathcal{O}$-module isomorphism between $\hat{T}(A)$ and $\hat{\mathcal{O}}$.

Lemma 2.3. *Given an $\mathcal{O}$-module isomorphism $\varphi: A_{\mathrm{tor}}\to B/\mathcal{O}$, the morphism*

$$\varphi^\dagger:\hat{T}(A)\to\hat{\mathcal{O}},\qquad f\mapsto\varprojlim_n n\cdot\varphi(f(1/n)),$$

where $n \cdot \varphi(f(1/n)) \in \mathcal{O}/n\mathcal{O}$, *is an* $\mathcal{O}$*-module isomorphism. Conversely, given an* $\mathcal{O}$*-module isomorphism* $\varphi^\dagger : \hat{T}(A) \to \hat{\mathcal{O}}$, *the morphism*

$$\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}, \qquad P \mapsto \frac{1}{\mathrm{ord}(P)}\varphi^\dagger(f) \in \hat{B}/\hat{\mathcal{O}} = B/\mathcal{O},$$

where f *is any element of* $\hat{T}(A)$ *such that* $f(1/\mathrm{ord}(P)) = P$, *is also an* $\mathcal{O}$*-module isomorphism. Moreover, this construction is inverse to the previous one.*

Proof. It is easy to check that one construction is inverse to the other. The other assertions follow from the fact that $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, B/\mathcal{O}) \simeq \hat{\mathcal{O}}$ by means of the isomorphism $f \mapsto \varprojlim_n n \cdot f(1/n)$. ∎

Since $(A, \imath)$ is defined over $\bar{\mathbb{Q}}$, we can fix a number field M and a model of A over M such that any endomorphism is defined over M. We denote such a model also by $(A, \imath)$ by abuse of notation. We fix a set $\mu = \{\mu_\sigma : ({}^\sigma A, {}^\sigma\imath) \to (A, \imath) : \sigma \in \mathrm{Gal}(M/L)\}$ of $\mathcal{O}$-isogenies and assume, after extending M if necessary, that every $\mathcal{O}$-isogeny in μ is also defined over M.

Consider the endomorphism on $B/\mathcal{O}$ given by

$$\varphi(P) \mapsto \varphi(\mu_\sigma({}^\sigma P)).$$

It commutes with the action of $\mathcal{O}$: indeed, for any $\alpha \in \mathcal{O}$,

$$\varphi(\mu_\sigma({}^\sigma(\imath(\alpha)P))) = \varphi(\mu_\sigma({}^\sigma\imath(\alpha)({}^\sigma P))) = \varphi(\imath(\alpha)\mu_\sigma({}^\sigma P)) = \alpha\varphi(\mu_\sigma({}^\sigma P)).$$

Hence it corresponds to an element of $\mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times$ since μ_σ has finite kernel. Once we identify $\mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times$ with $G(\mathbb{A}_{\mathrm{f}})$ acting on $B/\mathcal{O}$ on the right (provided that $\mathcal{O}$ acts on $B/\mathcal{O}$ on the left), we deduce that there exists $\rho^\mu_{(A,\imath,\varphi)}(\sigma) \in G(\mathbb{A}_{\mathrm{f}})$ such that

$$\varphi(\mu_\sigma({}^\sigma P)) = \varphi(P)\rho^\mu_{(A,\imath,\varphi)}(\sigma) \quad \text{for all } P \in A_{\mathrm{tor}}.$$

We have obtained a map

$$\rho^\mu_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \to G(\mathbb{A}_{\mathrm{f}}),$$

which may depend on the choice of the set μ of $\mathcal{O}$-isogenies. However, we can consider the quotient $G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)^\times$, where $\mathrm{End}^0(A,\imath)^\times$ is embedded in $G(\mathbb{A}_{\mathrm{f}})$ by means of the natural embedding

$$\varphi^* : \mathrm{End}^0(A,\imath)^\times \hookrightarrow G(\mathbb{A}_{\mathrm{f}}) = \mathrm{End}^0_{\mathcal{O}}(B/\mathcal{O})^\times, \qquad \varphi^*(\lambda) = \lambda^* = \varphi \circ \lambda \circ \varphi^{-1}.$$

The composition with the quotient map gives rise to a map

$$\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \to G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)^\times.$$

LEMMA 2.4. *The map* $\rho_{(A,\imath,\varphi)}$ *is independent of the choice of the set* μ *of* $\mathcal{O}$*-isogenies and of the choice of the model in the* $\bar{\mathbb{Q}}$*-isomorphism class of* $(A,\imath)$.

Proof. Let $\mu' = \{\mu'_\sigma : ({}^\sigma A, {}^\sigma\imath) \to (A,\imath) : \sigma \in \mathrm{Gal}(M/L)\}$ be another set of $\mathcal{O}$-isogenies. It can be defined over another Galois extension M' but we can extend both sets trivially. Then, for each $\sigma \in \mathrm{Gal}(M/L)$, $\lambda_\sigma := \frac{1}{\deg(\mu_\sigma)}\mu'_\sigma \circ \hat{\mu}_\sigma \in \mathrm{End}^0(A,\imath)^\times$. Hence

$$\begin{aligned}\varphi(P)\rho^{\mu'}_{(A,\imath,\varphi)}(\sigma) = \varphi(\mu'_\sigma({}^\sigma P)) = \varphi(\lambda_\sigma(\mu_\sigma({}^\sigma P))) = \varphi(\mu_\sigma({}^\sigma P))\lambda^*_\sigma \\ = \varphi(P)\rho^{\mu}_{(A,\imath,\varphi)}(\sigma)\lambda^*_\sigma.\end{aligned}$$

Thus $\rho^{\mu'}_{(A,\imath,\varphi)}(\sigma) = \rho^{\mu}_{(A,\imath,\varphi)}(\sigma)\lambda^*_\sigma$ and

$$\rho^{\mu'}_{(A,\imath,\varphi)}(\sigma)\mathrm{End}^0(A,\imath)^\times = \rho^{\mu}_{(A,\imath,\varphi)}(\sigma)\mathrm{End}^0(A,\imath)^\times,$$

which proves our first assertion.

Assume that we have another model $(A',\imath')$ over M'. Then we have an isomorphism $\eta : A' \to A$, defined over a bigger extension $N \supseteq M'M$, such that $\imath(\alpha)\circ\eta = \eta\circ\imath'(\alpha)$ for all $\alpha \in \mathcal{O}$. Since the isomorphism φ is chosen in the $\bar{\mathbb{Q}}$-isomorphism class of $(A,\imath)$, its realization φ' on $(A',\imath')$ satisfies $\varphi' = \varphi\circ\eta$. We compute

$$\varphi(\mu_\sigma({}^\sigma P)) = \varphi'(\eta^{-1}\circ\mu_\sigma\circ{}^\sigma\eta({}^\sigma(\eta^{-1}(P)))).$$

Since we have proved that $\rho_{(A',\imath',\varphi')}$ does not depend on the choice of the $\mathcal{O}$-isogenies, we can choose $\eta^{-1}\circ\mu_\sigma\circ{}^\sigma\eta$, obtaining the desired result $\rho_{(A',\imath',\varphi')} = \rho_{(A,\imath,\varphi)}$. ■

Note that in the non-CM case, $G(\mathbb{A}_\mathrm{f})/\mathrm{End}^0(A,\imath)^\times$ is a group. However, in the CM case, $\mathrm{End}(A,\imath)^0 = K$ is an imaginary quadratic field, hence $K^\times$ is not normal in $G(\mathbb{A}_\mathrm{f})$.

Given the embedding $\mathrm{End}^0(A,\imath)^\times \hookrightarrow G(\mathbb{A}_\mathrm{f})$ described above, let us denote by N_A the normalizer of $\mathrm{End}(A,\imath)$ in $G(\mathbb{A}_\mathrm{f})$. Note that $N_A = G(\mathbb{A}_\mathrm{f})$ in the non-CM case. Moreover, if $(A,\imath)$ has CM by the imaginary quadratic field K, then the ℓ component of N_A is $(N_A)_\ell = K_\ell^\times \cup jK_\ell^\times$ with $j^2 \in \mathbb{Q}^\times$ and $jk = \bar{k}j$ for all $k \in K_\ell^\times$. In any case, $N_A/\mathrm{End}^0(A,\imath)^\times$ is now a group.

Lemma 2.5. *The map $\rho_{(A,\imath,\varphi)}$ factors as*

$$\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \xrightarrow{\rho^N_{(A,\imath,\varphi)}} N_A/\mathrm{End}^0(A,\imath)^\times \hookrightarrow G(\mathbb{A}_\mathrm{f})/\mathrm{End}^0(A,\imath)^\times.$$

Moreover, $\rho^N_{(A,\imath,\varphi)}$ is a group homomorphism.

Proof. For the first statement, for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$ and $\lambda \in \mathrm{End}(A,\imath)$ we have

$$\rho^\mu_{(A,\imath,\varphi)}(\sigma)\lambda^*\rho^\mu_{(A,\imath,\varphi)}(\sigma)^{-1} \in \mathrm{End}(A,\imath)^0.$$

Indeed,

$$(\deg\mu_\sigma)\varphi(P)\rho^{\mu}_{(A,\imath,\varphi)}(\sigma)\lambda^*\rho^{\mu}_{(A,\imath,\varphi)}(\sigma)^{-1} = (\deg\mu_\sigma)\varphi(\lambda(\mu_\sigma({}^{\sigma}P)))\rho^{\mu}_{(A,\imath,\varphi)}(\sigma)^{-1}$$
$$= \varphi({}^{\sigma^{-1}}(\hat{\mu}_\sigma(\lambda(\mu_\sigma({}^{\sigma}P))))) = \varphi({}^{\sigma^{-1}}(\hat{\mu}_\sigma\lambda\mu_\sigma)(P)) = \varphi(P)({}^{\sigma^{-1}}(\hat{\mu}_\sigma\lambda\mu_\sigma))^*,$$

where clearly ${}^{\sigma^{-1}}(\hat{\mu}_\sigma\lambda\mu_\sigma) \in \mathrm{End}^0(A,\imath)$. Therefore $\rho^{\mu}_{(A,\imath,\varphi)}(\sigma) \in N_A$.

For the second statement, one checks that

$$\rho^{\mu}_{(A,\imath,\varphi)}(\sigma\tau)^{-1}\rho^{\mu}_{(A,\imath,\varphi)}(\sigma)\rho^{\mu}_{(A,\imath,\varphi)}(\tau)$$

acts on $\hat{T}A \otimes \mathbb{Q} := (\prod'_p T_pA)\otimes\mathbb{Q}$ in the same way as does

$$c_\mu(\sigma,\tau) = (1/\deg(\mu_{\sigma\tau}))\mu_\sigma{}^{\sigma}\mu_\tau\hat{\mu}_{\sigma\tau} \in (\mathrm{End}(A,\imath)\otimes_{\mathbb{Z}}\mathbb{Q})^\times = \mathrm{End}^0(A,\imath)^\times.$$

In particular, the quotient $\rho_{(A,\imath,\varphi)}(\sigma)$ is a group homomorphism. ∎

REMARK 2.6. Assume that the discriminant D equals 1, thus the quaternion algebra B is $\mathrm{M}_2(\mathbb{Q})$. An abelian surface with QM by $\mathcal{O} = \mathrm{M}_2(\mathbb{Z})$ is the product $A = E\times E$, where E is an elliptic curve. When E is defined over L (so clearly $A = E\times E$ is an abelian L-surface with QM), the representation $\rho_{(A,\imath,\varphi)}$ is just the quotient modulo $\mathrm{End}^0(A,\imath)^\times = \mathrm{End}^0(E)^\times$ of the classical action on the Tate module

$$\rho_E : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \to \mathrm{GL}_2(\hat{\mathbb{Z}}) = \prod_\ell \mathrm{GL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbb{A}_{\mathrm{f}}).$$

3. Shimura curves and isogeny classes. Assume that $\mathcal{O}_0$ is a maximal order in B, and let Γ be an open subgroup of $\hat{\mathcal{O}}_0^\times = G(\hat{\mathbb{Z}})$. We say that $\mathcal{O}_0$-module isomorphisms $\varphi,\varphi' : A_{\mathrm{tor}} \stackrel{\simeq}{\to} B/\mathcal{O}_0$ are *Γ-equivalent* if there exists $\gamma\in\Gamma$ such that $\varphi' = \varphi\gamma$. The *Shimura curve* X_Γ is the compactification of the coarse moduli space of triples $(A,\imath,\bar{\varphi})$, where $(A,\imath)$ is an abelian surface with QM by $\mathcal{O}_0$ and $\bar{\varphi}$ is the Γ-equivalence class of an $\mathcal{O}_0$-module isomorphism $\varphi : A_{\mathrm{tor}} \stackrel{\simeq}{\to} B/\mathcal{O}_0$. Such coarse moduli space is already compact unless $D = 1$. The curve X_Γ is defined over some number field L_Γ. If k is a field of characteristic zero, given a point $P\in X_\Gamma(\bar{k})$ corresponding to the isomorphism class of a triple $(A,\imath,\bar{\varphi})/\bar{k}$, its Galois conjugate ${}^{\sigma}P \in X_\Gamma(\bar{k})$, for any $\sigma\in\mathrm{Gal}(\bar{k}/k)$, corresponds to the isomorphism class of $({}^{\sigma}A, {}^{\sigma}\imath, {}^{\sigma}\bar{\varphi})$, where

$${}^{\sigma}\varphi : {}^{\sigma}A_{\mathrm{tor}} \stackrel{\simeq}{\to} B/\mathcal{O}_0, \qquad {}^{\sigma}\varphi({}^{\sigma}Q) = \varphi(Q).$$

Thus, a k-rational point P in X_Γ corresponds to the isomorphism class of a triple $(A,\imath,\bar{\varphi})/\bar{k}$ which is isomorphic to all its $\mathrm{Gal}(\bar{k}/k)$-conjugates.

The complex points of the Shimura curve are in correspondence with the double coset space

$$X_\Gamma(\mathbb{C}) = (\Gamma_\infty\Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q}) \cup \{\text{cusps}\}, \quad \Gamma_\infty = \left\{\begin{pmatrix} a & b\\ -b & a\end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})\right\},$$

where cusps only appear in case $D = 1$. The triple $(A_g, \imath_g, \bar{\varphi}_g)$ over $\mathbb{C}$ corresponding to $g = (g_\infty, g_\mathrm{f}) \in G(\mathbb{A})$ is $A_g := (B \otimes \mathbb{R})_{g_\infty}/I_{g_\mathrm{f}}$, where $I_{g_\mathrm{f}} = \hat{\mathcal{O}}_0 g_\mathrm{f} \cap B$ and $(B \otimes \mathbb{R})_{g_\infty} = \mathrm{M}_2(\mathbb{R})$ with complex structure

$$h_{g_\infty} : \mathbb{C} \to \mathrm{M}_2(\mathbb{R}), \quad i \mapsto g_\infty^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g_\infty;$$

the embedding $\imath_g : \mathcal{O}_0 \to \mathrm{End}(A_g)$ is given by $\imath_g(\alpha)(b \otimes z) = \alpha b \otimes z$; and $\bar{\varphi}_g$ is the Γ-equivalence class of $\varphi_g : (A_g)_\mathrm{tor} = B/I_{g_\mathrm{f}} \to B/\mathcal{O}_0$, $\varphi_g(b) = b g_\mathrm{f}^{-1}$. We compute that

$$\begin{aligned} &\mathrm{End}^0(A_g, \imath_g)^\times \\ &\quad = \{\gamma \in \mathrm{Aut}_B(B \otimes \mathbb{R}) : \gamma I_{g_\mathrm{f}} \otimes \mathbb{Q} = I_{g_\mathrm{f}} \otimes \mathbb{Q} \text{ and } \gamma h_{g_\infty} = h_{g_\infty}\gamma\} \\ &\quad = \{\gamma \in G(\mathbb{R}) : \gamma B = B \text{ and } \gamma h_{g_\infty}\gamma^{-1} = h_{g_\infty}\} \\ &\quad = \{\gamma \in G(\mathbb{Q}) : \gamma h_{g_\infty}\gamma^{-1} = h_{g_\infty}\} = \{\gamma \in G(\mathbb{Q}) : g_\infty \gamma g_\infty^{-1} \in \Gamma_\infty\}. \end{aligned}$$

Remark 3.1. In most of the literature, objects classified by the Shimura curve X_Γ are triples $(A, \imath, \bar{\psi})$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}_0$ as above and $\bar{\psi}$ is the Γ-equivalence class of an $\mathcal{O}_0$-module isomorphism $\psi : \hat{T}(A) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A_\mathrm{tor}) \xrightarrow{\simeq} \hat{\mathcal{O}}_0$. By Lemma 2.3, it is clear that this interpretation is equivalent to ours.

Remark 3.2. If $\Gamma = \Gamma_N = \ker(G(\hat{\mathbb{Z}}) \to G(\mathbb{Z}/N\mathbb{Z}))$, to give the Γ-equivalence class of an isomorphism $\varphi : A_\mathrm{tor} \to B/\mathcal{O}_0$ is equivalent to giving an isomorphism $\varphi_N : A[N] \to \mathcal{O}_0/N\mathcal{O}_0$, that is, a level-$N$-structure. This is the classical Shimura curve situation.

Let $\bar{k}$ be an algebraically closed field of characteristic 0. We say that triples $(A, \imath, \bar{\varphi})$ and $(A', \imath', \bar{\varphi}')$ over $\bar{k}$ are *isogenous* if $(A, \imath)$ and $(A', \imath')$ are isogenous.

Let $P \in X_\Gamma(\mathbb{C})$ be a point corresponding to $(A, \imath, \bar{\varphi})$. Denote by $[P]$ the $\mathbb{C}$-isogeny class of $(A, \imath, \bar{\varphi})/\mathbb{C}$ in X_Γ, that is, the set of points $Q \in X_\Gamma(\mathbb{C})$ parametrizing triples $(A', \imath', \bar{\varphi}')/\mathbb{C}$ where $(A', \imath')$ is isogenous to $(A, \imath)$.

Proposition 3.3. *Let $P = [g] = [g_\infty, 1] \in (\Gamma_\infty \Gamma \backslash G(\mathbb{A}))/G(\mathbb{Q}) \subseteq X_\Gamma(\mathbb{C})$. Then we have the following bijection:*

$$\psi_{g_\infty} : \Gamma \backslash G(\mathbb{A}_\mathrm{f})/\mathrm{End}^0(A_g, \imath_g)^\times \xrightarrow{\simeq} [P], \quad g_\mathrm{f} \mapsto [g_\infty, g_\mathrm{f}].$$

Proof. The non-CM case is described in [G-M, Lemma 1]; here we give a proof that works in any case. Recall that $(A_{g_\infty}, \imath_{g_\infty}, \bar{\varphi}_{g_\infty})$ is the triple corresponding to $P = [g_\infty, 1]$. For any $g_\mathrm{f} \in G(\mathbb{A}_\mathrm{f})$, there exists $n \in \mathbb{Z}$ such that $I_{g_\mathrm{f}} n \subseteq \mathcal{O}_0$. Therefore we have the isogeny

$$A_{g_\infty g_\mathrm{f}} = (B \otimes \mathbb{R})_{g_\infty}/I_{g_\mathrm{f}} \to (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}_0 = A_{g_\infty}, \quad b \mapsto nb,$$

which is clearly an $\mathcal{O}_0$-isogeny with respect to $\imath_{g_\infty}$ and $\imath_{g_\infty g_{\mathrm{f}}}$ since the inclusion $I_{g_{\mathrm{f}}}n \subseteq \mathcal{O}_0$ is a monomorphism of $\mathcal{O}_0$-modules. This implies that $[g_\infty, g_{\mathrm{f}}] \in [P]$ for all $g_{\mathrm{f}} \in G(\mathbb{A}_{\mathrm{f}})$.

Conversely, any $\mathcal{O}_0$-isogeny $(A_{g'_\infty g_{\mathrm{f}}}, \imath_{g'_\infty g_{\mathrm{f}}}) \to (A_{g_\infty}, \imath_{g_\infty})$ induces an equality of complex structures $(B\otimes\mathbb{R})_{g'_\infty} = (B\otimes\mathbb{R})_{g_\infty}$. This implies that $g'_\infty \in \Gamma_\infty g_\infty$. Therefore the corresponding point $[g'_\infty, g_{\mathrm{f}}]$ has a representantive of the form $[g_\infty, g'_{\mathrm{f}}]$ in the double coset space $(\Gamma_\infty \Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q})$.

We conclude that the map

$$\Gamma\backslash G(\mathbb{A}_{\mathrm{f}}) \to [P], \qquad g_{\mathrm{f}} \mapsto [g_\infty, g_{\mathrm{f}}],$$

is surjective. Finally, the result follows from the fact that $[g_\infty, g_{\mathrm{f}}] = [g_\infty, g'_{\mathrm{f}}]$ in $(\Gamma_\infty \Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q})$ if and only if there exists $\beta \in G(\mathbb{Q})$ such that $\Gamma g_{\mathrm{f}} = \Gamma g'_{\mathrm{f}}\beta$ and $g_\infty\beta \in \Gamma_\infty g_\infty$, hence $\beta \in \mathrm{End}^0(A_{g_\infty}, \imath_{g_\infty})^\times$. ∎

Remark 3.4. The above proposition asserts that the isogeny class $[P]$ corresponds to the fiber containing P of the natural map

$$X_\Gamma \supseteq (\Gamma_\infty \Gamma\backslash G(\mathbb{A}))/G(\mathbb{Q}) \to \Gamma_\infty\backslash G(\mathbb{R})/G(\mathbb{Q}), \qquad [g_\infty, g_{\mathrm{f}}] \mapsto [g_\infty].$$

4. Galois action on isogeny classes. Assume now that $(A,\imath)$ is an abelian L-surface with QM by $\mathcal{O}_0$, and let $(A,\imath,\bar\varphi)$ be a triple corresponding to the point $P \in X_\Gamma(\bar{\mathbb{Q}})$. First we show that any $(A',\imath')$ isogenous to $(A,\imath)$ is an abelian L-surface with QM.

Lemma 4.1. *Let $(A,\imath)$ be an abelian L-surface with QM and assume that $(A',\imath')$ over $\bar{\mathbb{Q}}$ is isogenous to $(A,\imath)$. Then $(A',\imath')$ is an abelian L-surface with QM.*

Proof. Let $\sigma \in \mathrm{Gal}(\bar L/L)$. Since $(A,\imath)$ is an abelian L-surface with QM, there exists an $\mathcal{O}_0$-isogeny $({}^\sigma A, {}^\sigma\imath) \xrightarrow{\mu_\sigma} (A,\imath)$. Fix an $\mathcal{O}_0$-isogeny $(A',\imath') \xrightarrow{\phi} (A,\imath)$ defined over $\bar{\mathbb{Q}}$ (such an isogeny exists since $(A,\imath)$ and $(A',\imath')$ are isogenous and both are defined over $\bar{\mathbb{Q}}$). Thus by conjugating ϕ by σ and composing with $\hat\phi\circ\mu_\sigma$, one obtains

$$({}^\sigma A', {}^\sigma\imath') \xrightarrow{{}^\sigma\phi} ({}^\sigma A, {}^\sigma\imath) \xrightarrow{\mu_\sigma} (A,\imath) \xrightarrow{\hat\phi} (A',\imath').$$

Hence $(A',\imath')/\bar{\mathbb{Q}}$ is an abelian L-surface with QM. ∎

Note that since P and so $(A,\imath)$ are defined over $\bar{\mathbb{Q}}$, the $\mathbb{C}$-isogeny class coincides with the $\bar{\mathbb{Q}}$-isogeny class $[P]$. Moreover, the above lemma implies that $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$ acts on $[P]$. Indeed, if $Q \in [P]$ corresponds to $(A',\imath',\bar\varphi')$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$, then ${}^\sigma Q$ parametrizes $({}^\sigma A', {}^\sigma\imath', {}^\sigma\bar\varphi')$. Since $(A',\imath')$ is an L-abelian surface with QM by the lemma, there exists an $\mathcal{O}_0$-isogeny $\mu'_\sigma : ({}^\sigma A', {}^\sigma\imath') \to (A',\imath')$. This implies $({}^\sigma A', {}^\sigma\imath')$ is isogenous to $(A,\imath)$, hence ${}^\sigma Q \in [P]$. The main theorem of this section relates this action to the map

$\rho_{(A,\imath,\varphi)}$ introduced in §2 by means of the characterization of $[P]$ given in Proposition 3.3.

THEOREM 4.2. *Assume that $P = [g_\infty, 1] \in X_\Gamma$ corresponds to a triple $(A, \imath, \bar{\varphi})$, where $(A, \imath)/\bar{\mathbb{Q}}$ is an abelian L-surface with QM, and $\bar{\varphi}$ is the Γ-equivalence class of the natural isomorphism*

$$\varphi : A_{\mathrm{tor}} = ((B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}_0)_{\mathrm{tor}} \to B/\mathcal{O}_0.$$

Then the map $\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{\mathbb{Q}}/L) \to G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)^\times$ constructed by means of φ satisfies

$${}^\sigma\psi_{g_\infty}([g_{\mathrm{f}}]) = \psi_{g_\infty}([g_{\mathrm{f}}\rho_{(A,\imath,\varphi)}(\sigma)]) \in [P]$$

for all $g_{\mathrm{f}} \in G(\mathbb{A}_{\mathrm{f}})$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$, where $[\cdot]$ denotes the class in the double coset space $\Gamma\backslash G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)^\times$.

REMARK 4.3. Note that, by Lemma 2.5, the image $\rho_{(A,\imath,\varphi)}(\sigma)$ lies in the commutator of $\mathrm{End}^0(A,\imath)$ in $G(\mathbb{A}_{\mathrm{f}})$. Thus the product $g_{\mathrm{f}}\rho_{(A,\imath,\varphi)}(\sigma)$ is well defined in $\Gamma\backslash G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)$.

Proof of Theorem 4.2. Recall that the abelian surface corresponding to $\psi_{g_\infty}([g_{\mathrm{f}}])$ is given by the complex torus $A_g = (B \otimes \mathbb{R})_{g_\infty}/I_{g_{\mathrm{f}}}$, where $I_{g_{\mathrm{f}}} = B \cap \hat{\mathcal{O}}_0 g_{\mathrm{f}}$ and $g = (g_\infty, g_{\mathrm{f}})$. Moreover, if we consider a representative of $[g_{\mathrm{f}}]$ such that $g_{\mathrm{f}}^{-1} \in \hat{\mathcal{O}}_0$, the $\mathcal{O}_0$-isogeny between $(A,\imath)$ and $(A_g, \imath_g)$ is given by

$$\phi_g : A = (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}_0 \to (B \otimes \mathbb{R})_{g_\infty}/I_{g_{\mathrm{f}}} = A_g, \quad b \mapsto b.$$

Also recall that a representative of $\bar{\varphi}_g$ is given by

$$\varphi_g : (A_g)_{\mathrm{tor}} = ((B \otimes \mathbb{R})_{g_\infty}/I_{g_{\mathrm{f}}})_{\mathrm{tor}} = B/I_{g_{\mathrm{f}}} \to B/\mathcal{O}_0, \quad b \mapsto b g_{\mathrm{f}}^{-1}.$$

Thus one checks that

$$\varphi_g \circ \phi_g = \varphi \cdot g_{\mathrm{f}}^{-1} : A_{\mathrm{tor}} \to B/\mathcal{O}_0. \tag{4.1}$$

For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$, the point ${}^\sigma\psi_{g_\infty}([g_{\mathrm{f}}])$ corresponds to the triple $({}^\sigma A_g, {}^\sigma\imath_g, {}^\sigma\bar{\varphi}_g)$. We have the following $\mathcal{O}_0$-isogenies:

$$({}^\sigma A_g, {}^\sigma\imath_g) \xleftarrow{{}^\sigma\phi_g} ({}^\sigma A, {}^\sigma\imath) \xrightarrow{\mu_\sigma} (A,\imath) \xrightarrow{\phi_g} (A_g, \imath_g),$$

thus $({}^\sigma A_g, {}^\sigma\imath_g)$ and $(A,\imath)$ are linked by the $\mathcal{O}_0$-isogeny ${}^\sigma\phi_g \circ \hat{\mu}_\sigma$. This implies that, as in the case of (4.1), we have a representative ${}^\sigma g_{\mathrm{f}} \in G(\mathbb{A}_{\mathrm{f}})$ of the double coset $\psi_{g_\infty}^{-1}({}^\sigma\psi_{g_\infty}([g_{\mathrm{f}}])) \in \Gamma\backslash G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A,\imath)$ satisfying ${}^\sigma\varphi_g \circ ({}^\sigma\phi_g \circ \hat{\mu}_\sigma) = \varphi \cdot {}^\sigma g_{\mathrm{f}}^{-1}$. Hence, for all $P \in A_{\mathrm{tor}}$,

$$\begin{aligned}\varphi(P)\rho_{(A,\imath,\varphi)}(\sigma)({}^\sigma g_{\mathrm{f}})^{-1} &= \varphi(\mu_\sigma({}^\sigma P))({}^\sigma g_{\mathrm{f}})^{-1} = \deg(\mu_\sigma)\,{}^\sigma\varphi_g({}^\sigma\phi_g({}^\sigma P)) \\ &= \deg(\mu_\sigma)\,{}^\sigma\varphi_g({}^\sigma(\phi_g(P))) = \deg(\mu_\sigma)\varphi_g(\phi_g(P)) = \deg(\mu_\sigma)\varphi(P)g_{\mathrm{f}}^{-1}.\end{aligned}$$

We conclude that $[g_{\mathrm{f}}\rho_{(A,\imath,\varphi)}(\sigma)] = [{}^\sigma g_{\mathrm{f}}] = \psi_{g_\infty}^{-1}({}^\sigma\psi_{g_\infty}([g_{\mathrm{f}}]))$. ∎

5. Change of moduli interpretation. In §2, we defined an abelian L-surface $(A, \imath)$ with QM by any Eichler order $\mathcal{O}$ and defined the corresponding representation $\rho_{(A,\imath,\varphi)}$ attached to a fixed $\mathcal{O}$-module isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$. However, we used a maximal order $\mathcal{O}_0$ to define the Shimura curve X_Γ and to describe its moduli interpretation as the space classifying triples $(A_0, \imath_0, \bar{\varphi}_0)$, where $(A_0, \imath_0)$ has QM by $\mathcal{O}_0$. In this section we shall change this moduli interpretation for some of these Shimura curves X_Γ in order to classify abelian surfaces with QM by $\mathcal{O}$.

Thus from now on, $\mathcal{O}$ will be an Eichler order in B of level N, and $\mathcal{O}_0$ a maximal order such that $\mathcal{O} \subseteq \mathcal{O}_0$. Fix the embedding $\lambda : \mathcal{O} \hookrightarrow \mathcal{O}_0$. Let Γ be now an open subgroup of $\hat{\mathcal{O}}^\times = (\mathcal{O} \otimes \hat{\mathbb{Z}})^\times$. Since $\hat{\mathcal{O}}^\times$ is an open subset of $G(\hat{\mathbb{Z}}) = \hat{\mathcal{O}}_0^\times$ by means of λ, the subgroup Γ is also an open subgroup of $G(\hat{\mathbb{Z}})$. Thus we can consider the Shimura curve X_Γ.

PROPOSITION 5.1. *We have an equivalence of moduli interpretations for the Shimura curve X_Γ. It classifies either*

(i) *the isomorphism classes of triples $(A_0, \imath_0, \bar{\varphi}_0)$, where $(A_0, \imath_0)$ is an abelian surface with QM by $\mathcal{O}_0$ and $\bar{\varphi}_0$ is the Γ-equivalence class of an $\mathcal{O}_0$-module isomorphism $\varphi_0 : (A_0)_{\mathrm{tor}} \to B/\mathcal{O}_0$, or*

(ii) *the isomorphism classes of triples $(A, \imath, \bar{\varphi})$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}$ and $\bar{\varphi}$ is the Γ-equivalence class of an $\mathcal{O}$-module isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$.*

In order to prove this proposition we will need the following lemma. Note that the embedding $\lambda : \mathcal{O} \hookrightarrow \mathcal{O}_0$ gives rise to a morphism $\lambda : B/\mathcal{O} \to B/\mathcal{O}_0$.

LEMMA 5.2. *There exists a one-to-one correspondence between triples $(A, \imath, \varphi)$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}$ and $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$ is an $\mathcal{O}$-module isomorphism, and triples $(A_0, \imath_0, \varphi_0)$, where $(A_0, \imath_0)$ is an abelian surface with QM by $\mathcal{O}_0$ and $\varphi_0 : (A_0)_{\mathrm{tor}} \to B/\mathcal{O}_0$ is an $\mathcal{O}_0$-module isomorphism. A triple $(A, \imath, \varphi)$ corresponds to $(A_0, \imath_0, \varphi_0)$ if there exists an isogeny $\phi : A \to A_0$ such that $\varphi_0 \circ \phi = \lambda \circ \varphi$ and $\phi \circ \imath(\alpha) = \imath_0(\lambda(\alpha)) \circ \phi$ for all $\alpha \in \mathcal{O}$.*

Proof. Given $(A, \imath, \varphi)$, consider the subgroup $C := \varphi^{-1}(\ker(B/\mathcal{O} \xrightarrow{\lambda} B/\mathcal{O}_0)) \subset A_{\mathrm{tor}}$. We can construct the abelian surface $A_0 = A/C$ and the corresponding isogeny $\phi : A \to A_0$. Since $\mathcal{O} \subseteq \mathcal{O}_0$, for all $\alpha \in \mathcal{O}$ we have $\alpha(\ker \lambda) \subseteq \ker \lambda$, hence $\imath(\alpha)C \subseteq C$ and the embedding $\imath$ gives rise to an embedding $\imath_0 : \mathcal{O} \hookrightarrow \mathrm{End}(A_0)$. The $\mathcal{O}$-module isomorphism φ gives rise to an $\mathcal{O}$-module isomorphism φ_0 that fits into the commutative diagram

$$\begin{array}{ccc} A_{\mathrm{tor}} & \xrightarrow{\varphi} & B/\mathcal{O} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\lambda} \\ (A_0)_{\mathrm{tor}} = A_{\mathrm{tor}}/C & \xrightarrow{\varphi_0} & B/\mathcal{O}_0 \end{array}$$

Hence $\varphi_0 \circ \phi = \lambda \circ \varphi$. Moreover, the fact that $(A_0)_{\mathrm{tor}} \simeq B/\mathcal{O}_0$ as $\mathcal{O}$-modules implies that $\imath_0$ can be extended to an embedding $\imath_0 : \mathcal{O}_0 \hookrightarrow \mathrm{End}(A_0)$. Thus $(A_0, \imath_0)$ has QM by $\mathcal{O}_0$. We have constructed the triple $(A_0, \imath_0, \varphi_0)$ corresponding to $(A, \imath, \varphi)$.

Finally, given $(A_0, \imath_0, \varphi_0)$, consider $\hat{C} := \varphi_0^{-1}(\ker(B/\mathcal{O}_0 \stackrel{\hat{\lambda}}{\to} B/\mathcal{O}))$, where $\hat{\lambda} : B/\mathcal{O}_0 \to B/\mathcal{O}$ is the well defined morphism $\hat{\lambda}(b + \mathcal{O}_0) = [\mathcal{O}_0 : \mathcal{O}]b + \mathcal{O}$. We define $A := A_0/\hat{C}$. Notice that $\imath_0(o)(\hat{C}) \subseteq \hat{C}$ for all $o \in \mathcal{O} \subset \mathcal{O}_0$. Hence $\imath_0$ gives rise to an embedding $\imath : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ and φ_0 provides an $\mathcal{O}$-module isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$. It is easy to check that this construction is inverse to the previous one, thus the result follows. ∎

Having the previous lemma, we can easily prove the above proposition:

Proof of Proposition 5.1. We know that the Shimura curve X_Γ classifies the isomorphism classes of triples $(A_0, \imath_0, \bar{\varphi}_0)$ as in (i). By the above lemma, given a representative φ_0 of the Γ-equivalence class $\bar{\varphi}_0$, there exists a triple $(A, \imath, \varphi)$, where φ is an $\mathcal{O}$-module isomorphism. It is clear that the Γ-equivalence class $\bar{\varphi}_0$ corresponds to the Γ-equivalence class $\bar{\varphi}$. ∎

Definition 5.3. A *triple with QM by* $\mathcal{O}$ is a triple $(A, \imath, \varphi)$, where $(A, \imath)$ is an abelian surface with QM by $\mathcal{O}$ and φ is an $\mathcal{O}$-module isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$. An *$L$-triple with QM by* $\mathcal{O}$ is a triple $(A, \imath, \varphi)$ with QM by $\mathcal{O}$ such that $(A, \imath)$ is an abelian L-surface with QM.

We denote the one-to-one correspondence of Lemma 5.2 by

$$\Lambda_{\mathcal{O}}^{\mathcal{O}_0} : \{\text{Triples with QM by } \mathcal{O}\}/\!\simeq \;\to \{\text{Triples with QM by } \mathcal{O}_0\}/\!\simeq.$$

Note that, given an L-triple $(A, \imath, \varphi)$ with QM by $\mathcal{O}$, one can construct the projective representation

$$\rho_{(A,\imath,\varphi)} : \mathrm{Gal}(\bar{L}/L) \to G(\mathbb{A}_{\mathrm{f}})/\mathrm{End}^0(A, \imath)^\times.$$

The following result relates the representations attached to triples associated by the correspondence $\Lambda_{\mathcal{O}}^{\mathcal{O}_0}$.

Lemma 5.4. *Let $(A, \imath, \varphi)$ be an L-triple with QM by $\mathcal{O}$ and assume that $\Lambda_{\mathcal{O}}^{\mathcal{O}_0}(A, \imath, \varphi) = (A_0, \imath_0, \varphi_0)$. Then $(A_0, \imath_0, \varphi_0)$ is an L-triple with QM by $\mathcal{O}_0$ and*

$$\rho_{(A,\imath,\varphi)} = \rho_{(A_0,\imath_0,\varphi_0)},$$

when we identify $G(\mathbb{A}_{\mathrm{f}}) = \mathrm{End}_{\mathcal{O}}^0(B/\mathcal{O})^\times = \mathrm{End}_{\mathcal{O}_0}^0(B/\mathcal{O}_0)^\times$ by means of λ.

Proof. We know that there exists an isogeny $\phi : A \to A_0$ such that $\varphi_0 \circ \phi = \lambda \circ \varphi$ and $\phi \circ \imath(\alpha) = \imath_0(\lambda(\alpha)) \circ \phi$ for all $\alpha \in \mathcal{O}$. Since $(A, \imath)$ is an abelian L-surface with QM, there exists a set $\mu = \{\mu_\sigma : ({}^\sigma A, {}^\sigma\imath) \to (A, \imath) : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)\}$ of $\mathcal{O}$-isogenies. The composition

$$\mu_\sigma^0 : {}^\sigma A_0 \xrightarrow{{}^\sigma\hat{\phi}} {}^\sigma A \xrightarrow{\mu_\sigma} A \xrightarrow{\phi} A_0$$

satisfies

$$\begin{aligned}\mu_\sigma^0 \circ {}^\sigma\imath_0(\lambda(o)) &= \phi \circ \mu_\sigma \circ {}^\sigma\hat{\phi} \circ {}^\sigma\imath_0(\lambda(o)) = \phi \circ \mu_\sigma \circ {}^\sigma\imath(o) \circ {}^\sigma\hat{\phi} \\ &= \phi \circ \imath(o) \circ \mu_\sigma \circ {}^\sigma\hat{\phi} = \imath_0(\lambda(o)) \circ \mu_\sigma^0 \qquad \text{for all } o \in \mathcal{O}.\end{aligned}$$

This implies that $(A_0, \imath_0)$ is an abelian L-surface with QM.

Moreover, for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$ and $P \in (A_0)_{\mathrm{tor}}$, we have

$$\begin{aligned}\varphi_0(\mu_\sigma^0({}^\sigma P)) &= \varphi_0(\phi \circ \mu_\sigma \circ {}^\sigma\hat{\phi}({}^\sigma P)) = \lambda(\varphi(\mu_\sigma({}^\sigma(\hat{\phi}(P))))) \\ &= \lambda(\varphi(\hat{\phi}(P)))\rho^\mu_{(A,\imath,\varphi)}(\sigma) = \deg(\phi)\varphi_0(P)\rho^\mu_{(A,\imath,\varphi)}(\sigma),\end{aligned}$$

since $\lambda(bg) = \lambda(b)g$ for all $b \in B/\mathcal{O}$ and $g \in \hat{\mathcal{O}}^{\mathrm{opp}} = \mathrm{End}_{\mathcal{O}}(B/\mathcal{O}) \subseteq \mathrm{End}_{\mathcal{O}_0}(B/\mathcal{O}_0)$. This implies that $\rho_{(A,\imath,\varphi)}(\sigma) = \rho_{(A_0,\imath_0,\varphi_0)}(\sigma)$. ∎

REMARK 5.5. As a consequence of this lemma, Theorem 4.2 also applies if we replace the maximal order $\mathcal{O}_0$ by a not necessarily maximal Eichler order $\mathcal{O}$, considering the moduli interpretation (ii) of Proposition 5.1.

6. Duality. In this section we describe the dual of an abelian surface with quaternionic multiplication and its associated Weil pairing.

Let $(A, \imath)/\mathbb{C}$ be an abelian surface with QM by $\mathcal{O}$. Denote by $A^\vee/\mathbb{C}$ its dual abelian surface. Denote by $\langle\,,\rangle$ the Weil pairing

$$\langle\,,\rangle : A_{\mathrm{tor}} \times A^\vee_{\mathrm{tor}} \to \{\zeta_n : n \in \mathbb{N}\},$$

where $\{\zeta_n : n \in \mathbb{N}\}$ is the group of roots of unity. This group is isomorphic to $\mathbb{Q}/\mathbb{Z}$ by means of the isomorphism

$$\psi : \{\zeta_n : n \in \mathbb{N}\} \to \mathbb{Q}/\mathbb{Z}, \quad e^{2\pi i m/n} \mapsto m/n.$$

Given $\mathcal{O}$, we have the two-sided ideal

$$\mathcal{O}^\# = \{b \in B : \mathrm{Tr}(b\mathcal{O}) \subseteq \mathbb{Z}\},$$

where Tr denotes the reduced trace. Given any left $\mathcal{O}$-ideal I, we denote by $\mathrm{Norm}(I)$ its reduced norm.

Given an isogeny $\mu : A \to B$ between abelian surfaces, we will denote by $\mu^\vee : B^\vee \to A^\vee$ the induced isogeny between their dual abelian surfaces.

PROPOSITION 6.1. *The dual abelian surface $A^\vee$ admits a quaternionic multiplication $\imath^\vee$ such that $(A^\vee, \imath^\vee)$ is isogenous to $(A, \imath)$ by means of an $\mathcal{O}$-isogeny $\varepsilon : (A, \imath) \to (A^\vee, \imath^\vee)$ of degree* $\mathrm{Norm}(\mathcal{O}^\#)^{-1}$. *The isogeny $\varepsilon^\vee$ satisfies $\varepsilon^\vee = -\varepsilon$ and the quaternionic multiplication $\imath^\vee$ satisfies $\imath^\vee(\alpha) = \imath(\bar{\alpha})^\vee$ for* *all $\alpha \in \mathcal{O}$. Moreover, for any $\mathcal{O}$-module isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$, there exists $u \in \hat{\mathbb{Z}}^\times$ such that the Weil pairing satisfies*

$$\psi(\langle P, \varepsilon(Q)\rangle) = u\,\mathrm{Tr}(\varphi(P)\overline{\varphi(Q)}) \quad \textit{for any } P, Q \in A_{\mathrm{tor}}.$$

Proof. We have seen that, as a complex torus, $A = (B \otimes \mathbb{R})_h/I$ for some left $\mathcal{O}$-ideal I and some complex structure $h : \mathbb{C} \to B \otimes \mathbb{R}$. Its dual complex torus corresponds to $(B \otimes \mathbb{R})^\vee_{h^\vee}/I^\vee$, where

$$\begin{aligned}(B \otimes \mathbb{R})^\vee &:= \{f : B \otimes \mathbb{R} \to \mathbb{C} : \mathbb{R}\text{-linear},\ f(h(i)v) = -if(v)\},\\ I^\vee &:= \{f \in (B \otimes \mathbb{R})^\vee : \mathrm{Im}\, f(I) \subseteq \mathbb{Z}\},\end{aligned}$$

and the complex structure $h^\vee : \mathbb{C}^\times \to \mathrm{Aut}_{\mathbb{R}}((B \otimes \mathbb{R})^\vee)$ is given by $h^\vee(z)f = zf$. The non-degenerate pairing $B \times B \to \mathbb{Q}$, $(b_1, b_2) \mapsto \mathrm{Tr}(b_2\bar{b}_1)$, provides the isomorphism

$$(B \otimes \mathbb{R})_h \to (B \otimes \mathbb{R})^\vee_{h^\vee}, \quad b \mapsto f_b(b') = i\,\mathrm{Tr}(b\bar{b}') + \mathrm{Tr}(b\bar{b}'h(i)) \in \mathbb{C},$$

since $\mathrm{Tr}(h(i)b_2\bar{b}_1\overline{h(i)}) = \mathrm{Tr}(b_2\bar{b}_1)$. Hence

$$A^\vee \simeq (B \otimes \mathbb{R})^\vee_{h^\vee}/I^\vee \simeq (B \otimes \mathbb{R})_h/I^\#, \quad I^\# = \{b \in B : \mathrm{Tr}(b\bar{I}) \subseteq \mathbb{Z}\}.$$

Since $I^\#$ is a left $\mathcal{O}$-ideal, we deduce that $A^\vee$ admits a quaternionic multiplication $\imath^\vee$, and $(A^\vee, \imath^\vee)$ is in the $\mathcal{O}$-isogeny class of $(A, \imath)$. An $\mathcal{O}$-isogeny $\varepsilon : A \to A^\vee$ is given by the inclusion $\frac{1}{\mathrm{Norm}(I)} I \subseteq I^\#$. Its kernel corresponds to the quotient $I^\#/\big(\frac{1}{\mathrm{Norm}(I)} I\big) \simeq \mathcal{O}^\#/\mathcal{O}$, hence $\deg(\varepsilon) = \mathrm{Norm}(\mathcal{O}^\#)^{-1}$. The isogeny ε is provided by the pairing

$$I \times I \to \mathbb{Z}, \quad (i_1, i_2) \mapsto \frac{1}{\mathrm{Norm}(I)}\,\mathrm{Tr}(i_2\bar{i}_1).$$

Since the pairing is symmetric, we obtain $\varepsilon^\vee = -\varepsilon$. Moreover, it follows directly from the above description that $\imath^\vee(\alpha) = \imath(\bar{\alpha})^\vee$.

Note that composing ε with any element $\alpha \in \mathcal{O}$ with trace zero we obtain a symmetric isogeny: indeed, $(\varepsilon \circ \imath(\alpha))^\vee = -\imath^\vee(\bar{\alpha}) \circ \varepsilon = \varepsilon \circ \imath(\alpha)$. This symmetric isogeny is given by a line bundle. Applying [Mum, Theorem 1, §IV.24] to such line bundles, we deduce that $\psi(\langle P, \varepsilon(Q)\rangle) = -\mathrm{Norm}(I)^{-1}\,\mathrm{Tr}(b\bar{b}')$ for any $P, Q \in A_{\mathrm{tor}}$ corresponding to $b, b' \in B/I$ respectively. Any isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$ is given by an element $g_{\mathrm{f}} \in G(\mathbb{A}_{\mathrm{f}})$ such that $\hat{\mathcal{O}}g_{\mathrm{f}} \cap B = I$, and the composition

$$\varphi : A_{\mathrm{tor}} \simeq B/I \xrightarrow{\simeq} B/\mathcal{O}, \quad b \mapsto bg_{\mathrm{f}}^{-1}.$$

We compute

$$\begin{aligned}\psi(\langle P, \varepsilon(Q)\rangle) &= \frac{-1}{\mathrm{Norm}(I)}\,\mathrm{Tr}(b\bar{b}') = \frac{-1}{\mathrm{Norm}(I)}\,\mathrm{Tr}(\varphi(P)g_{\mathrm{f}}\overline{g_{\mathrm{f}}}\overline{\varphi(Q)})\\ &= u\,\mathrm{Tr}(\varphi(P)\overline{\varphi(Q)}),\end{aligned}$$

where $u = -\mathrm{Norm}(g_{\mathrm{f}})/\mathrm{Norm}(I) \in \hat{\mathbb{Z}}^\times$. ∎

REMARK 6.2. We have seen in the above proof that, given an element $\mu \in \mathcal{O}$ with trace zero, the composition $\varepsilon \circ \imath(\mu)$ is a symmetric isogeny. The corresponding anti-symmetric pairing on the lattices is given by the natural pairing

$$E_\mu : I \times I \to \mathbb{Z}, \quad (i_1, i_2) \mapsto \frac{1}{\mathrm{Norm}(I)} \mathrm{Tr}(i_2 \mu \bar{i}_1).$$

The isogeny $\varepsilon \circ \imath(\mu)$ is a polarization if and only if the pairing E_μ defines a Riemann form. A simple computation shows that this is equivalent to the condition $\mu^2 < 0$.

REMARK 6.3. If $\mu : (A_0, \iota_0) \to (A, \iota)$ is an $\mathcal{O}$-isogeny, then

$$\mu^\vee \circ \iota^\vee(\alpha) = \mu^\vee \circ \iota(\bar{\alpha})^\vee = (\iota(\bar{\alpha}) \circ \mu)^\vee = (\mu \circ \iota_0(\bar{\alpha}))^\vee = \iota_0(\bar{\alpha})^\vee \circ \mu^\vee = \iota_0^\vee(\alpha) \circ \mu^\vee,$$

hence $\mu^\vee : (A^\vee, \iota^\vee) \to (A_0^\vee, \iota_0^\vee)$ is an $\mathcal{O}$-isogeny.

6.1. Weil pairing on abelian L-surfaces with QM. Let (A, ι) be an abelian L-surface with QM. Fix a model $(A, \imath)$ over some number field M, a set $\mu = \{\mu_\sigma : ({}^\sigma A, {}^\sigma \iota) \to (A, \iota)\}$ of $\mathcal{O}$-isogenies defined over M, and an $\mathcal{O}$-isomorphism $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$. In §2 we constructed the endomorphisms $\rho^\mu_{(A,\iota,\varphi)}(\sigma) \in G(\mathbb{A}_{\mathrm{f}})$, where $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L)$, satisfying $\varphi(\mu_\sigma({}^\sigma P)) = \varphi(P)\rho^\mu_{(A,\iota,\varphi)}(\sigma)$ for all $P \in A_{\mathrm{tor}}$.

Since $(A^\vee, \imath^\vee)$ is $\mathcal{O}$-isogenous to $(A, \imath)$, we know that $(A^\vee, \imath^\vee)$ is also an abelian L-surface with QM. Moreover, by the functorial description of the dual abelian surface, $(A^\vee, \imath^\vee)$ admits a model defined over M satisfying $({}^\sigma A)^\vee = {}^\sigma(A^\vee)$ and $\langle {}^\sigma P, {}^\sigma Q \rangle = {}^\sigma \langle P, Q \rangle$ for all $P \in A_{\mathrm{tor}}$, $Q \in A^\vee_{\mathrm{tor}}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

PROPOSITION 6.4. *The map $\varepsilon : (A, \imath) \to (A^\vee, \imath^\vee)$ is defined over M, and*

$$\mu_\sigma^\vee \circ \varepsilon \circ \mu_\sigma = \deg(\mu_\sigma)({}^\sigma \varepsilon) \quad \textit{for all } \sigma \in \mathrm{Gal}(M/L).$$

Proof. First we consider a Galois extension M'/M big enough such that ε is defined over M', and we extend the set of $\mathcal{O}$-isogenies in the natural way: for $\sigma \in \mathrm{Gal}(M'/L)$, we define $\mu_\sigma := \mu_{\pi(\sigma)}$, where $\pi : \mathrm{Gal}(M'/L) \to \mathrm{Gal}(M/L)$ is the natural projection. We consider the Weil restriction of scalars $X = \mathrm{Res}_L(A)$. It is an abelian variety defined over L and isomorphic over $\bar{\mathbb{Q}}$ to $X \simeq_{\bar{\mathbb{Q}}} \prod_{\sigma \in \mathrm{Gal}(M'/L)} {}^\sigma A$. For any $\tau \in \mathrm{Gal}(M'/L)$, the isogeny μ_τ gives rise to an endomorphism $\lambda_\tau \in \mathrm{End}(X)$ defined by

$$\lambda_\tau : \prod_\sigma {}^\sigma A \to \prod_\sigma {}^\sigma A, \quad (P_\sigma)_\sigma \mapsto ({}^\sigma \mu_\tau(P_{\sigma\tau}))_\sigma$$

By Remark 6.2, any $\mu \in \mathcal{O}$ satisfying $\mu^2 \in \mathbb{Q}^{<0}$ provides the polarization

$$\underline{\varepsilon \circ \mu} := ({}^\sigma \varepsilon \circ {}^\sigma \imath(\mu)) : X = \prod_\sigma {}^\sigma A \to \prod_\sigma {}^\sigma A^\vee = X^\vee.$$

Hence the Rosati involution of λ_τ with respect to $\underline{\varepsilon\circ\mu}$ is given by

$$\begin{aligned}\lambda_\tau^\dagger(P_\sigma)_\sigma &= \underline{\varepsilon\circ\mu}^{-1}\circ\lambda_\tau^\vee\circ\underline{\varepsilon\circ\mu}(P_\sigma)_\sigma = \underline{\varepsilon\circ\mu}^{-1}\circ\lambda_\tau^\vee({}^\sigma\varepsilon({}^\sigma\imath(\mu)(P_\sigma)))_\sigma\\ &= \underline{\varepsilon\circ\mu}^{-1}({}^\sigma\mu_\tau^\vee({}^\sigma\varepsilon({}^\sigma\imath(\mu)(P_\sigma)))_{\sigma\tau}) = (({}^{\sigma\tau}\varepsilon)^{-1}({}^\sigma\mu_\tau^\vee({}^\sigma\varepsilon(P_\sigma))))_{\sigma\tau}\\ &= ({}^\sigma({}^\tau\varepsilon^{-1}\circ\mu_\tau^\vee\circ\varepsilon)(P_\sigma))_{\sigma\tau},\end{aligned}$$

since the action of $\mathcal{O}$ commutes with μ_τ and ε. Thus $\lambda_\tau\circ\lambda_\tau^\dagger$ acts as the diagonal matrix with entries ${}^\sigma(\mu_\tau\circ{}^\tau\varepsilon^{-1}\circ\mu_\tau^\vee\circ\varepsilon)\in\mathrm{End}(A^\sigma,\imath^\sigma)$. Due to the fact that the Rosati involution is positive, we have $\mu_\tau\circ{}^\tau\varepsilon^{-1}\circ\mu_\tau^\vee\circ\varepsilon\in\mathbb{Q}^{>0}$. Since $\deg({}^\tau\varepsilon^{-1}\circ\mu_\tau^\vee\circ\varepsilon)=\deg(\mu_\tau)$, we conclude that ${}^\tau\varepsilon^{-1}\circ\mu_\tau^\vee\circ\varepsilon=\hat\mu_\tau$, hence $\mu_\sigma^\vee\circ\varepsilon\circ\mu_\sigma=\deg(\mu_\sigma)({}^\sigma\varepsilon)$. Since this equality is also true for any $\sigma\in\mathrm{Gal}(M'/M)$ where $\mu_\sigma=\mathrm{Id}$, we deduce that ε is defined over M. ∎

Let $\chi:\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to\hat{\mathbb{Z}}$ be the cyclotomic character that provides the Galois action on $\{\zeta_n : n\in\mathbb{N}\}\simeq\mathbb{Q}/\mathbb{Z}$, namely $\psi({}^\sigma\zeta_n)=\chi(\sigma)\psi(\zeta_n)$ for any root of unity ζ_n and $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Theorem 6.5. *We have*

$$\mathrm{Norm}(\rho^\mu_{(A,\imath,\varphi)}(\sigma))=\deg(\mu_\sigma)\chi(\sigma)\quad \textit{for all } \sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/L).$$

Proof. We have seen that there exists $u\in\mathbb{A}_\mathrm{f}^\times$ such that $\psi(\langle P,\varepsilon(Q)\rangle)=u\,\mathrm{Tr}(\varphi(P)\overline{\varphi(Q)})$. Thus we compute, for all $P,Q\in A_{\mathrm{tor}}$,

$$\begin{aligned}\deg(\mu_\sigma)\chi(\sigma)\psi(\langle P,\varepsilon(Q)\rangle)&=\deg(\mu_\sigma)\psi({}^\sigma\langle P,\varepsilon(Q)\rangle)=\psi(\langle{}^\sigma P,\deg(\mu_\sigma){}^\sigma\varepsilon({}^\sigma Q)\rangle)\\ &=\psi(\langle{}^\sigma P,\mu_\sigma^\vee\circ\varepsilon\circ\mu_\sigma({}^\sigma Q)\rangle)=\psi(\langle\mu_\sigma({}^\sigma P),\varepsilon(\mu_\sigma({}^\sigma Q))\rangle)\\ &=u\,\mathrm{Tr}(\varphi(\mu_\sigma({}^\sigma P))\overline{\varphi(\mu_\sigma({}^\sigma Q))})\\ &=u\,\mathrm{Tr}(\varphi(Q)\rho^\mu_{(A,\imath,\varphi)}(\sigma)\overline{\rho^\mu_{(A,\imath,\varphi)}(\sigma)}\,\overline{\varphi(P)})\\ &=\mathrm{Norm}(\rho^\mu_{(A,\iota,\varphi)}(\sigma))u\,\mathrm{Tr}(\varphi(P)\overline{\varphi(Q)})=\mathrm{Norm}(\rho^\mu_{(A,\iota,\varphi)}(\sigma))\psi(\langle P,\varepsilon(Q)\rangle),\end{aligned}$$

and the result follows. ∎

Corollary 6.6. *The composition*

$$\mathrm{Gal}(\bar{\mathbb{Q}}/L)\xrightarrow{\rho_{(A,\imath,\varphi)}}G(\mathbb{A}_\mathrm{f})/\mathrm{End}^0(A,\imath)^\times\xrightarrow{\mathrm{Norm}}\mathbb{A}_\mathrm{f}^\times/\mathbb{Q}^{>0}\simeq\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times\mathbb{R}^{>0}$$

factors through $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/(L\cap\mathbb{Q}^{\mathrm{ab}}))$ *and it is given by the restriction of the inverse of the Artin map.*

7. Complex multiplication abelian K-surfaces with QM. In this section we shall deal with the complex multiplication (CM) case. Hence, only for this section, we assume that the abelian surface $(A,\imath)$ with QM by $\mathcal{O}$ also has CM by K, which means that $\mathrm{End}^0(A,\imath)=K$ is an imaginary quadratic field. The following result describes the projective representation $\rho^N_{(A,\imath,\varphi)}$ (and therefore $\rho_{(A,\imath,\varphi)}$) in the CM case:

Proposition 7.1. *Assume that* $\mathrm{End}^0(A,\imath) = K$ *is an imaginary quadratic field, and let* $\mathbb{A}_{K,\mathrm{f}}$ *be the ring of finite adeles of* K. *Then:*

(i) *Any abelian surface* $(A',\imath')$ *with QM by* $\mathcal{O}$ *and CM by* K *is isogenous to* $(A,\imath)$.

(ii) *We can choose a representative of the isomorphism class of* $(A,\imath)$ *defined over* $\bar{\mathbb{Q}}$. *Moreover,* $(A,\imath)$ *is an abelian* $\mathbb{Q}$*-surface with QM.*

(iii) *Given any isomorphism* $\varphi : A_{\mathrm{tor}} \to B/\mathcal{O}$, *the restriction to* $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ *of the corresponding projective representation* $\rho^N_{(A,\imath,\varphi)}$ *factors through the inverse of the Artin map* $\mathrm{Art} : \mathbb{A}^\times_{K,\mathrm{f}}/K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$. *That is, we have the commutative diagram*

$$\begin{array}{ccccc} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & & \xrightarrow{\quad\rho^N_{(A,\imath,\varphi)}\quad} & & N_A/K^\times \\ \uparrow & & & & \uparrow \\ \mathrm{Gal}(\bar{\mathbb{Q}}/K) & \twoheadrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K) & \xrightarrow{\mathrm{Art}^{-1}} & \mathbb{A}^\times_{K,\mathrm{f}}/K^\times \end{array}$$

(iv) *The representation* $\rho^N_{(A,\imath,\varphi)}$ *factors through* $\mathrm{Gal}(K^{\mathrm{ab}}/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ *sending the complex conjugation* $\sigma_{\mathrm{c}} = [1,\sigma_0]$, *where* $\langle\sigma_0\rangle = \mathrm{Gal}(K/\mathbb{Q})$, *to the class* $jK^\times$, *where* $j \in N_A \cap B^\times$ *is any element satisfying* $j^2 \in \mathbb{Q}^\times$ *and* $jk = \bar{k}j$ *for all* $k \in \mathbb{A}^\times_{K,\mathrm{f}}$.

Proof. Consider first the open subgroup $\Gamma = \hat{\mathcal{O}}^\times$. We showed in §5 that the isomorphism classes of abelian surfaces with QM by $\mathcal{O}$ over $\mathbb{C}$ are classified by the non-cuspidal points of $X_{\hat{\mathcal{O}}^\times}$. Assume that $(A,\imath)$ with CM by K corresponds to $[g_\infty, g_{\mathrm{f}}] \in (\Gamma_\infty\hat{\mathcal{O}}^\times\backslash G(\mathbb{A}))/G(\mathbb{Q}) \subseteq X_{\hat{\mathcal{O}}^\times}(\mathbb{C})$. The natural composition

$$K^\times = \mathrm{End}^0(A,\imath)^\times = \{\gamma \in G(\mathbb{Q}) : g_\infty\gamma g_\infty^{-1} \in \Gamma_\infty\} \hookrightarrow G(\mathbb{Q}) = B^\times$$

provides an embedding $\psi_{(A,\imath)} : K \hookrightarrow B$. Given another $(A',\imath')$ with CM by K corresponding to $[g'_\infty, g'_{\mathrm{f}}]$, we obtain an analogous embedding $\psi_{(A',\imath')} : K \hookrightarrow B$. By Skolem–Noether, there exists $g \in G(\mathbb{Q})$ such that $\psi_{(A',\imath')} = g^{-1}\psi_{(A,\imath)}g$. This implies that $g'_\infty = g_\infty g$, hence $[g'_\infty, g'_{\mathrm{f}}] = [g_\infty, g'_{\mathrm{f}}g^{-1}]$. We conclude that $(A',\imath')$ is isogenous to $(A,\imath)$ by Proposition 3.3, thus (i) follows.

Note that the class of g_∞ in $\Gamma_\infty\backslash \mathrm{GL}_2(\mathbb{R}) \simeq \mathbb{C}\setminus\mathbb{R}$ corresponds to one of the two elements in $\mathbb{C}\setminus\mathbb{R}$ fixed by $\psi_{(A,\imath)}(K)$. This implies, by Shimura's Reciprocity Law [Shi], that all points in $[P]$ are defined over K^{ab}. In particular $(A,\imath)$ is defined over $\bar{\mathbb{Q}}$. Since $({}^\sigma A, {}^\sigma\imath)$ has CM by K for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we apply (i) to deduce that $(A,\imath)$ and $({}^\sigma A, {}^\sigma\imath)$ are $\mathcal{O}$-isogenous, which proves (ii).

Additionally, Shimura's Reciprocity Law describes the Galois action on the isogeny class of $(A,\imath)$. More precisely, for any open subgroup Γ, let $[P']$ be the isogeny class of $P' = [g_\infty, 1] \in X_\Gamma(K^{\mathrm{ab}})$ corresponding to $(A,\imath,\bar{\varphi})$ for some Γ-equivalence class $\bar{\varphi}$. Then Shimura's Reciprocity Law asserts that ${}^{\sigma}\psi_{g_\infty}([g_{\mathrm{f}}]) = \psi_{g_\infty}([g_{\mathrm{f}}\mathrm{Art}^{-1}(\sigma)])$ for all $\sigma \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$. We apply Theorem 4.2 to infer that $\rho_{(A,\imath,\varphi)}(\sigma) = \mathrm{Art}^{-1}(\sigma|K^{\mathrm{ab}})$ modulo $\Gamma \cap \mathbb{A}_{K,\mathrm{f}}^{\times}$ for any open subset Γ. Thus (iii) follows.

Finally, note that $jK^{\times}$ is an element in $N_A/K^{\times}$ of order 2, and $N_A/K^{\times} = A_{K,\mathrm{f}}^{\times}/K^{\times} \rtimes \langle jK^{\times}\rangle$. Let $\sigma_{\mathrm{c}} \in \mathrm{Aut}(\mathbb{C})$ denote the complex conjugation automorphism. Recall that $A = (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O}$, where the complex structure on $(B \otimes \mathbb{R})_{g_\infty}$ is given by $h_{g_\infty} : \mathbb{C} \hookrightarrow \mathrm{M}_2(\mathbb{R})$. It is easy to see that h_{g_∞} is the $\mathbb{R}$-extension of scalars of $\psi_{(A,\imath)} : K \hookrightarrow B$. Complex conjugation is the unique automorphism γ on $B \otimes \mathbb{R}$ such that $\gamma(h_{g_\infty}(z)) = h_{g_\infty}({}^{\sigma_{\mathrm{c}}}z)$ for all $z \in \mathbb{C}^{\times}$. Therefore γ corresponds to conjugation by j since $j^{-1}\psi_{(A,\imath)}(k)j = \psi_{(A,\imath)}({}^{\sigma_{\mathrm{c}}}k)$ for all $k \in K^{\times}$. This implies that ${}^{\sigma_{\mathrm{c}}}A = (B \otimes \mathbb{R})_{g_\infty j}/\mathcal{O}$, and rescaling j in such a way that $j \in \mathcal{O}$ if necessary, we have the isogeny

$$\mu_{\sigma_{\mathrm{c}}} : {}^{\sigma_{\mathrm{c}}}A = (B \otimes \mathbb{R})_{g_\infty j}/\mathcal{O} \to (B \otimes \mathbb{R})_{g_\infty}/\mathcal{O} = A, \qquad b \otimes x \mapsto jb \otimes x.$$

Hence, we have the diagram

$$\begin{array}{ccccc}
A_{\mathrm{tor}} & \xrightarrow{P \mapsto {}^{\sigma_{\mathrm{c}}}P} & {}^{\sigma_{\mathrm{c}}}A_{\mathrm{tor}} & \xrightarrow{\mu_{\sigma_{\mathrm{c}}}} & A_{\mathrm{tor}} \\
\varphi\downarrow & & {}^{\sigma_{\mathrm{c}}}\varphi\downarrow & & \downarrow\varphi \\
B/\mathcal{O} & \xrightarrow{b \mapsto b} & B/\mathcal{O} & \xrightarrow{b \mapsto jb} & B/\mathcal{O}
\end{array}$$

Thus, $\varphi(\mu_{\sigma_{\mathrm{c}}}({}^{\sigma_{\mathrm{c}}}P)) = \varphi(P)j$, which implies $\rho_{(A,\imath)}(\sigma_{\mathrm{c}}) = jK^{\times}$. Since $\rho_{(A,\imath,\varphi)}^{N}$ maps surjectively $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ to $\mathbb{A}_{K,\mathrm{f}}^{\times}/K^{\times}$ by (iii) and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})/\mathrm{Gal}(\bar{\mathbb{Q}}/K) \simeq \mathrm{Gal}(K/\mathbb{Q})$ is generated by the image of σ_{c}, part (iv) follows. ∎

8. Abelian $\mathbb{Q}$-surfaces as factors of abelian varieties of GL_2-type. In this section we will deal with abelian $\mathbb{Q}$-surfaces without complex multiplication. Thus, let $(A,\imath)$ be an abelian $\mathbb{Q}$-surface with QM by $\mathcal{O}$ such that $\mathrm{End}^0(A,\imath) = \mathbb{Q}$. We fix a set $\mu = \{\mu_\sigma : ({}^{\sigma}A, {}^{\sigma}\imath) \to (A,\imath) : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\}$ of $\mathcal{O}$-isogenies. Given μ, we can define the map

$$c_\mu : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{End}^0(A,\imath)^{\times} = \mathbb{Q}^{\times},$$
$$c_\mu(\sigma,\tau) = \frac{1}{\deg(\mu_{\sigma\tau})}\mu_\sigma\, {}^{\sigma}\mu_\tau \hat{\mu}_{\sigma\tau}.$$

Proposition 8.1. *The map c_μ is a 2-cocycle in $Z^2(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}^{\times})$. Its class in $H^2(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Q}^{\times})$ does not depend on the choice of μ or the choice of $(A,\imath)$ in a fixed $\mathcal{O}$-isogeny class. The class of c_μ in $H^2(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \bar{\mathbb{Q}}^{\times})$ coincides with the class of B in $\mathrm{Br}(\mathbb{Q})$, once we identify $\mathrm{Br}(\mathbb{Q}) \simeq H^2(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \bar{\mathbb{Q}}^{\times})$.*

Proof. Note that, for all $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the element ${}^{\sigma_1}c_\mu(\sigma_2, \sigma_3) \in \mathbb{Q}^\times$ commutes with μ_{σ_1}. This implies that

$$\begin{aligned} {}^{\sigma_1}c_\mu(\sigma_2,\sigma_3)c_\mu(\sigma_1,\sigma_2\sigma_3) &= \frac{1}{\deg(\mu_{\sigma_1\sigma_2\sigma_3})}\mu_{\sigma_1}{}^{\sigma_1}c_\mu(\sigma_2,\sigma_3)\,{}^{\sigma_1}\mu_{\sigma_2\sigma_3}\hat{\mu}_{\sigma_1\sigma_2\sigma_3} \\ &= \frac{\mu_{\sigma_1}{}^{\sigma_1}\mu_{\sigma_2}{}^{\sigma_1\sigma_2}\mu_{\sigma_3}{}^{\sigma_1}\hat{\mu}_{\sigma_2\sigma_3}{}^{\sigma_1}\mu_{\sigma_2\sigma_3}\hat{\mu}_{\sigma_1\sigma_2\sigma_3}}{\deg(\mu_{\sigma_2\sigma_3})\deg(\mu_{\sigma_1\sigma_2\sigma_3})} \\ &= \frac{\mu_{\sigma_1}{}^{\sigma_1}\mu_{\sigma_2}\hat{\mu}_{\sigma_1\sigma_2}\mu_{\sigma_1\sigma_2}{}^{\sigma_1\sigma_2}\mu_{\sigma_3}\hat{\mu}_{\sigma_1\sigma_2\sigma_3}}{\deg(\mu_{\sigma_1\sigma_2\sigma_3})\deg(\mu_{\sigma_1\sigma_2})} \\ &= c_\mu(\sigma_1,\sigma_2)c_\mu(\sigma_1\sigma_2,\sigma_3). \end{aligned}$$

Thus, c_μ is a 2-cocycle indeed.

If we choose $(A', \imath')$ in the $\mathcal{O}$-isogeny class of $(A, \imath)$ with a set $\mu' = \{\mu'_\sigma : ({}^\sigma A', {}^\sigma\imath') \to (A', \imath') : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\}$ of $\mathcal{O}$-isogenies, we obtain $c_\mu = c_{\mu'}\partial(\lambda)$, where

$$\lambda : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}^\times = \mathrm{End}^0(A,\imath)^\times, \quad \lambda(\sigma) = \frac{1}{\deg(\mu'_\sigma)\deg(\phi)}\mu_\sigma \circ {}^\sigma\hat{\phi} \circ \hat{\mu}'_\sigma \circ \phi,$$

for any $\mathcal{O}$-isogeny $\phi : (A, \imath) \to (A', \imath')$. Finally, the last assertion follows from [Pyle, Theorem 2.1]. ∎

Denote by $Q^\times$ the group $\bar{\mathbb{Q}}^\times$ with the trivial Galois action. By a theorem due to Tate [Rib, Theorem 6.3], the cohomology group $H^2(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), Q^\times)$ is trivial. This implies that $c_\mu = \partial(\alpha)$ for some $\alpha : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Q^\times$. Let E_α be the number field generated by the image of α.

By means of the set $\mu = \{\mu_\sigma : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\}$ of $\mathcal{O}$-isogenies and a 1-cocycle $\alpha : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Q^\times$ such that $c_\mu = \partial(\alpha)$, we shall construct an abelian variety A_{GL_2} of GL_2-type having A as a factor or contained in A.

Remark 8.2. If $B = \mathrm{M}_2(\mathbb{Q})$ and $A \simeq E \times E'$, where E is an elliptic curve defined over $\mathbb{Q}$, then either $A_{\mathrm{GL}_2} = E$, or A is a factor of A_{GL_2}.

Choose a model over a Galois extension $M/\mathbb{Q}$ such that the $\mathcal{O}$-isogenies $\mu_\sigma : ({}^\sigma A, {}^\sigma\imath) \to (A, \imath)$, with $\sigma \in \mathrm{Gal}(M/\mathbb{Q})$, are also defined over M. We consider the Weil restriction of scalars $X = \mathrm{Res}_{\mathbb{Q}}(A)$. It is an abelian variety defined over $\mathbb{Q}$ isomorphic over $\bar{\mathbb{Q}}$ to $\prod_{\sigma\in\mathrm{Gal}(M/\mathbb{Q})} {}^\sigma A$. For any $\tau \in \mathrm{Gal}(M/\mathbb{Q})$ and $o \in \mathcal{O}$, the isogeny μ_τ gives rise to an endomorphism $o\lambda_\tau \in \mathrm{End}(X)$ defined by

$$o\lambda_\tau : \prod_\sigma {}^\sigma A \to \prod_\sigma {}^\sigma A, \quad (P_\sigma)_\sigma \mapsto ({}^\sigma(\imath(o)\mu_\tau)(P_{\sigma\tau}))_\sigma.$$

The Galois action on the points of X is given by

$$\prod_{\sigma\in\mathrm{Gal}(M/\mathbb{Q})} {}^\sigma A \xrightarrow{\gamma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \prod_{\sigma\in\mathrm{Gal}(M/\mathbb{Q})} {}^\sigma A, \quad (P_\sigma)_\sigma \mapsto ({}^\gamma P_\sigma)_{\pi(\gamma)\sigma},$$

where $\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(M/\mathbb{Q})$ is the natural projection. We compute that, for $\tau \in \mathrm{Gal}(M/\mathbb{Q})$, $\gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $o \in \mathcal{O}$,

$$\begin{aligned} o\lambda_\tau({}^\gamma(P_\sigma)_\sigma) &= o\lambda_\tau(({}^\gamma P_\sigma)_{\pi(\gamma)\sigma}) = ({}^{\pi(\gamma)\sigma}(\imath(o)\mu_\tau)({}^\gamma P_{\sigma\tau}))_{\pi(\gamma)\sigma} \\ &= {}^\gamma({}^\sigma(\imath(o)\mu_\tau)(P_{\sigma\tau}))_\sigma = {}^\gamma(o\lambda_\tau((P_\sigma)_\sigma)). \end{aligned}$$

Hence $o\lambda_\tau \in \mathrm{End}_{\mathbb{Q}}(X)$.

LEMMA 8.3. *We have* $\mathrm{End}^0_{\mathbb{Q}}(X) = \prod_{\sigma\in\mathrm{Gal}(M/\mathbb{Q})} B\lambda_\sigma$.

Proof. The abelian variety X represents Weil's restriction of scalars functor $\mathrm{Res}_{\mathbb{Q}}(A)$, defined as the functor that maps a $\mathbb{Q}$-scheme S to $A(S_M)$, where $S_M = S \otimes_{\mathbb{Q}} M$ is the M-scheme obtained from S through extension of scalars. For $S = X$ we obtain

$$X(X) \otimes \mathbb{Q} = \mathrm{End}^0_{\mathbb{Q}}(X) = \mathrm{Hom}^0_M(X_M, A) = \prod_{\sigma\in\mathrm{Gal}(M/\mathbb{Q})} \mathrm{Hom}^0_M({}^\sigma A, A).$$

Hence the result follows from the fact that $\mathrm{Hom}^0_M({}^\sigma A, A) = \imath(B)\mu_\sigma$, which clearly maps to $B\lambda_\sigma$ under the above isomorphism. ∎

REMARK 8.4. Note that $(o\lambda_\tau)(o'\lambda_{\tau'}) = c_\mu(\tau,\tau')(oo'\lambda_{\tau\tau'})$ for any $\tau,\tau' \in \mathrm{Gal}(M/\mathbb{Q})$ and $o, o' \in \mathcal{O}$. Indeed,

$$\begin{aligned} (o\lambda_\tau)(o'\lambda_{\tau'})((P_\sigma)_\sigma) &= o\lambda_\tau(({}^\sigma(\imath(o')\mu_{\tau'})(P_{\sigma\tau'}))_\sigma) \\ &= ({}^\sigma(\imath(o)\mu_\tau)({}^{\sigma\tau}(\imath(o')\mu_{\tau'})(P_{\sigma\tau\tau'})))_\sigma = ({}^\sigma(\imath(o)\circ\mu_\tau\circ{}^\tau\imath(o')\circ{}^\tau\mu_{\tau'})(P_{\sigma\tau\tau'}))_\sigma \\ &= c_\mu(\tau,\tau')({}^\sigma(\imath(oo')\mu_{\tau\tau'})(P_{\sigma\tau\tau'}))_\sigma = c_\mu(\tau,\tau')(oo'\lambda_{\tau\tau'})((P_\sigma)_\sigma). \end{aligned}$$

Since $\lambda_\tau\lambda_{\tau^{-1}} \in \mathbb{Q}^\times$, we deduce that $\lambda_\tau \in \mathrm{Aut}_{\mathbb{Q}}(X)$ for all $\tau \in \mathrm{Gal}(M/\mathbb{Q})$.

Let $\hat{T}(X) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, X_{\mathrm{tor}})$ be the Tate module of X, and let $\hat{V}(X) = \hat{T}(X) \otimes \mathbb{Q}$. Since $X_{\mathrm{tor}} = \prod_\sigma {}^\sigma A_{\mathrm{tor}}$, we conclude that $\hat{T}(X) = \prod_\sigma \hat{T}({}^\sigma A) = \prod_\sigma \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, {}^\sigma A_{\mathrm{tor}})$.

LEMMA 8.5. *The morphism*

$$\varphi^\dagger : \hat{V}(X) \to \mathrm{End}^0_{\mathbb{Q}}(X) \otimes_{\mathcal{O}} \hat{\mathcal{O}}, \qquad (f_\sigma)_\sigma \mapsto \sum_\sigma \lambda_\sigma^{-1} \otimes \varprojlim_n \varphi(\mu_\sigma(f_\sigma(1/n))),$$

is an isomorphism of $\mathrm{End}^0_{\mathbb{Q}}(X)$*-modules.*

Proof. By Lemma 8.3, both $\hat{V}(X)$ and $\mathrm{End}^0_{\mathbb{Q}}(X) \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ are isomorphic to $(\hat{\mathcal{O}} \otimes \mathbb{Q})^{[M:\mathbb{Q}]}$. Moreover, $\varphi^\dagger$ is an isomorphism of $\mathcal{O}$-modules by Lemma 2.3. In order to prove the result, we have only to show that $\varphi^\dagger\lambda_\tau = \lambda_\tau\varphi^\dagger$ for all $\tau \in \mathrm{Gal}(M/\mathbb{Q})$. Indeed,

$$\begin{aligned}\varphi^\dagger\lambda_\tau(f_\sigma)_\sigma = \varphi^\dagger({}^\sigma\mu_\tau f_{\sigma\tau})_\sigma &= \sum_\sigma \lambda_\sigma^{-1}\otimes \varprojlim_n \varphi(\mu_\sigma({}^\sigma\mu_\tau(f_{\sigma\tau}(1/n))))\\ &= \sum_\sigma \lambda_\sigma^{-1}c_\mu(\sigma,\tau)\otimes \varprojlim_n \varphi(\mu_{\sigma\tau}(f_{\sigma\tau}(1/n)))\\ &= \sum_\sigma \lambda_\tau\lambda_{\sigma\tau}^{-1}\otimes \varprojlim_n \varphi(\mu_{\sigma\tau}(f_{\sigma\tau}(1/n))) = \lambda_\tau\varphi^\dagger(f_\sigma)_\sigma,\end{aligned}$$

by the above remark. Hence the result follows. ∎

Given the 2-cocycle c_μ associated to the set $\mu = \{\mu_\sigma : \sigma \in \mathrm{Gal}(M/\mathbb{Q})\}$ of $\mathcal{O}$-isogenies and the fixed 1-cocycle $\alpha : \mathrm{Gal}(M/\mathbb{Q}) \to E_\alpha^\times$ such that $c_\mu = \partial(\alpha)$, we claim that there is a ring homomorphism

$$\psi : \mathrm{End}^0_{\mathbb{Q}}(X) \to B\otimes_{\mathbb{Q}} E_\alpha, \quad b\lambda_\sigma \mapsto b\otimes\alpha(\sigma).$$

Indeed, for all $\sigma,\tau\in\mathrm{Gal}(M/\mathbb{Q})$ and $b,b'\in B$,

$$\begin{aligned}\psi((b\lambda_\sigma)(b'\lambda_\tau)) = \psi(c_\mu(\sigma,\tau)(bb'\lambda_{\sigma\tau})) &= bb'\otimes c_\mu(\sigma,\tau)\alpha(\sigma\tau)\\ &= (b\otimes\alpha(\sigma))(b'\otimes\alpha(\tau)) = \psi(b\lambda_\sigma)\psi(b'\lambda_\tau),\end{aligned}$$

by Remark 8.4.

Clearly the map ψ is surjective by definition. Consider the abelian subvariety $X_0/\mathbb{Q}$ defined by $X \supseteq X_0 := \{P\in X : \lambda(P) = 0 \text{ for all } \lambda\in N\}$, where $N = \ker(\psi)\cap\mathrm{End}_{\mathbb{Q}}(X)$. By construction, $\mathrm{End}^0_{\mathbb{Q}}(X_0) = \mathrm{End}^0_{\mathbb{Q}}(X)/\ker(\psi) = B\otimes_{\mathbb{Q}} E_\alpha$. By the previous lemma, we have $\hat{V}(X_0) = E_\alpha\otimes_{\mathbb{Z}}\hat{\mathcal{O}}$, where $\hat{V}(X_0) = \hat{T}(X_0)\otimes\mathbb{Q}$ and $\hat{T}(X_0) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z},(X_0)_{\mathrm{tor}})$ is its Tate module.

Lemma 8.6. *We have $B\otimes_{\mathbb{Q}} E_\alpha \simeq \mathrm{M}_2(E_\alpha)$.*

Proof. By Proposition 8.1, the 2-cocycle c_μ represents the class of B in the Brauer group $\mathrm{Br}(\mathbb{Q})$. Since c_μ is a coboundary when extended to E_α, it represents the trivial element of the Brauer group $\mathrm{Br}(E_\alpha)$. The fact that the cocycle representing B is trivial when extended to E_α implies the assertion (see [Pyle, §2]). ∎

The above lemma implies that X_0 is isogenous over $\mathbb{Q}$ to an abelian surface of the form $A^2_{\mathrm{GL}_2}$, where $\mathrm{End}^0_{\mathbb{Q}}(A_{\mathrm{GL}_2}) = E_\alpha$. Clearly, $\dim(A_{\mathrm{GL}_2}) = \dim(X_0)/2 = [E_\alpha:\mathbb{Q}]$, thus A_{GL_2} is an abelian surface of GL_2-type. More precisely, fix an isomorphism $\mathrm{End}^0_{\mathbb{Q}}(X_0)\simeq\mathrm{M}_2(E_\alpha)$, and let

$$\pi\in\mathrm{End}_{\mathbb{Q}}(X_0)\cap\left\{\begin{pmatrix}x&0\\0&0\end{pmatrix}\in B\otimes_{\mathbb{Q}}E_\alpha = \mathrm{End}^0_{\mathbb{Q}}(X_0) : x\in E_\alpha^\times\right\}.$$

Then $A_{\mathrm{GL}_2} := \pi(X_0)\subset X_0$ and we have the isogeny

$$A^2_{\mathrm{GL}_2} = \pi(X_0)^2 \to X_0, \qquad (P,Q)\mapsto P + \begin{pmatrix}0&1\\1&0\end{pmatrix}Q,$$

and its dual

$$X_0 \to \pi(X_0)^2 = A_{\mathrm{GL}_2}^2, \quad P \mapsto \left(\pi P, \pi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P\right).$$

Since $A_{\mathrm{GL}_2} \subset X_0 \subseteq X = \mathrm{Res}_{\mathbb{Q}}(A)$, we find that A is a factor of A_{GL_2} (or A_{GL_2} is a factor of A, see Remark 8.2). In particular, $\mathrm{End}^0(A_{\mathrm{GL}_2}) = \mathrm{M}_{d/2}(B)$, where $d = [E_\alpha : \mathbb{Q}]$ (we write formally $\mathrm{M}_{1/2}(\mathrm{M}_2(\mathbb{Q})) = \mathbb{Q}$ in case $E_\alpha = \mathbb{Q}$ and $B = \mathrm{M}_2(\mathbb{Q})$).

8.1. Galois representations attached to abelian varieties of GL_2-type. Let $A_{\mathrm{GL}_2}/\mathbb{Q}$ be an abelian variety of dimension d of GL_2-type such that $\mathrm{End}^0_{\bar{\mathbb{Q}}}(A_{\mathrm{GL}_2}) = \mathrm{M}_{d/2}(B)$. By definition $\mathrm{End}^0_{\mathbb{Q}}(A_{\mathrm{GL}_2}) = E$, a field extension of $\mathbb{Q}$ of degree d.

First we recall the classical construction of the Galois representation attached to A_{GL_2}: For any prime ℓ, the Tate module

$$T_\ell(A_{\mathrm{GL}_2}) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, (A_{\mathrm{GL}_2})_{\mathrm{tor}})$$

is a free $\mathbb{Z}_\ell$-module of rank $2d$. Its extension of scalars, $V_\ell(A_{\mathrm{GL}_2}) := T_\ell(A_{\mathrm{GL}_2}) \otimes \mathbb{Q}$, has a natural action of $\mathrm{End}^0_{\mathbb{Q}}(A_{\mathrm{GL}_2}) = E$, which provides it with the structure of a rank 2 $(E \otimes \mathbb{Q}_\ell)$-module. Given a basis $\varphi_1, \varphi_2 \in V_\ell(A_{\mathrm{GL}_2})$ as an $(E \otimes \mathbb{Q}_\ell)$-module, we obtain a representation

$$\rho^\ell_{\mathrm{GL}_2} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E \otimes \mathbb{Q}_\ell)^{\mathrm{opp}},$$

defined by $(\varphi_1({}^\sigma P), \varphi_2({}^\sigma P)) = (\varphi_1(P), \varphi_2(P))\rho^\ell_{\mathrm{GL}_2}(\sigma)$ for all $P \in (A_{\mathrm{GL}_2})_{\mathrm{tor}}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Note that we can consider the product of the subgroups $E_\alpha^\times G(\mathbb{A}_{\mathrm{f}})$ inside $\prod_\ell \mathrm{GL}_2(E_\alpha \otimes \mathbb{Q}_\ell)^{\mathrm{opp}}$, where E_α is embedded in $\prod_\ell \mathrm{GL}_2(E_\alpha \otimes \mathbb{Q}_\ell)^{\mathrm{opp}}$ diagonally and $G(\mathbb{A}_{\mathrm{f}})$ through the monomorphism

$$G(\mathbb{A}_{\mathrm{f}}) = (\hat{\mathcal{O}}^{\mathrm{opp}} \otimes \mathbb{Q})^\times \hookrightarrow \prod_\ell (B_\ell^{\mathrm{opp}} \otimes E_\alpha)^\times \simeq \prod_\ell \mathrm{GL}_2(E_\alpha \otimes \mathbb{Q}_\ell)^{\mathrm{opp}}.$$

The following result relates this representation $\rho^\ell_{\mathrm{GL}_2}$ to the representation $\rho^\mu_{(A,\imath,\varphi)}$ introduced in §2:

Theorem 8.7. *Let $(A,\imath)$ be an abelian $\mathbb{Q}$-surface with QM by $\mathcal{O}$. Assume that the abelian variety $A_{GL_2}/\mathbb{Q}$ of GL_2-type has been constructed by means of $(A,\imath)$, $\mu = \{\mu_\sigma : ({}^\sigma A, {}^\sigma\imath) \to (A,\imath) : \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\}$ and $\alpha : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to E_\alpha^\times$. Then there exist $(E_\alpha \otimes \mathbb{Q}_\ell)$-bases of the Tate modules of A_{GL_2} at every prime ℓ such that the product $\prod_\ell \rho^\ell_{\mathrm{GL}_2} =: \hat{\rho}_{\mathrm{GL}_2}$ of ℓ-adic representations factors through*

$$\hat{\rho}_{\mathrm{GL}_2} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\alpha^{-1}\rho^\mu_{(A,\imath,\varphi)}} E_\alpha^\times G(\mathbb{A}_{\mathrm{f}}) \subset \prod_\ell \mathrm{GL}_2(E_\alpha \otimes \mathbb{Q}_\ell)^{\mathrm{opp}}.$$

That is, $\hat{\rho}_{\mathrm{GL}_2}(\sigma) = \alpha(\sigma)^{-1}\rho^{\mu}_{(A,\imath,\varphi)}(\sigma) \in E_\alpha^\times G(\mathbb{A}_{\mathrm{f}}) \subseteq \prod_\ell \mathrm{GL}_2(E_\alpha \otimes \mathbb{Q}_\ell)^{\mathrm{opp}}$ *for all* $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. Fix a model $(A,\imath)$ over some Galois extension M as above. In order to construct A_{GL_2}, we consider the restriction of scalars $X = \mathrm{Res}_{\mathbb{Q}}(A)$. Recall the isomorphism $\varphi^\dagger : \hat{V}(X) \to \mathrm{End}^0_{\mathbb{Q}}(X) \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ of Lemma 8.5. Let res : $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(M/\mathbb{Q})$ be the natural quotient morphism. We compute, for all $\gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$,

$$\begin{aligned}
\varphi^\dagger({}^\gamma(f_\sigma)_\sigma) = \varphi^\dagger(({}^\gamma f_\sigma)_{\mathrm{res}(\gamma)\sigma}) &= \sum_\sigma \lambda^{-1}_{\mathrm{res}(\gamma)\sigma} \otimes \varprojlim_n \varphi(\mu_{\mathrm{res}(\gamma)\sigma}({}^\gamma f_\sigma(1/n))) \\
&= \sum_\sigma \lambda^{-1}_{\mathrm{res}(\gamma)\sigma} c_\mu(\mathrm{res}(\gamma),\sigma)^{-1} \otimes \varprojlim_n \varphi(\mu_{\mathrm{res}(\gamma)}({}^\gamma(\mu_\sigma(f_\sigma(1/n))))) \\
&= \sum_\sigma \lambda_\sigma^{-1}\lambda^{-1}_{\mathrm{res}(\gamma)} \otimes \varprojlim_n \varphi(\mu_\sigma(f_\sigma(1/n)))\rho^\mu_{(A,\imath,\varphi)}(\gamma) \\
&= \varphi^\dagger(f_\sigma)_\sigma(\lambda^{-1}_{\mathrm{res}(\gamma)} \otimes \rho^\mu_{(A,\imath,\varphi)}(\gamma)),
\end{aligned}$$

where the action of $\lambda^{-1}_{\mathrm{res}(\gamma)} \otimes \rho^\mu_{(A,\imath,\varphi)}(\gamma)$ on $\mathrm{End}^0_{\mathbb{Q}}(X) \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ is given by left translation.

Given α and the corresponding subvariety $X_0 \subseteq X$, we have seen that $\varphi^\dagger$ provides an isomorphism $\varphi^\dagger_\alpha : \hat{V}(X_0) \xrightarrow{\simeq} E_\alpha \otimes \hat{\mathcal{O}}$. Since $\psi(\lambda_{\mathrm{res}(\gamma)}) = \alpha(\gamma)$, the above calculation shows that the action of $\gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\hat{V}(X_0) \simeq E_\alpha \otimes \hat{\mathcal{O}}$ is given by right multiplication by $\alpha(\gamma)^{-1}\rho^\mu_{(A,\imath,\varphi)}(\gamma)$.

Finally, we know that $A_{\mathrm{GL}_2} = \pi(X_0) \subset X_0$, where $\pi \in E_\alpha \otimes \mathcal{O}$. Since right multiplication by $E_\alpha^\times G(\mathbb{A}_{\mathrm{f}})$ commutes with left multiplication by $E_\alpha \otimes \mathcal{O}$, we conclude that the action of $\gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\hat{V}(A_{\mathrm{GL}_2}) \simeq \pi(\hat{\mathcal{O}} \otimes E_\alpha)$ is also given by right multiplication by $\alpha(\gamma)^{-1}\rho^\mu_{(A,\imath,\varphi)}(\gamma)$. The fixed isomorphism $E_\alpha \otimes \mathcal{O}_\ell \simeq \mathrm{M}_2(E_\alpha \otimes \mathbb{Q}_\ell)$ provides a basis of the Tate module $\hat{V}(A_{\mathrm{GL}_2}) \simeq \pi(\mathcal{O}_\ell \otimes E_\alpha)$ as a rank 2 $(E_\alpha \otimes \mathbb{Q}_\ell)$-module. It is clear that the Galois action on $V_\ell(A_{\mathrm{GL}_2})$ is given by right multiplication by the ℓ-adic component of $\alpha^{-1}\rho^\mu_{(A,\imath,\varphi)}$, once we have identified $E_\alpha^\times G(\mathbb{Q}_\ell)$ inside $\mathrm{GL}_2(\mathbb{Q}_\ell \otimes E_\alpha)^{\mathrm{opp}}$. Thus the result follows. ∎

The following result, which is a direct consequence of the above theorem, is well known to experts.

COROLLARY 8.8. *Let* $\hat{\rho}_{\mathrm{GL}_2} = \prod_\ell \rho^\ell_{\mathrm{GL}_2}$ *be the Galois representation attached to the abelian variety of* GL_2*-type constructed by means of* $(A,\imath)$, μ *and* α. *Then its determinant is given by*

$$\det(\hat{\rho}_{\mathrm{GL}_2})(\sigma) = \chi(\sigma)\frac{\deg(\mu_\sigma)}{\alpha^2(\sigma)},$$

where χ *is the cyclotomic character.*

Proof. Since the determinant does not depend on the choice of the basis for the Tate module, we choose the basis of Theorem 8.7. Then the result follows from Theorems 8.7 and 6.5. ∎

REMARK 8.9. By the proof of Serre's conjecture on representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ [Ser, 3.2.4], $\rho^{\ell}_{\mathrm{GL}_2}$ is the ℓ-adic Galois representation associated with a modular newform in $S_1(N,\epsilon)$ with $N\in\mathbb{N}$ and $\epsilon(\sigma)=\deg(\mu_\sigma)/\alpha^2(\sigma)$ for all $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Recall that the representation $\rho_{(A,\imath,\varphi)}:\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to G(\mathbb{A}_{\mathrm{f}})/\mathbb{Q}^\times$ introduced in §2 is the projectivization of $\rho^{\mu}_{(A,\imath,\varphi)}$ modulo $\mathbb{Q}^\times$. Applying Theorem 8.7, we deduce the following corollary:

COROLLARY 8.10. *The representation $\rho_{(A,\imath,\varphi)}$ is the projectivization of the classical representation $\hat{\rho}_{\mathrm{GL}_2}:\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to E_\alpha^\times G(\mathbb{A}_{\mathrm{f}})$ attached to the abelian variety A_{GL_2} of* GL_2*-type modulo* $\mathrm{End}^0_{\mathbb{Q}}(A_{\mathrm{GL}_2})^\times=E_\alpha^\times$.

Acknowledgements. The author would like to thank Xevi Guitart for his comments and discussions throughout the development of this paper. The author also thanks the referee for carefully reading the paper and providing many helpful comments.

The author is supported in part by DGICYT Grant MTM2015-63829-P.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682152).

References

[G-M] X. Guitart and S. Molina, *Parametrization of abelian K-surfaces with quaternionic multiplication*, C. R. Math. Acad. Sci. Paris 347 (2009), 1325–1330.

[Mum] D. Mumford, *Abelian Varieties*, Tata Institute of Fundamental Research, Bombay, and Oxford Univ. Press, London, 1970.

[Pyle] E. Pyle, *Abelian varieties over $\mathbb{Q}$ with large endomorphism algebras and their simple components over $\overline{\mathbb{Q}}$*, in: Modular Curves and Abelian Varieties, Progr. Math. 224, Birkhäuser, Basel, 2004, 189–239.

[Rib] K. A. Ribet, *Abelian varieties over $\mathbb{Q}$ and modular forms*, in: Modular Curves and Abelian Varieties, Progr. Math. 224, Birkhäuser, Basel, 2004, 241–261.

[Ser] J.-P. Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. 54 (1987), 179–230.

[Shi] G. Shimura, *Construction of class fields and zeta functions of algebraic curves*, Ann. of Math. (2) 85 (1967), 58–159.

Santiago Molina
Centre de Recerca Matemática CRM
08176 Bellaterra, Barcelona, Spain
E-mail: santimolin@gmail.com