INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

DISSERTATIONES MATHEMATICAE

527

SAMEER CHAVAN, DEEPAK KUMAR PRADHAN and SHAILESH TRIVEDI

Multishifts on directed Cartesian products of rooted directed trees

WARSZAWA 2017

Sameer Chavan, Deepak Kumar Pradhan
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur, India
E-mail: chavan@iitk.ac.in, dpradhan@iitk.ac.in

Shailesh Trivedi
School of Mathematics
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi, Allahabad 211019, India
E-mail: shaileshtrivedi@hri.res.in

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound in Poland by HermanDruK, Warszawa
Nakład 200 egz.

Abstracted/Indexed in: Mathematical Reviews, Zentralblatt MATH, Science Citation Index Expanded, Journal Citation Reports/Science Edition, Google Science, Scopus, EBSCO Discovery Service.

Available online at **http://journals.impan.pl**

DOI: 10.4064/dm758-6-2017

ISSN 0012-3862

Contents

Abstract

We systematically develop the multivariable counterpart of the theory of weighted shifts on rooted directed trees. Capitalizing on the theory of products of directed graphs, we introduce and study the notion of multishifts on directed Cartesian products of rooted directed trees. This framework unifies the theory of weighted shifts on rooted directed trees and that of classical unilateral multishifts. Moreover, this setup brings out some new phenomena such as the appearance of a system of linear equations in the eigenvalue problem for the adjoint of a multishift. In the first half of the paper, we focus our attention mostly on the multivariable spectral theory and function theory including a finer analysis of various joint spectra and the wandering subspace property for multishifts. In the second half, we separate out two special classes of multishifts, which we refer to as torally balanced and spherically balanced multishifts. The classification of these two classes is closely related to toral and spherical polar decompositions of multishifts. Furthermore, we exhibit a family of spherically balanced multishifts on a d-fold directed Cartesian product $\mathscr{T}$ of rooted directed trees. These multishifts turn out to be multiplication d-tuples $\mathscr{M}_{z,a}$ on certain reproducing kernel Hilbert spaces $\mathscr{H}_a$ of vector-valued holomorphic functions defined on the unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$, which can be thought of as tree analogs of the multiplication d-tuples acting on the reproducing kernel Hilbert spaces associated with the kernels $1/(1-\langle z,w\rangle)^a$ $(z,w\in\mathbb{B}^d, a\in\mathbb{N})$. Indeed, the reproducing kernels associated with $\mathscr{H}_a$ are certain operator linear combinations of $1/(1-\langle z,w\rangle)^a$ and multivariable hypergeometric functions ${}_2F_1(\mathsf{d}_v+a+1,1,\mathsf{d}_v+2,\cdot)$ defined on $\mathbb{B}^d\times\mathbb{B}^d$, where d_v denotes the depth of a branching vertex v in $\mathscr{T}$. We also classify joint subnormal and joint hyponormal multishifts within the class of spherically balanced multishifts.

Acknowledgements. We express our gratitude to Dmitry Yakubovich for a careful reading and several useful suggestions. We also convey our sincere thanks to H. Turgay Kaptanoğlu, especially for drawing our attention to [6], where certain Hardy-type spaces are associated with some infinite acyclic, undirected, connected graphs. Further, we are grateful to Akash Anand for indicating a way to embed three-dimensional directed graphs into the plane making the presentation of diagrams more accessible. We also take this opportunity to thank the anonymous referee for several constructive suggestions which substantially improved the exposition of the paper. Finally, we thank the faculty and the administrative unit of Department of Mathematics and Statistics, IIT Kanpur and School of Mathematics, HRI, Allahabad for their warm hospitality during the preparation of the paper.

The work of the second author is supported through the NBHM Research Fellowship.

2010 *Mathematics Subject Classification*: Primary 47B37, 47A13; Secondary 47A10, 05C20.
Key words and phrases: directed tree, Cartesian product, tensor product, multishifts, circularity, wandering subspace, balanced multishifts, von Neumann inequality, subnormality.
Received 6 September 2016; revised 9 June 2017 and 21 October 2017.
Published online 15 December 2017.

1. Introduction

The investigations in the present work are related to the idea of shifts associated with discrete structures (e.g. directed trees), recently boosted in the theory of Hilbert space operators ([67], [27], [68], [28], [69], [70], [29], [30], [31], [35], [79], [26]). The significantly large class of weighted shift operators on directed trees contains all classical weighted shifts and has an overlap with that of composition operators. This interplay of graph theory and operator theory provides illuminating examples exhibiting subtle phenomena such as existence of nonhyponormal operators generating Stieltjes moment sequences and triviality of the domain of integral powers of densely defined subnormal operators ([68], [70], [30], [31]; refer also to [74], [76] for a systematic study of operator algebras associated with directed graphs). Further, this framework turns out to be a rich source of k-diagonal reproducing kernel Hilbert spaces of vector-valued holomorphic functions ([1], [35]).

The motivation for the present work primarily comes from multivariable operator theory, where the main objectives of study have been function theory and spectral theory of classical multishifts ([71], [42], [43], [49], [17], [18], [44], [45], [19], [20], [82], [83], [22], [14], [9], [60], [59], [58], [5], [84], [52], [54], [24], [72], [73], [36], [63]). The present work is an effort to develop the theory of weighted shifts on directed trees in several variables by implementing the methods of graph theory ([86], [92], [80], [62], [56], [66]; refer also to [10], [4], [6], [11], [12], where Hardy–Besov spaces associated with certain trees have been introduced and studied). The well-established theory of products of directed graphs provides a foundation for the study of multishifts. Various notions of product of directed graphs (e.g. Cartesian product, tensor product) lead naturally to interesting counterparts of classical shifts in one and several variables. One peculiar aspect of the directed Cartesian product of directed trees is that although it is not a directed tree, it admits a directed semi-tree structure. Interestingly, there is a natural shift operator on any directed semi-tree [79]. Thus a single discrete structure gives rise to at least two distinct notions of shifts. This is not so easy to reveal in the classical case. One of the advantages of this setup is that any disjoint decomposition of the set V of vertices induces a natural decomposition of the associated unweighted Lebesgue space $l^2(V)$, which in turn decomposes the multishift $S_{\boldsymbol\lambda}$ into known objects like tuples of compact operators and classical multishifts. On the other hand, there are numerous ways of decomposing V by defining equivalence relations on V in terms of siblings, generations etc. Obviously, every setup has its own set of problems. Here also in the context of subnormality of multishifts, one can ask for a finite and minimal subset W of the set V of vertices with the following property: A multishift is joint subnormal if and only if its moments are completely monotone at every vertex from W. The notion of joint branching index plays an important role in the answer to this problem.

In the remaining part of this section, we set some standard notation and also collect various notions and facts, which are central to the present text. For a set X, $\mathrm{card}(X)$ denotes the cardinality of X. We recall that $\mathbb{N}$ stands for the set of nonnegative integers, which forms a semigroup under addition. For a positive integer d, let $\mathbb{N}^d$ denote the d-fold Cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$. For $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ in $\mathbb{N}^d$, we write $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for all $j = 1, \ldots, d$, and we also use $\alpha! := \prod_{j=1}^d \alpha_j!$ and $|\alpha| := \sum_{j=1}^d \alpha_j$. Throughout this paper, we follow the conventions

$$\sum_{i=1}^{n-1} x_i = 0 \quad \text{if } n = 1, \qquad \prod_{j=0}^{n-1} y_j = 1 \quad \text{if } n = 0.$$

More generally, the sum over an empty set is understood to be 0, while the product over an empty set is always 1.

Let $\mathbb{C}$ denote the field of complex numbers. We set $tz := (tz_1, \ldots, tz_d)$ for $t \in \mathbb{C}$ and $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$. Whenever $\alpha \in \mathbb{N}^d$ or $z \in \mathbb{C}^d$, it is understood that $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $z = (z_1, \ldots, z_d)$. The complex conjugate of $z \in \mathbb{C}^d$ is defined by $\overline{z} := (\overline{z}_1, \ldots, \overline{z}_d)$. We denote by $\mathbb{B}^d_r$ the open ball in $\mathbb{C}^d$ centered at the origin and of radius $r > 0$:

$$\mathbb{B}^d_r := \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \|z\|_2 < r\},$$

where $\|z\|_2 := \sqrt{|z_1|^2 + \cdots + |z_d|^2}$ denotes the Euclidean norm of $z = (z_1, \ldots, z_d)$ in $\mathbb{C}^d$. The sphere $\{z \in \mathbb{C}^d : \|z\|_2 = r\}$ is denoted by $\partial\mathbb{B}^d_r$. For simplicity, the unit ball $\mathbb{B}^d_1$ and the unit sphere $\partial\mathbb{B}^d_1$ are denoted respectively by $\mathbb{B}^d$ and $\partial\mathbb{B}^d$. We denote by $\mathbb{D}^d_r$ the open polydisc centered at the origin and of polyradius $r = (r_1, \ldots, r_d)$ with $r_1, \ldots, r_d > 0$:

$$\mathbb{D}^d_r := \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| < r_1, \ldots, |z_d| < r_d\}.$$

The d-torus $\{z \in \mathbb{C}^d : |z_1| = r_1, \ldots, |z_d| = r_d\}$ is denoted by $\mathbb{T}^d_r$. Again, for simplicity, the unit polydisc $\mathbb{D}^d_1$ and the unit d-torus $\mathbb{T}^d_1$ are denoted respectively by $\mathbb{D}^d$ and $\mathbb{T}^d$. For a subset X of $\mathbb{C}^d$, the closure of X in $\|\cdot\|_2$ is denoted by $\mathrm{cl}(X)$.

Let $\mathcal{H}$ be a complex Hilbert space. The inner product on $\mathcal{H}$ will be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. If no confusion is likely, we suppress the subscript, and simply write $\langle\cdot,\cdot\rangle$. By a *subspace* of $\mathcal{H}$, we mean a closed linear manifold. Let W be a subset of $\mathcal{H}$. Then $\mathrm{span}\, W$ stands for the smallest linear manifold generated by W. In case W is singleton $\{w\}$, we use the convenient notation $[w]$ in place of $\mathrm{span}\,\{w\}$. By $\bigvee\{w : w \in W\}$, we understand the subspace generated by W. The orthogonal complement of a subspace $\mathcal{M}$ in $\mathcal{H}$ will be denoted by $\mathcal{M}^\perp$ or $\mathcal{H} \ominus \mathcal{M}$. The Hilbert space dimension of $\mathcal{M}$ is denoted by $\dim \mathcal{M}$. We use $P_{\mathcal{M}}$ to denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$, and I to denote the identity operator on $\mathcal{H}$. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, then we use $I|_{\mathcal{M}}$ to denote the identity operator on $\mathcal{M}$.

Unless stated otherwise, all the Hilbert spaces occurring in this paper are complex infinite-dimensional separable and for any such Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$ endowed with the operator norm. For $A \in B(\mathcal{H})$, the symbols $\ker A$ and $\mathrm{ran}\, A$ stand for the kernel and the range of A respectively. The Hilbert space adjoint of A will be denoted by A^*.

By a *commuting d-tuple* T on $\mathcal{H}$, we mean a tuple $(T_1, \ldots, T_d)$ of commuting bounded linear operators $T_1, \ldots, T_d$ on $\mathcal{H}$. In case $d = 1$, we denote the 1-tuple T by T_1. A commuting d-tuple T on $\mathcal{H}$ is said to be *doubly commuting* if $T_iT_j^* = T_j^*T_i$ for all $i, j = 1, \ldots, d$ with $i \neq j$. If $T = (T_1, \ldots, T_d)$ is a commuting d-tuple on $\mathcal{H}$, then we set T^* to denote $(T_1^*, \ldots, T_d^*)$ while T^α represents $T_1^{\alpha_1} \cdots T_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, where we use the convention that $A^0 = I$ for $A \in B(\mathcal{H})$.

Also, we find it convenient to introduce the following operators on $B(\mathcal{H})$. Given a commuting d-tuple T on $\mathcal{H}$, define $Q_T : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$Q_T(X) := \sum_{j=1}^{d} T_j^* X T_j \quad \text{for } X \in B(\mathcal{H}).$$

Further, the operator Q_T^n is inductively defined for all $n \in \mathbb{N}$ through $Q_T^0(X) := X$ and $Q_T^n(X) := Q_T(Q_T^{n-1}(X))$ $(n \geq 1)$ for $X \in B(\mathcal{H})$. It is easy to see that

$$Q_T^n(I) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} T^{*\alpha} T^\alpha \quad (n \in \mathbb{N}). \tag{1.1}$$

Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. We say that T is a

(i) *toral contraction* if $Q_{T_j}(I) \leq I$ for $j = 1, \ldots, d$;
(ii) *toral isometry* if $Q_{T_j}(I) = I$ for $j = 1, \ldots, d$;
(iii) *toral left invertible* d-tuple if $Q_{T_j}(I)$ is invertible for $j = 1, \ldots, d$;
(iv) *joint contraction* if $Q_T(I) \leq I$;
(v) *joint isometry* if $Q_T(I) = I$;
(vi) *joint left invertible* d-tuple if $Q_T(I)$ is invertible.

A commuting d-tuple $T = (T_1, \ldots, T_d)$ on $\mathcal{H}$ is *joint subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a commuting d-tuple $N = (N_1, \ldots, N_d)$ of normal operators $N_1, \ldots, N_d$ in $B(\mathcal{K})$ such that $N_jh = T_jh$ for every $h \in \mathcal{H}$ and $j = 1, \ldots, d$. It is well-known that a toral isometry and a joint isometry are joint subnormal [19, Propositions 1 and 2]. The notion of joint subnormal tuples is closely related to a classical notion from abstract harmonic analysis, namely, the notion of completely monotone functions.

Let ϕ be a real-valued map on $\mathbb{N}^d$. For $1 \leq j \leq d$, define the difference operators ∇_j by $(\nabla_j\phi)(\alpha) := \phi(\alpha) - \phi(\alpha + \epsilon_j)$ $(\alpha \in \mathbb{N}^d)$, where ϵ_j is the d-tuple with 1 in the jth place and zeros elsewhere. The operator ∇^β is inductively defined for every $\beta \in \mathbb{N}^d$ by $\nabla^0\phi := \phi$, $\nabla^{\beta+\epsilon_j}\phi := \nabla_j(\nabla^\beta\phi)$. A real-valued map ϕ on $\mathbb{N}^d$ is said to be *completely monotone* if $(\nabla^\beta\phi)(\alpha) \geq 0$ for all $\alpha, \beta \in \mathbb{N}^d$.

Remark 1.0.1. A toral contractive d-tuple T on $\mathcal{H}$ is joint subnormal if and only if $\phi(\alpha) := \|T^\alpha h\|^2$ $(\alpha \in \mathbb{N}^d)$ is completely monotone for every $h \in \mathcal{H}$ [17, Theorem 4.4].

A commuting d-tuple $T = (T_1, \ldots, T_d)$ is *joint hyponormal* if the $d \times d$ matrix $([T_j^*, T_i])_{1 \leq i,j \leq d}$ is positive definite, where $[A, B]$ stands for the commutator $AB - BA$ for A and B in $B(\mathcal{H})$. A joint subnormal tuple is always joint hyponormal [18], [45].

For all notions introduced above, we skip the prefixes toral or joint in case the dimension d is 1. *Although it has been a common practice to use interchangeably joint isometry with spherical isometry, we do not follow this practice.*

We briefly recall from [32] the definitions of toral and spherical Cauchy dual tuples. Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. Assume that T is toral left invertible. We refer to the d-tuple $T^{\mathfrak{t}} = (T_1^{\mathfrak{t}}, \ldots, T_d^{\mathfrak{t}})$ as the *toral Cauchy dual* of T, where

$$T_j^{\mathfrak{t}} := T_j(Q_{T_j}(I))^{-1} \quad (j = 1, \ldots, d). \tag{1.2}$$

Note that $(T^{\mathfrak{t}})^{\mathfrak{t}} = T$.

Assume that T is joint left invertible. We refer to the d-tuple $T^{\mathfrak{s}} = (T_1^{\mathfrak{s}}, \ldots, T_d^{\mathfrak{s}})$ as the *spherical Cauchy dual* of T, where

$$T_j^{\mathfrak{s}} := T_j(Q_T(I))^{-1} \quad (j = 1, \ldots, d). \tag{1.3}$$

Note that $(T^{\mathfrak{s}})^{\mathfrak{s}} = T$.

1.1. Multivariable spectral theory. In what follows, multivariable spectral theory will be central to the investigations in this paper. Therefore we include a brief account of Taylor's notion of invertibility and related concepts. We have freely drawn on [44] and [15] throughout this paper, particularly in the following discussion.

The classical spectral theory deals with the problem of finding $x \in \mathcal{H}$ such that $Tx = y$ for any given $y \in \mathcal{H}$, where $T \in B(\mathcal{H})$. Note that T is (boundedly) invertible if and only if the above problem is solvable for every $y \in \mathcal{H}$ with a unique $x \in \mathcal{H}$. Let us formulate an analog of the problem above for two operators $T_1, T_2 \in B(\mathcal{H})$ such that $T_1T_2 = T_2T_1$. One is interested in the notion of invertibility which will give a "unique" solution (x_1, x_2) of the problem of finding $x_1, x_2 \in \mathcal{H}$ such that $T_1x_1 + T_2x_2 = y$ for every given $y \in \mathcal{H}$. In this case, we cannot hope for uniqueness. Indeed, if (x_1, x_2) is a solution of this problem, then for any $h \in \mathcal{H}$, we may set $x_1' := x_1 - T_2h$ and $x_2' := x_2 + T_1h$, and verify that (x_1', x_2') is also a solution. Following [15], we will refer to (x_1', x_2') as the *tautological perturbation* of the solution (x_1, x_2).

Remark 1.1.1. There are no nontrivial tautological perturbations in case of a single operator.

One needs to determine what happens modulo tautological perturbations. This is where homology enters into the picture.

Given a Hilbert space $\mathcal{H}$, consider

$$\begin{aligned} \Lambda_0 &:= \mathcal{H}, \\ \Lambda_1 &:= \{(h_1, h_2) : h_1, h_2 \in \mathcal{H}\}, \\ \Lambda_2 &:= \{(h_{ij}) : h_{ij} \in \mathcal{H} \text{ for } 1 \le i, j \le 2,\ (h_{ij}) \text{ is skew symmetric}\}. \end{aligned}$$

Note that Λ_2 is isometrically isomorphic to $\mathcal{H}$ via $\left(\begin{smallmatrix} 0 & h \\ -h & 0 \end{smallmatrix}\right) \mapsto h$. Consider the sequence

$$K : \{0\} \xrightarrow{0} \Lambda_2 \xrightarrow{B_2} \Lambda_1 \xrightarrow{B_1} \Lambda_0 \xrightarrow{0} \{0\}, \tag{1.4}$$

where

$$B_2((h_{ij})) := (h_{ij}) \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = (T_2h_{12}, -T_1h_{12}),$$

$$B_1(h_1, h_2) := (h_1, h_2) \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = T_1h_1 + T_2h_2.$$

Note that K is a complex, that is, $B_1 \circ B_2 = 0$. Let us examine this complex:

1. $T_1x_1 + T_2x_2 = y$ has a solution if and only if $B_1(x_1, x_2) = y$ has a solution if and only if B_1 is surjective.
2. Given a solution $(x_1, x_2) \in \Lambda_1$ of $T_1x_1 + T_2x_2 = y$ for a given $y \in \mathcal{H}$, (x_1', x_2') is also a solution if and only if $(x_1 - x_1', x_2 - x_2') \in \ker B_1$.
3. $\ker B_2 = \ker T_1 \cap \ker T_2$.
4. (1.4) is exact if and only if $T_1\mathcal{H} + T_2\mathcal{H} = \mathcal{H}$, $\ker T_1 \cap \ker T_2 = \{0\}$ and the solution of $T_1x_1 + T_2x_2 = y$ for a given $y \in \mathcal{H}$ is unique up to tautological perturbations.

Let us now look at the Taylor invertibility in the general case. For that purpose, consider the coordinate linear functionals $e_1, \ldots, e_d$ on $\mathbb{C}^d$ with respect to the standard basis. Let $\Lambda^0(\mathbb{C}^d) := \mathbb{C}$ and let $\Lambda^1(\mathbb{C}^d)$ be the vector space with basis $\{e_1, \ldots, e_d\}$. Given $w, w' \in \Lambda^1(\mathbb{C}^d)$ define $w \wedge w'$ by

$$w \wedge w'(v_1, v_2) := w(v_1)w'(v_2) - w(v_2)w'(v_1) \quad (v_1, v_2 \in \mathbb{C}^d).$$

Note that $w \wedge w = 0$ and $w \wedge w' = -w' \wedge w$. Also note that any 2-form is a linear combination of $e_1 \wedge e_2$, $e_1 \wedge e_3$, $\ldots, e_{d-1} \wedge e_d$ ($\binom{d}{d-2}$ elements). One may now define inductively all higher ordered forms with the help of the following definition of wedge product: Let w (resp. w') denote a p-form on the p-fold Cartesian product $\mathbb{C}^{d^{(p)}}$ of $\mathbb{C}^d$ (resp. a q-form on $\mathbb{C}^{d^{(q)}}$). We define the *wedge product* $w \wedge w'$ as the $(p+q)$-form on $\mathbb{C}^{d^{(p+q)}}$ given by

$$w \wedge w'(v) := \frac{1}{p!q!} \sum_{\sigma \in \mathbb{S}_{p+q}} \operatorname{sgn}(\sigma) w(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) w'(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),$$

where $\mathbb{S}_n$ denotes the group of permutations on $\{1, \ldots, n\}$. For $i = 1, \ldots, d$, let $\Lambda^i(\mathbb{C}^d)$ be the vector space generated by i-forms. We define $\Lambda(\mathbb{C}^d)$ as the algebra over $\mathbb{C}$ consisting of $\Lambda^i(\mathbb{C}^d)$ $(i = 1, \ldots, d)$ with identity e_0 defined by $e_0 \wedge w = w$, where the multiplication is the wedge product. The vector space $\Lambda(\mathbb{C}^d)$ is 2^d-dimensional, and it can be endowed with an inner product $\langle \cdot, \cdot \rangle_\Lambda$ so that

$$\{e_0\} \cup \{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\}$$

forms an orthonormal basis.

There are natural operators E_i $(i = 0, \ldots, d)$, to be referred to as *creation operators*, acting on the finite-dimensional Hilbert space $\Lambda(\mathbb{C}^d)$ defined by $E_i(w) := e_i \wedge w$ $(i = 1, \ldots d)$, and $E_0(w) := w$. These operators satisfy the *anti-commutation relations*:

$$E_iE_j + E_jE_i = 0 \quad (1 \leq i, j \leq d).$$

Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$ and let $\Lambda(\mathcal{H}) := \mathcal{H} \otimes_{\mathbb{C}} \Lambda(\mathbb{C}^d)$ be a Hilbert space endowed with the inner product

$$\langle x \otimes w, y \otimes w' \rangle_{\Lambda(\mathcal{H})} := \langle x, y \rangle_{\mathcal{H}} \langle w, w' \rangle_{\Lambda(\mathbb{C}^d)}.$$

We set $\Lambda^i(\mathcal{H}) := \mathcal{H} \otimes_{\mathbb{C}} \Lambda^i(\mathbb{C}^d)$ for $i = 0, \ldots, d$. Consider the *boundary operator* $\partial_T : \Lambda(\mathcal{H}) \to \Lambda(\mathcal{H})$ given by

$$\partial_T(h \otimes w) := \sum_{i=1}^{d} T_i(h) \otimes E_i(w).$$

Note that ∂_T is a bounded linear operator on $\Lambda(\mathcal{H})$. Since T is commuting and $E_1, \ldots, E_d$ are anti-commuting, $\partial_T^2 = 0$. This allows us to define the *Koszul complex* $K(T)$ associated with T as

$$K(T) : \{0\} \xrightarrow{0} \Lambda^0(\mathcal{H}) \xrightarrow{\partial_{T,0}} \Lambda^1(\mathcal{H}) \xrightarrow{\partial_{T,1}} \Lambda^2(\mathcal{H}) \to \cdots \to \Lambda^{d-1}(\mathcal{H}) \xrightarrow{\partial_{T,d-1}} \Lambda^d(\mathcal{H}) \xrightarrow{0} \{0\}$$

where $\partial_{T,i} := \partial_T|_{\Lambda^i(\mathcal{H})}$ for $i = 0, \ldots, d-1$.

REMARK 1.1.2. If $K(T)$ is exact, then

1. $\ker \partial_{T,0} = \{0\}$, that is, $\bigcap_{i=1}^d \ker T_i = \{0\}$.
2. $\operatorname{ran} \partial_{T,p}$ is closed ($\Rightarrow \operatorname{ran} \partial_{T,p}^*$ is closed).
3. $\operatorname{ran} \partial_{T,d-1} = \Lambda^d(\mathcal{H})$, that is, $T_1\mathcal{H} + \cdots + T_d\mathcal{H} = \mathcal{H}$.

The *Taylor spectrum* (or *joint spectrum*) of T is defined as

$$\sigma(T) := \{\lambda \in \mathbb{C}^d : K(T - \lambda) \text{ is not exact}\}.$$

We also define the *point spectrum* of T as

$$\sigma_p(T) := \{\lambda \in \mathbb{C}^d : \partial_{T-\lambda,0} \text{ is not one-to-one}\},$$

and the *left spectrum* of T as

$$\sigma_l(T) := \{\lambda \in \mathbb{C}^d : \partial_{T-\lambda,0} \text{ is not bounded from below}\}.$$

REMARK 1.1.3. Note that $\sigma_p(T) \subseteq \sigma_l(T) \subseteq \sigma(T)$.

It turns out that the Taylor spectrum of T is a nonempty compact subset of $\mathbb{C}^d$, which has the spectral mapping property for polynomial mappings p from $\mathbb{C}^d$ into $\mathbb{C}^{d'}$ for any positive integer d'.

The *spectral radius for the Taylor spectrum* of a commuting d-tuple T on $\mathcal{H}$ is defined as

$$r(T) := \max\{\|z\|_2 : z \in \sigma(T)\}.$$

We recall the spectral radius formula for the Taylor spectrum of a commuting d-tuple T ([37], [82]):

$$r(T) = \lim_{n\to\infty} \|Q_T^n(I)\|^{1/(2n)}. \tag{1.5}$$

In particular, $\sigma(T) \subseteq \{w \in \mathbb{C}^d : \|w\|_2 \le r(T)\}$. It is also known from [36, Lemma 3.6] that the *inner radius* $m_\infty(T)$ for the left spectrum of T is given by

$$m_\infty(T) = \sup_{n\ge 1} \inf_{\substack{h\in\mathcal{H} \\ \|h\|=1}} \langle Q_T^n(I)h, h\rangle^{1/2n}. \tag{1.6}$$

Here by inner radius, we mean the largest nonnegative number r for which

$$\sigma_l(T) \subseteq \{w \in \mathbb{C}^d : r \le \|w\|_2 \le r(T)\}.$$

For $k = 0, \ldots, d$, let $H^k(T)$ denote the kth cohomology group appearing in the Koszul complex $K(T)$. We say that T is *Fredholm* if $H^k(T)$ is finite-dimensional for every $k = 0, \ldots, d$. The *Fredholm index* $\operatorname{ind}(T)$ of a Fredholm d-tuple T is the Euler characteristic of $K(T)$ given by

$$\mathrm{ind}(T) := \sum_{k=0}^{d}(-1)^k \dim H^k(T). \tag{1.7}$$

The *essential spectrum* of T is defined as

$$\sigma_e(T) := \{\lambda \in \mathbb{C}^d : T-\lambda \text{ is not Fredholm}\}.$$

Clearly, the essential spectrum is a subset of the Taylor spectrum. By the Atkinson–Curto Theorem [43], $\sigma_e(T) = \sigma(\pi(T))$, where π denotes the Calkin map and

$$\pi(T) := (\pi(T_1), \ldots, \pi(T_d)).$$

In particular, the essential spectrum is a nonempty compact set with the polynomial spectral mapping property.

1.2. Classical multishifts. For a given multisequence $\mathbf{w} = \{w_\alpha^{(j)} : 1 \le j \le d,\ \alpha \in \mathbb{N}^d\}$ of complex numbers and an orthonormal basis $\{e_\alpha\}_{\alpha\in\mathbb{N}^d}$ of a Hilbert space $\mathcal{H}$, we define the *d-variable unilateral weighted shift* $S_{\mathbf{w}} = (S_1, \ldots, S_d)$ as

$$S_j e_\alpha := w_\alpha^{(j)} e_{\alpha+\epsilon_j} \quad (1 \le j \le d).$$

For convenience, we refer to $S_{\mathbf{w}}$ as the *classical unilateral multishift* or sometimes simply the *classical multishift*. Notice that S_j commutes with S_k if and only if $w_\alpha^{(j)} w_{\alpha+\epsilon_j}^{(k)} = w_\alpha^{(k)} w_{\alpha+\epsilon_k}^{(j)}$ for all $\alpha \in \mathbb{N}^d$. Moreover, $S_1, \ldots, S_d$ are bounded if and only if

$$\sup\{|w_\alpha^{(j)}| : 1 \le j \le d,\ \alpha \in \mathbb{N}^d\} < \infty. \tag{1.8}$$

In this text, we always assume that the multisequence $\mathbf{w}$ consists of positive numbers and satisfies (1.8).

Let $S_{\mathbf{w}}$ be a classical multishift. Define $\gamma_\alpha := \|S_w^\alpha e_0\|$ ($\alpha \in \mathbb{N}^d$), where 0 is the d-tuple in $\mathbb{N}^d$ with all entries being zero. Consider the Hilbert space $H^2(\gamma)$ of formal power series $f(z) = \sum_{\alpha\in\mathbb{N}^d} a_\alpha z^\alpha$ such that

$$\|f\|_{H^2(\gamma)}^2 := \sum_{\alpha\in\mathbb{N}^d} |a_\alpha|^2 \gamma_\alpha^2 < \infty.$$

It is worth noting that $S_{\mathbf{w}}$ is unitarily equivalent to the d-tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on the corresponding space $H^2(\gamma)$ [71, Proposition 8].

Let us discuss some basic examples of classical multishifts.

Example 1.2.1. For integers $a, d > 0$, let $\mathcal{H}_{a,d}$ be the reproducing kernel Hilbert space of holomorphic functions on the open unit ball $\mathbb{B}^d$ with reproducing kernel

$$\kappa_{\mathcal{H}_{a,d}}(z,w) = \frac{1}{(1-\langle z, w\rangle)^a} \quad (z, w \in \mathbb{B}^d).$$

The multiplication d-tuple $M_{z,a}$ on $\mathcal{H}_{a,d}$ is unitarily equivalent to the weighted shift d-tuple $S_{\mathbf{w},a}$ with weight multisequence

$$w_{\alpha,a}^{(j)} = \sqrt{\frac{\alpha_j+1}{|\alpha|+a}} \quad (\alpha \in \mathbb{N}^d,\ j = 1, \ldots, d),$$

(see [58, proof of Lemma 4.4]). The spaces $\mathcal{H}_{d,d}, \mathcal{H}_{d+1,d}, \mathcal{H}_{1,d}$ are commonly known as the *Hardy space* $H^2(\partial\mathbb{B}^d)$, the *Bergman space* $A^2(\mathbb{B}^d)$, the *Drury–Arveson space* H^2_d respectively. The associated classical multishifts $S_{\mathbf{w},d}, S_{\mathbf{w},d+1}, S_{\mathbf{w},1}$ are referred to as the *Szegö d-shift*, the *Bergman d-shift*, and the *Drury–Arveson d-shift* respectively.

For ready reference, we record the following proposition about various spectral parts of $S_{\mathbf{w},a}$ (see [59, Proposition 2.6] and [36, Theorem 3.4]).

Proposition 1.2.2. *Let* $S_{\mathbf{w},a}$ *be as defined in Example* 1.2.1. *Then*

$$\sigma(S_{\mathbf{w},a}) = \mathrm{cl}(\mathbb{B}^d), \quad \sigma_p(S_{\mathbf{w},a}) = \emptyset, \quad \sigma_p(S^*_{\mathbf{w},a}) = \mathbb{B}^d, \quad \sigma_e(S_{\mathbf{w},a}) = \partial\mathbb{B}^d = \sigma_l(S_{\mathbf{w},a}).$$

We will later investigate the so-called tree analogs of $S_{\mathbf{w},a}$ (Section 1.4). We conclude this section with a brief discussion of multiplication tuples on a vector-valued reproducing kernel Hilbert space of holomorphic functions in several variables.

Let Ω be a nonempty bounded, open connected subset of $\mathbb{C}^d$ and let E be a separable Hilbert space. Let κ be a $B(E)$-valued positive definite function on $\Omega \times \Omega$. Consider the reproducing kernel Hilbert space $\mathscr{H}_\kappa$ associated with κ. If κ is jointly continuous, holomorphic in the first variable and conjugate holomorphic in the second variable, then by the Weierstrass convergence theorem, $\mathscr{H}_\kappa$ consists of E-valued holomorphic functions defined on Ω. Suppose that $\mathscr{H}_\kappa$ is z_j-invariant for every $j = 1, \ldots, d$. An application of the closed graph theorem shows that the linear operator $\mathscr{M}_{z_j}$ of multiplication by z_j defines a bounded linear operator on $\mathscr{H}_\kappa$ for every $j = 1, \ldots, d$. We refer to $\mathscr{M}_z = (\mathscr{M}_{z_1}, \ldots, \mathscr{M}_{z_d})$ as the *multiplication d-tuple* on $\mathscr{H}_\kappa$. Note that

$$\mathscr{M}^*_{z_j}\kappa(\cdot, w)f = \overline{w}_j\kappa(\cdot, w)f \quad \text{for } w \in \Omega,\ f \in E \text{ and } j = 1, \ldots, d.$$

In particular, $\{w \in \mathbb{C}^d : \overline{w} \in \Omega\} \subseteq \sigma_p(\mathscr{M}^*_z)$.

1.3. Weighted shifts on directed trees. In this section, we recall some basic concepts from the theory of directed graphs which will be frequently used in the subsequent chapters. The reader is referred to R. Diestel [53] for a detailed exposition of graph theory (see also [67] for a brief account of the theory of directed trees).

A *directed graph* is a pair $\mathscr{T} = (V, \mathcal{E})$, where V is a nonempty set and $\mathcal{E}$ is a nonempty subset of $V \times V \setminus \{(v, v) : v \in V\}$. An element of V (resp. $\mathcal{E}$) is called a *vertex* (resp. an *edge*) of $\mathscr{T}$. A finite sequence $\{v_i\}_{i=1}^n$ of distinct vertices is said to be a *circuit* in $\mathscr{T}$ if $n \geq 2$, $(v_i, v_{i+1}) \in \mathcal{E}$ for all $1 \leq i \leq n-1$ and $(v_n, v_1) \in \mathcal{E}$. We say that two distinct vertices u and v of $\mathscr{T}$ are *connected by a path* if there exists a finite sequence $\{v_i\}_{i=1}^n$ of distinct vertices of $\mathscr{T}$ $(n \geq 2)$ such that $u = v_1$, $v_n = v$ and (v_i, v_{i+1}) or $(v_{i+1}, v_i) \in \mathcal{E}$ for all $1 \leq i \leq n-1$. A directed graph $\mathscr{T}$ is said to be *connected* if any two distinct vertices of $\mathscr{T}$ can be connected by a path in $\mathscr{T}$. For a subset W of V, define

$$\mathsf{Chi}(W) := \bigcup_{u \in W} \{v \in V \colon (u, v) \in \mathcal{E}\}.$$

One may define inductively $\mathsf{Chi}^{\langle n \rangle}(W)$ for $n \in \mathbb{N}$ as follows:

$$\mathsf{Chi}^{\langle n \rangle}(W) := \begin{cases} W & \text{if } n = 0, \\ \mathsf{Chi}(\mathsf{Chi}^{\langle n-1 \rangle}(W)) & \text{if } n \geq 1. \end{cases}$$

Given $v \in V$, we write $\mathsf{Chi}(v) := \mathsf{Chi}(\{v\})$, $\mathsf{Chi}^{\langle n\rangle}(v) := \mathsf{Chi}^{\langle n\rangle}(\{v\})$. A member of $\mathsf{Chi}(v)$ is called a *child* of v. The *descendants* of a vertex $v \in V$ are given by

$$\mathsf{Des}(v) := \bigcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n\rangle}(v).$$

For a given vertex $v \in V$, consider the set $\mathsf{Par}(v) := \{u \in V : (u, v) \in \mathcal{E}\}$ (set of "generalized" parents). If $\mathsf{Par}(v)$ is singleton, then the unique vertex in $\mathsf{Par}(v)$ is called the *parent* of v, which we denote by $\mathsf{par}(v)$. Let

$$\mathsf{Root}(\mathscr{T}) := \{v \in V : \mathsf{Par}(v) = \emptyset\}.$$

An element of $\mathsf{Root}(\mathscr{T})$ is called a *root* of $\mathscr{T}$. If $\mathsf{Root}(\mathscr{T})$ is singleton, then its unique element is denoted by root. We set $V^{\circ} := V \setminus \mathsf{Root}(\mathscr{T})$. A directed graph $\mathscr{T} = (V, \mathcal{E})$ is called a *directed tree* if $\mathscr{T}$ has no circuits, $\mathscr{T}$ is connected and each vertex $v \in V^{\circ}$ has a unique parent.

Remark 1.3.1. It is well-known that every directed tree has at most one root [67, Proposition 2.1.1] (see Figure 1.1).

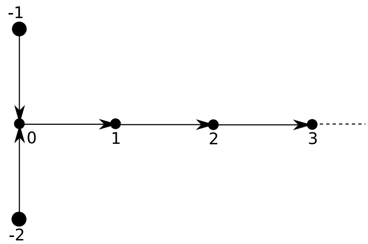

Fig. 1.1. A directed graph which is not a directed tree

The following example is borrowed from [67, Chapter 6].

Example 1.3.2. For a positive integer n_0 and $k_0 \in \mathbb{N}$, we define the directed tree $\mathscr{T}_{n_0,k_0} = (V, \mathcal{E})$ as follows:

$$\begin{aligned} V &= \{-1, \ldots, -k_0\} \cup \mathbb{N}, \\ \mathcal{E} &= \{(j, j+1) : j = -k_0, \ldots, -1\} \cup \{(0, j) : j = 1, \ldots, n_0\} \\ &\quad \cup \bigcup_{j=1}^{n_0} \{(j + (l-1)n_0, j + l n_0) : l \geq 1\} \end{aligned}$$

(see Figures 1.2 and 1.3 for the cases $(n_0, k_0) = (1, 0)$ and $(n_0, k_0) = (2, 0)$ respectively).

Fig. 1.2. The directed tree $\mathscr{T}_{1,0}$

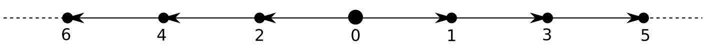

Fig. 1.3. The directed tree $\mathscr{T}_{2,0}$

A directed graph $\mathscr{T}$ is said to be

(i) *rooted* if it has a unique root;
(ii) *locally finite* if $\mathrm{card}(\mathsf{Chi}(u))$ is finite for all $u \in V$;
(iii) *leafless* if every vertex has at least one child.

Let $\mathscr{T} = (V, \mathcal{E})$ be a directed tree and let $l^2(V)$ stand for the Hilbert space of square summable complex functions on V equipped with the standard inner product. Note that the set $\{e_u\}_{u \in V}$ is an orthonormal basis of $l^2(V)$, where $e_u \in l^2(V)$ is the indicator function of $\{u\}$. Given a system $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of nonzero complex numbers, we define the *weighted shift operator* S_λ on $\mathscr{T}$ with weights λ by

$$\mathscr{D}(S_\lambda) := \{f \in l^2(V) \colon \Lambda_{\mathscr{T}} f \in l^2(V)\}, \quad S_\lambda f := \Lambda_{\mathscr{T}} f, \quad f \in \mathscr{D}(S_\lambda),$$

where $\Lambda_{\mathscr{T}}$ is the mapping defined on complex functions f on V by

$$(\Lambda_{\mathscr{T}} f)(v) := \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v \text{ is a root of } \mathscr{T}. \end{cases}$$

Unless stated otherwise, $\{\lambda_v\}_{v \in V^\circ}$ consists of nonzero complex numbers and S_λ belongs to $B(l^2(V))$. It may be concluded from [67, Proposition 3.1.7] that S_λ is an injective weighted shift on $\mathscr{T}$ if and only if $\mathscr{T}$ is leafless.

In what follows, we always assume that all the directed trees considered are leafless.

Let $\mathscr{T} = (V, \mathcal{E})$ be a rooted directed tree with root root. Then

$$V = \bigsqcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n \rangle}(\mathsf{root}) \quad \text{(disjoint union)} \tag{1.9}$$

[67, Corollary 2.1.5]. For $u \in V$, let d_u denote the unique integer in $\mathbb{N}$ (to be referred to as the *depth of u in $\mathscr{T}$*) such that $u \in \mathsf{Chi}^{\langle \mathsf{d}_u \rangle}(\mathsf{root})$. We use the convention that $\mathsf{Chi}^{\langle j \rangle}(\mathsf{root}) = \emptyset$ if $j < 0$. A similar convention holds for par. The *branching index* $k_{\mathscr{T}} \in \mathbb{N} \cup \{\infty\}$ of a rooted directed tree $\mathscr{T}$ is defined as

$$k_{\mathscr{T}} := \begin{cases} 1 + \sup\{\mathsf{d}_w : w \in V_\prec\} & \text{if } V_\prec \text{ is nonempty}, \\ 0 & \text{if } V_\prec \text{ is empty}, \end{cases}$$

where $V_\prec := \{u \in V : \mathrm{card}(\mathsf{Chi}(u)) \geq 2\}$. If $V_\prec$ is finite, then $k_{\mathscr{T}}$ is necessarily finite but the converse is not true in general [35, Remark 2].

Remark 1.3.3. Let $\mathscr{T}_{n_0,k_0}$ be as discussed in Example 1.3.2. Note that $\mathscr{T}_{n_0,k_0}$ is a locally finite, rooted directed tree with branching index

$$k_{\mathscr{T}} = \begin{cases} k_0 + 1 & \text{if } n_0 \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

If S_λ is a left invertible weighted shift on a rooted directed tree, then S_λ has the wandering subspace property. This fact was recorded in [35, Theorem 2.2(iii)]. But it turns out that this is a general feature of a (bounded) weighted shift on a rooted directed tree, and the left-invertibility is no longer required. We illustrate this fact in the following proposition.

PROPOSITION 1.3.4. *Let $\mathscr{T} = (V, \mathcal{E})$ be a leafless, rooted directed tree and $S_\lambda \in B(l^2(V))$ be a weighted shift on $\mathscr{T}$. Set $E := \ker S_\lambda^*$. Then*

$$\bigvee_{k\in\mathbb{N}} S_\lambda^k(E) = l^2(V). \tag{1.10}$$

Proof. Set $M := \bigvee_{k\in\mathbb{N}} S_\lambda^k(E)$. We claim that $e_v \in M$ for all $v \in \mathsf{Chi}^{\langle n\rangle}(\mathsf{root})$ and for all $n \in \mathbb{N}$. We prove this by induction on $n \in \mathbb{N}$. Recall from [67, Proposition 3.5.1(ii)] that

$$E = [e_{\mathsf{root}}] \oplus \bigoplus_{v\in V} \big(l^2(\mathsf{Chi}(v)) \ominus [\Gamma_v]\big), \tag{1.11}$$

where $\Gamma_v : \mathsf{Chi}(v) \to \mathbb{C}$ is given by $\Gamma_v = \sum_{u\in\mathsf{Chi}(v)} \lambda_u e_u = S_\lambda e_v$. Clearly, $e_{\mathsf{root}} \in M$. Thus the claim holds true for $n = 0$. Suppose it is true for some $n \in \mathbb{N}$. That is, $e_u \in M$ for all $u \in \mathsf{Chi}^{\langle n\rangle}(\mathsf{root})$. Let $v \in \mathsf{Chi}^{\langle n+1\rangle}(\mathsf{root})$. Then $v \in \mathsf{Chi}(u)$ for some $u \in \mathsf{Chi}^{\langle n\rangle}(\mathsf{root})$. By the induction hypothesis, $e_u \in M$. Since M is S_λ-invariant, $S_\lambda e_u = \Gamma_u \in M$ and hence $[\Gamma_u] \subseteq M$. Further, as $E \subseteq M$, $l^2(\mathsf{Chi}(u)) \ominus [\Gamma_u] \subseteq M$. Thus $l^2(\mathsf{Chi}(u)) \subseteq M$, which in turn implies that $e_v \in M$. Thus the claim stands verified. By (1.9), it follows that $e_v \in M$ for all $v \in V$, and hence $l^2(V) \subseteq M$. Thus (1.10) stands true. ∎

The argument above relies completely on the formula (1.11) for the kernel of S_λ^*. Clearly, this formula is associated with a system of linear equations corresponding to vertices from the branching set. In case of several variables, this correspondence becomes highly involved. This is one of the difficulties in the derivation of the wandering subspace property in several variables (see Theorem 4.0.1).

1.4. Overture. In this section, we briefly discuss some important aspects of this work. The exposition here is far from being complete, but it conveys some of the essential ideas presented in this text. Motivated by [36, Question 4.7] about the classification of so-called spherical tuples of higher multiplicity, we construct tree analogs of the multiplication d-tuples $M_{z,a}$ as discussed in Example 1.2.1. We outline this construction as follows.

Consider the directed Cartesian product $\mathscr{T} = (V, \mathcal{E})$ of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index and let $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ denote the multishift on $\mathscr{T}$ with weights given by

$$\lambda_w^{(j)} = \sqrt{\frac{1}{\operatorname{card}(\mathsf{Chi}_j(v))}}\sqrt{\frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+a}} \quad \text{for } w \in \mathsf{Chi}_j(v),\ v \in V \text{ and } j = 1, \ldots, d.$$

Here a is a positive integer and $\mathsf{d}_u \in \mathbb{N}^d$ denotes the depth of $u \in V$ in $\mathscr{T}$. It turns out that the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is unitarily equivalent to the multiplication d-tuple $\mathscr{M}_{z,a}$ acting on the reproducing kernel Hilbert space $\mathscr{H}_{a,d}$ of E-valued holomorphic functions on the open unit ball in $\mathbb{C}^d$, where E denotes the joint kernel of $S^*_{\boldsymbol\lambda_{\mathfrak{c}_a}}$. The associated reproducing kernel $\kappa_{\mathscr{H}_{a,d}} : \mathbb{B}^d \times \mathbb{B}^d \to B(E)$ is given by

$$\kappa_{\mathscr{H}_{a,d}}(z, w) = \frac{1}{(1 - \langle z, w\rangle)^a} P_{[e_{\mathsf{root}}]} + \sum_{\substack{F\in\mathscr{P}\\ F\neq\emptyset}} \sum_{u\in\Omega_F} \kappa_{u,F}(z, w),$$

where

$$\kappa_{u,F}(z,w) = \sum_{\alpha \in \mathbb{N}^d} \frac{\mathsf{d}_u!}{(\mathsf{d}_u+\alpha)!}\Big(\prod_{j=0}^{|\alpha|-1}(|\mathsf{d}_u|+a+j)\Big) z^{\alpha}\overline{w}^{\alpha}\, P_{\mathcal{L}_{u,F}}$$

with $P_{\mathcal{M}}$ being the orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{M}$ of $\mathcal{H}$. We refer the reader to Theorem 5.2.6 for a precise statement.

Remark 1.4.1. In case $\mathscr{T}_j = \mathscr{T}_{1,0}$ for $j = 1, \ldots, d$, $\mathcal{L}_{u,F} = \{0\}$ for every $u \in \Omega_F$ and every nonempty $F \in \mathscr{P}$, and hence we obtain the kernel

$$\frac{I_E}{(1-\langle z, w\rangle)^a} \quad (z, w \in \mathbb{B}^d).$$

Let us try to understand the above formula for $\kappa_{\mathscr{H}_{a,d}}(z,w)$. The following decomposition of the joint kernel E of $S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ is useful in this regard:

$$E = [e_{\mathsf{root}}] \oplus \bigoplus_{\substack{F\in\mathscr{P}\\ F\neq\emptyset}} \bigoplus_{u\in\Omega_F} \mathcal{L}_{u,F}.$$

Here $\mathscr{P}$ denotes the power set of $\{1, \ldots, d\}$ and Ω_F is a certain indexing set corresponding to $F \in \mathscr{P}$. In particular, the first series appearing in $\kappa_{\mathscr{H}_{a,d}}(z,w)$ corresponds to $[e_{\mathsf{root}}]$, while $\kappa_{u,F}$ corresponds to $\mathcal{L}_{u,F}$, where $\mathcal{L}_{u,F}$ is a subspace associated with a system of linear equations related to $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ (the reader is referred to Chapters 4 and 5 for a detailed discussion). In case $d = 1$, the reproducing kernel $\kappa_{\mathscr{H}_a} = \kappa_{\mathscr{H}_{a,1}}$ takes a concrete form:

$$\begin{aligned}\kappa_{\mathscr{H}_a}(z,w) &= \frac{1}{(1-z\overline{w})^a}\, P_{[e_{\mathsf{root}}]} \\ &\quad + \sum_{v\in V_{\prec}}\sum_{n=0}^{\infty} \frac{(\mathsf{d}_v+n+a)!(\mathsf{d}_v+1)!}{(\mathsf{d}_v+a)!(\mathsf{d}_v+n+1)!}\, z^n\overline{w}^n\, P_{l^2(\mathsf{Chi}(v))\ominus[\Gamma_v]} \quad (z, w \in \mathbb{D}),\end{aligned}$$

where $V_{\prec}$ denotes the set of branching vertices of V. One can rewrite this formula using the hypergeometric function ${}_2F_1(a, b, c, t)$ [87, p. 217]:

$$\begin{aligned}\kappa_{\mathscr{H}_a}(z,w) &= {}_2F_1(a,1,1,z\overline{w})P_{[e_{\mathsf{root}}]} \\ &\quad + \sum_{v\in V_{\prec}} {}_2F_1(\mathsf{d}_v+a+1,1,\mathsf{d}_v+2,z\overline{w})P_{l^2(\mathsf{Chi}(v))\ominus[\Gamma_v]} \quad (z, w \in \mathbb{D}).\end{aligned}$$

An alternative verification of this formula (based on Shimorin's analytic model) will be given in Chapter 5.

Remark 1.4.2. Below we analyze the cases in which $a = 1$ and $a = 2$ in the one-dimensional case. Note that $\kappa_{\mathscr{H}_1}$ is the Cauchy kernel $\frac{I_E}{1-z\overline{w}}$, while $\kappa_{\mathscr{H}_2}$ is given by

$$\begin{aligned}\kappa_{\mathscr{H}_2}(z,w) &= \sum_{n=0}^{\infty}(n+1)P_{[e_{\mathsf{root}}]}\, z^n\overline{w}^n \\ &\quad + \sum_{v\in V_{\prec}}\sum_{n=0}^{\infty}\left(\frac{\mathsf{d}_v+n+2}{\mathsf{d}_v+2}\right) z^n\overline{w}^n\, P_{l^2(\mathsf{Chi}(v))\ominus[\Gamma_v]} \quad (z, w \in \mathbb{D}).\end{aligned}$$

Note that $\kappa_{\mathscr{H}_2} = 1/(1-z\overline{w})^2$ in case $\mathscr{T} = \mathscr{T}_{1,0}$.

It turns out that $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ is a finitely multicyclic, essentially normal d-tuple with Taylor spectrum being equal to the closed unit ball $\mathrm{cl}(\mathbb{B}^d)$. However, we would like to emphasize here that $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ are, in general, not unitarily equivalent to orthogonal direct sums of any number of copies of the classical multishifts $S_{\mathbf{w},a}$. For instance, in case $d = 1$ and $a = 2$, the defect operator $I - 2S_{\mathbf{w},a}S^*_{\mathbf{w},a} + S^2_{\mathbf{w},a}S^{*2}_{\mathbf{w},a}$ is always an orthogonal projection of rank 1 [64, p. 618]. On the other hand, if $v \in V^\circ$ is such that $\mathrm{card}(\mathsf{sib}(\mathsf{par}(v))) = 1$ and $s_v := 1/\mathrm{card}(\mathsf{sib}(v)) < 1$, then

$$\langle (I - 2S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} + S^2_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}S^{*2}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}})^j e_v, e_v\rangle = (1 - s_v)^j \quad \text{for } j = 1, 2,$$

which shows that $I - 2S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} + S^2_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}S^{*2}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ is not even idempotent.

We conclude this chapter with a brief description of the layout of the present work. In Chapter 2, we discuss the theory of products of directed graphs in the context of directed trees. The motivation for this chapter comes from the theory of multishifts with which we are primarily concerned. In particular, we pay attention to two important notions, namely, directed Cartesian product and tensor product of directed trees. We will see the significance of the notion of tensor product of directed trees in the context of so-called spherically balanced multishifts later in Chapter 5.

In Chapter 3, we formally introduce the notion of multishifts $S_{\boldsymbol{\lambda}}$ on a directed Cartesian product $\mathscr{T}$ of finitely many rooted directed trees. Apart from various elementary properties of multishifts, we reveal the relation to the shift operator arising from the directed semi-tree structure of $\mathscr{T}$ as ensured in Chapter 2. The later half of this chapter deals with spectral properties of multishifts $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$. A particular attention is given to circularity and analyticity of $S_{\boldsymbol{\lambda}}$. Indeed, $S_{\boldsymbol{\lambda}}$ turns out to be strongly circular and separately analytic. These properties are then used to show that the point spectrum of $S_{\boldsymbol{\lambda}}$ is empty and the Taylor spectrum is Reinhardt. Further, we obtain a matrix decomposition of 2-variable multishifts and discuss some of its consequences for spectral theory. In particular, we locate essential spectra for a family of multishifts.

Chapter 4 is devoted to the description of the joint kernel of $S^*_{\boldsymbol{\lambda}}$. This in turn relies on decompositions of the vertex set of a product of directed trees and that of the underlying Hilbert space. It turns out that the problem of computing the joint kernel $\ker S^*_{\boldsymbol{\lambda}}$ of $S^*_{\boldsymbol{\lambda}}$ is equivalent to solving a system of (possibly infinitely many) linear equations. We illustrate this with the help of two instructive examples in which $\ker S^*_{\boldsymbol{\lambda}}$ is explicitly computed. The description of $\ker S^*_{\boldsymbol{\lambda}}$ enables us to derive the wandering subspace property for $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$. It is to be noted that the situation gets far simpler in the case of either one-variable weighted shifts or classical multishifts. As a consequence, we obtain a multivariable counterpart of Shimorin's model in this context, and use it to show that the multishifts which admit Shimorin's model belong to the Cowen–Douglas class.

In Chapter 5, we discuss two notions of balanced multishifts, namely, spherical and toral ones. We use the classification of torally balanced multishifts to obtain a local analog of von Neumann's inequality. The classification of spherically balanced multishifts is given in terms of certain integral representations. Unlike the classical case [33], several Reinhardt measures appear in this characterization. In the classification of spherically balanced multishifts, the notion of tensor product $\mathscr{T}^{\otimes}$ of directed trees appears naturally. Indeed, various properties of $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ are reflected in the corresponding properties of the one-variable shift on the component of $\mathscr{T}^{\otimes}$ containing the root. This correspondence allows us, in particular, to compute the spectral radius of the Taylor spectrum and the inner spectral radius of the left spectrum for $S_{\boldsymbol\lambda}$. In this chapter, we also discuss special classes of joint subnormal and joint hyponormal multishifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}$. In particular, we characterize these classes within the class of spherically balanced multishifts. We illustrate these results with a family of examples which can be thought of as tree analogs of the multiplication tuples on the reproducing kernel Hilbert spaces associated with the kernels $1/(1 - \langle z, w\rangle)^a$ $(z, w \in \mathbb{B}^d, a > 0)$.

2. Products of directed trees

In this chapter, we discuss two well-studied notions of product of directed trees, namely, the directed Cartesian product and the tensor product ([86], [92], [80], [62], [56]). These notions can certainly be introduced in the general context of (directed) graphs. However, since the main objects of the present study are multishifts on products of directed trees, we confine ourselves to directed trees.

We recall the assumption that all the directed trees under consideration are assumed to be leafless.

2.1. Directed Cartesian products of directed trees. The definition of the directed Cartesian product of two directed graphs has been introduced and studied by G. Sabidussi [86] (refer also to [56]). This notion readily generalizes to the case of finitely many directed trees as given below.

Definition 2.1.1. Let d be a positive integer and let $\mathscr{T}_j = (V_j, \mathcal{E}_j)$ $(j = 1, \ldots, d)$ be a collection of directed trees. The *directed Cartesian product* of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ is a directed graph $\mathscr{T} = (V, \mathcal{E})$, where $V := V_1 \times \cdots \times V_d$ and

$$\mathcal{E} := \big\{(v, w) \in V \times V : \text{there is a positive integer } k \in \{1, \ldots, d\} \\ \text{such that } v_j = w_j \text{ for } j \neq k \text{ and the edge } (v_k, w_k) \in \mathcal{E}_k\big\}.$$

We adhere here to the convention that $v \in V = V_1 \times \cdots \times V_d$ is always understood as $v = (v_1, \ldots, v_d)$ with $v_j \in V_j$ for $j = 1, \ldots, d$. We sometimes use the notation $\mathscr{T}_1 \times \cdots \times \mathscr{T}_d$ for the directed Cartesian product $\mathscr{T}$ of $\mathscr{T}_1, \ldots, \mathscr{T}_d$.

Remark 2.1.2. Note that $\mathcal{E}$ is precisely the collection of edges (v, w) such that w_k is a child of v_k for some k and $w_j = v_j$ for all $j \neq k$. In other words, $\mathsf{Chi}(v) = \{w \in V : (v, w) \in \mathcal{E}\}$. Further note that for a subset W of V,

$$\mathsf{Chi}(W) := \bigcup_{u \in W} \mathsf{Chi}(u).$$

Inductively for $n \in \mathbb{N}$, we set

$$\mathsf{Chi}^{\langle n\rangle}(W) := \begin{cases} W & \text{if } n = 0, \\ \mathsf{Chi}(\mathsf{Chi}^{\langle n-1\rangle}(W)) & \text{if } n \geq 1. \end{cases}$$

Remark 2.1.3. In case $d \geq 2$, $\mathscr{T} = (V, \mathcal{E})$ is never a directed tree. Indeed, $\mathrm{card}(\mathsf{Chi}(u) \cap \mathsf{Chi}(v)) \geq 1$ for some $u, v \in V$ with $u \neq v$. This can be seen as follows. Since $\mathscr{T}_1, \ldots, \mathscr{T}_d$ are leafless, for any $w = (w_1, \ldots, w_d) \in V$ consider $u = (u_1, w_2, \ldots, w_d)$ and $v =$

$(w_1, u_2, w_3, \ldots, w_d)$, where $u_j \in \mathsf{Chi}(w_j)$ for $j = 1, 2$. Then

$$(u_1, u_2, w_3, \ldots, w_d) \in \mathsf{Chi}(u) \cap \mathsf{Chi}(v).$$

REMARK 2.1.4. For $j = 1, \ldots, d$, let $\mathscr{T}_j$ be a rooted directed tree with root denoted by root_j. Then the directed Cartesian product $\mathscr{T}$ of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ is a rooted directed graph with root given by $\mathsf{root} = (\mathsf{root}_1, \ldots, \mathsf{root}_d) \in V$.

Below we discuss three basic examples of directed Cartesian products.

EXAMPLE 2.1.5. Let $\mathscr{T}_{1,0}$ be as discussed in Example 1.3.2 and let $\mathscr{T}_j = \mathscr{T}_{1,0}$ for all $j = 1, \ldots, d$. The directed Cartesian product $\mathscr{T} = \mathscr{T}_{1,0}^d$ of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ is given by $\mathscr{T} = (V, \mathcal{E})$, where $V = \mathbb{N}^d$ and

$$\begin{aligned}\mathcal{E} = \big\{(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d &: \text{there is a positive integer } k \in \{1, \ldots, d\} \\ &\quad \text{such that } \alpha_j = \beta_j \text{ for } j \neq k \text{ and } \beta_k = \alpha_k + 1\big\} \\ = \{(\alpha, \alpha + \epsilon_j) &\in \mathbb{N}^d \times \mathbb{N}^d : j = 1, \ldots, d\}\end{aligned}$$

(see Figure 2.1).

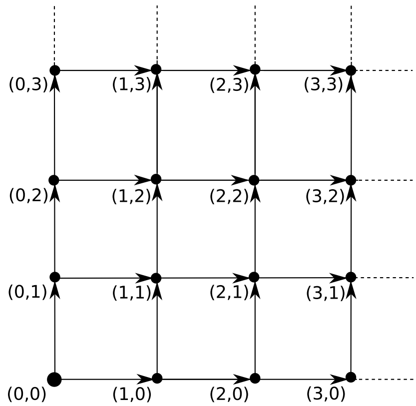

Fig. 2.1. The directed Cartesian product $\mathscr{T} = \mathscr{T}_{1,0} \times \mathscr{T}_{1,0}$

The directed graph $\mathscr{T}$ discussed above is the *d-finite Bargmann graph* in disguise, which was introduced in [79, Section 3].

EXAMPLE 2.1.6. Let $\mathscr{T}_{1,0}, \mathscr{T}_{2,0}$ be as discussed in Example 1.3.2. Then the directed Cartesian product $\mathscr{T} = \mathscr{T}_{2,0} \times \mathscr{T}_{1,0}$ is given by $\mathscr{T} = (V, \mathcal{E})$, where $V = \mathbb{N} \times \mathbb{N}$ and $((m, n), (k, l)) \in \mathcal{E}$ if and only if either $m = k$ and $l = n + 1$, or $n = l$ and

$$k = \begin{cases} m + 2 & \text{if } m \neq 0, \\ 1 \text{ or } 2 & \text{otherwise} \end{cases}$$

(see Figure 2.2).

EXAMPLE 2.1.7. Let $\mathscr{T}_{2,0}$ be as discussed in Example 1.3.2. Then the directed Cartesian product of $\mathscr{T}_{2,0}$ with itself is given by $\mathscr{T} = (V, \mathcal{E})$, where $V = \mathbb{N} \times \mathbb{N}$ and $((m, n), (k, l)) \in \mathcal{E}$

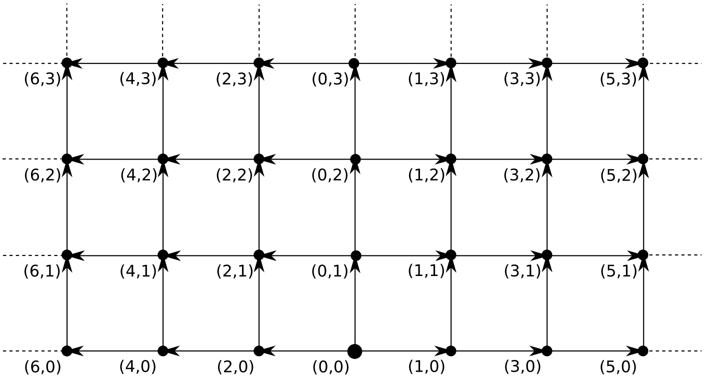

Fig. 2.2. The directed Cartesian product $\mathscr{T} = \mathscr{T}_{2,0} \times \mathscr{T}_{1,0}$

if and only if either $m = k$ and

$$l = \begin{cases} n+2 & \text{if } n \neq 0, \\ 1 \text{ or } 2 & \text{otherwise,} \end{cases}$$

or $n = l$ and

$$k = \begin{cases} m+2 & \text{if } m \neq 0, \\ 1 \text{ or } 2 & \text{otherwise} \end{cases}$$

(see Figure 2.3).

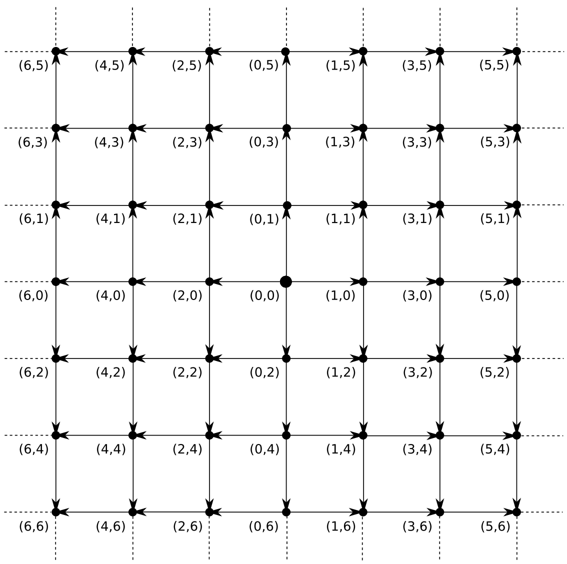

Fig. 2.3. The directed Cartesian product $\mathscr{T} = \mathscr{T}_{2,0} \times \mathscr{T}_{2,0}$

Definition 2.1.8. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. For $j = 1, \ldots, d$ and $v \in V$, we set

$$\mathsf{Chi}_j(v) := \big\{ w \in V : w_j \in \mathsf{Chi}(v_j) \text{ and } w_k = v_k \text{ for } k \neq j \big\}.$$

Further, for $W \subseteq V$, we define

$$\mathsf{Chi}_j(W) := \bigcup_{w \in W} \mathsf{Chi}_j(w).$$

For $k \in \mathbb{N}$, we denote $\mathsf{Chi}_j \overset{k}{\cdots} \mathsf{Chi}_j(W)$ by $\mathsf{Chi}_j^{\langle k \rangle}(W)$, where we understand that $\mathsf{Chi}_j^{\langle 0 \rangle}(W) = W$. Further, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $W \subseteq V$, we define

$$\mathsf{Chi}^{\ll \alpha \gg}(W) := \mathsf{Chi}_1^{\langle \alpha_1 \rangle} \cdots \mathsf{Chi}_d^{\langle \alpha_d \rangle}(W).$$

If $W = \{v\}$ for some $v \in V$, then we use the simpler notation $\mathsf{Chi}^{\ll \alpha \gg}(v)$.

Remark 2.1.9. It may be concluded from Remark 2.1.2 that

$$\mathsf{Chi}(v) = \{w \in V : (v, w) \in \mathcal{E}\} = \bigcup_{j=1}^{d} \mathsf{Chi}_j(v).$$

Further, $\mathsf{Chi}^{\ll \epsilon_j \gg}(v) = \mathsf{Chi}_j(v)$ for $j = 1, \ldots, d$.

The following lemma enriches the directed graph $\mathscr{T} = (V, \mathcal{E})$ with a tree-like structure.

Lemma 2.1.10. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Then:*

(i) $\mathsf{Chi}_j \mathsf{Chi}_i(v) = \mathsf{Chi}_i \mathsf{Chi}_j(v)$ *for all* $v \in V$ *and* $i, j = 1, \ldots, d$.

(ii) *For each* $\alpha \in \mathbb{N}^d$ *and distinct* $v, w \in V$,

$$\mathsf{Chi}^{\ll \alpha \gg}(v) \cap \mathsf{Chi}^{\ll \alpha \gg}(w) = \emptyset.$$

(iii) *For distinct* $\alpha, \beta \in \mathbb{N}^d$ *and* $v \in V$,

$$\mathsf{Chi}^{\ll \alpha \gg}(v) \cap \mathsf{Chi}^{\ll \beta \gg}(v) = \emptyset.$$

(iv) *For any* $n \in \mathbb{N}$ *and* $v \in V$,

$$\mathsf{Chi}^{\langle n \rangle}(v) = \bigsqcup_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| = n}} \mathsf{Chi}^{\ll \alpha \gg}(v).$$

(v) *For distinct* $m, n \in \mathbb{N}$ *and* $v \in V$,

$$\mathsf{Chi}^{\langle m \rangle}(v) \cap \mathsf{Chi}^{\langle n \rangle}(v) = \emptyset.$$

(vi) *If* $\mathscr{T}_1, \ldots, \mathscr{T}_d$ *are rooted directed trees, then*

$$\bigsqcup_{j \in \mathbb{N}} \mathsf{Chi}^{\langle j \rangle}(\mathsf{root}) = V = \bigsqcup_{\alpha \in \mathbb{N}^d} \mathsf{Chi}^{\ll \alpha \gg}(\mathsf{root}).$$

Proof. The conclusion in (i) follows from

$$\mathsf{Chi}_j \mathsf{Chi}_i(v) = \begin{cases} \{u \in V : u_i \in \mathsf{Chi}(v_i), u_j \in \mathsf{Chi}(v_j) \text{ and } u_k = v_k \text{ for } k \neq i, j\}, & i \neq j, \\ \{u \in V : u_i \in \mathsf{Chi}^{\langle 2 \rangle}(v_i) \text{ and } u_k = v_k \text{ for } k \neq i\}, & i = j, \end{cases}$$

for $i, j = 1, \ldots, d$. To see (ii), let $u \in \mathsf{Chi}^{\ll \alpha \gg}(v) \cap \mathsf{Chi}^{\ll \alpha \gg}(w)$. Then $u_j \in \mathsf{Chi}^{\langle \alpha_j \rangle}(v_j) \cap \mathsf{Chi}^{\langle \alpha_j \rangle}(w_j)$ for all $j = 1, \ldots, d$. In other words, $\mathsf{par}^{\langle \alpha_j \rangle}(u_j) = v_j$ and $\mathsf{par}^{\langle \alpha_j \rangle}(u_j) = w_j$ for all $j = 1, \ldots, d$, where $\mathsf{par}^{\langle k \rangle}(\cdot)$ is obtained by composing the partial function par with itself k times, $k \geq 1$ (refer to [67, p. 9]).

Since parent is unique, it follows that $v_j = w_j$ for all $j = 1, \ldots, d$. Thus $v = w$, proving (ii).

Let $u \in \mathsf{Chi}^{\ll\alpha\gg}(v) \cap \mathsf{Chi}^{\ll\beta\gg}(v)$. Then $u_j \in \mathsf{Chi}^{\langle\alpha_j\rangle}(v_j) \cap \mathsf{Chi}^{\langle\beta_j\rangle}(v_j)$ for all $j = 1, \ldots, d$. In other words, $\mathsf{par}^{\langle\alpha_j\rangle}(u_j) = v_j$ and $\mathsf{par}^{\langle\beta_j\rangle}(u_j) = v_j$ for all $j = 1, \ldots, d$. Now (1.9) implies that $\alpha_j = \beta_j$ for all $j = 1, \ldots, d$. Thus $\alpha = \beta$, which proves (iii).

We prove (iv) by induction on $n \in \mathbb{N}$. By Remark 2.1.9,

$$\mathsf{Chi}(v) = \bigsqcup_{j=1}^{d} \mathsf{Chi}_j(v) = \bigsqcup_{j=1}^{d} \mathsf{Chi}^{\ll\epsilon_j\gg}(v) = \bigsqcup_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=1}} \mathsf{Chi}^{\ll\alpha\gg}(v).$$

Thus the result holds for $n = 1$. Suppose it is true for some $n \geq 1$. Then

$$\begin{aligned}
\mathsf{Chi}^{\langle n+1\rangle}(v) &= \mathsf{Chi}\Big(\bigsqcup_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=n}} \mathsf{Chi}^{\ll\alpha\gg}(v)\Big) = \bigsqcup_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=n}} \mathsf{Chi}(\mathsf{Chi}^{\ll\alpha\gg}(v)) \\
&= \bigsqcup_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=n}} \bigcup_{j=1}^{d} \mathsf{Chi}^{\ll\epsilon_j\gg}(\mathsf{Chi}^{\ll\alpha\gg}(v)) \\
&= \bigcup_{j=1}^{d} \bigsqcup_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=n}} \mathsf{Chi}^{\ll\alpha+\epsilon_j\gg}(v) = \bigsqcup_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=n+1}} \mathsf{Chi}^{\ll\alpha\gg}(v),
\end{aligned}$$

where the last union is disjoint in view of (iii). Part (v) is now immediate from (iii) and (iv).

To see the last part, let $U := \bigsqcup_{\alpha\in\mathbb{N}^d} \mathsf{Chi}^{\ll\alpha\gg}(\mathsf{root})$ and $W := \bigsqcup_{j\in\mathbb{N}} \mathsf{Chi}^{\langle j\rangle}(\mathsf{root})$. Clearly, U and W are subsets of V. By (iv), $U = W$ and the unions in U and W are disjoint. Thus it suffices to check that $V \subseteq U$. Let $v = (v_1, \ldots, v_d) \in V$. Then there exists $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $v_j \in \mathsf{Chi}^{\langle\alpha_j\rangle}(\mathsf{root}_j)$ for $j = 1, \ldots, d$. This implies that $v \in \mathsf{Chi}_1^{\langle\alpha_1\rangle} \cdots \mathsf{Chi}_d^{\langle\alpha_d\rangle}(\mathsf{root}) = \mathsf{Chi}^{\ll\alpha\gg}(\mathsf{root})$. ■

DEFINITION 2.1.11. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $v \in V$. The unique $\mathsf{d}_v \in \mathbb{N}^d$ such that

$$v \in \mathsf{Chi}^{\ll\mathsf{d}_v\gg}(\mathsf{root})$$

will be referred to as the *depth of* v *in* $\mathscr{T}$ (see Lemma 2.1.10(vi)). The tth *generation* $\mathcal{G}_t$ of $\mathscr{T}$ is defined as

$$\mathcal{G}_t := \{v \in V : |\mathsf{d}_v| = t\}.$$

REMARK 2.1.12. Note that $\mathsf{d}_v = (\mathsf{d}_{v_1}, \ldots, \mathsf{d}_{v_d})$, where d_{v_j} denotes the depth of the vertex $v_j \in V_j$ in the directed tree $\mathscr{T}_j$ for $j = 1, \ldots, d$. Note further that by Lemma 2.1.10(iv), $\mathcal{G}_t = \mathsf{Chi}^{\langle t\rangle}(\mathsf{root})$.

Let us briefly discuss the notion of co-ordinate parent of a vertex in the directed Cartesian product of directed trees.

DEFINITION 2.1.13. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. For $j = 1, \ldots, d$ and $v \in V$, we set

$$\mathsf{par}_j(v) := \begin{cases} \{w \in V : w_j = \mathsf{par}(v_j) \text{ and } w_k = v_k \text{ for } k \neq j\} & \text{if } v_j \neq \mathsf{root}_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Further, for $W \subseteq V$, we define

$$\mathsf{par}_j(W) := \bigcup_{w \in W} \mathsf{par}_j(w).$$

For a positive integer k, we denote $\mathsf{par}_j \overset{k}{\cdots} \mathsf{par}_j(W)$ by $\mathsf{par}_j^{\langle k \rangle}(W)$. Moreover, we understand $\mathsf{par}_j^{\langle 0 \rangle}(W) = W$.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $W \subseteq V$, we define

$$\mathsf{par}^{\ll \alpha \gg}(W) := \mathsf{par}_1^{\langle \alpha_1 \rangle} \cdots \mathsf{par}_d^{\langle \alpha_d \rangle}(W).$$

In case $W = \{v\}$ for some $v \in V$, we use $\mathsf{par}^{\ll \alpha \gg}(v)$ for $\mathsf{par}^{\ll \alpha \gg}(\{v\})$.

Remark 2.1.14. Note that

$$\mathsf{Par}(v) = \bigsqcup_{j=1}^{d} \mathsf{par}_j(v). \tag{2.1}$$

In particular, $\mathrm{card}(\mathsf{Par}(v))$ is at most d.

Remark 2.1.15. Note that $\mathsf{par}_j \mathsf{par}_i(v) = \mathsf{par}_i \mathsf{par}_j(v)$ for all $v \in V$. Further, for $i, j = 1, \ldots, d$ with $i \neq j$,

$$\mathsf{Chi}_i(\mathsf{par}_j(v)) = \mathsf{par}_j(\mathsf{Chi}_i(v)).$$

Although the directed Cartesian product of directed trees need not be a directed tree, the following result shows that it has many structural similarities to a directed tree. In particular, it always admits a directed semi-tree structure in the sense of [79, Definition 2.6]. This has been observed in the special context of Example 2.1.5 in [79, Lemmas 3.1, 3.2, 3.5(ii)].

Theorem 2.1.16. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Then:*

(i) *$\mathscr{T}$ has no circuits.*
(ii) *$\mathscr{T}$ is connected.*
(iii) *$\mathscr{T}$ can have at most one root.*
(iv) $\mathrm{card}(\mathsf{Chi}(u) \cap \mathsf{Chi}(v)) \leq 1$ *for any distinct $u, v \in V$.*

Proof. To see (i), suppose that $\{v^{(i)}\}_{i=1}^n \subseteq V$ is a circuit in $\mathscr{T}$. Then $(v^{(1)}, v^{(2)}) \in \mathcal{E}$ implies that $v^{(2)} \in \mathsf{Chi}(v^{(1)})$. Similarly, $(v^{(2)}, v^{(3)}) \in \mathcal{E}$ implies that $v^{(3)} \in \mathsf{Chi}(v^{(2)}) \subseteq \mathsf{Chi}^{\langle 2 \rangle}(v^{(1)})$. Consequently, $v^{(n)} \in \mathsf{Chi}^{\langle n-1 \rangle}(v^{(1)})$. Finally, $(v^{(n)}, v^{(1)}) \in \mathcal{E}$ implies that $v^{(1)} \in \mathsf{Chi}^{\langle n \rangle}(v^{(1)})$. Also, since $v^{(1)} \in \mathsf{Chi}^{\langle 0 \rangle}(v^{(1)})$, in view of Lemma 2.1.10(v), we arrive at the contradiction that $v^{(1)} \in \mathsf{Chi}^{\langle 0 \rangle}(v^{(1)}) \cap \mathsf{Chi}^{\langle n \rangle}(v^{(1)})$.

To prove (ii), let $v, w \in V$. Since each $\mathscr{T}_j$ $(1 \leq j \leq d)$ is connected, for each v_j and w_j there is a finite sequence of vertices $\{u_{j,k}\}_{k=1}^{\alpha_j} \subseteq V_j$ such that $u_{j,1} = v_j$, $u_{j,\alpha_j} = w_j$ and $(u_{j,k}, u_{j,k+1})$ or $(u_{j,k+1}, u_{j,k})$ is in $\mathcal{E}_j$ $(k = 1, \ldots, \alpha_j - 1)$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$. We construct a sequence $\{u^{(k)}\}_{k=1}^{|\alpha|}$ of vertices in V as follows. Set

$$u^{(1)} = v, \quad u^{(2)} = (u_{1,2}, v_2, \ldots, v_d), \ \ldots, \ u^{(\alpha_1)} = (u_{1,\alpha_1}, v_2, \ldots, v_d),$$
$$u^{(\alpha_1+1)} = (u_{1,\alpha_1}, u_{2,2}, v_3, \ldots, v_d), \ \ldots, \ u^{(\alpha_1+\alpha_2)} = (u_{1,\alpha_1}, u_{2,\alpha_2}, v_3, \ldots, v_d), \ \ldots,$$
$$u^{(|\alpha|)} = (u_{1,\alpha_1}, u_{2,\alpha_2}, \ldots, u_{d,\alpha_d}) = w.$$

It is evident from the construction that $(u^{(j)}, u^{(j+1)})$ or $(u^{(j+1)}, u^{(j)}) \in \mathcal{E}$. This proves that $\mathscr{T}$ is connected.

If at least one of the directed trees is rootless, then for all $v \in V$, $\mathsf{par}_j(v) \neq \emptyset$ for some $j = 1, \ldots, d$. It follows from (2.1) that $\mathsf{Par}(v) \neq \emptyset$, and hence $\mathsf{Root}(\mathscr{T}) = \emptyset$. Thus we may assume that all the directed trees are rooted. Suppose that $\mathscr{T}$ has a root $v \neq \mathsf{root}$. By Lemma 2.1.10(vi), there exists some $\alpha \in \mathbb{N}^d \setminus \{0\}$ such that $v \in \mathsf{Chi}^{\langle\langle \alpha \rangle\rangle}(\mathsf{root})$. But then $\mathsf{Par}(v) \neq \emptyset$, which contradicts the definition of the root.

Let $u, v \in V$ and $u \neq v$. Without loss of generality, we may assume that $u_1 \neq v_1$. Suppose that $s, w \in \mathsf{Chi}(u) \cap \mathsf{Chi}(v)$. Then $s \in \mathsf{Chi}_i(u) \cap \mathsf{Chi}_k(v)$ and $w \in \mathsf{Chi}_j(u) \cap \mathsf{Chi}_l(v)$ for some $1 \leq i, j, k, l \leq d$. Note that $i \neq k$, for if $i = k$, then $u_1 \neq v_1$ would imply that $\mathsf{Chi}_i(u) \cap \mathsf{Chi}_k(v) = \emptyset$. Similarly, $j \neq l$. We treat only the case in which $i < k$ and $j < l$. Since $s \in \mathsf{Chi}_i(u)$ we have $s = (u_1, \ldots, u_{i-1}, s_i, u_{i+1}, \ldots, u_d)$, and $w \in \mathsf{Chi}_j(u)$ implies that $w = (u_1, \ldots, u_{j-1}, w_j, u_{j+1}, \ldots, u_d)$. Let $\mathsf{par}(u_k) = \hat{u}_k$ and $\mathsf{par}(u_l) = \hat{u}_l$. Since $i \neq k$ and $j \neq l$, it follows from $\mathsf{par}_k(s) = v = \mathsf{par}_l(w)$ that

$$(u_1, \ldots, s_i, \ldots, \hat{u}_k, \ldots, u_d) = v = (u_1, \ldots, w_j, \ldots, \hat{u}_l, \ldots, u_d). \tag{2.2}$$

In case $i = j$, we obtain $s_i = w_i$ in view of (2.2). But then as $s, w \in \mathsf{Chi}_i(u)$, we must have $s = w$. Let $i \neq j$. Then from (2.2), either $s_i = u_i$ or $s_i = \hat{u}_l$. As $s \in \mathsf{Chi}_i(u)$, we must have $s_i = \hat{u}_l$, and hence $i = l$. Therefore, $s = \mathsf{par}_i(u)$. Thus $s \in \mathsf{Chi}_i(u) \cap \mathsf{par}_i(u)$. This is not possible. Hence the case $i \neq j$ cannot occur. This proves that $s = w$ and hence (iv) stands verified. ∎

There are a couple of possible generalizations of the notion of branching vertex in several variables. However, the following toral analog seems to be suitable to the investigations in the present text.

Definition 2.1.17. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. A vertex $v = (v_1, \ldots, v_d) \in V$ is called a *branching vertex* of $\mathscr{T}$ if $\mathrm{card}(\mathsf{Chi}(v_j)) \geq 2$ for all $j = 1, \ldots, d$. The set of all branching vertices of $\mathscr{T}$ is denoted by $V_{\prec}$.

Remark 2.1.18. If $V_{\prec}^{(j)}$ is the set of branching vertices of $\mathscr{T}_j$, then

$$V_{\prec} = V_{\prec}^{(1)} \times \cdots \times V_{\prec}^{(d)}.$$

Proposition 2.1.19. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. If $\mathscr{T}_j$ has finite branching index $k_{\mathscr{T}_j}$ for $j = 1, \ldots, d$, then for $k_{\mathscr{T}} = (k_{\mathscr{T}_1}, \ldots, k_{\mathscr{T}_d})$, one has*

$$\mathsf{Chi}^{\langle\langle k_{\mathscr{T}} \rangle\rangle}(V_{\prec}) \cap V_{\prec} = \emptyset.$$

Proof. Assume that each $\mathscr{T}_j$ has finite branching index $k_{\mathscr{T}_j}$ and let $v \in \mathsf{Chi}^{\langle\langle k_{\mathscr{T}} \rangle\rangle}(V_{\prec})$. Then $v \in \mathsf{Chi}_1^{\langle k_{\mathscr{T}_1} \rangle} \cdots \mathsf{Chi}_d^{\langle k_{\mathscr{T}_d} \rangle}(w)$ for some $w \in V_{\prec}$. That is, $v = (v_1, \ldots, v_d)$ and $v_j \in \mathsf{Chi}^{\langle k_{\mathscr{T}_j} \rangle}(w_j)$ with $w_j \in V_{\prec}^{(j)}$. But then $v_j \notin V_{\prec}^{(j)}$. Hence $v \notin V_{\prec}$. This shows that $\mathsf{Chi}^{\langle k_{\mathscr{T}} \rangle}(V_{\prec}) \cap V_{\prec} = \emptyset$. ∎

The multiindex $k_{\mathscr{T}} \in \mathbb{N}^d$ appearing in Proposition 2.1.19 will be referred to as the *joint branching index* of $\mathscr{T}$. Also, we say that $\mathscr{T}$ *has finite joint branching index* if $|k_{\mathscr{T}}|$ is finite.

We conclude this section with a brief discussion on the notion of siblings of a vertex.

Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. For $u \in V$ and $j = 1, \ldots, d$, we set

$$\mathsf{sib}_j(u) := \begin{cases} \mathsf{Chi}_j(\mathsf{par}_j(u)) & \text{if } u_j \neq \mathsf{root}_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

For $W \subseteq V$, we define $\mathsf{sib}_j(W) := \bigcup_{u \in W} \mathsf{sib}_j(u)$.

Remark 2.1.20. Let $1 \leq i, j \leq d$ and $v \in V$. Then $\mathsf{sib}_i\mathsf{sib}_j(v) = \mathsf{sib}_j\mathsf{sib}_i(v)$. Further, $\mathsf{sib}_i\mathsf{sib}_i(v) = \mathsf{sib}_i(v)$.

For future reference, we record the following simple yet useful observation.

Proposition 2.1.21. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Then*

$$\operatorname{card}(\mathsf{sib}_i(v)) \operatorname{card}(\mathsf{sib}_j(\mathsf{par}_i(v))) = \operatorname{card}(\mathsf{sib}_j(v)) \operatorname{card}(\mathsf{sib}_i(\mathsf{par}_j(v)))$$

for every $v \in \mathsf{Chi}_i(\mathsf{Chi}_j(w))$, $w \in V$, and $i, j = 1, \ldots, d$.

Proof. Clearly, the identity holds for $i = j$. In case $i \neq j$, the conclusion follows from the fact that $\operatorname{card}(\mathsf{sib}_j(\mathsf{par}_i(v))) = \operatorname{card}(\mathsf{sib}(v_j)) = \operatorname{card}(\mathsf{sib}_j(v))$. ∎

2.2. Tensor products of rooted directed trees. In this section, we discuss another notion of product of two directed trees, to be referred to as tensor product. This notion was introduced by P. Weichsel [92] for undirected graphs, and later extended to directed graphs by M. McAndrew [80] (refer also to [62]). In the literature, the tensor product is also known as categorical product, Kronecker product, cardinal product, weak direct product and even Cartesian product. We emphasize that the notions of Cartesian product and tensor product, as discussed in this text, are different.

The definition of the tensor product of two directed graphs extends naturally to finitely many directed trees.

Definition 2.2.1. Let d be a positive integer and let $\mathscr{T}_j = (V_j, \mathcal{E}_j)$ $(j = 1, \ldots, d)$ be a collection of rooted directed trees. The *tensor product* of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ is a directed graph $\mathscr{T}^{\otimes} = (V, \mathcal{E}^{\otimes})$, where $V := V_1 \times \cdots \times V_d$ and

$$\mathcal{E}^{\otimes} := \big\{(\mathfrak{v}, \mathfrak{w}) \in V \times V : (\mathfrak{v}_j, \mathfrak{w}_j) \in \mathcal{E}_j \text{ for all } j = 1, \ldots, d\big\}.$$

Caution. Since the set of vertices of the tensor product is same as that of the directed Cartesian product, we use scripted letters throughout the text to distinguish the vertices (except the root) of the tensor product from those of the directed Cartesian product.

Remark 2.2.2. Note that $(\mathfrak{v}, \mathfrak{w}) \in \mathcal{E}^{\otimes}$ if and only if $\mathfrak{w} \in \mathsf{Chi}_1 \cdots \mathsf{Chi}_d(\mathfrak{v})$. Thus $\mathsf{Chi}(\mathfrak{v}) = \mathsf{Chi}_1 \cdots \mathsf{Chi}_d(\mathfrak{v})$ (with reference to the directed graph $\mathscr{T}^{\otimes}$). This should be compared with the expression for $\mathsf{Chi}(\cdot)$ (with reference to the directed graph $\mathscr{T}$) as given in Remark 2.1.9.

Let $\mathscr{T}^{\otimes} = (V, \mathcal{E}^{\otimes})$ be the tensor product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Define vertices $\mathfrak{v}, \mathfrak{w} \in V$ to be equivalent if either $\mathfrak{v} = \mathfrak{w}$, or $\mathfrak{v}$ and $\mathfrak{w}$ can be connected by a path in $\mathscr{T}^{\otimes}$. Note that this defines an equivalence relation. A *component* of $\mathscr{T}^{\otimes}$ is an equivalence class of this relation. It turns out that each component of $\mathscr{T}^{\otimes}$ is a rooted directed tree. We illustrate this in more detail in the following theorem.

THEOREM 2.2.3. *Let $\mathscr{T}^{\otimes} = (V, \mathcal{E}^{\otimes})$ be the tensor product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $\mathscr{T}^{\otimes}_{\mathsf{root}} = (V^{\otimes}, \mathcal{F})$ denote the (unique) component of $\mathscr{T}^{\otimes}$ that contains* root. *Let $\mathscr{T}$ be the directed Cartesian product of $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Set*

$$\mathsf{Root}^{\otimes} := \{\mathfrak{v} \in V : \mathfrak{v}_j = \mathsf{root}_j \textit{ for at least one } j = 1, \ldots, d\}.$$

Then:

(i) *If $\mathfrak{v} \in \mathsf{Root}^{\otimes}$, then there does not exist any $\mathfrak{u} \in V$ such that $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{E}^{\otimes}$.*
(ii) *For each $\mathfrak{v} \in V \setminus \mathsf{Root}^{\otimes}$, there is a unique $\mathfrak{u} \in V$ such that $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{E}^{\otimes}$. In other words, each $\mathfrak{v} \in V \setminus \mathsf{Root}^{\otimes}$ has a parent.*
(iii) *$\mathscr{T}^{\otimes}$ has no circuits.*
(iv) *No two distinct vertices in $\mathsf{Root}^{\otimes}$ can be connected by a path in $\mathscr{T}^{\otimes}$.*
(v) *There is a bijective correspondence between the collection of components of $\mathscr{T}^{\otimes}$ and the elements of $\mathsf{Root}^{\otimes}$. In particular, $\mathscr{T}^{\otimes}$ contains countably many components.*
(vi) *Each component is a rooted directed tree with root coming from $\mathsf{Root}^{\otimes}$.*
(vii) *$\mathscr{T}^{\otimes}$ is locally finite if and only if $\mathscr{T}$ is locally finite.*
(viii) *$\mathscr{T}^{\otimes}$ is leafless.*
(ix) *If $\mathscr{T}$ is of finite joint branching index $k_{\mathscr{T}} = (k_{\mathscr{T}_1}, \ldots, k_{\mathscr{T}_d})$, then the branching index $k_{\mathscr{T}^{\otimes}_{\mathsf{root}}}$ of $\mathscr{T}^{\otimes}_{\mathsf{root}}$ equals $\max\{k_{\mathscr{T}_j} : 1 \le j \le d\}$.*
(x) *For each $v \in V$, there exists $\mathfrak{v} \in V^{\otimes}$ such that $|\mathsf{d}_v| = \mathsf{d}_{\mathfrak{v}}$, where d_v is the depth of v in $\mathscr{T}$ and $\mathsf{d}_{\mathfrak{v}}$ is the depth of $\mathfrak{v}$ in the directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$.*

REMARK 2.2.4. The conclusion of (v) above is in contrast with the situation occurring in the case of undirected trees, where the tensor product of two undirected trees has exactly two components (see [66, Theorem 5.29]).

Proof of Theorem 2.2.3. Let $\mathfrak{v} \in \mathsf{Root}^{\otimes}$. Then $\mathfrak{v}_j = \mathsf{root}_j$ for some $j = 1, \ldots, d$. Now, if $\mathfrak{u} \in V$ such that $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{E}^{\otimes}$, then $\mathfrak{u}_j = \mathsf{par}(\mathfrak{v}_j)$, which is not possible. This proves (i).

To see (ii), let $\mathfrak{v} \in V \setminus \mathsf{Root}^{\otimes}$. Then $\mathfrak{v}_j \neq \mathsf{root}_j$ for all $j = 1, \ldots, d$. Consider $\mathfrak{u} \in V$ with $\mathfrak{u}_j = \mathsf{par}(\mathfrak{v}_j)$ for all $j = 1, \ldots, d$. Then $(\mathfrak{u}, \mathfrak{v}) \in \mathcal{E}^{\otimes}$. This proves the existence as well as the uniqueness of $\mathfrak{u}$.

To see (iii), suppose there is a finite sequence $\{\mathfrak{w}^{(i)}\}_{i=1}^{n}$ $(n \ge 2)$ of distinct vertices such that $(\mathfrak{w}^{(i)}, \mathfrak{w}^{(i+1)}) \in \mathcal{E}^{\otimes}$ for all $1 \le i \le n-1$ and $(\mathfrak{w}^{(n)}, \mathfrak{w}^{(1)}) \in \mathcal{E}^{\otimes}$. Then $\mathfrak{w}^{(1)}_j \in \mathsf{Chi}^{\langle n \rangle}(\mathfrak{w}^{(1)}_j)$ for all $1 \le j \le d$. This contradicts the fact that $\mathscr{T}_j$ has no circuits.

Let $\mathfrak{u}$ and $\mathfrak{v}$ be distinct vertices in $\mathsf{Root}^{\otimes}$. Suppose there exists a finite sequence $\{\mathfrak{w}^{(i)}\}_{i=1}^{n} \subseteq V$ $(n \ge 2)$ of distinct vertices such that $\mathfrak{w}^{(1)} = \mathfrak{u}$, $\mathfrak{w}^{(n)} = \mathfrak{v}$ and $(\mathfrak{w}^{(i)}, \mathfrak{w}^{(i+1)})$ or $(\mathfrak{w}^{(i+1)}, \mathfrak{w}^{(i)}) \in \mathcal{E}^{\otimes}$ $(i = 1, \ldots, n-1)$. By (i), $(\mathfrak{w}^{(2)}, \mathfrak{w}^{(1)})$ cannot belong to $\mathcal{E}^{\otimes}$. Hence, we must have $(\mathfrak{w}^{(1)}, \mathfrak{w}^{(2)}) \in \mathcal{E}^{\otimes}$. Thus $\mathfrak{w}^{(2)} \in V \setminus \mathsf{Root}^{\otimes}$. Next, if $(\mathfrak{w}^{(3)}, \mathfrak{w}^{(2)}) \in \mathcal{E}^{\otimes}$, then by (ii), $\mathfrak{w}^{(3)} = \mathfrak{w}^{(1)}$. This contradicts the vertices $\mathfrak{w}^{(1)}, \ldots, \mathfrak{w}^{(n)}$ being distinct. Thus $(\mathfrak{w}^{(2)}, \mathfrak{w}^{(3)}) \in \mathcal{E}^{\otimes}$. By arguing similarly, one can see that $(\mathfrak{w}^{(i)}, \mathfrak{w}^{(i+1)}) \in \mathcal{E}^{\otimes}$ for all $1 \le i \le n-1$. Thus $\mathfrak{v}_j \in \mathsf{Chi}^{\langle n-1 \rangle}(\mathfrak{u}_j)$ for each $j = 1, \ldots, d$. Hence $\mathfrak{v} \notin \mathsf{Root}^{\otimes}$, which is a contradiction. This proves (iv).

Suppose that $\mathfrak{v} \in V$ belongs to some component $\mathcal{C}$. There exist $k_1, \ldots, k_d \in \mathbb{N}$ such that $\mathfrak{v}_j \in \mathsf{Chi}^{\langle k_j \rangle}(\mathsf{root}_j)$ for $j = 1, \ldots, d$. Let $k = \min\{k_j : 1 \leq j \leq d\}$. Then $\mathfrak{u} = \mathsf{par}_1^{\langle k \rangle} \cdots \mathsf{par}_d^{\langle k \rangle}(\mathfrak{v}) \in \mathsf{Root}^{\otimes}$, and since $\mathcal{C}$ is connected, $\mathfrak{u} \in \mathcal{C}$. Thus each component contains an element from $\mathsf{Root}^{\otimes}$. Further, (iv) implies that each component contains at most one element from $\mathsf{Root}^{\otimes}$. Clearly, as each element of $\mathsf{Root}^{\otimes}$ belongs to some component, the correspondence between the collection of components of $\mathscr{T}^{\otimes}$ and the elements of $\mathsf{Root}^{\otimes}$ is bijective. This completes the verification of (v).

Let $\mathcal{C}$ be any component of $\mathscr{T}^{\otimes}$. From (v), there exists a unique $\mathfrak{v} \in \mathsf{Root}^{\otimes}$ such that $\mathfrak{v} \in \mathcal{C}$. Further, (i) implies that $\mathfrak{v}$ has no parent, in particular, in the subgraph $\mathcal{C}$. Thus $\mathfrak{v}$ is a root for $\mathcal{C}$. Clearly, $\mathcal{C}$ is connected, and by (ii), each $\mathfrak{u} \in \mathcal{C}$, with $\mathfrak{u} \neq \mathfrak{v}$, has a parent. Also, by (iii), $\mathcal{C}$ has no circuits. This proves (vi).

Let $\mathfrak{v}$ be a vertex in V. By Remark 2.2.2, $\mathsf{Chi}(\mathfrak{v}) = \mathsf{Chi}_1 \cdots \mathsf{Chi}_d(\mathfrak{v})$. It follows that

$$\operatorname{card}(\mathsf{Chi}(\mathfrak{v})) = \prod_{j=1}^{d} \operatorname{card}(\mathsf{Chi}_j(\mathfrak{v})) = \prod_{j=1}^{d} \operatorname{card}(\mathsf{Chi}(\mathfrak{v}_j)).$$

Thus $\mathscr{T}^{\otimes}$ is locally finite if and only if $\mathscr{T}$ is locally finite.

(viii) is an obvious consequence of the fact (observed in Remark 2.2.2) that $\mathsf{Chi}(\mathfrak{v}) = \mathsf{Chi}_1 \cdots \mathsf{Chi}_d(\mathfrak{v})$ for all $\mathfrak{v} \in V$.

To see (ix), first note that $V^{\otimes} = \bigsqcup_{k=0}^{\infty} \mathsf{Chi}^{\langle k \rangle}(\mathsf{root}) = \bigsqcup_{k=0}^{\infty} \mathsf{Chi}_1^{\langle k \rangle} \cdots \mathsf{Chi}_d^{\langle k \rangle}(\mathsf{root})$. Hence, if $\mathfrak{v}$ is any vertex of $\mathscr{T}^{\otimes}_{\mathsf{root}}$, then there exists a unique $k \in \mathbb{N}$ such that $\mathfrak{v}_j \in \mathsf{Chi}^{\langle k \rangle}(\mathsf{root}_j)$ for all $1 \leq j \leq d$. Thus $\mathsf{d}_{\mathfrak{v}} = \mathsf{d}_{\mathfrak{v}_j}$ for all $1 \leq j \leq d$. Next, observe that $\operatorname{card}(\mathsf{Chi}(\mathfrak{v})) \geq 2$ if and only if there exists a positive integer j $(1 \leq j \leq d)$ such that $\operatorname{card}(\mathsf{Chi}(\mathfrak{v}_j)) \geq 2$. With these two observations, it is easy to see that

$$\sup\{\mathsf{d}_{\mathfrak{v}} : \operatorname{card}(\mathsf{Chi}(\mathfrak{v})) \geq 2\} = \max_{1 \leq j \leq d} \sup\{\mathsf{d}_{\mathfrak{v}_j} : \operatorname{card}(\mathsf{Chi}(\mathfrak{v}_j)) \geq 2\}.$$

This implies that $k_{\mathscr{T}^{\otimes}_{\mathsf{root}}} = \max\{k_{\mathscr{T}_j} : 1 \leq j \leq d\}$. We leave the verification of the last part to the reader. ■

We will see later that certain weighted shifts acting on the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$ arise naturally in an integral representation of so-called spherically balanced multishifts originating from the directed Cartesian product of directed trees.

Remark 2.2.5. Note that $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is an isomorphic invariant for $\mathscr{T}^{\otimes}$. In fact, let $\{\tilde{\mathscr{T}}_j : 1 \leq j \leq d\}$ be a collection of rooted directed trees with respective roots $\tilde{\mathsf{root}}_j$ and let $\tilde{\mathscr{T}}^{\otimes}$ be the tensor product of $\tilde{\mathscr{T}}_1, \ldots, \tilde{\mathscr{T}}_d$. Let $\tilde{\mathscr{T}}^{\otimes}_{\tilde{\mathsf{root}}}$ be the component of $\tilde{\mathscr{T}}^{\otimes}$ containing $\tilde{\mathsf{root}}$. Suppose that ϕ_j is an isomorphism between $\mathscr{T}_j$ and $\tilde{\mathscr{T}}_j$. Then $(\mathfrak{v}_1, \ldots, \mathfrak{v}_d) \mapsto (\phi_1(\mathfrak{v}_1), \ldots, \phi_d(\mathfrak{v}_d))$ defines an isomorphism between $\mathscr{T}^{\otimes}$ and $\tilde{\mathscr{T}}^{\otimes}$, and hence $\mathscr{T}^{\otimes}_{\mathsf{root}}$ and $\tilde{\mathscr{T}}^{\otimes}_{\tilde{\mathsf{root}}}$ become isomorphic.

For future reference, we describe $\mathscr{T}^{\otimes}_{\mathsf{root}}$ in Examples 2.1.5, 2.1.6, 2.1.7 (see Figures 2.4, 2.5, 2.6 respectively). In this regard, the reader is advised to recall the definition of $\mathscr{T}_{n_0,k_0}$ as given in Example 1.3.2.

Example 2.2.6. Let $\mathscr{T}_{1,0}$ be the tree as discussed in Example 2.1.5. Let $d = 2$ and $\mathscr{T}_j = \mathscr{T}_{1,0}$ for $j = 1, 2$. Then

$$\mathsf{Root}^{\otimes} = \{(i, 0), (0, j) : i, j \geq 0\}.$$

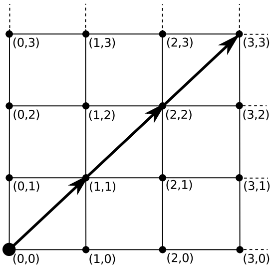

Fig. 2.4. In $\mathscr{T}_{1,0} \otimes \mathscr{T}_{1,0}$, $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is represented with bold-faced edges

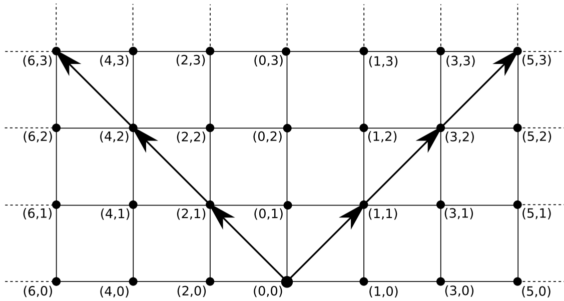

Fig. 2.5. In $\mathscr{T}_{2,0} \otimes \mathscr{T}_{1,0}$, $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is represented with bold-faced edges

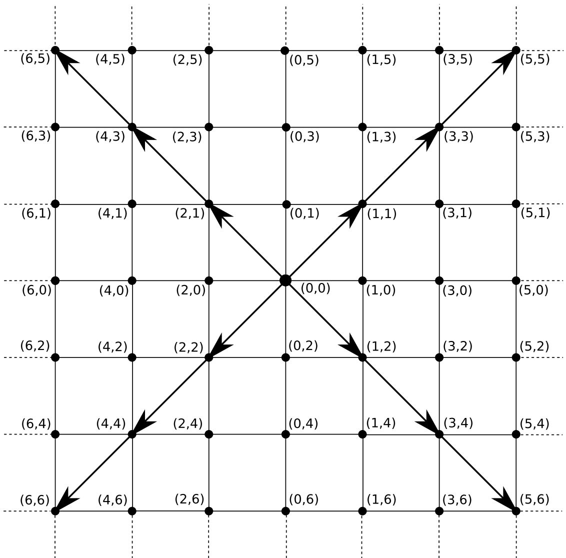

Fig. 2.6. In $\mathscr{T}_{2,0} \otimes \mathscr{T}_{2,0}$, $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is represented with bold-faced edges

For $i \geq 1$, the components $\mathcal{C}_{(i,0)} = (V_{(i,0)}, \mathcal{E}_{(i,0)})$ containing $(i, 0)$ are given by

$$V_{(i,0)} = \{(i+k, k) : k \geq 0\} \quad \text{and} \quad \mathsf{Chi}((i+k, k)) = \{(i+k+1, k+1)\}$$

for all $k \geq 0$. A similar description of $\mathcal{C}_{(0,j)}$ is obtained for $j \geq 1$. Further, the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$, with set of vertices $V^{\otimes}$, is given by

$$V^{\otimes} = \{(k,k) : k \geq 0\} \quad \text{and} \quad \mathsf{Chi}((k,k)) = \{(k+1, k+1)\}$$

for all $k \geq 0$. Note that $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is isomorphic to $\mathscr{T}_{1,0}$ via $(k,k) \mapsto k$.

Example 2.2.7. Let $\mathscr{T}_1 = \mathscr{T}_{2,0}$, $\mathscr{T}_2 = \mathscr{T}_{1,0}$ (see Example 2.1.6). Then

$$\mathsf{Root}^{\otimes} = \{(i,0), (0,j) : i,j \geq 0\}.$$

For $i \geq 1$, the components $\mathcal{C}_{(i,0)} = (V_{(i,0)}, \mathcal{E}_{(i,0)})$ containing $(i,0)$ are given by

$$V_{(i,0)} = \{(i+2k, k) : k \geq 0\} \quad \text{and} \quad \mathsf{Chi}((i+2k,k)) = \{(i+2k+2, k+1)\}$$

for all $k \geq 0$. Further, for $j \geq 1$, the components $\mathcal{C}_{(0,j)} = (V_{(0,j)}, \mathcal{E}_{(0,j)})$ containing $(0,j)$ are given by

$$V_{(0,j)} = \{(0,j), (1, j+1), (2, j+1)\} \cup \{(2k+1, j+k+1), (2k, j+k) : k \geq 1\}$$

and $\mathsf{Chi}((0,j)) = \{(1,j+1),(2,j+1)\}$, $\mathsf{Chi}((k,l)) = \{(k+2, l+1)\}$ for all $k, l \geq 1$. Moreover, the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$, with set of vertices $V^{\otimes}$, is given by

$$V^{\otimes} = \{(2k+1, k+1), (2k, k) : k \geq 0\}$$

and $\mathsf{Chi}((0,0)) = \{(1,1),(2,1)\}$, $\mathsf{Chi}((k,l)) = \{(k+2, l+1)\}$ for all $k, l \geq 1$. Note that $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is isomorphic to $\mathscr{T}_{2,0}$ via $(k,l) \mapsto k$.

Example 2.2.8. Let $\mathscr{T}_1 = \mathscr{T}_{2,0} = \mathscr{T}_2$. Then

$$\mathsf{Root}^{\otimes} = \{(i,0), (0,j) : i,j \geq 0\}.$$

For $j \geq 1$, the components $\mathcal{C}_{(0,j)} = (V_{(0,j)}, \mathcal{E}_{(0,j)})$ containing $(0,j)$ are given by

$$V_{(0,j)} = \{(0,j), (1, j+2), (2, j+2)\} \cup \{(2k+1, j+2k+2), (2k, j+2k) : k \geq 1\}$$

and $\mathsf{Chi}((0,j)) = \{(1,j+2),(2,j+2)\}$, $\mathsf{Chi}((k,l)) = \{(k+2, l+2)\}$ for all $k, l \geq 1$. Similar description for $\mathcal{C}_{(i,0)}$ is obtained for all $i \geq 1$. Further, the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$, with set of vertices $V^{\otimes}$, is given by

$$V^{\otimes} = \{(k,k) : k \geq 0\} \cup \{(2k-1, 2k), (2k, 2k-1) : k \geq 1\}$$

and $\mathsf{Chi}((0,0)) = \{(1,1),(1,2),(2,1),(2,2)\}$, $\mathsf{Chi}((k,l)) = \{(k+2, l+2)\}$ for all $k, l \geq 1$. It can be seen that $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is isomorphic to the rooted directed tree $\mathscr{T}_{4,0}$.

We revisit the above examples once again in the context of spherically balanced multishifts in Chapter 5.

3. Multishifts on products of rooted directed trees

In this chapter, we introduce and study the notion of multishifts on directed Cartesian product of finitely many rooted directed trees. In particular, we discuss some basic properties of multishifts such as boundedness, commutativity, circularity and analyticity. These are then used to describe various spectral parts of $S_{\boldsymbol\lambda}$ including the Taylor spectrum.

In this paper, we are interested in the tree counterpart of classical unilateral multishifts. Hence all the directed trees discussed in the remaining part of this text will be rooted.

3.1. Definition and elementary properties. Let $\mathscr{T}_j = (V_j, \mathcal{E}_j)$ $(j = 1, \ldots, d)$ be rooted directed trees and let $\mathscr{T} = (V, \mathcal{E})$ be their directed Cartesian product. For a vertex $v \in V$, let $e_v : V \to \mathbb{C}$ denote the indicator function of the set $\{v\}$. Consider the complex Hilbert space $l^2(V)$ of square summable complex functions on V equipped with the standard inner product. Note that $l^2(V)$ admits the orthonormal basis $\{e_v : v \in V\}$. We always assume that $\mathrm{card}(V) = \aleph_0$. For a nonempty subset W of V, $l^2(W)$ may be considered as a subspace of $l^2(V)$. Indeed, if one sets $\tilde{f} = f$ on W and 0 otherwise, then the mapping $U : l^2(W) \to l^2(V)$ given by $Uf = \tilde{f}$ is an isometric homomorphism.

Remark 3.1.1. Consider the category $\mathcal{T}$ of directed Cartesian products of finitely many directed trees with morphisms being directed graph homomorphisms (or directed graph isomorphisms). Note that l^2 defines a *covariant functor* from $\mathcal{T}$ into the category $\mathcal{C}$ of Hilbert spaces with bounded linear operators (resp. unitaries) as morphisms. Indeed, any graph homomorphism (resp. isomorphism) ϕ induces a bounded linear operator (resp. unitary) $l^2(\phi)$ given by

$$l^2(\phi)(e_v) = e_{\phi(v)},$$

which satisfies $l^2(\phi \circ \psi) = l^2(\phi) \circ l^2(\psi)$.

Definition 3.1.2. Given a system $\boldsymbol\lambda = \{\lambda_v^{(j)} : v \in V^\circ,\ j = 1, \ldots, d\}$ of complex numbers, we define the *multishift* $S_{\boldsymbol\lambda}$ *on* $\mathscr{T}$ with weights $\boldsymbol\lambda$ as the d-tuple of operators $S_1, \ldots, S_d$ on $l^2(V)$ given by

$$\mathscr{D}(S_j) := \{f \in l^2(V) \colon \varLambda_{\mathscr{T}}^{(j)} f \in l^2(V)\}, \qquad S_j f := \varLambda_{\mathscr{T}}^{(j)} f, \quad f \in \mathscr{D}(S_j),$$

where $\varLambda_{\mathscr{T}}^{(j)}$ is the mapping defined on complex functions f on V by

$$(\varLambda_{\mathscr{T}}^{(j)} f)(v) := \begin{cases} \lambda_v^{(j)} \cdot f(\mathsf{par}_j(v)) & \text{if } v_j \in V_j^\circ, \\ 0 & \text{if } v_j \text{ is a root of } \mathscr{T}_j. \end{cases}$$

We note here that not all weights $\lambda_v^{(j)}$ in the system $\boldsymbol{\lambda}$ are used in the above definition. For instance, if $v \in \mathsf{Chi}_1(\mathsf{root})$ then $\lambda_v^{(2)}$ will not appear in the definition of $S_{\boldsymbol{\lambda}}$. Further, from here onwards, we assume that $\boldsymbol{\lambda}$ consists of nonzero complex numbers.

Remark 3.1.3. If $e_v \in \mathscr{D}(S_j)$, then

$$S_j e_v = \sum_{w \in \mathsf{Chi}_j(v)} \lambda_w^{(j)} e_w. \tag{3.1}$$

Example 3.1.4 (Classical multishifts). Consider the directed Cartesian product $\mathscr{T}$ of d copies of $\mathscr{T}_{1,0}$ as discussed in Example 2.1.5. Assume that S_j is bounded for $j = 1, \ldots, d$. Then

$$S_j e_\alpha = \sum_{\beta \in \mathsf{Chi}_j(\alpha)} \lambda_\beta^{(j)} e_\beta = \lambda_{\alpha+\epsilon_j}^{(j)} e_{\alpha+\epsilon_j}.$$

If one sets $w_\alpha^{(j)} := \lambda_{\alpha+\epsilon_j}^{(j)}$, then $S_{\boldsymbol{\lambda}}$ is nothing but the classical multishift $S_{\mathbf{w}}$ with weight multisequence $\{w_\alpha^{(j)} : \alpha \in \mathbb{N}^d,\ j = 1, \ldots, d\}$.

Lemma 3.1.5. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}} = (S_1, \ldots, S_d)$ be a multishift on $\mathscr{T}$. Then, for $j = 1, \ldots, d$, the following statements hold:*

(i) *S_j is a bounded linear operator on $l^2(V)$ if and only if*

$$\sup_{v \in V} \sum_{w \in \mathsf{Chi}_j(v)} |\lambda_w^{(j)}|^2 < \infty.$$

(ii) *S_j is injective.*

Proof. We argue along the lines of [67, Propositions 3.1.8 and 3.1.7]. By Lemma 2.1.10(ii), $\{e_w\}_{w \in \mathsf{Chi}_j(v)}$ is orthogonal for every $v \in V$ and $j = 1, \ldots, d$. The first part now follows from (3.1). To see (ii), suppose that $S_j f = 0$ for some $f \in l^2(V)$. Then

$$\sum_{v \in V} |f(v)|^2 \sum_{w \in \mathsf{Chi}_j(v)} |\lambda_w^{(j)}|^2 = \|S_j f\|^2 = 0.$$

Since $\boldsymbol{\lambda}$ consists of nonzero complex numbers, the above equality holds if and only if either $f(v) = 0$ or $\mathsf{Chi}_j(v) = \emptyset$. However, by assumption $\mathscr{T}_1, \ldots, \mathscr{T}_d$ are leafless, and hence $f(v) = 0$ for all $v \in V$. ■

Unless stated otherwise, S_j belongs to $B(l^2(V))$ for every $j = 1, \ldots, d$.

If $S_{\boldsymbol{\lambda}} = (S_1, \ldots, S_d)$ is the multishift on $\mathscr{T}$, then the Hilbert space adjoint S_j^* of S_j is given by

$$(S_j^* f)(v) = \sum_{w \in \mathsf{Chi}_j(v)} \overline{\lambda}_w^{(j)} f(w), \qquad f \in l^2(V).$$

In particular, for all $v \in V$,

$$S_j^* e_v = \begin{cases} \overline{\lambda}_v^{(j)} e_{\mathsf{par}_j(v)} & \text{if } \mathsf{par}_j(v) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1.6. Note that $S_j^* e_{\mathsf{root}} = 0$ for all $j = 1, \ldots, d$. In particular, 0 belongs to the point spectrum of $S_{\boldsymbol{\lambda}}^*$.

In the following proposition, we collect several elementary properties of $S_{\boldsymbol\lambda}$.

Proposition 3.1.7. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a multishift on $\mathscr{T}$. For $j = 1, \ldots, d$, $w \in V$, let $\beta(j, w, 0) := 1$ and*

$$\beta(j, w, n) := \lambda_w^{(j)} \lambda_{\mathsf{par}_j(w)}^{(j)} \cdots \lambda_{\mathsf{par}_j^{\langle n-1\rangle}(w)}^{(j)} \qquad (n \geq 1).$$

Also, let $\alpha^{(0)} = 0 \in \mathbb{N}^d$ and $\alpha^{(j)} = (\alpha_1, \ldots, \alpha_j, 0, \ldots, 0) \in \mathbb{N}^d$ for $j = 1, \ldots, d$. Then:

(i) *$S_{\boldsymbol\lambda}$ is commuting if and only if for all $i, j = 1, \ldots, d$ and for all $v \in V$,*

$$\lambda_u^{(j)} \lambda_{\mathsf{par}_j(u)}^{(i)} = \lambda_u^{(i)} \lambda_{\mathsf{par}_i(u)}^{(j)} \qquad \text{for all } u \in \mathsf{Chi}_j \mathsf{Chi}_i(v). \tag{3.2}$$

(ii) *$S_{\boldsymbol\lambda}$ is doubly commuting if and only if (3.2) holds and for all $v \in V$ and $i, j = 1, \ldots, d$ with $i \neq j$:*

$$\overline{\lambda}_v^{(j)} \lambda_{\mathsf{par}_j(u)}^{(i)} = \lambda_u^{(i)} \overline{\lambda}_u^{(j)} \qquad \text{for all } u \in \mathsf{Chi}_i(v). \tag{3.3}$$

If, in addition, $S_{\boldsymbol\lambda}$ is commuting then:

(iii) *For all $1 \leq i, j \leq d$, all $v \in V$ and all $n \geq 1$,*

$$\beta(j, \mathsf{par}_i(v), n) \lambda_v^{(i)} = \beta(j, v, n) \lambda_{\mathsf{par}_j^{\langle n\rangle}(v)}^{(i)}. \tag{3.4}$$

(iv) *For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and all $v \in V$,*

$$S_{\boldsymbol\lambda}^{\alpha} e_v = \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \prod_{j=1}^{d} \beta(j, \mathsf{par}^{\ll \alpha^{(j-1)} \gg}(w), \alpha_j) e_w. \tag{3.5}$$

(v) *For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and all $v \in V$,*

$$S_{\boldsymbol\lambda}^{*\alpha} e_v = \prod_{j=1}^{d} \overline{\beta}(j, \mathsf{par}^{\ll \alpha^{(j-1)} \gg}(v), \alpha_j) e_{\mathsf{par}^{\ll \alpha \gg}(v)},$$

where $\overline{\beta}(\cdot) = \overline{\beta(\cdot)}$.

(vi) *For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and all $v \in V$,*

$$S_{\boldsymbol\lambda}^{*\alpha} S_{\boldsymbol\lambda}^{\alpha} e_v = \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \prod_{j=1}^{d} |\beta(j, \mathsf{par}^{\ll \alpha^{(j-1)} \gg}(w), \alpha_j)|^2 e_v.$$

(vii) *The multishift $S_{\boldsymbol\lambda}$ is toral left invertible if and only if*

$$\inf_{1 \leq j \leq d} \inf_{v \in V} \sum_{w \in \mathsf{Chi}_j(v)} |\lambda_w^{(j)}|^2 > 0.$$

(viii) *The multishift $S_{\boldsymbol\lambda}$ is joint left invertible if and only if*

$$\inf_{v \in V} \sum_{j=1}^{d} \sum_{w \in \mathsf{Chi}_j(v)} |\lambda_w^{(j)}|^2 > 0.$$

(ix) *If $\alpha \neq \beta$ in $\mathbb{N}^d$, then $\langle S_{\boldsymbol\lambda}^{\alpha} e_v, S_{\boldsymbol\lambda}^{\beta} e_v \rangle = 0$ for every $v \in V$.*

(x) *If $v \neq w$ in V, then $\langle S_{\boldsymbol\lambda}^{\alpha} e_v, S_{\boldsymbol\lambda}^{\alpha} e_w \rangle = 0$ for every $\alpha \in \mathbb{N}^d$.*

Proof. Let $i, j = 1, \ldots, d$ and $v \in V$. Then

$$\begin{aligned} S_j S_i e_v = S_j \sum_{w \in \mathsf{Chi}_i(v)} \lambda^{(i)}_w e_w &= \sum_{w \in \mathsf{Chi}_i(v)} \sum_{u \in \mathsf{Chi}_j(w)} \lambda^{(i)}_w \lambda^{(j)}_u e_u \\ &= \sum_{w \in \mathsf{Chi}_i(v)} \sum_{u \in \mathsf{Chi}_j(w)} \lambda^{(i)}_{\mathsf{par}_j(u)} \lambda^{(j)}_u e_u. \end{aligned} \tag{3.6}$$

By symmetry,

$$S_i S_j e_v = \sum_{w \in \mathsf{Chi}_j(v)} \sum_{u \in \mathsf{Chi}_i(w)} \lambda^{(i)}_u \lambda^{(j)}_{\mathsf{par}_i(u)} e_u. \tag{3.7}$$

By Lemma 2.1.10(i), $\mathsf{Chi}_j \mathsf{Chi}_i(v) = \mathsf{Chi}_i \mathsf{Chi}_j(v)$. Hence, by evaluating at $u \in \mathsf{Chi}_j \mathsf{Chi}_i(v)$, we infer from (3.6) and (3.7) that $S_{\boldsymbol\lambda}$ is commuting if and only if (3.2) holds. To see (ii), note that

$$S_i S^*_j e_v = \sum_{w \in \mathsf{Chi}_i(\mathsf{par}_j(v))} \overline{\lambda}^{(j)}_v \lambda^{(i)}_w e_w \quad \text{and} \quad S^*_j S_i e_v = \sum_{w \in \mathsf{Chi}_i(v)} \lambda^{(i)}_w \overline{\lambda}^{(j)}_w e_{\mathsf{par}_j(w)}.$$

By Remark 2.1.15, $\mathsf{Chi}_i(\mathsf{par}_j(v)) = \mathsf{par}_j(\mathsf{Chi}_i(v))$. Therefore, by arguing as above, $S_i S^*_j e_v = S^*_j S_i e_v$ for all $v \in V$ if and only if (3.3) holds. This proves (i) and (ii). Also, (iii) may be deduced by repeated applications of (3.2).

We prove (iv) by induction on $|\alpha|$ for $\alpha \in \mathbb{N}^d$. In case $|\alpha| = 0$, it is easily verified that (3.5) holds. Let $n \in \mathbb{N}$ and suppose that (3.5) holds for all $\alpha \in \mathbb{N}^d$ with $|\alpha| = n$. Let $\alpha \in \mathbb{N}^d$ and $|\alpha| = n + 1$. Then $\alpha = \gamma + \epsilon_i$ for some $1 \leq i \leq d$ and some $\gamma \in \mathbb{N}^d$ with $|\gamma| = n$. Therefore, for $v \in V$,

$$\begin{aligned} S^\alpha_{\boldsymbol\lambda} e_v = S^{\epsilon_i}_{\boldsymbol\lambda} S^\gamma_{\boldsymbol\lambda} e_v &= S_i \sum_{w \in \mathsf{Chi}^{\langle\langle \gamma \rangle\rangle}(v)} \prod_{j=1}^d \beta(j, \mathsf{par}^{\langle\langle \gamma^{(j-1)} \rangle\rangle}(w), \gamma_j) e_w \\ &= \sum_{w \in \mathsf{Chi}^{\langle\langle \gamma \rangle\rangle}(v)} \prod_{j=1}^d \beta(j, \mathsf{par}^{\langle\langle \gamma^{(j-1)} \rangle\rangle}(w), \gamma_j) \sum_{u \in \mathsf{Chi}_i(w)} \lambda^{(i)}_u e_u \\ &= \sum_{u \in \mathsf{Chi}^{\langle\langle \alpha \rangle\rangle}(v)} \prod_{j=1}^d \beta(j, \mathsf{par}^{\langle\langle \gamma^{(j-1)} \rangle\rangle}(\mathsf{par}_i(u)), \gamma_j) \lambda^{(i)}_u e_u. \end{aligned}$$

In view of $\mathsf{Chi}_i(\mathsf{Chi}^{\langle\langle \gamma \rangle\rangle}(v)) = \mathsf{Chi}^{\langle\langle \alpha \rangle\rangle}(v)$, the last equality may be justified by pointwise evaluation. It now suffices to check that

$$\prod_{j=1}^d \beta(j, \mathsf{par}^{\langle\langle \gamma^{(j-1)} \rangle\rangle}(\mathsf{par}_i(u)), \gamma_j) \lambda^{(i)}_u = \prod_{j=1}^d \beta(j, \mathsf{par}^{\langle\langle \alpha^{(j-1)} \rangle\rangle}(u), \alpha_j). \tag{3.8}$$

This follows from repeated applications of (3.4). Indeed,

$$\begin{aligned} \beta(1, \mathsf{par}_i(u), \gamma_1) \lambda^{(i)}_u &\overset{(3.4)}{=} \beta(1, u, \gamma_1) \lambda^{(i)}_{\mathsf{par}^{\langle \gamma_1 \rangle}_1(u)}, \\ \beta(2, \mathsf{par}^{\langle \gamma_1 \rangle}_1(\mathsf{par}_i(u)), \gamma_2) \lambda^{(i)}_{\mathsf{par}^{\langle \gamma_1 \rangle}_1(u)} &\overset{(3.4)}{=} \beta(2, \mathsf{par}^{\langle \gamma_1 \rangle}_1(u), \gamma_2) \lambda^{(i)}_{\mathsf{par}^{\langle \gamma_2 \rangle}_2 \mathsf{par}^{\langle \gamma_1 \rangle}_1(u)}. \end{aligned}$$

Continuing in a similar manner and using the facts that $\gamma + \epsilon_i = \alpha$ and $\mathsf{par}_i(\mathsf{par}_j(u)) = \mathsf{par}_j(\mathsf{par}_i(u))$, we obtain (3.8). This proves (iv). The proof of (v) is along the lines of (iv); we skip the details. Note that (vi) is a consequence of (iv) and (v). We leave the routine verifications of (vii) and (viii) to the reader. Finally, (ix) may be deduced from (3.5) and Lemma 2.1.10(iii), while (x) is an immediate consequence of (3.5) and Lemma 2.1.10(ii). ■

The following result says that there is no loss of generality if we assume that the weight system of $S_{\boldsymbol\lambda}$ is a subset of the positive real line.

Corollary 3.1.8. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr{T}$. Then $S_{\boldsymbol\lambda}$ is unitarily equivalent to $S_{|\boldsymbol\lambda|}$, where*

$$|\boldsymbol\lambda| = \{|\lambda_v^{(j)}| : v \in V^\circ,\ j = 1, \ldots, d\}.$$

Proof. We combine ideas from [71, Corollary 2] and [67, Theorem 3.2.1]. For simplicity, we treat the case $d = 2$. The proof for $d \geq 3$ is the same. By Proposition 3.1.7(i), for all $i, j = 1, 2$ and all $v \in V$,

$$\arg_u^{(j)} + \arg_{\mathsf{par}_j(u)}^{(i)} = \arg_u^{(i)} + \arg_{\mathsf{par}_i(u)}^{(j)} \qquad \text{for all } u \in \mathsf{Chi}_j\mathsf{Chi}_i(v), \tag{3.9}$$

where $\arg_v^{(j)}$ denotes the principal argument of $\lambda_v^{(j)}$. For $\{\vartheta_v\}_{v\in V} \subset \mathbb{R}$, define the unitary operator $U_\vartheta : l^2(V) \to l^2(V)$ by

$$U_\vartheta e_v = \exp(i\vartheta_v) e_v, \qquad v \in V.$$

Let (T_1, T_2) denote the commuting 2-tuple $S_{|\boldsymbol\lambda|}$. With this notation, the system $S_j U_\vartheta = U_\vartheta T_j$ $(j = 1, 2)$ is equivalent to

$$\vartheta_w - \vartheta_{\mathsf{par}_j(w)} = \arg_w^{(j)}, \qquad w \in \mathsf{Chi}_j(V) \text{ and } j = 1, 2. \tag{3.10}$$

We will show that the above system has a solution. Let $\vartheta_{\mathsf{root}} = 0$. Its clear that ϑ_w can be defined recursively using (3.10). To see that ϑ_w is well-defined, it suffices to check that

$$\arg_w^{(1)} + \vartheta_{\mathsf{par}_1(w)} = \arg_w^{(2)} + \vartheta_{\mathsf{par}_2(w)}, \qquad w \in \mathsf{Chi}_1(V) \cap \mathsf{Chi}_2(V),$$

whenever (3.10) holds for $\mathsf{par}_j(w)$ $(j = 1, 2)$. Note that

$$\begin{aligned}\arg_w^{(2)} + \vartheta_{\mathsf{par}_2(w)} &\overset{(3.10)}{=} \arg_w^{(2)} + (\vartheta_{\mathsf{par}_1(\mathsf{par}_2(w))} + \arg_{\mathsf{par}_2(w)}^{(1)}) \\ &\overset{(3.9)}{=} \arg_w^{(1)} + (\vartheta_{\mathsf{par}_2(\mathsf{par}_1(w))} + \arg_{\mathsf{par}_1(w)}^{(2)}) \overset{(3.10)}{=} \arg_w^{(1)} + \vartheta_{\mathsf{par}_1(w)}. \ ■\end{aligned}$$

From here onwards, we assume that the weights from $\boldsymbol\lambda$ appearing in the definition of $S_{\boldsymbol\lambda}$ are always positive.

Given a positive integer d, we set

$$\mathcal{H}^{\oplus d} := \mathcal{H} \oplus \overset{d}{\cdots} \oplus \mathcal{H}.$$

For a commuting d-tuple $T = (T_1, \ldots, T_d)$ on $\mathcal{H}$, consider the linear transformation $D_T : \mathcal{H} \to \mathcal{H}^{\oplus d}$ given by

$$D_T h := (T_1 h, \ldots, T_d h) \qquad \text{for } h \in \mathcal{H}.$$

Note that the kernel of D_T is precisely the joint kernel $\ker T := \bigcap_{j=1}^d \ker T_j$ of T. We write $\ker D_T$ and $\ker T$ interchangeably.

Corollary 3.1.9. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. For $i \in \mathbb{N}$, let $T^{(i)}$ denote the commuting d-tuple $(S_1^{*i}, \ldots, S_d^{*i})$. Then $\bigcup_{i\in\mathbb{N}} \ker D_{T^{(i)}}$ is dense in $l^2(V)$.*

Proof. Since $\{\ker D_{T^{(i)}}\}_{i\in\mathbb{N}}$ is an increasing sequence of subspaces of $l^2(V)$, it suffices to show that $\mathcal{M} := \bigcup_{i\in\mathbb{N}} \ker D_{T^{(i)}}$ contains e_v for every $v \in V$. For any $v \in V$, by Proposition 3.1.7(v), $e_v \in \ker D_{T^{(i)}}$ for all $i > |\mathsf{d}_v|$, where d_v is the depth of v in $\mathscr{T}$. ■

The following proposition is motivated by the description of the kernel of the adjoint of a weighted shift on a directed tree as given in [67].

Proposition 3.1.10. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Then, for $j = 1, \ldots, d$, we have*

$$\ker S_j^* = \bigoplus_{\substack{v\in V\\ v_j\in V_{\prec}^{(j)}}} \{l^2(\mathsf{Chi}_j(v)) \ominus [\Gamma_v^{(j)}]\} \oplus \bigvee\{e_v : v \in V \text{ and } v_j = \mathsf{root}_j\}, \tag{3.11}$$

where $\Gamma_v^{(j)} : \mathsf{Chi}_j(v) \to \mathbb{C}$ is given by $\Gamma_v^{(j)}(u) = \lambda_u^{(j)} (= (S_j e_v)(u))$.

Proof. This follows immediately from [67, Proposition 3.5.1(ii)] and [35, (2.2)]. ■

Remark 3.1.11. Note that $\ker S_j^*$ is infinite-dimensional whenever $d > 1$.

It is desirable to have a description similar to (3.11) for the joint kernel of $S^*_{\boldsymbol\lambda}$. The following example shows that the situation is more intriguing than it seems.

Example 3.1.12. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of two rooted directed trees $\mathscr{T}_1, \mathscr{T}_2$. Assume that there exists a vertex $v = (v_1, v_2) \in V$ such that v_1 has two children, say $\dot v_1$ and $\ddot v_1$, and v_2 has only one child $\dot v_2$. Then

$$\mathsf{Chi}(v) = \{(\dot v_1, v_2), (\ddot v_1, v_2), (v_1, \dot v_2)\},$$

and hence $l^2(\mathsf{Chi}(v))$ is 3-dimensional. Let $f \in l^2(\mathsf{Chi}(v)) \ominus [\Gamma_v^{(1)}, \Gamma_v^{(2)}]$, and write

$$f = \alpha e_{(\dot v_1, v_2)} + \beta e_{(\ddot v_1, v_2)} + \gamma e_{(v_1, \dot v_2)}$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. We claim that

$$l^2(\mathsf{Chi}(v)) \ominus [\Gamma_v^{(1)}, \Gamma_v^{(2)}] \nsubseteq E,$$

where E denotes the joint kernel of $S^*_{\boldsymbol\lambda}$. Assume to the contrary that $f \in E$. Note that $S_2^* f = 0$ implies

$$\alpha\lambda^{(2)}_{(\dot v_1, v_2)} e_{(\dot v_1, \mathsf{par}(v_2))} + \beta\lambda^{(2)}_{(\ddot v_1, v_2)} e_{(\ddot v_1, \mathsf{par}(v_2))} + \gamma\lambda^{(2)}_{(v_1, \dot v_2)} e_{(v_1, v_2)} = 0,$$

which is true only if $\alpha = \beta = \gamma = 0$, that is, $f = 0$. On the other hand, $[\Gamma_v^{(1)}, \Gamma_v^{(2)}]$ is at most two-dimensional (as $\Gamma_v^{(1)}, \Gamma_v^{(2)}$ could be linearly dependent), and hence

$$\dim(l^2(\mathsf{Chi}(v)) \ominus [\Gamma_v^{(1)}, \Gamma_v^{(2)}]) \geq 1.$$

Thus the claim stands verified.

As is evident from the preceding discussion, the exact description of the joint kernel of $S^*_{\boldsymbol\lambda}$ is not as simple as in the case $d = 1$, and hence we postpone it to Chapter 4. For the time being, let us see that the joint kernel of $S^*_{\boldsymbol\lambda}$ can be finite-dimensional in many interesting situations (cf. Remark 3.1.11).

Proposition 3.1.13. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$ and let E denote the joint kernel of $S^*_{\boldsymbol\lambda}$. Then*

$$\bigvee_{j=1}^{d} \bigoplus_{v \in D_j} \{l^2(\mathsf{Chi}_j(v)) \ominus [\Gamma^{(j)}_v]\} \oplus [e_{\mathsf{root}}] \subseteq E \subseteq \bigvee \{e_v : v \in F_1 \times \cdots \times F_d\}, \tag{3.12}$$

where $D_j := \{v \in V : v_j \in V^{(j)}_{\prec} \text{ and } v_i = \mathsf{root}_i \text{ for } i \neq j\}$, $\Gamma^{(j)}_v : \mathsf{Chi}_j(v) \to \mathbb{C}$ is given by $\Gamma^{(j)}_v(u) = \lambda^{(j)}_u$, and $F_j := \mathsf{Chi}(V^{(j)}_{\prec}) \cup \{\mathsf{root}_j\}$ $(j = 1, \ldots, d)$.

Proof. To see the first inclusion, let $f \in l^2(\mathsf{Chi}_j(v)) \ominus [\Gamma^{(j)}_v]$ for some $v \in D_j$ and for a fixed $j = 1, \ldots, d$. Thus $f = \sum_{u \in \mathsf{Chi}_j(v)} f(u) e_u$ satisfies $\langle f, \Gamma^{(j)}_v \rangle = 0$. Now, for any $i \neq j$,

$$S^*_i f = \sum_{u \in \mathsf{Chi}_j(v)} f(u) \lambda^{(i)}_u e_{\mathsf{par}_i(u)} = 0$$

since $v \in D_j$. Further,

$$S^*_j f = \sum_{u \in \mathsf{Chi}_j(v)} f(u) \lambda^{(j)}_u e_{\mathsf{par}_j(u)} = \sum_{u \in \mathsf{Chi}_j(v)} f(u) \lambda^{(j)}_u e_v = \langle f, \Gamma^{(j)}_v \rangle e_v = 0,$$

where we have used $\langle f, \Gamma^{(j)}_v \rangle = \sum_{u \in \mathsf{Chi}_j(v)} f(u) \lambda^{(j)}_u$.

To see the second inclusion, let $f \in E$ be such that $f = \sum_{v \in V} f(v) e_v$. Then, for $j = 1, \ldots, d$,

$$\begin{aligned} S^*_j f &= \sum_{v \in V} f(v) \lambda^{(j)}_v e_{\mathsf{par}_j(v)} \\ &= \sum_{\substack{v \in V \\ \mathrm{card}(\mathsf{sib}_j(v)) = 1}} f(v) \lambda^{(j)}_v e_{\mathsf{par}_j(v)} + \sum_{\substack{v \in V \\ \mathrm{card}(\mathsf{sib}_j(v)) \geq 2}} f(v) \lambda^{(j)}_v e_{\mathsf{par}_j(v)}. \end{aligned}$$

Note that $e_{\mathsf{par}_j(v)}$ is orthogonal to $e_{\mathsf{par}_j(w)}$ if $v \neq w$, and $\mathrm{card}(\mathsf{sib}_j(v)) = 1 = \mathrm{card}(\mathsf{sib}_j(w))$. Since $S^*_j f = 0$, we obtain $f(v) = 0$ for every $v \in V$ such that $\mathrm{card}(\mathsf{sib}_j(v)) = 1$. Thus $f(v) \neq 0$ implies that either $\mathrm{card}(\mathsf{sib}_j(v))$ is 0 for all $j = 1, \ldots, d$ or is more than 1 for all j. However, $\mathrm{card}(\mathsf{sib}_j(v)) \geq 2$ if and only if $v_j \in \mathsf{Chi}(V^{(j)}_{\prec})$. Further, $\mathrm{card}(\mathsf{sib}_j(v)) = 0$ if and only if $v_j = \mathsf{root}_j$. This completes the proof. ∎

Corollary 3.1.14. *Let $\mathscr{T}$, $S_{\boldsymbol\lambda}$ and E be as in the preceding proposition. If $\mathscr{T}$ is locally finite with finite set of branching vertices, then E is finite-dimensional. Moreover,*

$$1 + \sum_{j=1}^{d} \sum_{v_j \in V^{(j)}_{\prec}} \big(\mathrm{card}(\mathsf{Chi}(v_j)) - 1\big) \leq \dim E \leq \prod_{j=1}^{d} \big(\mathrm{card}(\mathsf{Chi}(V^{(j)}_{\prec})) + 1\big). \tag{3.13}$$

Proof. This is obvious from (3.12). ∎

Remark 3.1.15. In Example 2.1.5, the formula (3.13) holds with equalities at all places (with $\dim E = 1$). On the other hand, in Example 2.1.6, equality holds only at the left end of (3.13) (with $\dim E = 2$). Further, in Example 2.1.7, equality may or may not hold even at the left end of (3.13) (with $\dim E = 3$ or 4). The last two assertions may be deduced from Examples 4.1.8 and 4.1.9.

Corollary 3.1.16. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr{T}$ and let E denote the joint kernel of $S^*_{\boldsymbol\lambda}$. Then E is finite-dimensional if and only if $\mathscr{T}$ is locally finite with finite joint branching index.*

Proof. If E is infinite-dimensional then by (3.12), the cardinality of $F_j = \mathsf{Chi}(V^{(j)}_{\prec}) \cup \{\mathsf{root}_j\}$ must be infinite for some $j = 1, \ldots, d$. It follows that either $\mathscr{T}_j$ is not locally finite or $V^{(j)}_{\prec}$ is infinite. To see the converse, suppose that for some $j = 1, \ldots, d$, $\mathscr{T}_j$ is either not locally finite or of infinite branching index. By Proposition 3.1.13,

$$\mathcal{M} := \bigoplus_{v \in D_j} \{l^2(\mathsf{Chi}_j(v)) \ominus [\Gamma^{(j)}_v]\} \subseteq E,$$

where $D_j = \{v \in V : v_j \in V^{(j)}_{\prec} \text{ and } v_i = \mathsf{root}_i \text{ for } i \neq j\}$. Note that $l^2(\mathsf{Chi}_j(v)) \ominus [\Gamma^{(j)}_v]$ is nonzero for every $v \in D_j$. If $\mathscr{T}_j$ is not locally finite, then $l^2(\mathsf{Chi}_j(v))$ is infinite-dimensional for some $v \in D_j$. If $\mathscr{T}_j$ is of infinite branching index, then D_j is infinite. In any case, $\mathcal{M}$ and hence E is infinite-dimensional. ∎

We have already seen in Theorem 2.1.16 that the directed Cartesian product of directed trees admits a directed semi-tree structure. Since there is a notion of shifts $\mathscr{S}_{\boldsymbol\delta}$ on directed semi-trees [79], it is thus natural to reveal the relation between $\mathscr{S}_{\boldsymbol\delta}$ on a directed semi-tree $\mathscr{T}$ and the multishift $S_{\boldsymbol\lambda}$. To see this, let us recall the notion of shift on a directed semi-tree from [79, Section 5].

Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Given a system $\boldsymbol\delta = \{\delta_{(u,v)} : (u, v) \in \mathcal{E}\}$ of positive numbers, define the *weighted shift operator* $\mathscr{S}_{\boldsymbol\delta}$ on $\mathscr{T}$ with weights $\boldsymbol\delta$ by

$$\mathscr{D}(\mathscr{S}_{\boldsymbol\delta}) := \{f \in l^2(V) \colon \varDelta_{\mathscr{T}} f \in l^2(V)\}, \quad \mathscr{S}_{\boldsymbol\delta} f := \varDelta_{\mathscr{T}} f, \quad f \in \mathscr{D}(\mathscr{S}_{\boldsymbol\delta}),$$

where $\varDelta_{\mathscr{T}}$ is the mapping defined on complex functions f on V by

$$(\varDelta_{\mathscr{T}} f)(v) := \begin{cases} \sum_{u \in \mathsf{Par}(v)} \delta_{(u,v)} f(u) & \text{if } v \in V \setminus \{\mathsf{root}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us see the precise relation between $S_{\boldsymbol\lambda}$ and $\mathscr{S}_{\boldsymbol\delta}$.

Proposition 3.1.17. *Let $\mathscr{S}_{\boldsymbol\delta}$ be the weighted shift operator on the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Set $\lambda^{(j)}_v := \delta_{(u,v)}$ for $v \in \mathsf{Chi}_j(u)$, and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be the multishift on $\mathscr{T}$ (possibly unbounded). Then $\bigcap_{j=1}^d \mathscr{D}(S_j) \subseteq \mathscr{D}(\mathscr{S}_{\boldsymbol\delta})$. If, in addition, $S_1, \ldots, S_d$ are bounded linear operators on $l^2(V)$, then $\mathscr{S}_{\boldsymbol\delta}$ is bounded. In this case, $\mathscr{S}_{\boldsymbol\delta} = \sum_{j=1}^d S_j$.*

Proof. Let $f \in \bigcap_{j=1}^d \mathscr{D}(S_j)$. Then

$$\begin{aligned}(\mathscr{S}_{\boldsymbol\delta} f)(v) &= \sum_{u \in \mathsf{Par}(v)} \delta_{(u,v)} f(u) \overset{(2.1)}{=} \sum_{j=1}^d \sum_{u = \mathsf{par}_j(v)} \delta_{(u,v)} f(u) \\ &= \sum_{j=1}^d \lambda^{(j)}_v f(\mathsf{par}_j(v)) = \sum_{j=1}^d (S_j f)(v).\end{aligned}$$

This shows that $f \in \mathscr{D}(\mathscr{S}_{\boldsymbol\delta})$. If $S_1, \ldots, S_d$ are bounded on $l^2(V)$, then $\mathscr{D}(\mathscr{S}_{\boldsymbol\delta}) = l^2(V)$. However, $\mathscr{S}_{\boldsymbol\delta}$ is always closed [79, Proposition 5.1], and hence $\mathscr{S}_{\boldsymbol\delta}$ is a bounded linear operator in this case. ■

The above result shows that $\sum_{j=1}^d S_j$ can be realized as the shift on $\mathscr{T}$ endowed with a directed semi-tree structure.

In the remaining part of this chapter, we obtain some basic properties of multishifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}$. These include circularity and analyticity. We also obtain a matrix decomposition for $S_{\boldsymbol\lambda}$ in dimension $d = 2$. All these results are then used to examine various spectral parts of $S_{\boldsymbol\lambda}$ such as point spectrum, Taylor spectrum, and essential spectrum. The discussion to follow relies heavily on the multivariable spectral theory as expounded in [44].

3.2. Strong circularity and Taylor spectrum. Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. We say that T is *circular* if for every $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, there exists a unitary operator Γ_θ on $\mathcal{H}$ such that

$$\Gamma_\theta^* T_j \Gamma_\theta = \exp(i\theta_j) T_j \quad \text{for all } j = 1, \ldots, d.$$

We say that T is *strongly circular* if in addition Γ_θ can be chosen to be a strongly continuous unitary representation of $\mathbb{R}^d$ in the following sense: for every $h \in \mathcal{H}$, the function $\theta \mapsto \Gamma_\theta h$ is continuous on $\mathbb{R}^d$.

The above notion in dimension $d = 1$ has been introduced and studied in [16]. These operators have been extensively studied thereafter (refer to [57], [81], [88], [23]). The fact that any classical multishift is circular is first obtained in [71, Corollary 3]. This may also be deduced from [36, Lemma 2.14].

The following generalizes [71, Corollary 3]. Unlike the method of proof of [67, Theorem 3.3.1], where the unitary Γ_θ comes from solution of a system of equations, our proof exhibits a formula for Γ_θ.

Proposition 3.2.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr{T}$. Then $S_{\boldsymbol\lambda}$ is strongly circular.*

Proof. Let $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$. For $f = \sum_{v \in V} f(v) e_v \in l^2(V)$, define $\Gamma_\theta : l^2(V) \to l^2(V)$ by

$$\Gamma_\theta f := \sum_{v \in V} \exp(-i\mathsf{d}_v \cdot \theta) f(v) e_v,$$

where d_v is the depth of v in $\mathscr{T}$ and $\alpha \cdot \theta := \sum_{j=1}^d \alpha_j \theta_j$ for $\alpha \in \mathbb{N}^d$. Clearly, Γ_θ is unitary with inverse $\Gamma_{-\theta}$. Note that $\mathsf{d}_w = \mathsf{d}_v + \epsilon_j$ if $w \in \mathsf{Chi}_j(v)$ for any $v \in V$ and $j = 1, \ldots, d$. It follows that

$$\begin{aligned}(\Gamma_\theta^* S_j \Gamma_\theta) f &= (\Gamma_\theta^* S_j) \sum_{v \in V} \exp(-i\mathsf{d}_v \cdot \theta) f(v) e_v \\ &= \Gamma_\theta^* \Big(\sum_{v \in V} \exp(-i\mathsf{d}_v \cdot \theta) f(v) \sum_{w \in \mathsf{Chi}_j(v)} \lambda_w^{(j)} e_w \Big) \\ &= \sum_{v \in V} \exp(-i\mathsf{d}_v \cdot \theta) f(v) \sum_{w \in \mathsf{Chi}_j(v)} \lambda_w^{(j)} \exp(i\mathsf{d}_w \cdot \theta) e_w\end{aligned}$$

$$= \sum_{v\in V} \exp(-i\mathsf{d}_v\cdot\theta) f(v) \sum_{w\in \mathsf{Chi}_j(v)} \lambda_w^{(j)} \exp(i(\mathsf{d}_v+\epsilon_j)\cdot\theta) e_w$$
$$= \sum_{v\in V} \exp(i\epsilon_j\cdot\theta) f(v) \sum_{w\in \mathsf{Chi}_j(v)} \lambda_w^{(j)} e_w = \exp(i\theta_j) S_j f.$$

Now we show that for any $f \in l^2(V)$, $\theta \mapsto \Gamma_\theta f$ is continuous. Let $\{\theta^{(n)}\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^d$ which converges to θ. Then for $f = \sum_{v\in V} f(v)e_v \in l^2(V)$,

$$\|(\Gamma_{\theta^{(n)}} - \Gamma_\theta) f\|^2 = \sum_{v\in V} |\exp(-i\mathsf{d}_v\cdot\theta^{(n)}) - \exp(-i\mathsf{d}_v\cdot\theta)|^2 |f(v)|^2.$$

Let $\epsilon > 0$. Since $f \in l^2(V)$, there is a finite subset W of V such that $\sum_{v\in V\setminus W} |f(v)|^2 < \epsilon$. Further, as $\theta \mapsto \exp(-i\mathsf{d}_u\cdot\theta)$ is continuous for each $u \in W$, there exists a positive integer $n(u)$ such that $\big|\exp(-i\mathsf{d}_u\cdot\theta^{(n)}) - \exp(-i\mathsf{d}_u\cdot\theta)\big|^2 < \epsilon$ for all $n \geq n(u)$. Let $n(0) := \max\{n(u) : u \in W\} < \infty$. Then, for all $n \geq n(0)$,

$$\begin{aligned}\|(\Gamma_{\theta^{(n)}} - \Gamma_\theta) f\|^2 &= \sum_{v\in W} |\exp(-i\mathsf{d}_v\cdot\theta^{(n)}) - \exp(-i\mathsf{d}_v\cdot\theta)|^2 |f(v)|^2 \\ &\quad + \sum_{v\in V\setminus W} |\exp(-i\mathsf{d}_v\cdot\theta^{(n)}) - \exp(-i\mathsf{d}_v\cdot\theta)|^2 |f(v)|^2 \\ &< \epsilon^2 \Big(\sum_{v\in W} |f(v)|^2\Big) + 4\epsilon. \ \blacksquare\end{aligned}$$

The following is immediate from the spectral mapping property of Taylor spectrum and the preceding proposition.

Corollary 3.2.2. *The Taylor spectrum of a commuting multishift $S_{\boldsymbol\lambda}$ has polycircular symmetry, that is, $\zeta\cdot w \in \sigma(S_{\boldsymbol\lambda})$ for any $w \in \sigma(S_{\boldsymbol\lambda})$ and any $\zeta \in \mathbb{T}^d$. In particular, the Taylor spectrum of $S_{\boldsymbol\lambda}$ coincides with that of $S^*_{\boldsymbol\lambda}$.*

Remark 3.2.3. Note that point spectrum, left spectrum and essential spectrum also have polycircular symmetry.

A special case of the following result, in which $\mathscr{T}$ is the directed Cartesian product of $\mathscr{T}_{1,0}$ with itself, has been obtained in [36].

Proposition 3.2.4. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr{T}$. Then the Taylor spectrum of $S_{\boldsymbol\lambda}$ is connected.*

Proof. The idea of this proof is similar to that of [36, Lemma 3.8]. By Remark 3.1.6, 0 belongs to the point spectrum $\sigma_p(S^*_{\boldsymbol\lambda})$ of $S^*_{\boldsymbol\lambda}$. Hence 0 belongs to the Taylor spectrum of $S^*_{\boldsymbol\lambda}$. In view of Corollary 3.2.2, it suffices to check that $\sigma(S^*_{\boldsymbol\lambda})$ is connected. Let K_1 be the connected component of $\sigma(S^*_{\boldsymbol\lambda})$ containing 0 and let $K_2 = \sigma(S^*_{\boldsymbol\lambda}) \setminus K_1$. By the Shilov Idempotent Theorem [44, Application 5.24], there exist invariant subspaces $\mathcal{M}_1, \mathcal{M}_2$ of $S^*_{\boldsymbol\lambda}$ such that $l^2(V) = \mathcal{M}_1 \dot{+} \mathcal{M}_2$ (vector space direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$) and $\sigma(S^*_{\boldsymbol\lambda}|_{\mathcal{M}_i}) = K_i$ for $i = 1, 2$.

For every $i \in \mathbb{N}$, let $T^{(i)}$ denote the commuting d-tuple $(S_1^{*i}, \ldots, S_d^{*i})$. Let $h \in \ker(D_{T^{(i)}})$ for fixed $i \in \mathbb{N}$. Then $h = x + y$ for $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_2$. It follows that $S_j^{*i}x = 0$ and $S_j^{*i}y = 0$ for all $j = 1, \ldots, d$. If y is nonzero, then $0 \in \sigma_p(T^{(i)}|_{\mathcal{M}_2}) \subseteq \sigma(T^{(i)}|_{\mathcal{M}_2})$, and hence by the spectral mapping property [44], $0 \in \sigma(T^{(1)}|_{\mathcal{M}_2}) = \sigma(S^*_{\boldsymbol{\lambda}}|_{\mathcal{M}_2})$. Since $0 \notin K_2$, we must have $y = 0$. It follows that $\mathcal{M}_1$ contains the linear manifold $\bigcup_{i\in\mathbb{N}} \ker(D_{T^{(i)}})$, which is dense in $l^2(V)$ by Corollary 3.1.9. Hence $\mathcal{M}_1 = l^2(V)$. Thus the Taylor spectrum of $S^*_{\boldsymbol{\lambda}}$ is equal to K_1. In particular, the Taylor spectrum of $S_{\boldsymbol{\lambda}}$ is connected. ∎

A connected subset Ω of $\mathbb{C}^d$ is said to be *Reinhardt* if it is invariant under the action of the d-torus $\mathbb{T}^d$, that is, $\zeta \cdot z := (\zeta_1 z_1, \ldots, \zeta_d z_d)$ belongs to Ω whenever $z \in \Omega$ and $\zeta \in \mathbb{T}^d$.

Combining Proposition 3.2.1 with the preceding result, we obtain the following fact.

Corollary 3.2.5. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Then the Taylor spectrum of a commuting multishift on $\mathscr{T}$ is Reinhardt.*

Remark 3.2.6. Suppose that the Taylor spectrum $\sigma(S_{\boldsymbol{\lambda}})$ has spherical symmetry in the following sense: $Uz \in \sigma(S_{\boldsymbol{\lambda}})$ whenever $z \in \sigma(S_{\boldsymbol{\lambda}})$ for every $d \times d$ unitary matrix U. Then $\sigma(S_{\boldsymbol{\lambda}})$ must be a closed ball centered at the origin. Indeed, $0 \in \sigma(S_{\boldsymbol{\lambda}})$ since e_{root} belongs to the joint kernel of $S^*_{\boldsymbol{\lambda}}$ in view of Remark 3.1.6. The desired conclusion now follows from the fact that every spherically symmetric, compact Reinhardt set containing 0 is a closed ball centered at 0.

3.3. Analyticity and point spectrum. A commuting d-tuple $T = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{H}$ is called *analytic* if

$$\bigcap_{\alpha \in \mathbb{N}^d} \operatorname{ran} T^{\alpha} = \{0\}.$$

Just as in the classical case, the multishifts on $\mathscr{T}$ are analytic. Indeed, they are separately analytic in the following sense.

Proposition 3.3.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Then for each $j = 1, \ldots, d$, S_j is analytic.*

Proof. Let $j = 1, \ldots, d$ be fixed. For $n \in \mathbb{N}$, let

$$M_n := \bigvee \{e_v : v \in \mathsf{Chi}^{\langle\langle n\epsilon_j \rangle\rangle}(V)\},$$

and note that by (3.5), $\operatorname{ran} S_j^n \subseteq M_n$. It now suffices to check that $\bigcap_{n=0}^{\infty} M_n = \{0\}$. To see this, note that if $f \in M_n$, then $f(u) = 0$ for every $u \in V$ such that $u_j \in \bigcup_{i=0}^{n-1} \mathsf{Chi}^{\langle i \rangle}(\mathsf{root}_j)$. However, $\bigcup_{i=0}^{\infty} \mathsf{Chi}^{\langle i \rangle}(\mathsf{root}_j) = V_j$, and hence for any $f \in \bigcap_{n=0}^{\infty} M_n$, we must have $f(u) = 0$ for any $u \in V$. ∎

The next corollary generalizes [71, Theorem 15], where the method of proof relies on the description of the commutant of S_j.

Corollary 3.3.2. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Then for each $j = 1, \ldots, d$, the spectrum of S_j equals $\operatorname{cl}(\mathbb{D}_{r(S_j)})$, where $r(T)$ denotes the spectral radius of a bounded linear operator T.*

Proof. By [35, Lemma 5.2], the spectrum of S_j is connected. Since S_j is circular (Proposition 3.2.1) and $0 \in \sigma(S_j^*)$ (Remark 3.1.6), the spectrum of S_j must be the disc $\operatorname{cl}(\mathbb{D}_{r(S_j)})$. ∎

Corollary 3.3.3. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Then the commuting multishift $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ is analytic.*

Proof. Note that $\bigcap_{\alpha \in \mathbb{N}^d} \operatorname{ran} S^{\alpha}_{\boldsymbol\lambda} \subseteq \bigcap_{k \in \mathbb{N}} \operatorname{ran} S^k_j$ for any $j = 1, \ldots, d$. The desired conclusion now follows from Proposition 3.3.1. ■

Corollary 3.3.4. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Then for each $j = 1, \ldots, d$, the point spectrum of S_j is empty. In particular, the joint kernel of $S_{\boldsymbol\lambda}$ is trivial.*

Proof. By Lemma 3.1.5(ii), S_j is injective. Also, if $S_j f = wf$ for some nonzero $w \in \mathbb{C}$ then $f \in \bigcap_{k \in \mathbb{N}} \operatorname{ran} S^k_j = \{0\}$, and hence $f = 0$. This proves that the point spectrum of S_j is empty. ■

Remark 3.3.5. Note that none of $S_1, \ldots, S_d$ can be normal, that is, $S^*_j S_j \neq S_j S^*_j$ for every $j = 1, \ldots, d$. In view of Remark 3.1.6, this may be deduced from the fact that for any normal operator T, the kernel of T and the kernel of T^* coincide.

Corollary 3.3.6. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Then for each $j = 1, \ldots, d$,*

$$\bigvee_{k \in \mathbb{N}} \ker S^{*k}_j = l^2(V) = \bigvee_{\alpha \in \mathbb{N}^d} \ker S^{*\alpha}_{\boldsymbol\lambda}.$$

Proof. After taking orthogonal complement, the first equality may be deduced from the analyticity of S_j, while the second one follows from the analyticity of $S_{\boldsymbol\lambda}$. ■

3.4. A matrix decomposition and essential spectrum. In this section, we discuss a matrix decomposition of 2-shifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ (cf. [35, Lemma 5.3]). The building blocks in this decomposition include classical 2-shifts and 2-tuples with entries being weighted shifts on directed trees. We will use this decomposition to relate the spectral parts of $S_{\boldsymbol\lambda}$ with the spectral parts of the building blocks appearing in the matrix decomposition of $S_{\boldsymbol\lambda}$. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_j = (V_j, \mathcal{E}_j)$, $j = 1, 2$. Assume that $\mathscr{T}$ is locally finite with finite joint branching index $k_{\mathscr{T}} = (k_{\mathscr{T}_1}, k_{\mathscr{T}_2})$. Let us observe the following:

(A) Fix $v_1 \in V_1$. If $W = \{v_1\} \times V_2$, then $l^2(W)$ is invariant under S_2. Moreover,

$$P_{l^2(W)} S_1|_{l^2(W)} = 0,$$

and $S_2|_{l^2(W)}$ is unitarily equivalent to a weighted shift on the directed tree $\mathscr{T}_2$. A similar observation holds for $V_1 \times \{v_2\}$ for any $v_2 \in V_2$.

(B) Fix $j \in \{1, 2\}$. Define $G_j := \{\mathsf{root}_j\}$ if $V^{(j)}_{\prec} = \emptyset$. Otherwise, let

$$G_j := \{v_j \in \mathsf{Chi}(V^{(j)}_{\prec}) : \operatorname{card}(\mathsf{Chi}^{\langle n \rangle}(v_j)) = 1 \text{ for all } n \geq 1\}.$$

Let $v \in G_1 \times G_2$. Then $L_v := \bigsqcup_{\alpha \in \mathbb{N}^2} \mathsf{Chi}^{\ll \alpha \gg}(v)$ is a directed graph isomorphic to $\mathscr{T}_{1,0} \times \mathscr{T}_{1,0}$. Note first that for any two distinct vertices $u_j, v_j \in G_j$, there is no positive integer n_j such that $\mathsf{Chi}^{\langle n_j \rangle}(u_j) = \{v_j\}$. Indeed, if $\mathsf{Chi}^{\langle n_j \rangle}(u_j) = \{v_j\}$ then $\mathsf{Chi}^{\langle n_j - 1 \rangle}(u_j) \cap V^{(j)}_{\prec} = \{\mathsf{par}(v_j)\}$, and hence $\operatorname{card}(\mathsf{Chi}^{\langle n_j \rangle}(u_j)) \geq 2$, a contradiction.

We next check that for distinct $v, w \in G_1 \times G_2$, $L_v \cap L_w = \emptyset$. Without loss of generality, assume that $v_1 \neq w_1$. Suppose that $u \in L_v \cap L_w$. Then $\mathsf{Chi}^{\langle \alpha_1 \rangle}(v_1) \cap \mathsf{Chi}^{\langle \beta_1 \rangle}(w_1) = \{u_1\}$ for some $\alpha_1, \beta_1 \in \mathbb{N}$, and hence $\mathsf{Chi}^{\langle \alpha_1 \rangle}(v_1) = \mathsf{Chi}^{\langle \beta_1 \rangle}(w_1)$. It follows that either $\mathsf{Chi}^{\langle \beta_1' \rangle}(w_1) = \{v_1\}$ or $\mathsf{Chi}^{\langle \alpha_1' \rangle}(v_1) = \{w_1\}$ for some $\alpha_1', \beta_1' \in \mathbb{N}$. This contradicts the above observation.

(C) Let $W_j := \bigcup_{n=1}^{\infty} \mathsf{par}^{\langle n \rangle}(G_j)$ for $j = 1, 2$. Note that W_1, W_2 are finite sets. Consider the disjoint sets

$$F_1 := \bigsqcup_{w_1 \in W_1} \{w_1\} \times V_2, \quad F_2 := \bigsqcup_{w_2 \in W_2} (V_1 \setminus W_1) \times \{w_2\}.$$

(D) Note that $V = F_1 \sqcup F_2 \sqcup F_3$, where

$$F_3 := \bigsqcup_{v \in G_1 \times G_2} L_v.$$

This gives the decomposition $l^2(V) = l^2(F_1) \oplus l^2(F_2) \oplus l^2(F_3)$, where

$$l^2(F_1) = \bigoplus_{w_1 \in W_1} \mathcal{N}_{w_1} \quad \text{and} \quad \mathcal{N}_{w_1} := l^2(\{w_1\} \times V_2),$$
$$l^2(F_2) = \bigoplus_{w_2 \in W_2} \mathcal{M}_{w_2} \quad \text{and} \quad \mathcal{M}_{w_2} := l^2((V_1 \setminus W_1) \times \{w_2\}),$$
$$l^2(F_3) = \bigoplus_{v \in G_1 \times G_2} l^2(L_v).$$

We now decompose (S_1, S_2) with respect to the decomposition $l^2(V) = l^2(F_1) \oplus l^2(F_2) \oplus l^2(F_3)$. Indeed, $S_1 = (A_{ij})_{1 \leq i,j \leq 3}$ and $S_2 = (B_{ij})_{1 \leq i,j \leq 3}$, where

(a) $A_{1i} = 0$ $(i = 2, 3)$, $A_{23} = 0 = A_{32}$, $B_{1i} = 0$ $(i = 2, 3)$, $B_{2j} = 0$ $(j = 1, 3)$, $B_{31} = 0$,
(b) $A_{11} = 0$ if $\mathsf{Chi}(W_1) \cap W_1 = \emptyset$ (if and only if $k_{\mathscr{T}_1} \leq 1$), and otherwise of infinite rank, $B_{22} = 0$ if $\mathsf{Chi}(W_2) \cap W_2 = \emptyset$ (if and only if $k_{\mathscr{T}_2} \leq 1$), and otherwise of infinite rank,
(c) A_{21} is the matrix with generic entry $P_{\mathcal{M}_{w_2}} S_1|_{\mathcal{N}_{w_1}}$ (finite rank operator),
(d) A_{22} is the diagonal matrix with generic entry $S_1|_{\mathcal{M}_{w_2}}$, B_{11} is the diagonal matrix with generic entry $S_2|_{\mathcal{N}_{w_1}}$ (one-variable shifts on directed trees),
(e) A_{33} is the diagonal matrix with generic entry $S_1|_{l^2(L_v)}$, B_{33} is the diagonal matrix with generic entry $S_2|_{l^2(L_v)}$ (entries of the classical multishift S_w),
(f) A_{31} is the matrix with generic entry $P_{l^2(L_v)} S_1|_{\mathcal{N}_{w_1}}$, B_{32} is the matrix with generic entry $P_{l^2(L_v)} S_2|_{\mathcal{M}_{w_2}}$ (infinite rank nonshifts).

Since S_1 and S_2 are commuting, a plain calculation shows that

$$A_{21}B_{11} = 0, \quad A_{31}B_{11} = B_{32}A_{21} + B_{33}A_{31},$$
$$A_{33}B_{32} = B_{32}A_{22}, \quad A_{33}B_{33} = B_{33}A_{33}.$$

Thus the building blocks for $S_{\boldsymbol{\lambda}}$ consist of 2-tuples of the form (A_{11}, B_{11}) or (A_{22}, B_{22}) for single variable weighted shifts B_{11}, A_{22} on directed trees, commuting classical multishifts (A_{33}, B_{33}), finite rank 2-tuple $(A_{21}, 0)$, and infinite rank nonshifts $(A_{31}, 0)$, $(0, B_{32})$.

It is worth noting that the situation in case $d = 1$ is entirely different in the sense that all nondiagonal entries in the matrix decomposition of S_λ are of finite rank (see [35, Lemma 5.3]).

Before we see applications of the above decomposition, we would like to discuss convergence of nets associated with directed Cartesian products of directed trees. Let $\mathscr{T}_j = (V_j, \mathcal{E}_j)$ $(j = 1, \ldots, d)$ be rooted directed trees and let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Define a relation $\leq$ on V as follows:

$$v \leq w \quad \text{if} \quad \mathsf{d}_v \leq \mathsf{d}_w,$$

where d_v denotes the depth of v in $\mathscr{T}$. Note that V is a partially ordered set with partial order relation $\leq$ (that is, $\leq$ is reflexive and transitive). Note that given two vertices $v, w \in V$, there exists $u \in V$ such that $v \leq u$ and $w \leq u$. In this text, we will be interested in the nets $\{\lambda_v\}_{v \in V}$ of complex numbers induced by the above partial order (the reader is referred to [75] for the definition and elementary facts pertaining to nets).

Remark 3.4.1. One can also endow V with the following partial order relations:

1. $v \leq w$ if d_v is less than or equal to d_w with respect to the dictionary ordering,
2. $v \leq w$ if $|\mathsf{d}_v| \leq |\mathsf{d}_w|$.

Note that convergence of nets in (1) is weaker than and that in (2) is stronger than the convergence defined prior to the remark. All these notions agree in case $d = 1$.

We now see an application of the matrix decomposition of multishifts as discussed above.

Proposition 3.4.2. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \mathscr{T}_2$ of finite joint branching index $k_{\mathscr{T}} = (k_{\mathscr{T}_1}, k_{\mathscr{T}_2})$. Let $S_{\boldsymbol\lambda}$ be the commuting multishift on $\mathscr{T}$. Then:*

(i) $\sigma(S_{\boldsymbol\lambda}) \subseteq \sigma((A_{11}, B_{11})) \cup \sigma((A_{22}, B_{22})) \cup \sigma((A_{33}, B_{33}))$.
(ii) $\sigma_l((A_{33}, B_{33})) \subseteq \sigma_l(S_{\boldsymbol\lambda})$.

Assume further that $\max\{k_{\mathscr{T}_1}, k_{\mathscr{T}_2}\} \leq 1$. *Then:*

(iii) $\sigma(S_{\boldsymbol\lambda}) \subseteq (\{0\} \times \sigma(B_{11})) \cup (\sigma(A_{22}) \times \{0\}) \cup \sigma((A_{33}, B_{33}))$.
(iv) *If, in addition,*

$$\lim_{u_2 \in \mathsf{Des}(v_2)} \lambda^{(1)}_{(v_1, u_2)} = 0 = \lim_{u_1 \in \mathsf{Des}(v_1)} \lambda^{(2)}_{(u_1, v_2)} \quad \text{for all } v \in G_1 \times G_2, \tag{3.14}$$

then $\sigma_e(S_{\boldsymbol\lambda})$ is the union of the essential spectra of finitely many 2-tuples of the form $(0, U_\lambda)$ or $(U_\lambda, 0)$ for a weighted shift U_λ on a directed tree, and the essential spectra of finitely many commuting classical 2-variable shifts.

Proof. Note that (i) and (iii) are particular consequences of part (b) of (D) of the previous decomposition and [42, Lemmas 4.4 and 4.5]. To see (ii), note that if any commuting d-tuple T on $\mathcal{H}$ is bounded below then so is its restriction to any joint invariant subspace $\mathcal{M}$ of $\mathcal{H}$. Applying this fact to $T := S_{\boldsymbol\lambda} - \omega$ $(\omega \in \mathbb{C}^2)$ and $\mathcal{M} := l^2(F_3)$ yields the conclusion in (ii).

To see the remaining part, assume further that (3.14) holds. We first note that $S_{\boldsymbol\lambda}$ is a commuting compact perturbation of an orthogonal direct sum of finitely many 2-tuples of the form $(0, U_\lambda)$ or $(U_\lambda, 0)$ for a single variable weighted shift U_λ on a directed tree, and finitely many commuting classical 2-variable shifts. This may be seen once we observe that $A_{31} = P_{l^2(L_v)} S_1|_{\mathcal{N}_{w_1}}$ and $B_{32} = P_{l^2(L_v)} S_2|_{\mathcal{M}_{w_2}}$ are compact for every $v \in G_1 \times G_2$. But this follows from (3.14). To complete the proof, in view of the Atkinson–Curto Theorem [43, Theorem 2], we need the fact that the essential spectrum $\sigma_e(A\oplus B)$ of the orthogonal direct sum of A and B is the union of $\sigma_e(A)$ and $\sigma_e(B)$, where A and B denote commuting d-tuples of bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. We include an elementary verification of this fact. Note that the boundary operators $\partial_{A\oplus B}$ appearing in the Koszul complex of $A\oplus B$ are the orthogonal direct sums of the boundary operators ∂_A and ∂_B appearing in the Koszul complexes of A and B respectively (refer to Section 1.1). That is, $\partial_{A\oplus B}+\partial^*_{A\oplus B} = (\partial_A+\partial^*_A)\oplus(\partial_B+\partial^*_B)$. On the other hand, by [44, Theorem 6.2], a d-tuple T is Fredholm if and only if $\partial_T+\partial^*_T$ is Fredholm. The desired conclusion is now immediate. ■

We illustrate the previous result with an example.

Example 3.4.3. Let $\mathscr{T} = \mathscr{T}_{2,0} \times \mathscr{T}_{1,0}$ be as discussed in Example 2.1.6. Note that $G_1 = \{1,2\}$, $G_2 = \{0\}$, $W_1 = \{0\}$, $W_2 = \emptyset$, $F_1 = \{0\} \times V_2$, $F_2 = \emptyset$, $F_3 = L_{(1,0)} \cup L_{(2,0)}$.

Let $S_{\boldsymbol\lambda}$ be a multishift on $\mathscr{T}$ with weights $\boldsymbol\lambda$ such that

$$\lim_{k\to\infty} \lambda^{(1)}_{(1,k)} = \lim_{k\to\infty} \lambda^{(1)}_{(2,k)} = 0.$$

By the above result, the essential spectrum $\sigma_e(S_{\boldsymbol\lambda})$ is equal to the union of the essential spectra of $(0, U_\lambda), S_{\mathbf{w}^{(1)}}, S_{\mathbf{w}^{(2)}}$. In particular, this is applicable to the commuting multishift $S_{\boldsymbol\lambda} = (S_1, S_2)$ with weights given by

$$\lambda^{(1)}_{(m,n)} = \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_1(m,n))}}\sqrt{\frac{[m/2]}{[m/2]+n}}, \qquad \lambda^{(2)}_{(m,n)} = \sqrt{\frac{n}{[m/2]+n}},$$

where $m, n \in \mathbb{N}$, and

$$\left[\frac{m}{2}\right] = \begin{cases} m/2 & \text{if } m \text{ is an even integer,} \\ (m+1)/2 & \text{otherwise.}\end{cases}$$

Note that none of S_1, S_2 is compact. Further, U_λ is the unilateral unweighted shift with essential spectrum the unit circle $\mathbb{T}$ (refer to [88]). By spectral mapping property, the essential spectrum of $(0, U_\lambda)$ is $\{0\}\times\mathbb{T}$. Further, $S_{w^{(1)}}$ is the 2-variable classical multishift with weights

$$w^{(1)}_{(2k+1,l)} = \sqrt{\frac{k+1}{k+l+1}}, \qquad w^{(2)}_{(2k-1,l+1)} = \sqrt{\frac{l+1}{k+l+1}} \qquad (k \geq 1, l \geq 0).$$

It is easy to see that $S_{\mathbf{w}^{(1)}}$ is unitarily equivalent to the Drury–Arveson 2-shift, and hence the essential spectrum of $S_{\mathbf{w}^{(1)}}$ equals the unit sphere $\partial\mathbb{B}^2$ in $\mathbb{C}^2$ (Proposition 1.2.2). Similarly, the essential spectrum of $S_{\mathbf{w}^{(2)}}$ is $\partial\mathbb{B}^2$. It follows that $\sigma_e(S_{\boldsymbol\lambda}) = \partial\mathbb{B}^2$. Also, since the left spectrum of the Drury–Arveson shift is the unit sphere (Proposition 1.2.2), it may be concluded from Proposition 3.4.2(ii) that $\sigma_l(S_{\boldsymbol\lambda})$ contains $\partial\mathbb{B}^2$.

Remark 3.4.4. We will see later in Chapter 5 that $\sigma_l(S_{\boldsymbol\lambda})$ is contained in the unit sphere in $\mathbb{C}^2$ (see (5.30)).

We conclude this section with a brief discussion of how to construct noncompact multishifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ having Taylor spectra with empty interior. One such family of classical multishifts has been exhibited in [51, Example 2]. One may capitalize on this example to construct examples of multishifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{2,0} \times \mathscr{T}_{2,0}$ with Taylor spectra of empty interior (see Proposition 3.4.2(iii)).

4. Wandering subspace property

Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on a Hilbert space $\mathcal{H}$. A subspace $\mathcal{W}$ of $\mathcal{H}$ is said to be *wandering* for T if $T^\alpha \mathcal{W}$ is orthogonal to $\mathcal{W}$ for every $\alpha \in \mathbb{N}^d \setminus \{0\}$. Note that the joint kernel $E = \bigcap_{j=1}^d \ker T_j^*$ of T^* is always a wandering subspace for T. Following [89, Definition 2.4], we say that T has the *wandering subspace property* if $\mathcal{H} = [E]_T$, where

$$[E]_T := \bigvee_{\alpha \in \mathbb{N}^d} T^\alpha E.$$

The main result of this chapter ensures the wandering subspace property for a multishift $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ under some modest assumptions. Unlike the cases either of classical multishifts or of one-variable weighted shifts on rooted directed trees, this fact lies deeper.

Theorem 4.0.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr{T}$. Then $S_{\boldsymbol\lambda}$ possesses wandering subspace property.*

The proof of Theorem 4.0.1, as presented below, relies heavily on the analysis of the joint kernel of $S^*_{\boldsymbol\lambda}$ carried out in the next section. Before we start preparing for the proof of this result, we would like to discuss Shimorin's approach to the wandering subspace property for left invertible analytic operators. Note that the wandering subspace property for a left invertible analytic operator T on $\mathcal{H}$ is a simple consequence of the duality relation

$$\Big(\bigcap_{k \in \mathbb{N}} T'^k \mathcal{H}\Big)^\perp = [\ker T^*]_T, \tag{4.1}$$

which in turn relies on the identity

$$I - T^n T'^{*n} = \sum_{k=0}^{n-1} T^k (I - TT'^*) T'^{*k}, \tag{4.2}$$

where $T' = T(T^*T)^{-1}$ denotes the Cauchy dual of T and $I - TT'^*$ is the orthogonal projection $P_{\ker T^*}$ onto $\ker T^*$. In order not to distract the reader from the main line of development, we have relegated the discussion of some of the difficulties arising in finding a multivariable counterpart of Shimorin's approach to the Appendix.

4.1. The joint kernel and a system of linear equations. In this section, we show that finding the joint kernel of $S^*_{\boldsymbol\lambda}$ is equivalent to solving a certain system of linear equations. This information is then used to derive the wandering subspace property for $S_{\boldsymbol\lambda}$.

We now introduce a framework suitable for decomposing the joint kernel of $S^*_{\boldsymbol\lambda}$ into smaller subspaces of $l^2(V)$. These are induced by a system of linear equations arising from the action of $S^*_{\boldsymbol\lambda}$.

For a set A, let $\mathscr{P}(A)$ denote the collection of all subsets of A. In case $A = \{1, \dots, d\}$, we sometimes write just $\mathscr{P}$ for $\mathscr{P}(\{1, \dots, d\})$.

Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \dots, \mathscr{T}_d$. Consider the set-valued function $\Phi : \mathscr{P} \to \mathscr{P}(V)$ given by $\Phi(F) = \Phi_F$, where

$$\Phi_F := \{v \in V : v_j \in V_j^\circ \text{ if } j \in F, \text{ and } v_j = \mathsf{root}_j \text{ if } j \notin F\}, \qquad F \in \mathscr{P}. \tag{4.3}$$

Note that $\Phi_F \cap \Phi_G = \emptyset$ if $F \neq G$. Further, if $v \in V$ then $v \in \Phi_F$ for

$$F = \big\{j \in \{1, \dots, d\} : v_j \neq \mathsf{root}_j\big\}.$$

This shows that

$$V = \bigsqcup_{F \in \mathscr{P}} \Phi_F. \tag{4.4}$$

Let $F := \{i_1, \dots, i_k\} \subseteq \{1, \dots, d\}$ be fixed. For $u \in \Phi_F$, define

$$\mathsf{sib}_F(u) := \mathsf{sib}_{i_1} \cdots \mathsf{sib}_{i_k}(u). \tag{4.5}$$

As a convention, we set $\mathsf{sib}_\emptyset(u) = \{u\}$ for all $u \in V$.

Define a relation $\sim$ on Φ_F by $u \sim v$ if $u \in \mathsf{sib}_F(v)$, and note that $\sim$ is an equivalence relation. Moreover, for any $u \in \Phi_F$, the equivalence class containing u is precisely $\mathsf{sib}_F(u)$. An application of the axiom of choice [61] allows us to form a set Ω_F (to be referred to as an *indexing set corresponding to* F) by picking up exactly one element from each equivalence class $\mathsf{sib}_F(u)$. Thus we have the disjoint union

$$\Phi_F = \bigsqcup_{u \in \Omega_F} \mathsf{sib}_F(u). \tag{4.6}$$

This combined with (4.4) yields

$$V = \bigsqcup_{F \in \mathscr{P}} \bigsqcup_{u \in \Omega_F} \mathsf{sib}_F(u).$$

As a consequence, we obtain the following decomposition of $l^2(V)$.

Proposition 4.1.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \dots, \mathscr{T}_d$. Then*

$$l^2(V) = \bigoplus_{F \in \mathscr{P}} \bigoplus_{u \in \Omega_F} l^2(\mathsf{sib}_F(u)),$$

where Ω_F is the indexing set corresponding to F and $\mathsf{sib}_F(u)$ is given by (4.5).

In order to describe the joint kernel E of $S^*_{\boldsymbol\lambda}$, one needs to understand the subspace $l^2(\mathsf{sib}_F(u))$. Before that, let us make some definitions.

Definition 4.1.2. For $F \in \mathscr{P}$ and $v = (v_1, \dots, v_d) \in V$, let v_F denote the d-tuple with jth coordinate given by

$$(v_F)_j := \begin{cases} v_j & \text{if } j \in F, \\ \mathsf{root}_j & \text{if } j \notin F. \end{cases}$$

Further, for fixed $1 \leq i \leq d$ such that $i \notin F$, and $u_i \in V_i$, we define $v_F|u_i$ to be $(w_1, \ldots, w_d)$, where

$$w_j = \begin{cases} u_i & \text{if } j = i, \\ (v_F)_j & \text{otherwise.} \end{cases}$$

REMARK 4.1.3. Note that v_F is obtained from v by replacing the jth coordinate by root_j whenever $j \notin F$. On the other hand, $v_F|u_i$ is obtained from v_F by replacing its ith coordinate by u_i.

For subsets F, G of $\{1, \ldots, d\}$ such that $G \subseteq F$, and $u \in \Phi_F$, we define

$$\mathsf{sib}_{F,G}(u) := \{v_G : v \in \mathsf{sib}_F(u)\}. \tag{4.7}$$

REMARK 4.1.4. Note that different vertices v in $\mathsf{sib}_F(u)$ may correspond to a single $v_G \in \mathsf{sib}_{F,G}(u)$.

LEMMA 4.1.5. *Let $F \in \mathscr{P}$ and let $i \in F$. For $u \in \Phi_F$ and $G := F \setminus \{i\}$, we have:*

(i) $\mathsf{sib}_F(u) = \bigsqcup_{v_G \in \mathsf{sib}_{F,G}(u)} \mathsf{sib}_i(v_G|u_i)$.
(ii) *For all $v_G \in \mathsf{sib}_{F,G}(u)$, $\mathrm{card}(\mathsf{sib}_i(v_G|u_i))$ is constant.*
(iii) $\mathrm{card}(\mathsf{sib}_{F,G}(u)) = \prod_{j \in F,\, j \neq i} \mathrm{card}(\mathsf{sib}_j(u))$.
(iv) $\mathrm{card}(\mathsf{sib}_F(u)) = \prod_{j \in F} \mathrm{card}(\mathsf{sib}_j(u))$.

Proof. Let $v_G, w_G \in \mathsf{sib}_{F,G}(u)$ be such that $v_G \neq w_G$. Then there exists $j \in G$ such that $v_j \neq w_j$. Suppose that $\eta \in \mathsf{sib}_i(v_G|u_i) \cap \mathsf{sib}_i(w_G|u_i)$. Then $v_j = \eta_j = w_j$, which is a contradiction. Hence $\mathsf{sib}_i(v_G|u_i) \cap \mathsf{sib}_i(w_G|u_i) = \emptyset$ if $v_G \neq w_G$. Next, observe that $\mathsf{sib}_i(v_G|u_i) \subseteq \mathsf{sib}_F(u)$ for all $v_G \in \mathsf{sib}_{F,G}(u)$. To see the other inclusion in (i), note that if $w \in \mathsf{sib}_F(u)$ then $w \in \mathsf{sib}_i(w_G|u_i)$. This completes the proof of (i). Next, (ii) follows from the fact that $\mathrm{card}(\mathsf{sib}_i(v_G|u_i)) = \mathrm{card}(\mathsf{sib}(u_i))$; (iii) follows from (4.7); and (iv) is immediate from (4.5). ∎

The following lemma describes the action of $S^*_{\boldsymbol\lambda}$ on $l^2(\mathsf{sib}_F(u))$.

LEMMA 4.1.6. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Let $F \in \mathscr{P}$, $i \in F$ and $G := F \setminus \{i\}$. If $u \in \Phi_F$, then for any $f \in l^2(\mathsf{sib}_F(u))$,*

$$S^*_i(f) = \sum_{v_G \in \mathsf{sib}_{F,G}(u)} \Big(\sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w)\lambda^{(i)}_w \Big) e_{\mathsf{par}_i(v_G|u_i)}. \tag{4.8}$$

Proof. Let $u \in \Phi_F$. By Lemma 4.1.5(i), we obtain the orthogonal decomposition

$$l^2(\mathsf{sib}_F(u)) = \bigoplus_{v_G \in \mathsf{sib}_{F,G}(u)} l^2(\mathsf{sib}_i(v_G|u_i)).$$

Let $f \in l^2(\mathsf{sib}_F(u))$. Then $f = \sum_{v_G \in \mathsf{sib}_{F,G}(u)} \sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w) e_w \in l^2(V)$. It follows that

$$\begin{aligned} S^*_i(f) &= \sum_{v_G \in \mathsf{sib}_{F,G}(u)} \sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w)\lambda^{(i)}_w e_{\mathsf{par}_i(w)} \\ &= \sum_{v_G \in \mathsf{sib}_{F,G}(u)} \Big(\sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w)\lambda^{(i)}_w \Big) e_{\mathsf{par}_i(v_G|u_i)}, \end{aligned}$$

where we have used the fact that $\mathsf{par}_i(\mathsf{sib}_i(v)) = \mathsf{par}_i(v)$ for any $v \in V$. ∎

Fix $i \in F$ and let $G := F \setminus \{i\}$. In view of (4.8), finding a solution of $S_i^*(f) = 0$, $f \in l^2(\mathsf{sib}_F(u))$ amounts to solving the following system of $N_{i,u,F}$ equations in $M_{i,u,F}$ unknowns:

$$\boxed{\sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w)\lambda_w^{(i)} = 0, \ v_G \in \mathsf{sib}_{F,G}(u) \text{ and } G = F \setminus \{i\},} \tag{4.9}$$

where, in view of Lemma 4.1.5, $N_{i,u,F}, M_{i,u,F} \in \mathbb{N} \cup \{\infty\}$ are given by

$$N_{i,u,F} = \mathrm{card}(\mathsf{sib}_{F,G}(u)) = \prod_{j \in F, j \neq i} \mathrm{card}(\mathsf{sib}_j(u)),$$

$$M_{i,u,F} = \mathrm{card}(\mathsf{sib}_i(v_G|u_i))N_{i,u,F} = \mathrm{card}(\mathsf{sib}_F(u)) = \prod_{j \in F} \mathrm{card}(\mathsf{sib}_j(u)).$$

Note that $M_{u,F} := M_{i,u,F}$ is independent of i. Further, by Lemma 4.1.5(i), the set of unknowns in (4.9) is equal to $\{f(w) : w \in \mathsf{sib}_F(u)\}$ for each $i \in F$. Thus varying i over F, we get the system of $N(u, F) := \sum_{i \in F} N_{i,u,F}$ equations in $M_{u,F}$ unknowns given by

$$\boxed{\sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w)\lambda_w^{(i)} = 0, \ i \in F, \ v_G \in \mathsf{sib}_{F,G}(u) \text{ and } G = F \setminus \{i\}.} \tag{4.10}$$

Let $\mathcal{L}_{u,F}$ denote the linear manifold of $l^2(\mathsf{sib}_F(u))$ given by

$$\mathcal{L}_{u,F} := \{f \in l^2(\mathsf{sib}_F(u)) : f \text{ is a solution of (4.10)}\}. \tag{4.11}$$

If $\mathscr{T}_1, \ldots, \mathscr{T}_d$ are locally finite then $\mathcal{L}_{u,F}$ is a subspace. In this case, by Proposition 4.1.1, the joint kernel E of $S_{\boldsymbol{\lambda}}^*$ is given by

$$E = [e_{\mathsf{root}}] \oplus \bigoplus_{\substack{F \in \mathscr{P} \\ F \neq \emptyset}} \bigoplus_{u \in \Omega_F} \mathcal{L}_{u,F}. \tag{4.12}$$

REMARK 4.1.7. Let us discuss the system (4.10) in following special cases:

1. In case $S_{\boldsymbol{\lambda}}$ is the classical multishift S_w, the system (4.10) has only the trivial solution, and hence $E = [e_0]$.
2. In case $d = 1$, $\mathscr{P} = \{\emptyset, \{1\}\}$, and hence (4.10) takes the form

   $$\sum_{w \in \mathsf{sib}(u)} f(w)\lambda_w = 0 \quad (u \in V^\circ).$$

   However, linear equations associated with vertices outside $\mathsf{Chi}(V_\prec)$ have trivial solutions, and hence

   $$E = [e_{\mathsf{root}}] \oplus \bigoplus_{u \in \mathsf{Chi}(V_\prec)} \mathcal{L}_{u,\{1\}}.$$

   This expression should be compared with (1.11).

To understand the above description of the joint kernel of $S_{\boldsymbol{\lambda}}^*$, we include a couple of instructive examples.

EXAMPLE 4.1.8. Let $\mathscr{T}$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1 = \mathscr{T}_{2,0}, \mathscr{T}_2 = \mathscr{T}_{1,0}$ as described in Example 2.1.6. Note that

$$\mathscr{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

It follows from (4.3) that

$$\Phi_{\emptyset} = \{(0,0)\}, \quad \Phi_{\{1\}} = \{(i,0) : i \geq 1\},$$
$$\Phi_{\{2\}} = \{(0,j) : j \geq 1\}, \quad \Phi_{\{1,2\}} = \{(i,j) : i,j \geq 1\}.$$

Let us now find $\mathsf{sib}_F(u)$ for $F \in \mathscr{P}$ and $u \in \Phi_F$. By convention, $\mathsf{sib}_{\emptyset}((0,0)) = \{(0,0)\}$. Note that

$$\mathsf{sib}_{\{1\}}((1,0)) = \{(1,0),(2,0)\} = \mathsf{sib}_{\{1\}}((2,0)), \quad \mathsf{sib}_{\{1\}}((i,0)) = \{(i,0)\} \quad (i \geq 3),$$
$$\mathsf{sib}_{\{2\}}((0,j)) = \{(0,j)\} \quad \text{for all } j \geq 1,$$
$$\mathsf{sib}_{\{1,2\}}((i,j)) = \begin{cases} \{(1,j),(2,j)\} & \text{if } i \in \{1,2\} \text{ and } j \geq 1, \\ \{(i,j)\} & \text{if } i \geq 3, j \geq 1. \end{cases}$$

One may form Ω_F by picking up one element from each equivalence class $\mathsf{sib}_F(u)$ as follows:

$$\Omega_{\emptyset} = \{(0,0)\}, \quad \Omega_{\{1\}} = \{(1,0)\} \cup \{(i,0) : i \geq 3\}, \quad \Omega_{\{2\}} = \{(0,j) : j \geq 1\},$$
$$\Omega_{\{1,2\}} = \{(1,j),(i,j) : i \geq 3,\ j \geq 1\}.$$

Let us calculate $\mathsf{sib}_{F,G}(u)$ for possible choices of F, G, and $u \in \Omega_F$. If $F = \{1\}$, then $G = \emptyset$. In this case,

$$\mathsf{sib}_{\{1\},\emptyset}(1,0) = \{(0,0)\} = \mathsf{sib}_{\{1\},\emptyset}(i,0) \quad (i \geq 3).$$

This together with (4.9) yields the following equations:

$$f(1,0)\lambda^{(1)}_{(1,0)} + f(2,0)\lambda^{(1)}_{(2,0)} = 0, \quad f(i,0)\lambda^{(1)}_{(i,0)} = 0 \quad (i \geq 3).$$

In case $F = \{2\}$, $G = \emptyset$ and $\mathsf{sib}_{\{2\},\emptyset}(0,j) = \{(0,0)\}$ for $j \geq 1$, and hence we obtain the equations

$$f(0,j)\lambda^{(2)}_{(0,j)} = 0 \quad (j \geq 1).$$

In case $F = \{1,2\}$, $G = \{1\}$ or $\{2\}$. Then for all $i \geq 3$ and $j \geq 1$,

$$\mathsf{sib}_{\{1,2\},\{2\}}(1,j) = \{(0,j)\}, \quad \mathsf{sib}_{\{1,2\},\{1\}}(1,j) = \{(1,0),(2,0)\},$$
$$\mathsf{sib}_{\{1,2\},\{2\}}(i,j) = \{(0,j)\}, \quad \mathsf{sib}_{\{1,2\},\{1\}}(i,j) = \{(i,0)\}.$$

This gives the following equations:

$$\left.\begin{aligned} f(1,j)\lambda^{(1)}_{(1,j)} + f(2,j)\lambda^{(1)}_{(2,j)} &= 0, \\ f(1,j)\lambda^{(2)}_{(1,j)} = 0, \quad f(2,j)\lambda^{(2)}_{(2,j)} &= 0, \\ f(i,j)\lambda^{(1)}_{(i,j)} = 0, \quad f(i,j)\lambda^{(2)}_{(i,j)} &= 0 \end{aligned}\right\} \tag{4.13}$$

for $i \geq 3$ and $j \geq 1$. Solving (4.13), we get $f(1,0) = \alpha\lambda^{(1)}_{(2,0)}$, $f(2,0) = -\alpha\lambda^{(1)}_{(1,0)}$ for $\alpha \in \mathbb{C}$, $f(i,j) = 0$ for all $i,j \geq 1$, $f(0,j) = 0$ for all $j \geq 1$ and $f(i,0) = 0$ for all $i \geq 3$. Thus,

$$E = [e_{\mathsf{root}}] \oplus [\lambda^{(1)}_{(2,0)}e_{(1,0)} - \lambda^{(1)}_{(1,0)}e_{(2,0)}].$$

The situation in the preceding example resembles the situation occurring in dimension $d = 1$ (cf. (1.11)). Below we present an example which gives an idea of the complications which can occur in dimension more than 1.

EXAMPLE 4.1.9. Consider the directed Cartesian product $\mathscr{T} = (V, \mathcal{E})$ of the directed tree $\mathscr{T}_{2,0}$ with itself (see Example 2.1.7). Note that

$$\mathscr{P} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

It follows that

$$\Phi_{\emptyset} = \{(0,0)\}, \quad \Phi_{\{1\}} = \{(i,0) : i \geq 1\},$$
$$\Phi_{\{2\}} = \{(0,i) : i \geq 1\}, \quad \Phi_{\{1,2\}} = \{(i,j) : i,j \geq 1\}.$$

Let us now find $\mathsf{sib}_F(u)$ for $F \in \mathscr{P}$ and $u \in \Phi_F$. By convention, $\mathsf{sib}_{\emptyset}((0,0)) = \{(0,0)\}$. Note that

$$\mathsf{sib}_{\{1\}}((1,0)) = \{(1,0),(2,0)\} = \mathsf{sib}_{\{1\}}((2,0)), \quad \mathsf{sib}_{\{1\}}((i,0)) = \{(i,0)\} \quad (i \geq 3).$$

Similarly,

$$\mathsf{sib}_{\{2\}}((0,1)) = \{(0,1),(0,2)\} = \mathsf{sib}_{\{2\}}((0,2)), \quad \mathsf{sib}_{\{2\}}((0,i)) = \{(0,i)\} \quad (i \geq 3).$$

Further,

$$\mathsf{sib}_{\{1,2\}}((i,j)) = \begin{cases} \{(1,1),(1,2),(2,1),(2,2)\} & \text{if } i,j \in \{1,2\}, \\ \{(1,j),(2,j)\} & \text{if } i \in \{1,2\} \text{ and } j \geq 3, \\ \{(i,1),(i,2)\} & \text{if } i \geq 3 \text{ and } j \in \{1,2\}, \\ \{(i,j)\} & \text{if } i,j \geq 3. \end{cases}$$

One may form Ω_F by picking up one element from each $\mathsf{sib}_F(u)$ as follows:

$$\Omega_{\emptyset} = \{(0,0)\}, \quad \Omega_{\{1\}} = \{(1,0)\} \cup \{(i,0) : i \geq 3\}, \quad \Omega_{\{2\}} = \{(0,1)\} \cup \{(0,j) : j \geq 3\},$$
$$\Omega_{\{1,2\}} = \{(1,1)\} \cup \{(i,1),(1,j),(i,j) : i,j \geq 3\}.$$

Let us calculate $\mathsf{sib}_{F,G}(u)$ for possible choices of F, G, and $u \in \Omega_F$. If $F = \{1\}$, then $G = \emptyset$. In this case,

$$\mathsf{sib}_{\{1\},\emptyset}(1,0) = \{(0,0)\}, \quad \mathsf{sib}_{\{1\},\emptyset}(i,0) = \{(0,0)\} \quad (i \geq 3).$$

This yields

$$\sum_{w \in \mathsf{sib}_1(v_{\emptyset}|1)} f(w)\lambda^{(1)}_w = 0, \quad v_{\emptyset} \in \mathsf{sib}_{\{1\},\emptyset}(1,0) = \{(0,0)\},$$
$$\sum_{w \in \mathsf{sib}_1(v_{\emptyset}|i)} f(w)\lambda^{(1)}_w = 0, \quad v_{\emptyset} \in \mathsf{sib}_{\{1\},\emptyset}(i,0) = \{(0,0)\} \ (i \geq 3),$$

which is the same as

$$f(1,0)\lambda^{(1)}_{(1,0)} + f(2,0)\lambda^{(1)}_{(2,0)} = 0, \quad f(i,0)\lambda^{(1)}_{(i,0)} = 0 \quad (i \geq 3).$$

Similarly, in case $F = \{2\}$, we obtain

$$f(0,1)\lambda^{(2)}_{(0,1)} + f(0,2)\lambda^{(2)}_{(0,2)} = 0, \quad f(0,j)\lambda^{(2)}_{(0,j)} = 0 \quad (j \geq 3).$$

If $F = \{1, 2\}$ then $G = \{1\}$ or $\{2\}$, and hence for $i, j \geq 3$,

$$\begin{aligned}
&\mathsf{sib}_{\{1,2\},\{2\}}(1,1) = \{(0,1),(0,2)\}, && \mathsf{sib}_{\{1,2\},\{1\}}(1,1) = \{(1,0),(2,0)\},\\
&\mathsf{sib}_{\{1,2\},\{2\}}(i,1) = \{(0,1),(0,2)\}, && \mathsf{sib}_{\{1,2\},\{1\}}(i,1) = \{(i,0)\},\\
&\mathsf{sib}_{\{1,2\},\{2\}}(1,j) = \{(0,j)\}, && \mathsf{sib}_{\{1,2\},\{1\}}(1,j) = \{(1,0),(2,0)\},\\
&\mathsf{sib}_{\{1,2\},\{2\}}(i,j) = \{(0,j)\}, && \mathsf{sib}_{\{1,2\},\{1\}}(i,j) = \{(i,0)\}.
\end{aligned}$$

Thus we obtain the following equations for $i, j \geq 3$:

$$\begin{aligned}
\sum_{w \in \mathsf{sib}_1(v_{\{2\}}|1)} f(w)\lambda_w^{(1)} &= 0, \quad v_{\{2\}} \in \{(0,1),(0,2)\},\\
\sum_{w \in \mathsf{sib}_2(v_{\{1\}}|1)} f(w)\lambda_w^{(2)} &= 0, \quad v_{\{1\}} \in \{(1,0),(2,0)\},\\
\sum_{w \in \mathsf{sib}_1(v_{\{2\}}|i)} f(w)\lambda_w^{(1)} &= 0, \quad v_{\{2\}} \in \{(0,1),(0,2)\},\\
\sum_{w \in \mathsf{sib}_2(v_{\{1\}}|1)} f(w)\lambda_w^{(2)} &= 0, \quad v_{\{1\}} \in \{(i,0)\},\\
\sum_{w \in \mathsf{sib}_1(v_{\{2\}}|1)} f(w)\lambda_w^{(1)} &= 0, \quad v_{\{2\}} \in \{(0,j)\},\\
\sum_{w \in \mathsf{sib}_2(v_{\{1\}}|j)} f(w)\lambda_w^{(2)} &= 0, \quad v_{\{1\}} \in \{(1,0),(2,0)\},\\
\sum_{w \in \mathsf{sib}_1(v_{\{2\}}|i)} f(w)\lambda_w^{(1)} &= 0, \quad v_{\{2\}} \in \{(0,j)\},\\
\sum_{w \in \mathsf{sib}_2(v_{\{1\}}|j)} f(w)\lambda_w^{(2)} &= 0, \quad v_{\{1\}} \in \{(i,0)\},
\end{aligned}$$

which is the same as

$$\begin{aligned}
f(1,1)\lambda_{(1,1)}^{(1)} + f(2,1)\lambda_{(2,1)}^{(1)} &= 0, \quad f(1,2)\lambda_{(1,2)}^{(1)} + f(2,2)\lambda_{(2,2)}^{(1)} = 0,\\
f(1,1)\lambda_{(1,1)}^{(2)} + f(1,2)\lambda_{(1,2)}^{(2)} &= 0, \quad f(2,1)\lambda_{(2,1)}^{(2)} + f(2,2)\lambda_{(2,2)}^{(2)} = 0,\\
f(i,1)\lambda_{(i,1)}^{(1)} &= 0, \quad f(i,2)\lambda_{(i,2)}^{(1)} = 0,\\
f(i,1)\lambda_{(i,1)}^{(2)} + f(i,2)\lambda_{(i,2)}^{(2)} &= 0,\\
f(1,j)\lambda_{(1,j)}^{(1)} + f(2,j)\lambda_{(2,j)}^{(1)} &= 0,\\
f(1,j)\lambda_{(1,j)}^{(2)} &= 0, \quad f(2,j)\lambda_{(2,j)}^{(2)} = 0,\\
f(i,j)\lambda_{(i,j)}^{(1)} &= 0,\\
f(i,j)\lambda_{(i,j)}^{(2)} &= 0.
\end{aligned}$$

Let $W := \{(i,j) \in V : i \geq 3 \text{ or } j \geq 3\} \cup \{(0,0)\}$. Then $f \in E \ominus [e_{\mathsf{root}}]$ if and only if $f \in l^2(V \setminus W)$ satisfies the following systems of equations:

$$\begin{aligned} L_{(1,0),\{1\}}[f(1,0), f(2,0)]^{\mathsf{T}} &= 0, \\ L_{(0,1),\{2\}}[f(0,1), f(0,2)]^{\mathsf{T}} &= 0, \\ L_{(1,1),\{1,2\}}[f(1,1), f(2,1), f(1,2), f(2,2)]^{\mathsf{T}} &= 0, \end{aligned}$$

where X^{T} denotes the transpose of a row vector X. Here $L_{(1,0),\{1\}} = [\lambda^{(1)}_{(1,0)}, \lambda^{(1)}_{(2,0)}]$, $L_{(0,1),\{2\}} = [\lambda^{(2)}_{(0,1)}, \lambda^{(2)}_{(0,2)}]$, and

$$L_{(1,1),\{1,2\}} = \begin{bmatrix} \lambda^{(1)}_{(1,1)} & \lambda^{(1)}_{(2,1)} & 0 & 0 \\ 0 & 0 & \lambda^{(1)}_{(1,2)} & \lambda^{(1)}_{(2,2)} \\ \lambda^{(2)}_{(1,1)} & 0 & \lambda^{(2)}_{(1,2)} & 0 \\ 0 & \lambda^{(2)}_{(2,1)} & 0 & \lambda^{(2)}_{(2,2)} \end{bmatrix}.$$

Note that the rank of $L_{(1,1),\{1,2\}}$ is at least 3. By Schur's formula [93, Theorem 1.1], the determinant of $L_{(1,1),\{1,2\}}$ is zero if and only if

$$\lambda^{(1)}_{(1,2)}\lambda^{(1)}_{(2,1)}\lambda^{(2)}_{(1,1)}\lambda^{(2)}_{(2,2)} = \lambda^{(1)}_{(1,1)}\lambda^{(1)}_{(2,2)}\lambda^{(2)}_{(1,2)}\lambda^{(2)}_{(2,1)}.$$

Thus any $f \in E$ takes the form

$$\begin{aligned} f &= f(0,0)e_{(0,0)} + \sum_{v \in V \setminus W} f(v)e_v \\ &= f(0,0)e_{(0,0)} + f(1,0)(e_{(1,0)} + a_1 e_{(2,0)}) + f(0,1)(e_{(0,1)} + a_2 e_{(0,2)}) + g_{(1,1)}, \end{aligned}$$

where $g_{(1,1)}$ is given by

$$g_{(1,1)} = \begin{cases} 0 & \text{if } \operatorname{rank} L_{(1,1),\{1,2\}} = 4, \\ f(1,1)(e_{(1,1)} + b_1 e_{(2,1)} + b_2 e_{(1,2)} + b_3 e_{(2,2)}) & \text{otherwise.} \end{cases}$$

Further, the scalars a_1, a_2, b_1, b_2, b_3 are given by

$$a_1 = -\frac{\lambda^{(1)}_{(1,0)}}{\lambda^{(1)}_{(2,0)}}, \quad a_2 = -\frac{\lambda^{(2)}_{(0,1)}}{\lambda^{(2)}_{(0,2)}},$$

$$b_1 = -\frac{\lambda^{(1)}_{(1,1)}}{\lambda^{(1)}_{(2,1)}}, \quad b_2 = -\frac{\lambda^{(2)}_{(1,1)}}{\lambda^{(2)}_{(1,2)}}, \quad b_3 = \frac{\lambda^{(2)}_{(2,1)}}{\lambda^{(2)}_{(2,2)}} \frac{\lambda^{(1)}_{(1,1)}}{\lambda^{(1)}_{(2,1)}}.$$

Thus the dimension of E is either 3 or 4.

As a key step in the proof of Theorem 4.0.1, we need to calculate the orthogonal complement of the subspace $\mathcal{L}_{u,F}$ of $l^2(\mathsf{sib}_F(u))$.

LEMMA 4.1.10. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Let $\mathcal{L}_{u,F}$ be as given in* (4.11)*. Then*

$$l^2(\mathsf{sib}_F(u)) \ominus \mathcal{L}_{u,F} = \bigvee \{S_i e_{v_G|\mathsf{par}(u_i)} : v_G \in \mathsf{sib}_{F,G}(u) \text{ with } G = F \setminus \{i\},\ i \in F\}.$$

Proof. For $v_G \in \mathsf{sib}_{F,G}(u)$, note that

$$S_i e_{v_G|\mathsf{par}(u_i)} = \sum_{w \in \mathsf{Chi}_i(v_G|\mathsf{par}(u_i))} \lambda^{(i)}_w e_w = \sum_{w \in \mathsf{sib}_i(v_G|u_i)} \lambda^{(i)}_w e_w.$$

Thus, for $f \in l^2(\mathsf{sib}_F(u))$, by Lemma 4.1.5(i),

$$\begin{aligned}\langle f, S_i e_{v_G|\mathsf{par}(u_i)}\rangle &= \Big\langle \sum_{\eta_G \in \mathsf{sib}_{F,G}(u)} \sum_{w \in \mathsf{sib}_i(\eta_G|u_i)} f(w) e_w, \sum_{w \in \mathsf{sib}_i(v_G|u_i)} \lambda_w^{(i)} e_w \Big\rangle \\ &= \sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w) \lambda_w^{(i)}.\end{aligned}$$

In particular, $f \in l^2(\mathsf{sib}_F(u))$ is orthogonal to $S_i e_{v_G|\mathsf{par}(u_i)}$ for every $v_G \in \mathsf{sib}_{F,G}(u)$ and $i \in F$ if and only if f satisfies the system (4.10). The latter holds if and only if $f \in \mathcal{L}_{u,F}$. This yields the desired formula. ∎

We are now ready to complete the derivation of the wandering subspace property for $S_{\boldsymbol\lambda}$.

Proof of Theorem 4.0.1. Let E denote the joint kernel of $S^*_{\boldsymbol\lambda}$. Since $e_{\mathsf{root}} \in E$, it is enough to see that for every nonempty $F \subseteq \{1, \ldots, d\}$,

$$l^2(V_F) \subseteq [E]_{S_{\boldsymbol\lambda}} = \bigvee_{\alpha \in \mathbb{N}^d} \{S^\alpha_{\boldsymbol\lambda} f : f \in E\},$$

where $V_F = \bigsqcup_{G \in \mathscr{P}(F)} \Phi_G$. Fix a nonempty subset F of $\{1, \ldots, d\}$. For $l = 0, \ldots, \mathrm{card}(F)$, set

$$\mathscr{F}_l := \{e_v \in l^2(V) : v \in \Phi_G \text{ with } G \in \mathscr{P}(F) \text{ and } \mathrm{card}(G) \le l\}.$$

Since $\bigvee \mathscr{F}_{\mathrm{card}(F)} = l^2(V_F)$, it suffices to check that

$$\mathscr{F}_{l-1} \subseteq [E]_{S_{\boldsymbol\lambda}} \ \Rightarrow\ \mathscr{F}_l \subseteq [E]_{S_{\boldsymbol\lambda}},\ l = 1, \ldots, \mathrm{card}(F).$$

To this end, fix $1 \le l \le \mathrm{card}(F)$, and assume that $\mathscr{F}_{l-1} \subseteq [E]_{S_{\boldsymbol\lambda}}$. Let $G \in \mathscr{P}(F)$ with $\mathrm{card}(G) = l$. In particular,

$$e_{v_{G\setminus\{i\}}} \in [E]_{S_{\boldsymbol\lambda}} \quad \text{for every } i \in G \text{ and } v \in \Phi_G. \tag{4.14}$$

We must check that $e_v \in [E]_{S_{\boldsymbol\lambda}}$ for every $v \in \Phi_G$.

We prove the following statement by induction on $k \in \mathbb{N}$:

$$\text{For every } i \in G \text{ and } v \in \Phi_G,\ e_{v_{G\setminus\{i\}}|w_i} \in [E]_{S_{\boldsymbol\lambda}} \text{ for all } w_i \in \mathsf{Chi}^{\langle k\rangle}(\mathsf{root}_i). \tag{4.15}$$

In view of (4.14), this holds trivially for $k = 0$ since $v_{G\setminus\{i\}}|\mathsf{root}_i = v_{G\setminus\{i\}}$. Let us assume the inductive statement for an integer $k \ge 0$ and let $w_i \in \mathsf{Chi}^{\langle k+1\rangle}(\mathsf{root}_i)$. By induction hypothesis, $e_{v_{G\setminus\{i\}}|\mathsf{par}(w_i)} \in [E]_{S_{\boldsymbol\lambda}}$ for every $i \in G$ and $v \in \Phi_G$. It follows from Lemma 4.1.10 with $u := v_{G\setminus\{i\}}|w_i$ that

$$l^2(\mathsf{sib}_G(u)) \ominus \mathcal{L}_{u,G} = \bigvee \{S_i e_{v_{G\setminus\{i\}}|\mathsf{par}(w_i)} : v_{G\setminus\{i\}} \in \mathsf{sib}_{G,G\setminus\{i\}}(u),\ i \in G\} \subseteq [E]_{S_{\boldsymbol\lambda}}.$$

But we already know that $\mathcal{L}_{u,G} \subseteq E$, and hence

$$e_{v_{G\setminus\{i\}}|w_i} \in l^2(\mathsf{sib}_G(u)) = \mathcal{L}_{u,G} \oplus (\mathcal{L}_{u,G})^\perp \subseteq [E]_{S_{\boldsymbol\lambda}}.$$

This completes the verification of (4.15).

To complete the proof, let $v \in \Phi_G$. Thus $v = (v_1, \ldots, v_d)$ with $v_j \in V_j^\circ$ for $j \in G$ and $v_j = \mathsf{root}_j$ for $j \notin G$. Since $v_i \in \mathsf{Chi}^{\langle \mathsf{d}_{v_i}\rangle}(\mathsf{root}_i)$, we obtain $e_v = e_{v_{G\setminus\{i\}}|v_i} \in [E]_{S_{\boldsymbol\lambda}}$. ∎

In the remaining part of this section, we discuss some immediate consequences of Theorem 4.0.1.

Let $T = (T_1, \ldots, T_d)$ be a commuting d-tuple on a Hilbert space $\mathcal{H}$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be *cyclic* for T if

$$\mathcal{H} = \bigvee\{T^\alpha h : h \in \mathcal{M},\, \alpha \in \mathbb{N}^d\}.$$

We say that T is *finitely multicyclic* if there exists a finite-dimensional cyclic subspace for T.

Corollary 4.1.11. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr{T}$. If $\mathscr{T}$ has finite joint branching index, then $S_{\boldsymbol\lambda}$ is finitely multicyclic with the cyclic subspace being the joint kernel E of $S^*_{\boldsymbol\lambda}$. In this case, $\dim\ker(S^*_\lambda - \omega) \le \dim E$ for every $\omega \in \mathbb{C}^d$.*

Proof. Assume that $\mathscr{T}$ has finite joint branching index. The first part follows from Theorem 4.0.1 and Corollary 3.1.14. The idea of the proof of the second part seems to be known (see, for instance, [65] for the case $d = 1$). Let $\omega \in \mathbb{C}^d$. By Theorem 4.0.1,

$$\bigvee_{\alpha\in\mathbb{N}^d} S^\alpha_{\boldsymbol\lambda}(E) = l^2(V).$$

Since $S^\alpha_{\boldsymbol\lambda}$ is a finite linear combination of terms of the form $(S_{\boldsymbol\lambda} - \omega)^\beta$ for $\beta \in \mathbb{N}^d$, we must have

$$\bigvee_{\alpha\in\mathbb{N}^d} (S_{\boldsymbol\lambda} - \omega)^\alpha(E) = l^2(V).$$

If P_ω denotes the orthogonal projection of $l^2(V)$ onto $\ker(S^*_{\boldsymbol\lambda} - \omega)$, then $P_\omega(S_j - \overline{\omega}_j) = 0$ for any $j = 1, \ldots, d$. It follows that

$$\ker(S^*_{\boldsymbol\lambda} - \omega) = P_\omega l^2(V) = P_\omega \bigvee_{\alpha\in\mathbb{N}^d} (S_{\boldsymbol\lambda} - \overline{\omega})^\alpha(E) = P_\omega E + P_\omega\Big(\bigvee_{\substack{\alpha\in\mathbb{N}^d\\ \alpha\neq 0}} (S_{\boldsymbol\lambda} - \overline{\omega})^\alpha(E)\Big)$$

$$= P_\omega E + \bigvee_{j=1}^{d} \bigvee_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_j\neq 0}} P_\omega(S_j - \overline{\omega}_j)^{\alpha_j}(S_{\boldsymbol\lambda} - \overline{\omega})^{\alpha-\alpha_j\epsilon_j}(E) = P_\omega E,$$

since $P_\omega(S_j - \overline{\omega}_j)^k = 0$ for $k \neq 0$ and $j = 1, \ldots, d$. Hence the dimension of $\ker(S^*_{\boldsymbol\lambda} - \omega)$ is at most $\dim E$. ∎

Recall that $l^2_{\mathcal{M}}(\mathbb{N}^d)$ is defined as the Hilbert space of square-summable multisequences $\{h_\alpha\}_{\alpha\in\mathbb{N}^d}$ in $\mathcal{M}$, where $\mathcal{M}$ is a nonzero complex Hilbert space. If $\{W^{(j)}_\alpha\}_{\alpha\in\mathbb{N}^d} \subseteq B(\mathcal{M})$ for $j = 1, \ldots, d$, then the linear operator W_j in $l^2_{\mathcal{M}}(\mathbb{N}^d)$ is defined by $W_j(h_\alpha)_{\alpha\in\mathbb{N}^d} = (k_\alpha)_{\alpha\in\mathbb{N}^d}$ for $(h_\alpha)_{\alpha\in\mathbb{N}^d} \in \mathcal{D}$, where

$$k_\alpha = \begin{cases} W^{(j)}_{\alpha-\epsilon_j} h_{\alpha-\epsilon_j} & \text{if } \alpha_j \ge 1,\\ 0 & \text{if } \alpha_j = 0 \end{cases}$$

and $\mathcal{D} := \{(h_\alpha)_{\alpha\in\mathbb{N}^d} \in l^2_{\mathcal{M}}(\mathbb{N}^d) : (k_\alpha)_{\alpha\in\mathbb{N}^d} \in l^2_{\mathcal{M}}(\mathbb{N}^d)\}$. If we use the convention that $W^{(j)}_\alpha = 0 = h_\alpha$ whenever $\alpha_j < 0$, then the definition of W_j can be rewritten as $W_j(h_\alpha)_{\alpha\in\mathbb{N}^d} = (W^{(j)}_{\alpha-\epsilon_j} h_{\alpha-\epsilon_j})_{\alpha\in\mathbb{N}^d}$. We refer to the d-tuple $W = (W_1, \ldots, W_d)$ as an *operator valued multishift with operator weights* $\{W^{(j)}_\alpha : \alpha \in \mathbb{N}^d,\, j = 1, \ldots, d\}$ (for one-variable counterpart of operator valued multishift, the reader is referred to [78]). Note that

W is unitarily equivalent to the classical multishift S_w in case $\mathcal{M}$ is the one-dimensional complex Hilbert space.

COROLLARY 4.1.12. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda}$ be a toral left invertible multishift on $\mathscr{T}$ and let E denote the joint kernel of $S^*_{\boldsymbol\lambda}$. If the multisequence $\{S^\alpha_{\boldsymbol\lambda} E\}_{\alpha\in\mathbb{N}^d}$ of linear manifolds of $l^2(V)$ is mutually orthogonal, then $S_{\boldsymbol\lambda}$ is unitarily equivalent to a commuting operator valued multishift W on $l^2_E(\mathbb{N}^d)$ with invertible weights.*

Proof. The proof relies on the technique employed in [7, Theorem 3.8]. Assume that the multisequence $\{S^\alpha_{\boldsymbol\lambda} E\}_{\alpha\in\mathbb{N}^d}$ is mutually orthogonal. Since the operator $S_{\boldsymbol\lambda}$ is toral left invertible, $S^\alpha_{\boldsymbol\lambda}$ is left invertible for every $\alpha \in \mathbb{N}^d$. Hence $\mathcal{M}_\alpha := S^\alpha_{\boldsymbol\lambda}(E)$ ($\alpha \in \mathbb{N}^d$) is a subspace of $l^2(V)$ and $\dim \mathcal{M}_\alpha = \dim E$ for every $\alpha \in \mathbb{N}^d$. For $\alpha \in \mathbb{N}^d$, let $U_\alpha : \mathcal{M}_\alpha \to E$ be any isometric isomorphism. By Theorem 4.0.1, $l^2(V) = \bigoplus_{\alpha\in\mathbb{N}^d} S^\alpha_{\boldsymbol\lambda}(E) = \bigoplus_{\alpha\in\mathbb{N}^d} \mathcal{M}_\alpha$. We can now define the isometric isomorphism $U : l^2(V) \to l^2_E(\mathbb{N}^d)$ by

$$U(\oplus_{\alpha\in\mathbb{N}^d} h_\alpha) := (U_\alpha h_\alpha)_{\alpha\in\mathbb{N}^d}, \quad \oplus_{\alpha\in\mathbb{N}^d} h_\alpha \in l^2(V).$$

Consider the operator valued multishift W on $l^2_{\mathcal{M}_0}(\mathbb{N}^d)$ with weights

$$\{W^{(j)}_\alpha := U_{\alpha+\epsilon_j} S_j|_{\mathcal{M}_\alpha} U^{-1}_\alpha\}_{\alpha\in\mathbb{N}^d} \subseteq B(E) \quad \text{for } j = 1, \ldots, d.$$

Then for $\oplus_{\alpha\in\mathbb{N}^d} h_\alpha \in l^2(V)$ with at most finitely many nonzero terms h_α,

$$\begin{aligned} US_j(\oplus_{\alpha\in\mathbb{N}^d} h_\alpha) &= U(\oplus_{\alpha\in\mathbb{N}^d} S_j|_{\mathcal{M}_\alpha} h_\alpha) = (U_\alpha S_j|_{\mathcal{M}_{\alpha-\epsilon_j}} h_{\alpha-\epsilon_j})_{\alpha\in\mathbb{N}^d} \\ &= (W^{(j)}_{\alpha-\epsilon_j} U_{\alpha-\epsilon_j} h_{\alpha-\epsilon_j})_{\alpha\in\mathbb{N}^d} = W_j U(\oplus_{\alpha\in\mathbb{N}^d} h_\alpha). \end{aligned}$$

This shows that US_jU^* agrees with W_j on a dense linear manifold of $l^2_E(\mathbb{N}^d)$, and hence W_j must be a bounded linear operator on $l^2_E(\mathbb{N}^d)$. Since $S_{\boldsymbol\lambda}$ is commuting, so is W. Finally, since $S_{\boldsymbol\lambda}$ is toral left invertible, each $W^{(j)}_\alpha$ is invertible in $B(E)$. ∎

REMARK 4.1.13. The converse of the above result is trivially true. We note further that in case E is finite-dimensional, the conclusion of the corollary holds without the assumption of toral left invertibility of $S_{\boldsymbol\lambda}$. A slight modification of the argument above together with the injectivity of $S_1, \ldots, S_d$, as ensured by Corollary 3.3.4, yields the desired conclusion.

4.2. Multishifts admitting Shimorin's analytic model. In this section, we discuss a subclass of multishifts $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ satisfying a kernel condition (cf. [32, (5.3)]). In particular, we obtain an analytic model for this class and discuss its applications to multivariable spectral theory. It turns out that Shimorin's analytic model [89] does not naturally extend to all toral left invertible multishifts. Indeed, the toral left invertible multishifts admitting Shimorin's model must belong to the aforementioned class (see Remark 4.2.5). Before we introduce this class, we need a lemma pertaining to toral Cauchy dual of multishifts.

LEMMA 4.2.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a toral left invertible multishift on $\mathscr{T}$. Then the toral Cauchy dual $S^{\mathfrak{t}}_{\boldsymbol\lambda} = (S^{\mathfrak{t}}_1, \ldots, S^{\mathfrak{t}}_d)$ of $S_{\boldsymbol\lambda}$ is given by*

$$S^{\mathfrak{t}}_j e_v = \frac{1}{\|S_j e_v\|^2} \sum_{w\in\mathsf{Chi}_j(v)} \lambda^{(j)}_w e_w \quad \textit{for all } v \in V \textit{ and } j = 1, \ldots, d.$$

In particular, $S^{\mathfrak{t}}_{\boldsymbol\lambda}$ *is a multishift with weights*

$$\{\lambda^{(j)}_w/\|S_j e_v\|^2 : w \in \mathsf{Chi}_j(v),\ v \in V,\ j = 1, \ldots, d\}.$$

Proof. This follows from the definition of $S^{\mathfrak{t}}_j$ (see (1.2)). ■

DEFINITION 4.2.2. Let $T = (T_1, \ldots, T_d)$ be a toral left invertible d-tuple on $\mathcal{H}$ and let E denote the joint kernel of T^*. Let $T^{\mathfrak{t}}$ be the toral Cauchy dual of T. We say that T *satisfies kernel condition* ($\mathfrak{K}$) if

$$E \subseteq \ker T_j^* {T^{\mathfrak{t}}}^{\alpha}_{[j]} \quad \text{for all } j = 1, \ldots, d \text{ and all } \alpha \in \mathbb{N}^d, \tag{$\mathfrak{K}$}$$

where, for $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$,

$${T^{\mathfrak{t}}}^{\alpha}_{[j]} = \begin{cases} I & \text{if } d = 1, \\ \prod_{i \neq j} T_i^{\mathfrak{t}\alpha_i} & \text{if } d \geq 2. \end{cases}$$

REMARK 4.2.3. Note that the kernel condition ($\mathfrak{K}$) is satisfied in any one of the following cases:

(i) $d = 1$.
(ii) T is doubly commuting.
(iii) T is a commuting operator valued multishift W.

In dimension 2, $S_{\boldsymbol\lambda}$ satisfies the kernel condition ($\mathfrak{K}$) if and only if

$$E \subseteq (\ker S_1^* {S_2^{\mathfrak{t}}}^{\alpha_2}) \cap (\ker S_2^* {S_1^{\mathfrak{t}}}^{\alpha_1}) \quad \text{for all } (\alpha_1, \alpha_2) \in \mathbb{N}^2. \tag{4.16}$$

In general, $S_{\boldsymbol\lambda}$ does not satisfy ($\mathfrak{K}$). Indeed, a rather tedious calculation shows that for $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{2,0} \times \mathscr{T}_{2,0}$, as discussed in Example 2.1.7, $f := \lambda^{(1)}_{(2,0)} e_{(1,0)} - \lambda^{(1)}_{(1,0)} e_{(2,0)} \in E$ does not belong to $\ker S_1^* S_2^{\mathfrak{t}}$ for a suitable choice of weights $\boldsymbol\lambda$.

The following provides a multivariable counterpart of [35, Theorem 2.2].

THEOREM 4.2.4. *Let* $\mathscr{T} = (V, \mathcal{E})$ *be the directed Cartesian product of locally finite, rooted directed trees* $\mathscr{T}_1, \ldots, \mathscr{T}_d$ *and let* $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ *be a toral left invertible multishift on* $\mathscr{T}$. *Let E be the joint kernel of* $S^*_{\boldsymbol\lambda}$. *Assume that the toral Cauchy dual* $S^{\mathfrak{t}}_{\boldsymbol\lambda} = (S^{\mathfrak{t}}_1, \ldots, S^{\mathfrak{t}}_d)$ *of* $S_{\boldsymbol\lambda}$ *is commuting and let*

$$r := (r(S^{\mathfrak{t}}_1)^{-1}, \ldots, r(S^{\mathfrak{t}}_d)^{-1}),$$

where $r(T)$ *denotes the spectral radius of a bounded linear operator* T. *If* $S_{\boldsymbol\lambda}$ *satisfies the kernel condition* ($\mathfrak{K}$), *then there exist a reproducing kernel Hilbert space* $\mathscr{H}$ *of E-valued holomorphic functions defined on the polydisc* $\mathbb{D}^d_r$ *and a unitary* $U : l^2(V) \to \mathscr{H}$ *such that* $US_j = \mathscr{M}_{z_j} U$ *for* $j = 1, \ldots, d$. *If, in addition,* $\mathscr{T}$ *has finite joint branching index* $k_{\mathscr{T}}$, *then the reproducing kernel* κ *of* $\mathscr{H}$ *is given by*

$$\kappa_{\mathscr{H}}(z, w) = \sum_{\substack{\alpha, \beta \in \mathbb{N}^d \\ |\alpha_j - \beta_j| \leq k_{\mathscr{T}_j} \\ j = 1, \ldots, d}} P_E {S^{\mathfrak{t}}_{\boldsymbol\lambda}}^{*\alpha} {S^{\mathfrak{t}}_{\boldsymbol\lambda}}^{\beta}|_E z^\alpha \overline{w}^\beta \quad (z, w \in \mathbb{D}^d_r). \tag{4.17}$$

Proof. The proof relies on Shimorin's technique as presented in [89] and the wandering subspace property of $S_{\boldsymbol\lambda}$ as obtained in Theorem 4.0.1. Assume that $S_{\boldsymbol\lambda}$ satisfies the kernel condition $(\mathfrak{K})$. For $f \in l^2(V)$, define

$$U_f(z) := \sum_{\alpha\in\mathbb{N}^d} (P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} f) z^\alpha, \quad z \in \mathbb{C}^d.$$

Then the power series U_f converges absolutely on the polydisc $\mathbb{D}_r^d$ of polyradius r. Let $\mathscr{H}$ denote the complex vector space of E-valued holomorphic functions of the form U_f. Thus $U : l^2(V) \to \mathscr{H}$ defines a map from $l^2(V)$ onto $\mathscr{H}$ given by $U(f) = U_f$. Now we show that U is injective.

To this end, let $U_f = 0$ for some $f \in l^2(V)$. Then $\sum_{\alpha\in\mathbb{N}^d}(P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} f)z^\alpha = 0$, which implies that $P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} f = 0$ for all $\alpha \in \mathbb{N}^d$. Note that $S_{\boldsymbol\lambda}^{\mathfrak{t}}$ is also a multishift and the joint kernel of $S_{\boldsymbol\lambda}^{\mathfrak{t}*}$ is equal to E (since $\ker S_j^{\mathfrak{t}*} = \ker S_j^*$). Hence by Theorem 4.0.1, we get $\bigvee_{\alpha\in\mathbb{N}^d} S_{\boldsymbol\lambda}^{\mathfrak{t}^\alpha}(E) = l^2(V)$. By taking orthogonal complement on both sides, we get $\bigcap_{\alpha\in\mathbb{N}^d}(S_{\boldsymbol\lambda}^{\mathfrak{t}^\alpha}(E))^\perp = \{0\}$. It is easy to see that $(S_{\boldsymbol\lambda}^{\mathfrak{t}^\alpha}(E))^\perp = \ker P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}}$ for any $\alpha \in \mathbb{N}^d$. Hence $\bigcap_{\alpha\in\mathbb{N}^d} \ker P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} = \{0\}$. Since $P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} f = 0$ for all $\alpha \in \mathbb{N}^d$, we must have $f \in \bigcap_{\alpha\in\mathbb{N}^d} \ker P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}}$. This shows that $f = 0$, and hence U is injective.

We now define the inner product on $\mathscr{H}$ as $\langle U_f, U_g\rangle = \langle f, g\rangle_{l^2(V)}$ for all $f, g \in l^2(V)$. Since $\|U_f - U_g\| = \|f - g\|_{l^2(V)}$ for all $f, g \in l^2(V)$, by a standard Cauchy-sequence argument, $\mathscr{H}$ is complete. Thus it becomes a Hilbert space and U a unitary. Also, for $f \in l^2(V)$,

$$\begin{aligned}
(US_jf)(z) &= \sum_{\alpha\in\mathbb{N}^d} (P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} S_j f) z^\alpha \\
&= \sum_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_j=0}} (P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} S_j f) z^\alpha + \sum_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_j\geq 1}} (P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} S_j f) z^\alpha \\
&= \sum_{\substack{\alpha\in\mathbb{N}^d\\ \alpha_j=0}} (P_E S_{\boldsymbol\lambda[j]}^{\mathfrak{t}^{*\alpha}} S_j f) z^\alpha + \sum_{\alpha\in\mathbb{N}^d} (P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha+\epsilon_j}} S_j f) z^{\alpha+\epsilon_j} \\
&= \sum_{\alpha\in\mathbb{N}^d} (P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha+\epsilon_j}} S_j f) z^{\alpha+\epsilon_j}, \qquad (4.18)
\end{aligned}$$

where we have used the kernel condition $(\mathfrak{K})$ to get the last equality. Since the toral Cauchy dual $S_{\boldsymbol\lambda}^{\mathfrak{t}}$ is commuting and $S_j^{\mathfrak{t}*} S_j = I$, the sum on the right hand side of (4.18) is equal to $z_j \sum_{\alpha\in\mathbb{N}^d}(P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} f)z^\alpha = z_j U_f(z)$. Thus we get $US_j = \mathscr{M}_{z_j} U$.

We skip the verification of

$$\kappa_{\mathscr{H}}(z, w) = \sum_{\alpha,\beta\in\mathbb{N}^d} P_E S_{\boldsymbol\lambda}^{\mathfrak{t}^{*\alpha}} S_{\boldsymbol\lambda}^{\mathfrak{t}^\beta}|_E z^\alpha \overline{w}^\beta = P_E \prod_{i=1}^d (I - z_i S_i^{\mathfrak{t}^*})^{-1} \prod_{j=1}^d (I - \bar{w}_j S_j^{\mathfrak{t}})^{-1}|_E$$

for $z, w \in \mathbb{D}_r^d$, since it is along the lines of [89, Proposition 2.13]. It is now easy to see that

$$\langle U_f, \kappa_{\mathscr{H}}(\cdot, w)g\rangle_{\mathscr{H}} = \langle U_f(w), g\rangle \quad (f, g \in E,\ w \in \mathbb{D}_r^d).$$

Thus $\mathscr{H}$ is a reproducing kernel Hilbert space with kernel κ.

Assume further that $\mathscr{T}$ has finite joint branching index $k_{\mathscr{T}}$. To check that κ has the form given in (4.17), let $\alpha, \beta \in \mathbb{N}^d$ be such that $|\alpha_j - \beta_j| > k_{\mathscr{T}_j}$ for some $j =$

$1,\ldots,d$. In view of Proposition 3.1.13, it suffices to check that $P_E S_{\boldsymbol\lambda}^{\mathsf{t}^{*\alpha}} S_{\boldsymbol\lambda}^{\mathsf{t}^{\beta}} e_v = 0$ for all $v \in F_1 \times \cdots \times F_d$, where $F_j := \mathsf{Chi}(V_{\prec}^{(j)}) \cup \{\mathsf{root}_j\}$ $(j = 1,\ldots,d)$. To see this, let $v \in F_1 \times \cdots \times F_d$. Since the depth d_v of v equals $(\mathsf{d}_{v_1},\ldots,\mathsf{d}_{v_d})$ with d_{v_j} being the depth of v_j in $\mathscr{T}_j$, we obtain $0 \le \mathsf{d}_{v_j} \le k_{\mathscr{T}_j}$ for every $j = 1,\ldots,d$. An application of Proposition 3.1.7(vi) shows that

$$S_{\boldsymbol\lambda}^{\mathsf{t}^{*\alpha}} S_{\boldsymbol\lambda}^{\mathsf{t}^{\beta}} e_v = \sum_{u \in \mathsf{par}^{\langle\langle\alpha\rangle\rangle}(\mathsf{Chi}^{\langle\langle\beta\rangle\rangle}(v))} \gamma_u e_u$$

for some $\gamma_u \in \mathbb{C}$. It follows that $\mathsf{d}_u = \mathsf{d}_v + \beta - \alpha$ for $u \in \mathsf{par}^{\langle\langle\alpha\rangle\rangle}(\mathsf{Chi}^{\langle\langle\beta\rangle\rangle}(v))$. Note that $\beta_j - \alpha_j = \mathsf{d}_{u_j} - \mathsf{d}_{v_j} \ge \mathsf{d}_{u_j} - k_{\mathscr{T}_j}$, and hence

$$\beta_j - \alpha_j + |\beta_j - \alpha_j| > \beta_j - \alpha_j + k_{\mathscr{T}_j} \ge \mathsf{d}_{u_j} \ge 0.$$

This shows that $\beta_j - \alpha_j > 0$, and consequently

$$\mathsf{d}_{u_j} = \mathsf{d}_{v_j} + \beta_j - \alpha_j = \mathsf{d}_{v_j} + |\alpha_j - \beta_j| > k_{\mathscr{T}_j}.$$

Thus $u \notin F_1 \times \cdots \times F_d$, and hence by Proposition 3.1.13, $P_E S_{\boldsymbol\lambda}^{\mathsf{t}^{*\alpha}} S_{\boldsymbol\lambda}^{\mathsf{t}^{\beta}} e_v = 0$. ■

REMARK 4.2.5. The kernel condition $(\mathfrak{K})$ is used only in obtaining the intertwining relation $US_j = \mathscr{M}_{z_j} U$. Conversely, if one assumes that relation then the calculations in (4.18) show that the kernel condition $(\mathfrak{K})$ is also necessary.

It would be of interest to know the maximum value of r for which $\kappa_{\mathscr{H}}(z,w)$ converges on $\mathbb{D}_r^d \times \mathbb{D}_r^d$. Unfortunately, we do not know this even in the one-dimensional case (see [35, (2.3)]). Before we proceed to the next result, it is convenient to introduce some terminology.

Let $S_{\boldsymbol\lambda}$ be a toral left invertible multishift on $\mathscr{T}$. We refer to the pair $(\mathscr{M}_z, \kappa_{\mathscr{H}})$ as *Shimorin's analytic model.*

If one relaxes the toral left invertibility of $S_{\boldsymbol\lambda}$ then it may happen that the interior of the Taylor spectrum of $S_{\boldsymbol\lambda}$ is empty [51, Example 2]. This is not possible otherwise.

COROLLARY 4.2.6. *If $S_{\boldsymbol\lambda}$ admits Shimorin's analytic model, then the polydisc $\mathbb{D}_r^d$ is contained in the point spectrum of $S_{\boldsymbol\lambda}^*$, where $r := (r(S_1^{\mathsf{t}})^{-1},\ldots,r(S_d^{\mathsf{t}})^{-1})$.*

REMARK 4.2.7. Since $\mathrm{cl}(\sigma_p(S_{\boldsymbol\lambda}^*)) \subseteq \sigma(S_{\boldsymbol\lambda}^*) = \sigma(S_{\boldsymbol\lambda})$, we must have

$$(r(S_1^{\mathsf{t}})^{-2} + \cdots + r(S_d^{\mathsf{t}})^{-2})^{1/2} \le r(S_{\boldsymbol\lambda}),$$

where $r(S_{\boldsymbol\lambda})$ is the spectral radius of $S_{\boldsymbol\lambda}$.

Let us analyze Theorem 4.2.4 in case $S_{\boldsymbol\lambda}$ is a doubly commuting toral isometry.

COROLLARY 4.2.8. *Let $\mathscr{T} = (V,\mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1,\ldots,\mathscr{T}_d$, let $S_{\boldsymbol\lambda}$ be a toral isometry multishift on $\mathscr{T}$ and let E denote the joint kernel of $S_{\boldsymbol\lambda}^*$. Then $S_{\boldsymbol\lambda}$ is doubly commuting if and only if $S_{\boldsymbol\lambda}$ is unitarily equivalent to the multiplication d-tuple $\mathscr{M}_z$ on the E-valued Hardy space of the unit polydisc $\mathbb{D}^d$. In particular,*

$$\kappa_{\mathscr{H}}(z,w) = \prod_{j=1}^{d} \frac{I_E}{1 - z_j \overline{w}_j} \qquad (z, w \in \mathbb{D}^d),$$

where I_E denotes the identity operator.

Proof. Assume that $S_{\boldsymbol\lambda}$ is doubly commuting. Since $S_{\boldsymbol\lambda}$ is a toral isometry, $S^{\mathfrak t}_{\boldsymbol\lambda} = S_{\boldsymbol\lambda}$. Thus

$$P_E S^{\mathfrak{t}^{*\alpha}}_{\boldsymbol\lambda} S^{\mathfrak{t}^{\beta}}_{\boldsymbol\lambda}|_E = \delta_{\alpha\beta} I_E \quad (\alpha, \beta \in \mathbb{N}^d),$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta. It follows from (4.17) that κ has the desired form. The conclusion can now be drawn from Theorem 4.2.4 and the fact that the reproducing kernel uniquely determines the reproducing kernel Hilbert space [13]. We leave the converse to the interested reader. ■

Now we discuss a large class of multishifts $S_{\boldsymbol\lambda}$ (not covered by Remark 4.2.3) which always satisfy the kernel condition ($\mathfrak{K}$).

Corollary 4.2.9. *Let $\mathscr{T}_1 = (V_1, \mathcal{E}_1)$ be a locally finite, rooted directed tree and let $\mathscr{T}_2 = \mathscr{T}_{1,0}$ be the rooted directed trees as described in Example* 2.1.5*. Consider the directed Cartesian product $\mathscr{T}$ of $\mathscr{T}_1$ and $\mathscr{T}_2$. Let $S_{\boldsymbol\lambda} = (S_1, S_2)$ be a toral left invertible multishift on $\mathscr{T}$ and let E be the joint kernel of $S^*_{\boldsymbol\lambda}$. Assume that the toral Cauchy dual $S^{\mathfrak t}_{\boldsymbol\lambda} = (S^{\mathfrak t}_1, S^{\mathfrak t}_2)$ of $S_{\boldsymbol\lambda}$ is commuting and let $r := (r(S^{\mathfrak t}_1)^{-1}, r(S^{\mathfrak t}_2)^{-1})$, where $r(T)$ denotes the spectral radius of a bounded linear operator T. Then $S_{\boldsymbol\lambda}$ admits Shimorin's analytic model $(\mathscr{M}_z, \kappa_{\mathscr{H}})$. If, in addition, $\mathscr{T}_1$ has finite branching index $k_{\mathscr{T}_1}$, then the reproducing kernel κ of $\mathscr{H}$ is given by*

$$\begin{aligned}\kappa_{\mathscr{H}}(z, w) = &\sum_{\alpha \in \mathbb{N}^2} P_E S^{\mathfrak{t}^{*\alpha}}_{\boldsymbol\lambda} S^{\mathfrak{t}^{\alpha}}_{\boldsymbol\lambda}|_E \ z^\alpha \overline{w}^\alpha \\ &+ \sum_{\substack{\alpha,\beta \in \mathbb{N}^2 \\ 0<|\alpha_1-\beta_1| \leq k_{\mathscr{T}_1} \\ \alpha_2 = \beta_2}} P_E S^{\mathfrak{t}^{*\alpha}}_{\boldsymbol\lambda} S^{\mathfrak{t}^{\beta}}_{\boldsymbol\lambda}|_E \ z^\alpha \overline{w}^\beta \quad (z, w \in \mathbb{D}^2_r).\end{aligned}$$

Proof. We first compute the joint kernel E of $S^*_{\boldsymbol\lambda}$. The argument is similar to that of Example 4.1.8. Note that $\mathscr{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Also,

$$\begin{aligned}\Phi_\emptyset &= \{(\mathsf{root}_1, 0)\}, \quad \Phi_{\{1\}} = \{(v, 0) : v \in V^\circ_1\}, \\ \Phi_{\{2\}} &= \{(\mathsf{root}_1, j) : j \geq 1\}, \quad \Phi_{\{1,2\}} = \{(v, j) : v \in V^\circ_1,\ j \geq 1\}.\end{aligned}$$

Note further that

$$\begin{aligned}\mathsf{sib}_\emptyset((\mathsf{root}_1, 0)) &= \{(\mathsf{root}_1, 0)\}, \\ \mathsf{sib}_{\{1\}}((v, 0)) &= \{(w, 0) : w \in \mathsf{sib}(v)\} \ (v \in V^\circ_1), \\ \mathsf{sib}_{\{2\}}((\mathsf{root}_1, j)) &= \{(\mathsf{root}_1, j)\} \quad \text{for all } j \geq 1, \\ \mathsf{sib}_{\{1,2\}}((v, j)) &= \{(w, j) : w \in \mathsf{sib}(v)\} \quad (v \in V^\circ_1,\ j \geq 1).\end{aligned}$$

Let us form Ω_F by picking up one element from each of the equivalence classes $\mathsf{sib}_F(u)$ for every $F \in \mathscr{P}$. We next calculate $\mathsf{sib}_{F,G}(u)$ for possible choices of F, G, and $u \in \Omega_F$. If $F = \{1\}$, then $G = \emptyset$. In this case,

$$\mathsf{sib}_{\{1\},\emptyset}(v, 0) = \{(\mathsf{root}_1, 0)\} \quad (v \in V^\circ_1).$$

This together with (4.9) yields the following equations:

$$\sum_{w\in \mathsf{Chi}(v)} f(w,0)\lambda^{(1)}_{(w,0)} = 0 \quad (v \in V^{(1)}_{\prec}),$$

$$f(w,0)\lambda^{(1)}_{(w,0)} = 0 \quad (w \in \mathsf{Chi}(V_1 \setminus V^{(1)}_{\prec})).$$

In case $F = \{2\}$, we have $G = \emptyset$ and $\mathsf{sib}_{\{2\},\emptyset}(\mathsf{root}_1, j) = \{(\mathsf{root}_1, 0)\}$ for $j \geq 1$, and hence

$$f(\mathsf{root}_1, j)\lambda^{(2)}_{(\mathsf{root}_1,j)} = 0 \quad (j \geq 1).$$

In case $F = \{1, 2\}$, we get $G = \{1\}$ or $\{2\}$. Then for all $w \in V_1^\circ$ and $j \geq 1$,

$$\mathsf{sib}_{\{1,2\},\{2\}}(w,j) = \{(\mathsf{root}_1, j)\}, \quad \mathsf{sib}_{\{1,2\},\{1\}}(w,j) = \{(u,0) : u \in \mathsf{sib}(w)\}.$$

This gives the following equations for $j \geq 1$:

$$\sum_{u\in \mathsf{Chi}(w)} f(u,j)\lambda^{(1)}_{(u,j)} = 0 (w \in V^{(1)}_{\prec}), \quad f(w,j)\lambda^{(1)}_{(w,j)} = 0 \quad (w \in \mathsf{Chi}(V_1 \setminus V^{(1)}_{\prec})),$$

$$f(w,j)\lambda^{(2)}_{(w,j)} = 0 \quad (w \in \mathsf{Chi}(V^{(1)}_{\prec})),$$

$$f(w,j)\lambda^{(1)}_{(w,j)} = 0, \quad f(w,j)\lambda^{(2)}_{(w,j)} = 0 \quad (w \in \mathsf{Chi}(V_1 \setminus V^{(1)}_{\prec})).$$

Solving all the above equations, we get

$$E = [e_{(\mathsf{root}_1,0)}] \bigoplus_{w\in V^{(1)}_{\prec}} (l^2(\mathsf{Chi}(w) \times \{0\}) \ominus [\Gamma^{(1)}_{(w,0)}]),$$

where $\Gamma^{(1)}_{(w,0)} : \mathsf{Chi}_1((w,0)) \to \mathbb{C}$ defined as $\Gamma^{(1)}_{(w,0)}(u,0) = \lambda^{(1)}_{(u,0)}$.

We next check that $S_{\boldsymbol\lambda}$ satisfies the kernel condition $(\mathfrak{K})$. Since $E \subseteq \ker S_2^* S_1^{\mathsf{t}^{\alpha_1}}$ for all $\alpha_1 \in \mathbb{N}$, in view of (4.16), it suffices to check that

$$E \subseteq \ker S_1^* S_2^{\mathsf{t}^{\alpha_2}} \qquad \text{for all } \alpha_2 \in \mathbb{N}.$$

Clearly, $e_{(\mathsf{root}_1,0)} \in \ker S_1^* S_2^{\mathsf{t}^{\alpha_2}}$. For $w \in V^{(1)}_{\prec}$, let

$$f = \sum_{u\in \mathsf{Chi}(w)} f((u,0))e_{(u,0)} \in l^2(\mathsf{Chi}(w) \times \{0\})$$

be such that

$$\sum_{u\in \mathsf{Chi}(w)} f((u,0))\lambda^{(1)}_{(u,0)} = 0 \quad (w \in V^{(1)}_{\prec}). \tag{4.19}$$

Note that

$$S_1^* S_2^{\mathsf{t}^{\alpha_2}} f = S_1^* \sum_{u\in \mathsf{Chi}(w)} \frac{f((u,0))}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} e_{(u,\alpha_2)} = \bigg(\sum_{u\in \mathsf{Chi}(w)} \frac{f((u,0))}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} \lambda^{(1)}_{(u,\alpha_2)}\bigg) e_{(w,\alpha_2)}.$$

Thus $S_1^* S_2^{\mathsf{t}^{\alpha_2}} f = 0$ if and only if

$$\sum_{u\in \mathsf{Chi}(w)} \frac{f((u,0))}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} \lambda^{(1)}_{(u,\alpha_2)} = 0.$$

Note that any general solution of (4.19) is of the form

$$f = \sum_{\substack{u \in \mathsf{Chi}(w) \\ u \neq v}} f((u,0))\Big(e_{(u,0)} - e_{(v,0)} \frac{\lambda^{(1)}_{(u,0)}}{\lambda^{(1)}_{(v,0)}}\Big)$$

for some fixed $v \in \mathsf{Chi}(w)$. It follows that

$$\sum_{u \in \mathsf{Chi}(w)} \frac{f((u,0))}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} \lambda^{(1)}_{(u,\alpha_2)} = \sum_{\substack{u \in \mathsf{Chi}(w) \\ u \neq v}} f((u,0))\bigg(\frac{\lambda^{(1)}_{(u,\alpha_2)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} - \frac{\lambda^{(1)}_{(u,0)}}{\lambda^{(1)}_{(v,0)}} \frac{\lambda^{(1)}_{(v,\alpha_2)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(v,k)}}\bigg),$$

and hence it suffices to see that for every $u \in \mathsf{Chi}(w) \setminus \{v\}$,

$$\frac{\lambda^{(1)}_{(u,\alpha_2)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}} - \frac{\lambda^{(1)}_{(u,0)}}{\lambda^{(1)}_{(v,0)}} \frac{\lambda^{(1)}_{(v,\alpha_2)}}{\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(v,k)}} = 0.$$

However, by repeated applications of (3.2), we obtain

$$\lambda^{(1)}_{(u,\alpha_2)}\Big(\lambda^{(1)}_{(v,0)} \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(v,k)}\Big) = \lambda^{(1)}_{(u,\alpha_2)}\Big(\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(w,k)} \lambda^{(1)}_{(v,\alpha_2)}\Big),$$
$$\lambda^{(1)}_{(v,\alpha_2)}\Big(\lambda^{(1)}_{(u,0)} \prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(u,k)}\Big) = \lambda^{(1)}_{(v,\alpha_2)}\Big(\prod_{k=1}^{\alpha_2} \lambda^{(2)}_{(w,k)} \lambda^{(1)}_{(u,\alpha_2)}\Big),$$

which shows that $S_{\boldsymbol\lambda}$ satisfies the kernel condition ($\mathfrak{K}$). The desired conclusion now follows from Theorem 4.2.4 once we observe that $k_{\mathscr{T}_2} = 0$. ∎

REMARK 4.2.10. We discuss two special cases of the preceding corollary.

(i) In case $\mathscr{T}_1$ is $\mathscr{T}_{1,0}$, $S_{\boldsymbol\lambda}$ is nothing but the classical multishift and the associated kernel $\kappa_{\mathscr{H}}(z,w)$ is diagonal.
(ii) In case $\mathscr{T}_1$ is $\mathscr{T}_{2,0}$, we have $k_{\mathscr{T}_1} = 1$, and hence the kernel $\kappa_{\mathscr{H}}(z,w)$ is given by

$$\kappa_{\mathscr{H}}(z,w) = \sum_{\alpha \in \mathbb{N}^2} P_E S^{\mathfrak{t}^{*\alpha}}_{\boldsymbol\lambda} S^{\mathfrak{t}^{\alpha}}_{\boldsymbol\lambda}|_E \, z^\alpha \overline{w}^\alpha + \sum_{\alpha \in \mathbb{N}^2} P_E S^{\mathfrak{t}^{*\alpha}}_{\boldsymbol\lambda} S^{\mathfrak{t}^{\alpha+\epsilon_1}}_{\boldsymbol\lambda}|_E \, z^\alpha \overline{w}^{\alpha+\epsilon_1}$$
$$+ \sum_{\alpha \in \mathbb{N}^2} P_E S^{\mathfrak{t}^{*\alpha+\epsilon_1}}_{\boldsymbol\lambda} S^{\mathfrak{t}^{\alpha}}_{\boldsymbol\lambda}|_E \, z^{\alpha+\epsilon_1} \overline{w}^\alpha \quad (z, w \in \mathbb{D}^2_r).$$

We conclude this chapter with one application to the Cowen–Douglas theory.

Let Ω be an open connected subset of $\mathbb{C}^d$. For a positive integer n, let $B_n(\Omega)$ denote the set of all commuting d-tuples T on $\mathcal{H}$ satisfying the following conditions:

(1) For every point $\omega = (\omega_1, \ldots, \omega_d) \in \Omega$, we have:
   (a) the map $D_{T-\omega}(x) = ((T_j - \omega_j)x)$ from $\mathcal{H}$ into $\mathcal{H}^{\oplus d}$ has closed range.
   (b) $\dim \ker(T - \omega) = n$.
(2) The subspace $\bigvee_{\omega \in \Omega} \ker(T - \omega)$ of $\mathcal{H}$ equals $\mathcal{H}$.

We will call the set $B_n(\Omega)$ the *Cowen–Douglas class of degree n with respect to Ω* (refer to [41], [49] for the basic theory of the Cowen–Douglas class in one and several variables).

Corollary 4.2.11. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ with finite joint branching index. If $S_{\boldsymbol\lambda}$ is a toral left invertible, Fredholm multishift which satisfies the kernel condition ($\mathfrak{K}$), then $S^*_{\boldsymbol\lambda}$ belongs to the Cowen–Douglas class $B_{\dim E}(\mathbb{D}^d_r)$ for some $r > 0$, where E denotes the joint kernel of $S^*_{\boldsymbol\lambda}$.*

Proof. Suppose that $S_{\boldsymbol\lambda}$ is a toral left invertible multishift satisfying ($\mathfrak{K}$). By Corollary 3.1.16, $\dim E$ is finite. Also, by Theorem 4.0.1, the toral Cauchy dual $S^{\mathfrak{t}}_{\boldsymbol\lambda}$ has the wandering subspace property. It may now be concluded from [32, Theorem 5.4] that for any $s \leq r$,

$$l^2(V) = \bigvee_{\omega \in \mathbb{D}^d_s} \ker(S^*_{\boldsymbol\lambda} - \omega) \quad \text{and} \quad \dim \ker(S^*_{\boldsymbol\lambda} - \omega) \geq \dim E.$$

However, by Corollary 4.1.11, we get $\dim \ker(S^*_{\boldsymbol\lambda} - \omega) = \dim E$. Thus it only remains to check (1)(a) of the definition of the Cowen–Douglas class. However, by our assumption, 0 lies in the open set $\mathbb{C}^d \setminus \sigma_e(S_{\boldsymbol\lambda})$. Thus for some $0 < t_j < r_j$ $(j = 1, \ldots, d)$, $\mathbb{D}^d_t \subseteq \mathbb{C}^d \setminus \sigma_e(S_{\boldsymbol\lambda})$, and hence (1)(a) follows for every $w \in \mathbb{D}^d_t$. ∎

5. Special classes of multishifts

In this chapter, we discuss two classes of so-called balanced multishifts, namely torally balanced multishifts and spherically balanced multishifts (cf. [33], [77]). These substantially generalize the classes of toral and spherical isometries ([19], [52], [55], [3]). In particular, we introduce tree analogs of the classical multishifts $S_{\mathbf{w},a}$ as discussed in Example 1.2.1. We show that these multishifts are unitarily equivalent to multiplication tuples acting on reproducing kernel Hilbert spaces of vector valued holomorphic functions defined on the unit ball. We also provide a compact formula for the associated reproducing kernels involving finitely many hypergeometric functions. We further investigate some known classes of multishifts which include mainly the well-studied class of joint subnormal tuples ([50] [17], [19], [90], [39], [20], [55], [9], [60]), and the comparatively less understood class of joint hyponormal tuples ([18], [47], [48], [45], [46]). In particular, we emphasize characterizations of these classes within the class of spherically balanced multishifts.

5.1. Torally balanced multishifts. Before we introduce the class of torally balanced multishifts, let us understand the somewhat related class of multishifts with commuting toral Cauchy dual. The latter class admits a polar decomposition in the following sense.

Proposition 5.1.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a toral left invertible multishift on $\mathscr{T}$ with toral Cauchy dual $S^{\mathfrak{t}}_{\boldsymbol\lambda}$. Then $S^{\mathfrak{t}}_{\boldsymbol\lambda}$ is commuting if and only if there exist a toral isometry multishift $U_{\boldsymbol\theta} = (U_1, \ldots, U_d)$ on $\mathscr{T}$ and a commuting d-tuple $D = (D_1, \ldots, D_d)$ of diagonal, positive, invertible bounded linear operators on $l^2(V)$ such that*

$$S_j = U_j D_j, \quad j = 1, \ldots, d. \tag{5.1}$$

Further, this decomposition is unique.

Proof. Let us first see the uniqueness of the above decomposition. Indeed, if (5.1) holds then for $j = 1, \ldots, d$,

$$S_j^* S_j = D_j U_j^* U_j D_j = D_j^2,$$

and hence D_j must be the positive square root of $S_j^* S_j$. Also, since D_j is invertible, $U_j = S_j D_j^{-1}$ for $j = 1, \ldots, d$.

By Proposition 3.1.7(i) and Lemma 4.2.1, $S^{\mathfrak{t}}_{\boldsymbol\lambda}$ is commuting if and only if

$$\|S_j e_{\mathsf{par}_j(v)}\| \, \|S_i e_{\mathsf{par}_i \mathsf{par}_j(v)}\| = \|S_i e_{\mathsf{par}_i(v)}\| \, \|S_j e_{\mathsf{par}_j \mathsf{par}_i(v)}\| \tag{5.2}$$

for all $v \in V$ and $i, j = 1, \ldots, d$. Consider now the multishift $U_{\boldsymbol\theta} = (U_1, \ldots, U_d)$ with weights given by

$$\theta_w^{(j)} := \frac{\lambda_w^{(j)}}{\|S_j e_v\|} \quad \text{for } w \in \mathsf{Chi}_j(v),\ v \in V \text{ and } j = 1, \ldots, d. \tag{5.3}$$

Since $S_iS_j = S_jS_i$ $(i, j = 1, \ldots, d)$, by an application of Proposition 3.1.7(i), $S_{\boldsymbol\lambda}^{\mathfrak t}$ is commuting if and only if $U_iU_j = U_jU_i$ $(i, j = 1, \ldots, d)$.

To see the sufficiency part, assume that $S_{\boldsymbol\lambda}^{\mathfrak t}$ is commuting. Thus $U_{\boldsymbol\theta}$ is commuting. Also, since $U_j^*U_j = I$ for $j = 1, \ldots, d$, $U_{\boldsymbol\theta}$ is a toral isometry. Thus $S_{\boldsymbol\lambda}$ admits the decomposition (5.1), where D_j $(j = 1, \ldots, d)$ is given by

$$D_j e_v := \|S_j e_v\| e_v, \qquad v \in V.$$

Further, by Proposition 3.1.7(vii), $D_1, \ldots, D_d$ are diagonal, positive, invertible bounded linear operators.

Finally, since $S_{\boldsymbol\lambda}^{\mathfrak t}$ is commuting if and only if $U_{\boldsymbol\theta}$ is commuting, the necessity follows from the uniqueness of (5.1). ∎

For the sake of convenience, we refer to $U_{\boldsymbol\theta}$ and $D = (D_1, \ldots, D_d)$ as the *toral isometry part* and the *diagonal part* of the multishift $S_{\boldsymbol\lambda}$ respectively.

REMARK 5.1.2. Let $S_{\mathbf w}$ be a toral left invertible classical multishift. Then the operator tuple $S_{\mathbf w}^{\mathfrak t}$ toral Cauchy dual to $S_{\mathbf w}$ is given by

$$S_j^{\mathfrak t} e_\alpha = \frac{1}{w_\alpha^{(j)}} e_{\alpha+\epsilon_j} \quad (1 \le j \le d).$$

Note that $S_{\mathbf w}^{\mathfrak t}$ is also a commuting d-variable weighted shift with weight multisequence

$$\{1/w_\alpha^{(j)} : 1 \le j \le d,\ \alpha \in \mathbb N^d\}.$$

Thus the toral Cauchy dual of a classical multishift $S_{\mathbf w}$ always commutes. Indeed, (5.2) is equivalent to the commutativity of $S_{\mathbf w}$ in this case. Moreover, the toral isometry part can be identified with the multiplication tuple $\mathscr M_z$ of the Hardy space of the unit polydisc (commonly known as the *Cauchy d-shift*). Indeed, $\theta_v^{(j)} = 1$ for all $v \in V^\circ$ and $j = 1, \ldots, d$.

Since the toral isometry part and the diagonal part in the above decomposition need not commute, prima facie the relation between the moments $\{\|S_{\boldsymbol\lambda}^\alpha e_v\|^2\}_{\alpha\in\mathbb N^d}$ of $S_{\boldsymbol\lambda}$ and that of the toral isometry part $U_{\boldsymbol\theta}$ is not visible. Nevertheless, there is a subclass of multishifts for which we get a nice formula for $\{\|S_{\boldsymbol\lambda}^\alpha e_v\|^2\}_{\alpha\in\mathbb N^d}$ (see (5.9) below).

DEFINITION 5.1.3. Let $\mathscr T = (V, \mathcal E)$ be the directed Cartesian product of rooted directed trees $\mathscr T_1, \ldots, \mathscr T_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr T$. For $j = 1, \ldots, d$, define $\mathfrak C_j : V \to (0, \infty)$ by

$$\mathfrak C_j(v) := \|S_j e_v\| \quad (v \in V,\ i, j = 1, \ldots, d).$$

We say that $S_{\boldsymbol\lambda}$ is *torally balanced* if for each $j = 1, \ldots, d$, $\mathfrak C_j$ is constant on every generation $\mathcal G_t$, $t \in \mathbb N$. We denote the constant value of $\mathfrak C_j(v)$ by $c_{|\mathsf d_v|}^{(j)}$, where $\mathsf d_v$ is the depth of v in $\mathscr T$ (see Definition 2.1.11).

REMARK 5.1.4. Note that $S_{\boldsymbol\lambda}$ is a toral isometry if and only if

$$\sum_{w\in\mathsf{Chi}_j(v)} (\lambda_w^{(j)})^2 = 1 \quad \text{for all } v \in V \text{ and } j = 1, \ldots, d.$$

Clearly, any toral isometry multishift is torally balanced with $\mathfrak{C}_j$ being the constant function with value 1 for every $j = 1, \ldots, d$.

The following proposition leads to an interesting family of torally balanced multishifts. In particular, any directed Cartesian product of locally finite rooted directed trees supports a toral isometry (cf. [67, Proposition 8.1.3]).

Proposition 5.1.5. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. For $i = 1, \ldots, d$, let $c = \{c(t, i)\}_{t \in \mathbb{N}}$ be a bounded sequence of positive real numbers such that*

$$c(t,i)c(t-1,j) = c(t,j)c(t-1,i) \quad \text{for all integers } t \geq 1 \text{ and } i,j = 1, \ldots, d. \tag{5.4}$$

Consider the multishift $S_{\boldsymbol{\lambda}_c} = (S_1, \ldots, S_d)$ with weights

$$\lambda_w^{(j)} = \sqrt{\frac{c(|\mathsf{d}_v|, j)}{\mathrm{card}(\mathsf{Chi}_j(v))}} \quad \text{for } w \in \mathsf{Chi}_j(v),\ v \in V \text{ and } j = 1, \ldots, d.$$

Then $S_{\boldsymbol{\lambda}_c}$ defines a torally balanced multishift. In case $c(t,j) = 1$ for all $j = 1, \ldots, d$ and $t \in \mathbb{N}$, $S_{\boldsymbol{\lambda}_c}$ is a toral isometry.

Proof. Since $\{c(t,j)\}_{t \in \mathbb{N}}$ is a bounded sequence, by Lemma 3.1.5(i), S_j defines a bounded linear operator on $l^2(V)$ for every $j = 1, \ldots, d$. Let $w \in V$ and $i, j = 1, \ldots, d$. By Proposition 2.1.21, for every $v \in \mathsf{Chi}_i(\mathsf{Chi}_j(w))$, we get

$$\mathrm{card}(\mathsf{sib}_i(v))\,\mathrm{card}(\mathsf{sib}_j(\mathsf{par}_i(v))) = \mathrm{card}(\mathsf{sib}_j(v))\,\mathrm{card}(\mathsf{sib}_i(\mathsf{par}_j(v))). \tag{5.5}$$

We now check the commutativity of $S_{\boldsymbol{\lambda}_c}$. Note that for $w \in \mathsf{Chi}_j(v)$, $\lambda_w^{(j)}$ can be rewritten as $\lambda_w^{(j)} = \sqrt{c(|\mathsf{d}_w| - 1, j)/\mathrm{card}(\mathsf{sib}_j(w))}$. Now for $u \in \mathsf{Chi}_i\mathsf{Chi}_j(v)$, we have

$$\begin{aligned}
\lambda_u^{(i)}\lambda_{\mathsf{par}_i(u)}^{(j)} &= \sqrt{\frac{c(|\mathsf{d}_u| - 1, i)}{\mathrm{card}(\mathsf{sib}_i(u))}}\sqrt{\frac{c(|\mathsf{d}_{\mathsf{par}_i(u)}| - 1, j)}{\mathrm{card}(\mathsf{sib}_j(\mathsf{par}_i(u)))}} \\
&= \sqrt{\frac{c(|\mathsf{d}_u| - 1, i)}{\mathrm{card}(\mathsf{sib}_i(u))}}\sqrt{\frac{c(|\mathsf{d}_u| - 2, j)}{\mathrm{card}(\mathsf{sib}_j(\mathsf{par}_i(u)))}}, \\
\lambda_u^{(j)}\lambda_{\mathsf{par}_j(u)}^{(i)} &= \sqrt{\frac{c(|\mathsf{d}_u| - 1, j)}{\mathrm{card}(\mathsf{sib}_j(u))}}\sqrt{\frac{c(|\mathsf{d}_{\mathsf{par}_j(u)}| - 1, i)}{\mathrm{card}(\mathsf{sib}_i(\mathsf{par}_j(u)))}} \\
&= \sqrt{\frac{c(|\mathsf{d}_u| - 1, j)}{\mathrm{card}(\mathsf{sib}_j(u))}}\sqrt{\frac{c(|\mathsf{d}_u| - 2, i)}{\mathrm{card}(\mathsf{sib}_i(\mathsf{par}_j(u)))}}.
\end{aligned}$$

This together with Proposition 3.1.7(i), (5.4) and (5.5) shows that $S_{\boldsymbol{\lambda}_c}$ is commuting. Further,

$$\|S_j e_v\|^2 = \sum_{w \in \mathsf{Chi}_j(v)} (\lambda_w^{(j)})^2 = \sum_{w \in \mathsf{Chi}_j(v)} \frac{c(|\mathsf{d}_v|, j)}{\mathrm{card}(\mathsf{Chi}_j(v))} = c(|\mathsf{d}_v|, j),$$

which is a function of $|\mathsf{d}_v|$. This shows that $S_{\boldsymbol{\lambda}_c}$ is torally balanced. The last assertion is immediate from the above equality. ∎

Remark 5.1.6. Assume that $S_{\boldsymbol\lambda_c}$ is toral left invertible. Then $S^{\mathfrak t}_{\boldsymbol\lambda_c}$ is commuting. Indeed, the weights of $S^{\mathfrak t}_{\boldsymbol\lambda_c}$ are

$$\Big\{\frac{1}{\sqrt{c(|\mathsf{d}_v|, j)}}\,\frac{1}{\sqrt{\mathrm{card}(\mathsf{Chi}_j(v))}} : w \in \mathsf{Chi}_j(v),\ v \in V \text{ and } j = 1, \ldots, d\Big\}.$$

By arguing as above, it can be seen that $S^{\mathfrak t}_{\boldsymbol\lambda_c}$ is commuting.

Example 5.1.7. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. For positive $a, b \in \mathbb{R}$, let

$$c_{a,b}(t, j) = \frac{t + b}{t + a} \qquad (t \in \mathbb{N},\ j = 1, \ldots, d).$$

Then $S_{\boldsymbol\lambda_c}$ is a torally balanced multishift on $\mathscr{T}$. In case $\mathscr{T}$ is the d-fold directed Cartesian product of $\mathscr{T}_{1,0}$ with itself, the choice $a = 1 = b$ yields the Cauchy d-shift. In case $b = 1$, we denote $c_{a,b}$ simply by c_a.

In dimension $d = 1$, the multishifts $S_{\boldsymbol\lambda_c}$ with $b = 1$ can be realized as multiplication operators on reproducing kernel Hilbert spaces. Indeed, these shifts can be looked upon as tree counterpart of Agler-type shifts [2]. This is made precise in the following proposition.

Proposition 5.1.8. *Let $\mathscr{T} = (V, \mathcal{E})$ be a locally finite rooted directed tree of finite branching index. For a positive integer a, let $S_{\lambda_{c_a}}$ denote the weighted shift on $\mathscr{T}$ with weights given by*

$$\lambda_u = \sqrt{\frac{\mathsf{d}_v + 1}{\mathsf{d}_v + a}}\,\frac{1}{\sqrt{\mathrm{card}(\mathsf{Chi}(v))}} \qquad \text{for } u \in \mathsf{Chi}(v),\ v \in V,$$

where d_v denotes the depth of v in $\mathscr{T}$. Then $S_{\lambda_{c_a}}$ is unitarily equivalent to the multiplication operator $\mathscr{M}_{z,a}$ on a reproducing kernel Hilbert space $\mathscr{H}_a$ of E-valued holomorphic functions on unit disc $\mathbb{D}$, where $E := \ker S^*_{\lambda_{c_a}}$ (see (1.11)). Moreover, the reproducing kernel $\kappa_{\mathscr{H}_a}$ associated with $\mathscr{H}_a$ is given by

$$\begin{aligned}\kappa_{\mathscr{H}_a}(z, w) = {}& \sum_{n=0}^{\infty} \binom{n + a - 1}{n} z^n \overline{w}^n P_{[e_{\mathsf{root}}]} \\ &+ \sum_{v \in V_{\prec}} \sum_{n=0}^{\infty} \frac{(\mathsf{d}_v + n + a)!(\mathsf{d}_v + 1)!}{(\mathsf{d}_v + a)!(\mathsf{d}_v + n + 1)!} z^n \overline{w}^n P_{l^2(\mathsf{Chi}(v)) \ominus [\Gamma_v]} \qquad (z, w \in \mathbb{D}),\end{aligned}$$

where $\Gamma_v : \mathsf{Chi}(v) \to \mathbb{C}$ is given by $\Gamma_v = \sum_{u \in \mathsf{Chi}(v)} \lambda_u e_u$.

Proof. Note that

$$\inf_{v \in V} \sum_{u \in \mathsf{Chi}(v)} \lambda_u^2 = \inf_{v \in V} \frac{\mathsf{d}_v + 1}{\mathsf{d}_v + a} = \frac{1}{a}.$$

It now follows from Proposition 3.1.7(vii) that $S_{\lambda_{c_a}}$ is left invertible. Hence the first part follows from Remark 4.2.3 and Theorem 4.2.4. To see the remaining part, we need the following identity:

$$\sum_{u \in \mathsf{Chi}^{\langle k \rangle}(v)} \prod_{l=0}^{k-1} \frac{1}{s_{l,u}} = 1 \qquad \text{for } v \in V \text{ and } k \geq 1, \tag{5.6}$$

where $s_{i,v} := \mathrm{card}(\mathsf{sib}(\mathsf{par}^{\langle i\rangle}(v)))$ for a nonnegative integer i and $v \in V$. We prove this by induction on $k \geq 1$. For $k = 1$, (5.6) follows from the fact that $\mathrm{card}(\mathsf{Chi}(v)) = \mathrm{card}(\mathsf{sib}(u))$, where $u \in \mathsf{Chi}(v)$. Suppose that (5.6) holds for some $k \geq 1$. Then

$$\sum_{u\in\mathsf{Chi}^{\langle k+1\rangle}(v)} \prod_{l=0}^{k} \frac{1}{s_{l,u}} = \sum_{\eta\in\mathsf{Chi}^{\langle k\rangle}(v)} \sum_{u\in\mathsf{Chi}(\eta)} \prod_{l=0}^{k} \frac{1}{s_{l,u}} = \sum_{\eta\in\mathsf{Chi}^{\langle k\rangle}(v)} \sum_{u\in\mathsf{Chi}(\eta)} \frac{1}{s_{0,u}} \prod_{l=1}^{k} \frac{1}{s_{l-1,\mathsf{par}(u)}}$$

$$= \sum_{\eta\in\mathsf{Chi}^{\langle k\rangle}(v)} \Big(\sum_{u\in\mathsf{Chi}(\eta)} \frac{1}{s_{0,u}}\Big) \prod_{l=0}^{k-1} \frac{1}{s_{l,\eta}} = \sum_{\eta\in\mathsf{Chi}^{\langle k\rangle}(v)} \prod_{l=0}^{k-1} \frac{1}{s_{l,\eta}} = 1,$$

where the last equality follows from the induction hypothesis.

Note that the weights of $S_{\lambda_{c_a}}$ can be rewritten as

$$\lambda_v = \sqrt{\frac{\mathsf{d}_v}{\mathsf{d}_v + a - 1}}\, \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}(v))}} \quad \text{for } v \in V^\circ.$$

Let $S_{\lambda'_{c_a}}$ denote the Cauchy dual of $S_{\lambda_{c_a}}$. By Lemma 4.2.1, the weights λ' of $S_{\lambda'_{c_a}}$ are given by

$$\lambda'_v = \sqrt{\frac{\mathsf{d}_v + a - 1}{\mathsf{d}_v}}\, \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}(v))}} \quad \text{for all } v \in V^\circ.$$

Now for $v \in V$ and $j, k \geq 1$, an application of Proposition 3.1.7(iv)&(v) yields

$$S^k_{\lambda'_{c_a}} e_v = \sqrt{\frac{(\mathsf{d}_v + k + a - 1)!\mathsf{d}_v!}{(\mathsf{d}_v + a - 1)!(\mathsf{d}_v + k)!}} \sum_{u\in\mathsf{Chi}^{\langle k\rangle}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{s_{l,u}}} e_u,$$

$$S^{*j}_{\lambda'_{c_a}} e_v = \sqrt{\frac{(\mathsf{d}_v + a - 1)!(\mathsf{d}_v - j)!}{(\mathsf{d}_v + a - j - 1)!\mathsf{d}_v!}} \prod_{i=0}^{j-1} \frac{1}{\sqrt{s_{i,v}}} e_{\mathsf{par}^{\langle j\rangle}(v)}.$$

For positive integers j, k and $v \in V$ such that $\mathsf{par}^{\langle j-k\rangle}(v)$ is nonempty, set

$$\beta_{j,k}(v, a) := \prod_{i=0}^{j-k-1} \frac{1}{\sqrt{s_{i,v}}} \sqrt{\frac{(\mathsf{d}_v + k + a - 1)!\mathsf{d}_v!}{(\mathsf{d}_v + a - 1)!(\mathsf{d}_v + k)!}} \times \sqrt{\frac{(\mathsf{d}_v + k + a - 1)!(\mathsf{d}_v + k - j)!}{(\mathsf{d}_v + k + a - j - 1)!(\mathsf{d}_v + k)!}}.$$

Let $v \in V$ and $j \geq k$. It is easily seen that if $\mathsf{par}^{\langle j-k\rangle}(v)$ is empty, then $S^{*j}_{\lambda'_{c_a}} S^k_{\lambda'_{c_a}} e_v = 0$. Otherwise

$$S^{*j}_{\lambda'_{c_a}} S^k_{\lambda'_{c_a}} e_v = \sqrt{\frac{(\mathsf{d}_v + k + a - 1)!\mathsf{d}_v!}{(\mathsf{d}_v + a - 1)!(\mathsf{d}_v + k)!}} \sum_{u\in\mathsf{Chi}^{\langle k\rangle}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{s_{l,u}}} S^{*j}_{\lambda'_{c_a}} e_u$$

$$= \sqrt{\frac{(\mathsf{d}_v + k + a - 1)!\mathsf{d}_v!}{(\mathsf{d}_v + a - 1)!(\mathsf{d}_v + k)!}} \sum_{u\in\mathsf{Chi}^{\langle k\rangle}(v)} \prod_{l=0}^{k-1} \frac{1}{\sqrt{s_{l,u}}}$$

$$\times \sqrt{\frac{(\mathsf{d}_u + a - 1)!(\mathsf{d}_u - j)!}{(\mathsf{d}_u + a - j - 1)!\mathsf{d}_u!}} \prod_{i=0}^{j-1} \frac{1}{\sqrt{s_{i,u}}} e_{\mathsf{par}^{\langle j \rangle}(u)}$$

$$= \beta_{j,k}(v,a) \sum_{u \in \mathsf{Chi}^{\langle k \rangle}(v)} \prod_{l=0}^{k-1} \frac{1}{s_{l,u}} e_{\mathsf{par}^{\langle j-k \rangle}(v)} = \beta_{j,k}(v,a) e_{\mathsf{par}^{\langle j-k \rangle}(v)},$$

where the last equality follows from (5.6).

Let E be the kernel of S^*_{λ, c_a} and let $f \in E$. Note that $f = f_{\mathsf{root}} + \sum_{v \in V_{\prec}} f_v$, where $f_{\mathsf{root}} = \gamma e_{\mathsf{root}}$ for some $\gamma \in \mathbb{C}$ and $f_v = \sum_{u \in \mathsf{Chi}(v)} f(u) e_u$ such that $\sum_{u \in \mathsf{Chi}(v)} f(u) \lambda_u = 0$ for $v \in V_{\prec}$. Since λ_u is constant on $\mathsf{Chi}(v)$, we obtain

$$\sum_{u \in \mathsf{Chi}(v)} f(u) = 0 \quad \text{for all } v \in V_{\prec}. \tag{5.7}$$

It follows that for $v \in V_{\prec}$ and $j > k$,

$$\begin{aligned} S^{*j}_{\lambda'_{c_a}} S^k_{\lambda'_{c_a}} \sum_{u \in \mathsf{Chi}(v)} f(u) e_u &= \sum_{u \in \mathsf{Chi}(v)} f(u) S^{*j}_{\lambda'_{c_a}} S^k_{\lambda', c_a} e_u \\ &= \sum_{u \in \mathsf{Chi}(v)} f(u) \beta_{j,k}(u,a) e_{\mathsf{par}^{\langle j-k \rangle}(u)} = 0, \end{aligned}$$

where we have used (5.7) and the fact that $\beta_{j,k}(u,a)$ is constant on $\mathsf{Chi}(v)$. Almost the same calculations show that

$$S^{*k}_{\lambda'_{c_a}} S^k_{\lambda'_{c_a}} f_v = \frac{(\mathsf{d}_v + k + a)!(\mathsf{d}_v + 1)!}{(\mathsf{d}_v + a)!(\mathsf{d}_v + k + 1)!} f_v \quad \text{for every } k \in \mathbb{N}.$$

It may now be concluded from (4.17) that the reproducing kernel $\kappa_{\mathscr{H}_a}$ takes the required form. Finally, since E is finite-dimensional (Corollary 3.1.16), we note that for every $w \in \mathbb{D}$, $\kappa_{\mathscr{H}_a}(\cdot, w)$ is a sum of finitely many power series in z converging on $\mathbb{D}$. ∎

We now classify all torally balanced multishifts. For this, we need to calculate their moments.

Lemma 5.1.9. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda}$ be a toral left invertible multishift on $\mathscr{T}$ with commuting toral Cauchy dual $S^{\mathfrak{t}}_{\boldsymbol\lambda}$ and let $U_{\boldsymbol\theta}$ be the toral isometry part of $S_{\boldsymbol\lambda}$ governed by* (5.1). *If $S_{\boldsymbol\lambda}$ is torally balanced, then, for any $v \in V$ and $\alpha \in \mathbb{N}^d$,*

$$S^{\alpha}_{\boldsymbol\lambda} e_v = \prod_{j=1}^{d} \Big(\prod_{k=0}^{\alpha_j - 1} c^{(j)}_{|\mathsf{d}_v| + \sum_{i=1}^{j-1} \alpha_i + k} \Big) U^{\alpha}_{\boldsymbol\theta} e_v, \tag{5.8}$$

where $c^{(j)}_{|\mathsf{d}_v|}$ denotes the constant value of $\mathfrak{C}_j(v)$ for $v \in V$ and $j = 1, \ldots, d$. In this case,

$$\|S^{\alpha}_{\boldsymbol\lambda} e_v\|^2 = \prod_{j=1}^{d} \Big(\prod_{k=0}^{\alpha_j - 1} c^{(j)}_{|\mathsf{d}_v| + \sum_{i=1}^{j-1} \alpha_i + k} \Big)^2 \quad (v \in V,\ \alpha \in \mathbb{N}^d). \tag{5.9}$$

Proof. Assume that $S_{\boldsymbol\lambda}$ is torally balanced. Let us first verify

$$S^n_j e_v = \Big(\prod_{k=0}^{n-1} c^{(j)}_{|\mathsf{d}_v| + k} \Big) U^n_j e_v \quad (n \geq 1,\ j = 1, \ldots, d)$$

by induction on $n \geq 1$. Note that

$$\begin{aligned} S_j e_v &= \sum_{w \in \mathsf{Chi}_j(v)} \lambda_w^{(j)} e_w = \sum_{w \in \mathsf{Chi}_j(v)} \frac{\lambda_w^{(j)}}{\mathfrak{C}_j(\mathsf{par}_j(w))} \mathfrak{C}_j(\mathsf{par}_j(w)) e_w \\ &\overset{(5.3)}{=} \mathfrak{C}_j(v) \sum_{w \in \mathsf{Chi}_j(v)} \theta_w^{(j)} e_w = c^{(j)}_{|\mathsf{d}_v|} U_j e_v. \end{aligned} \tag{5.10}$$

This verifies the base case $n = 1$. Assume the induction hypothesis for $n \geq 1$, and consider

$$\begin{aligned} S_j^{n+1} e_v &= \prod_{k=0}^{n-1} c^{(j)}_{|\mathsf{d}_v|+k} S_j U_j^n e_v = \prod_{k=0}^{n-1} c^{(j)}_{|\mathsf{d}_v|+k} S_j \sum_{w \in \mathsf{Chi}_j^{\langle n \rangle}(v)} \theta_w^{(j)} \cdots \theta^{(j)}_{\mathsf{par}_j^{\langle n-1 \rangle}(w)} e_w \\ &\overset{(5.10)}{=} \prod_{k=0}^{n-1} c^{(j)}_{|\mathsf{d}_v|+k} \sum_{w \in \mathsf{Chi}_j^{\langle n \rangle}(v)} \theta_w^{(j)} \cdots \theta^{(j)}_{\mathsf{par}_j^{\langle n-1 \rangle}(w)} c^{(j)}_{|\mathsf{d}_w|} U_j e_w \\ &= \prod_{k=0}^{n} c^{(j)}_{|\mathsf{d}_v|+k} \sum_{w \in \mathsf{Chi}_j^{\langle n \rangle}(v)} \theta_w^{(j)} \cdots \theta^{(j)}_{\mathsf{par}_j^{\langle n-1 \rangle}(w)} U_j e_w \quad (\text{since } \mathsf{d}_w = \mathsf{d}_v + n\epsilon_j) \\ &= \Big(\prod_{k=0}^{n} c^{(j)}_{|\mathsf{d}_v|+k} \Big) U_j^{n+1} e_v. \end{aligned}$$

This completes the inductive argument. A similar inductive argument on $m \geq 1$ together with the commutativity of $S_{\boldsymbol\lambda}^{\mathfrak{t}}$ yields

$$S_i^m S_j^n e_v = \prod_{k=0}^{m-1} c^{(i)}_{|\mathsf{d}_v|+k} \prod_{k=0}^{n-1} c^{(j)}_{|\mathsf{d}_v|+m+k} U_i^m U_j^n e_v$$

for $m, n \in \mathbb{N}$ and $1 \leq i, j \leq d$. The desired conclusion may now be deduced from the above identity by a finite inductive argument. Finally, we note that (5.9) follows from (5.8) and the fact that $U_{\boldsymbol\theta}$ is a toral isometry. ■

Definition 5.1.10. Let $\mathbf{c} := \{c(t, j) : t \in \mathbb{N},\ j = 1, \ldots, d\}$ be a bounded multisequence of positive real numbers such that (5.4) holds. For $s \in \mathbb{N}$, set

$$\gamma_{\alpha,s} := \prod_{j=1}^{d} \prod_{k=0}^{\alpha_j - 1} c(s + \alpha_1 + \cdots + \alpha_{j-1} + k, j) \quad (\alpha \in \mathbb{N}^d).$$

We refer to the Hilbert space of formal series $H^2(\gamma_s)$ as *the Hilbert space associated with* $\mathbf{c}$, where $\gamma_s = \{\gamma_{\alpha,s}\}_{\alpha \in \mathbb{N}^d}$.

Remark 5.1.11. Note that the multiplication d-tuple on $H^2(\gamma_s)$ is a commuting d-tuple of bounded linear operators $M_{z_1}, \ldots, M_{z_d}$.

We are now in a position to present the main result of this section.

Theorem 5.1.12. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. For $v \in V$, let*

$$f = \sum_{\beta \in \mathbb{N}^d} a_\beta S_{\boldsymbol\lambda}^{\beta} e_v \in l^2(V), \qquad \tilde{f}(w) = \sum_{\beta \in \mathbb{N}^d} a_\beta w^\beta.$$

Let $S_{\boldsymbol\lambda}$ be a toral left invertible multishift on $\mathscr{T}$ with commuting toral Cauchy dual $S^{\mathfrak{t}}_{\boldsymbol\lambda}$. Then $S_{\boldsymbol\lambda}$ is a torally balanced multishift on $\mathscr{T}$ if and only if for every $v \in V$, there exists a Hilbert space $H^2(\gamma_{|\mathsf{d}_v|})$ of formal power series in the variables $w_1, \ldots, w_d$ associated with a bounded multisequence $\mathbf{c}$ such that

$$\|f\|_{l^2(V)} = \|\tilde{f}\|_{H^2(\gamma_{|\mathsf{d}_v|})}.$$

Proof. Suppose that $S_{\boldsymbol\lambda}$ is a torally balanced multishift. Set $c(t,j) := c^{(j)}_t$ for $t \in \mathbb{N}$ and $j = 1, \ldots, d$, where $c^{(j)}_t$ denotes the constant value of $\mathfrak{C}_j(v)$ on the generation $\mathcal{G}_t$. Consider the Hilbert space $H^2(\gamma_{|\mathsf{d}_v|})$ of formal power series in $w_1, \ldots, w_d$. By taking norms on both sides of (5.8) we obtain, for every $\alpha \in \mathbb{N}^d$,

$$\begin{aligned}\|S^{\alpha}_{\boldsymbol\lambda} e_v\|_{l^2(V)} &= \prod_{j=1}^{d} \Big(\prod_{k=0}^{\alpha_j - 1} c^{(j)}_{|\mathsf{d}_v| + \sum_{i=1}^{j-1} \alpha_i + k} \Big) \|U^{\alpha}_{\boldsymbol\theta} e_v\|_{l^2(V)} \\ &= \gamma_{\alpha, |\mathsf{d}_v|} = \|w^{\alpha}\|_{H^2(\gamma_{|\mathsf{d}_v|})},\end{aligned} \tag{5.11}$$

where we have used the fact that U_θ is a toral isometry. By orthogonality of $\{S^{\alpha}_{\boldsymbol\lambda} e_v\}_{\alpha \in \mathbb{N}^d}$ (Proposition 3.1.7(ix)), the above formula holds for all pairs f and $\tilde{f}$. To see the converse, let $f = S_j e_v$ and $\tilde{f} = w_j$ in $\|f\|_{l^2(V)} = \|\tilde{f}\|_{H^2(\gamma_{|\mathsf{d}_v|})}$ to obtain $\|S_j e_v\| = c(|\mathsf{d}_v|, j)$, which is clearly constant on $\mathcal{G}_{|\mathsf{d}_v|}$. ∎

Here we present a local analog of von Neumann's inequality for torally balanced multishifts.

Corollary 5.1.13. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \mathscr{T}_2$ and let $S_{\boldsymbol\lambda}$ be a toral left invertible, torally balanced multishift with commuting toral Cauchy dual 2-tuple $S^{\mathfrak{t}}_{\boldsymbol\lambda}$. If $S_{\boldsymbol\lambda}$ is a toral contraction, then for any positive integer k and scalars a_β for $\beta \in \mathbb{N}^2$ with $|\beta| \le k$,*

$$\sup_{v \in V} \Big\| \sum_{|\beta| \le k} a_\beta S^{\beta}_{\boldsymbol\lambda} e_v \Big\| \le \sup_{z \in \mathbb{D}^2} \Big| \sum_{|\beta| \le k} a_\beta z^{\beta} \Big|.$$

Proof. Assume that $S_{\boldsymbol\lambda}$ is a toral contraction. Fix $v \in V$. By the preceding theorem, there exists a Hilbert space $H^2(\gamma_{|\mathsf{d}_v|})$ of formal power series in the variables w_1, w_2 such that

$$\Big\| \sum_{|\beta| \le k} a_\beta S^{\beta}_{\boldsymbol\lambda} e_v \Big\|_{l^2(V)} = \Big\| \sum_{|\beta| \le k} a_\beta w^{\beta} \Big\|_{H^2(\gamma_{|\mathsf{d}_v|})} = \Big\| \sum_{|\beta| \le k} a_\beta M^{\beta}_w 1 \Big\|_{H^2(\gamma_{|\mathsf{d}_v|})},$$

where M_w denotes the 2-tuple of operators of multiplication by the coordinate functions w_1, w_2. However, since $S_{\boldsymbol\lambda}$ is a toral contraction, by (5.11), so is M_w. It follows from $\|1\|_{H^2(\gamma_{|\mathsf{d}_v|})} = 1$ that

$$\Big\| \sum_{|\beta| \le k} a_\beta S^{\beta}_{\boldsymbol\lambda} e_v \Big\|_{l^2(V)} \le \Big\| \sum_{|\beta| \le k} a_\beta M^{\beta}_w \Big\| \|1\|_{H^2(\gamma_{|\mathsf{d}_v|})} = \Big\| \sum_{|\beta| \le k} a_\beta M^{\beta}_w \Big\|.$$

By Ando's dilation theorem [8], von Neumann's inequality holds for any torally contractive classical multishift 2-tuple. Hence

$$\Big\| \sum_{|\beta| \le k} a_\beta S^{\beta}_{\boldsymbol\lambda} e_v \Big\|_{l^2(V)} \le \sup_{z \in \mathbb{D}^2} \Big| \sum_{|\beta| \le k} a_\beta z^{\beta} \Big|.$$

Taking supremum over $v \in V$ on the left hand side, we get the desired inequality. ∎

Remark 5.1.14. By a recent result of M. Hartz [63, Theorem 1.1], the result above holds for any number of variables.

5.2. Spherically balanced multishifts

Definition 5.2.1. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Define $\mathfrak{C} : V \to (0, \infty)$ by

$$\mathfrak{C}(v) := \sum_{j=1}^{d} \|S_j e_v\|^2 \quad \text{for } v \in V. \tag{5.12}$$

We say that $S_{\boldsymbol\lambda}$ is *spherically balanced* if $\mathfrak{C}$ is constant on every generation $\mathcal{G}_t$, $t \in \mathbb{N}$. We then denote the constant value of $\mathfrak{C}(v)$ by $\mathfrak{C}_{|\mathsf{d}_v|}$, where d_v is the depth of v in $\mathscr{T}$.

Remark 5.2.2. Note that $S_{\boldsymbol\lambda}$ is a joint isometry if and only if

$$\sum_{j=1}^{d} \|S_j e_v\|^2 = \sum_{j=1}^{d} \sum_{w \in \mathsf{Chi}_j(v)} (\lambda_w^{(j)})^2 = 1 \quad \text{for all } v \in V.$$

It is now clear that every joint isometry multishift is spherically balanced with $\mathfrak{C}$ being the constant function 1.

The following proposition yields examples of spherically balanced multishifts apart from joint isometries.

Proposition 5.2.3. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $\mathfrak{c} = \{c_t\}_{t \in \mathbb{N}}$ be a bounded sequence of positive real numbers. Consider the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}}} = (S_1, \ldots, S_d)$ with weights*

$$\lambda_w^{(i)} = \sqrt{\frac{c_{|\mathsf{d}_v|}}{\operatorname{card}(\mathsf{Chi}_i(v))}} \sqrt{\frac{\mathsf{d}_{v_i} + 1}{|\mathsf{d}_v| + d}} \quad \textit{for } w \in \mathsf{Chi}_i(v),\ v \in V \textit{ and } i = 1, \ldots, d.$$

Then $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ defines a spherically balanced multishift. In case $c_t = 1$ for all $t \in \mathbb{N}$, $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ is a joint isometry.

Proof. One may conclude from Lemma 3.1.5(i) that $S_1, \ldots, S_d$ are bounded linear operators on $l^2(V)$ whenever $\{c_t\}_{t \in \mathbb{N}}$ is bounded. Let $v \in V$ and $i, j = 1, \ldots, d$. First we check the commutativity of $S_{\boldsymbol\lambda_{\mathfrak{c}}}$. Note that for $w \in \mathsf{Chi}_i(v)$, $\lambda_w^{(i)}$ can be rewritten as

$$\lambda_w^{(i)} = \sqrt{\frac{c_{|\mathsf{d}_w|-1}}{\operatorname{card}(\mathsf{sib}_i(w))}} \sqrt{\frac{\mathsf{d}_{w_i}}{|\mathsf{d}_w| + d - 1}}.$$

Now for $u \in \mathsf{Chi}_i \mathsf{Chi}_j(v)$, we have

$$\lambda_u^{(i)} \lambda_{\mathsf{par}_i(u)}^{(j)} = \sqrt{\frac{c_{|\mathsf{d}_u|-1}}{\operatorname{card}(\mathsf{sib}_i(u))}} \sqrt{\frac{\mathsf{d}_{u_i}}{|\mathsf{d}_u| + d - 1}} \times \sqrt{\frac{c_{|\mathsf{d}_u|-2}}{\operatorname{card}(\mathsf{sib}_j(\mathsf{par}_i(u)))}} \sqrt{\frac{\mathsf{d}_{\mathsf{par}_i(u)_j}}{|\mathsf{d}_{\mathsf{par}_i(u)}| + d - 1}},$$

$$\lambda_u^{(j)}\lambda_{\mathsf{par}_j(u)}^{(i)} = \sqrt{\frac{c_{|\mathsf{d}_u|-1}}{\mathrm{card}(\mathsf{sib}_j(u))}}\sqrt{\frac{\mathsf{d}_{u_j}}{|\mathsf{d}_u|+d-1}} \times \sqrt{\frac{c_{|\mathsf{d}_u|-2}}{\mathrm{card}(\mathsf{sib}_i(\mathsf{par}_j(u)))}}\sqrt{\frac{\mathsf{d}_{\mathsf{par}_j(u)_i}}{|\mathsf{d}_{\mathsf{par}_j(u)}|+d-1}}.$$

Since $\mathsf{d}_{u_i} = \mathsf{d}_{\mathsf{par}_j(u)_i}$ for $i \neq j$, by (5.5), $S_{\boldsymbol\lambda_\mathfrak{c}}$ is commuting. Note further that

$$\sum_{j=1}^d \|S_j e_v\|^2 = \sum_{j=1}^d \sum_{w\in\mathsf{Chi}_j(v)} (\lambda_w^{(j)})^2 = \sum_{j=1}^d \sum_{w\in\mathsf{Chi}_j(v)} \frac{c_{|\mathsf{d}_v|}}{\mathrm{card}(\mathsf{Chi}_j(v))}\frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+d} = c_{|\mathsf{d}_v|}\sum_{j=1}^d \frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+d} = c_{|\mathsf{d}_v|},$$

which is a function of $|\mathsf{d}_v|$. Thus $S_{\boldsymbol\lambda_\mathfrak{c}}$ is spherically balanced. The above calculation also shows that $S_{\boldsymbol\lambda_\mathfrak{c}}$ is a joint isometry if and only if $c_t = 1$ for all $t \in \mathbb{N}$. ∎

Remark 5.2.4. The choice $c_t = \frac{t+d}{t+1}\frac{t+2}{t+1}$ ($t \in \mathbb{N}$) with $\mathscr{T}$ being the d-fold directed Cartesian product of $\mathscr{T}_{1,0}$ with itself yields the Dirichlet d-shift on the unit ball [60, Example 1].

Below we discuss a family of examples of spherically balanced multishifts, which is a tree analog of the multiplication d-tuples on the reproducing kernel Hilbert spaces associated with the reproducing kernels $1/(1-\langle z,w\rangle)^a$ defined on the unit ball in $\mathbb{C}^d$, where a is a positive number.

Example 5.2.5. Let $\mathscr{T} = (V,\mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1,\ldots,\mathscr{T}_d$. For a positive real number a, consider the sequence $\mathfrak{c}_a = \{c_{a,t}\}_{t\in\mathbb{N}}$ given by

$$c_{a,t} = \frac{t+d}{t+a} \quad (t \in \mathbb{N}).$$

Then the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ on $\mathscr{T}$ is spherically balanced. In case $\mathscr{T}$ is the d-fold directed Cartesian product of $\mathscr{T}_{1,0}$ with itself then the choices $a = d$, $a = d+1$, $a = 1$ yield Szegö d-shift, Bergman d-shift, Drury–Arveson d-shift respectively on the unit ball (see Example 1.2.1).

We refer to the multishifts $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ on $\mathscr{T}$ as the *tree analog of Szegö d-shift, Bergman d-shift, Drury–Arveson d-shift* respectively in case $a = d, a = d+1, a = 1$. It is worth noting that $S_{\boldsymbol\lambda_{\mathfrak{c}_d}}$ and $S_{\boldsymbol\lambda_{\mathfrak{c}_{d+1}}}$ are joint contractions while $S_{\boldsymbol\lambda_{\mathfrak{c}_1}}$ is a *row contraction* (that is, $S^*_{\boldsymbol\lambda_{\mathfrak{c}_1}}$ is a joint contraction).

Although there is no known satisfactory counterpart of Shimorin's analytic model for joint left invertible analytic tuples, we are able to show, as in the classical case, that the multishifts $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ on $\mathscr{T}$ can be realized as multiplication tuples $\mathscr{M}_z$ on reproducing kernel Hilbert spaces at least in case the joint kernel E of $S^*_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is finite-dimensional.

Theorem 5.2.6. *Let $\mathscr{T} = (V,\mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1,\ldots,\mathscr{T}_d$ of finite joint branching index. Let $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ be as introduced in Example* 5.2.5 *and let E denote the joint kernel of $S^*_{\boldsymbol\lambda_{\mathfrak{c}_a}}$. Then $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is unitarily equivalent to the multiplication d-tuple $\mathscr{M}_{z,a} = (\mathscr{M}_{z_1},\ldots,\mathscr{M}_{z_d})$ on a reproducing kernel Hilbert space*

$\mathscr{H}_{a,d}$ of E-valued holomorphic functions defined on the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. Further, the reproducing kernel $\kappa_{\mathscr{H}_{a,d}} : \mathbb{B}^d \times \mathbb{B}^d \to B(E)$ associated with $\mathscr{H}_{a,d}$ is given by

$$\kappa_{\mathscr{H}_{a,d}}(z,w) = \sum_{\alpha \in \mathbb{N}^d} \frac{a(a+1)\cdots(a+|\alpha|-1)}{\alpha!} z^{\alpha}\overline{w}^{\alpha} P_{[e_{\mathsf{root}}]} + \sum_{\substack{F \in \mathscr{P} \\ F \neq \emptyset}} \sum_{u \in \Omega_F} \kappa_{u,F}(z,w),$$

where $\kappa_{u,F}(z,w)$ is given by

$$\kappa_{u,F}(z,w) = \sum_{\alpha \in \mathbb{N}^d} \frac{\mathsf{d}_u!}{(\mathsf{d}_u+\alpha)!} \prod_{j=0}^{|\alpha|-1} (|\mathsf{d}_u| + a + j) z^{\alpha}\overline{w}^{\alpha} P_{\mathcal{L}_{u,F}}, \tag{5.13}$$

with $P_{\mathcal{M}}$ being the orthogonal projection of $\mathcal{H}$ onto a subspace $\mathcal{M}$ of $\mathcal{H}$.

The proof of the above theorem, as presented below, turns out to be more involved than that of Proposition 5.1.8. Perhaps one reason may be that no counterpart of Shimorin's model is known for joint left invertible analytic tuples. This proof relies on the description of the joint kernel E as provided in Chapter 4. In addition, a key observation required is the following lemma.

Lemma 5.2.7. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index. Let $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ be as introduced in Example 5.2.5 and let E denote the joint kernel of $S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$. Then:*

(i) *E is invariant under $S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ and $S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}|_E$ is boundedly invertible for every $\alpha \in \mathbb{N}^d$.*

(ii) *The multisequence $\{S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} E\}_{\alpha \in \mathbb{N}^d}$ of subspaces of $l^2(V)$ is mutually orthogonal.*

Proof. (i) In view of (4.12) and $S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v = \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\|^2 e_v$ $(v \in V)$, it suffices to check that for every $F \in \mathscr{P}$ and $u \in \Omega_F$, the function $v \mapsto \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\|^2$ is constant on $\mathsf{sib}_F(u)$ with value $\|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_u\|^2$. For $j = 1, \ldots, d$, $w \in V$, let $\beta(j, w, 0) = 1$ and

$$\beta(j,w,n) = \lambda^{(j)}_w \lambda^{(j)}_{\mathsf{par}_j(w)} \cdots \lambda^{(j)}_{\mathsf{par}_j^{\langle n-1\rangle}(w)} \quad (n \geq 1).$$

It is easy to see using $\lambda^{(i)}_w = \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_i(w))}} \sqrt{\frac{\mathsf{d}_{w_i}}{|\mathsf{d}_w|+a-1}}$ that

$$\beta(j,w,n)^2 = \Big(\prod_{k=0}^{n-1} \frac{1}{\mathrm{card}(\mathsf{sib}_j \mathsf{par}_j^{\langle k\rangle}(w))}\Big) \frac{\mathsf{d}_{w_j}!}{(\mathsf{d}_{w_j}-n)!} \frac{(|\mathsf{d}_w| - n + a - 1)!}{(|\mathsf{d}_w| + a - 1)!} \tag{5.14}$$

for $n \geq 1$ and $j = 1, \ldots, d$. Fix $\alpha \in \mathbb{N}^d$ and let $1 \leq i_1 < \cdots < i_k \leq d$ be integers such that $\alpha_{i_1}, \ldots, \alpha_{i_k}$ are the only nonzero entries in α. A routine verification using Proposition 3.1.7(vi) and (5.14) shows that

$$\begin{aligned} S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v &= \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \prod_{j=1}^{k} \beta\big(i_j, \mathsf{par}^{\ll \alpha^{(i_j-1)} \gg}(w), \alpha_{i_j}\big)^2 e_v \\ &= \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \Bigg(\prod_{j=1}^{k} \frac{\mathsf{d}_{w_{i_j}}!}{(\mathsf{d}_{w_{i_j}} - \alpha_{i_j})!} \frac{(|\mathsf{d}_{\mathsf{par}^{\ll \alpha^{(i_j-1)} \gg}(w)}| - \alpha_{i_j} + a - 1)!}{(|\mathsf{d}_{\mathsf{par}^{\ll \alpha^{(i_j-1)} \gg}(w)}| + a - 1)!} \\ &\qquad \times \prod_{j=1}^{k} \prod_{l=0}^{\alpha_{i_j}-1} \frac{1}{\mathrm{card}(\mathsf{sib}_{i_j}(\mathsf{par}_{i_j}^{\langle l\rangle} \mathsf{par}_{i_{j-1}}^{\langle \alpha_{i_{j-1}}\rangle} \cdots \mathsf{par}_{i_1}^{\langle \alpha_{i_1}\rangle}(w)))} \Bigg) e_v \end{aligned}$$

$$= \prod_{j=1}^{k} \frac{(\mathsf{d}_{v_{i_j}} + \alpha_{i_j})!}{\mathsf{d}_{v_{i_j}}!} \frac{(|\mathsf{d}_v| + \sum_{m=j+1}^{k} \alpha_{i_m} + a - 1)!}{(|\mathsf{d}_v| + \sum_{m=j}^{k} \alpha_{i_m} + a - 1)!}$$
$$\times \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \prod_{j=1}^{k} \prod_{l=0}^{\alpha_{i_j}-1} \frac{1}{\mathrm{card}(\mathsf{sib}_{i_j}(\mathsf{par}_{i_j}^{\langle l \rangle} \mathsf{par}_{i_{j-1}}^{\langle \alpha_{i_{j-1}} \rangle} \cdots \mathsf{par}_{i_1}^{\langle \alpha_{i_1} \rangle}(w)))} \, e_v,$$

where we have used $|\mathsf{d}_{\mathsf{par}^{\ll \alpha^{(i_j-1)} \gg}(w)}| = |\mathsf{d}_w| - \sum_{l=1}^{j-1} \alpha_{i_l}$, $\mathsf{d}_w = \mathsf{d}_v + \alpha$ for any $w \in \mathsf{Chi}^{\ll \alpha \gg}(v)$. As in the proof of Proposition 5.1.8, one can verify by induction on $|\alpha| \geq 1$ that

$$\sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \prod_{j=1}^{d} \prod_{l=0}^{\alpha_j - 1} \frac{1}{\mathrm{card}(\mathsf{sib}_j(\mathsf{par}_j^{\langle l \rangle} \mathsf{par}_{j-1}^{\langle \alpha_{j-1} \rangle} \cdots \mathsf{par}_1^{\langle \alpha_1 \rangle}(w)))} = 1.$$

This yields

$$S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v = \prod_{j=1}^{k} \frac{(\mathsf{d}_{v_{i_j}} + \alpha_{i_j})!}{\mathsf{d}_{v_{i_j}}!} \frac{(|\mathsf{d}_v| + \sum_{m=j+1}^{k} \alpha_{i_m} + a - 1)!}{(|\mathsf{d}_v| + \sum_{m=j}^{k} \alpha_{i_m} + a - 1)!} e_v \quad (v \in V). \tag{5.15}$$

Since depth is constant on $\mathsf{sib}_F(u)$ and E is finite-dimensional (Corollary 3.1.16), (i) is immediate from this formula. For future reference, we also note the following expression for moments of $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ deduced from (5.15):

$$\begin{aligned}\|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\|^2 &= \prod_{j=1}^{d} \frac{(\mathsf{d}_{v_j} + \alpha_j)!}{\mathsf{d}_{v_j}!} \frac{(|\mathsf{d}_v| + \sum_{i=j+1}^{d} \alpha_i + a - 1)!}{(|\mathsf{d}_v| + \sum_{i=j}^{d} \alpha_i + a - 1)!} \\ &= \bigg(\prod_{j=1}^{d} \frac{(\mathsf{d}_{v_j} + \alpha_j)!}{\mathsf{d}_{v_j}!} \bigg) \frac{1}{(|\mathsf{d}_v| + a)(|\mathsf{d}_v| + a + 1) \cdots (|\mathsf{d}_v| + a + |\alpha| - 1)}\end{aligned} \tag{5.16}$$

for all $v \in V$.

(ii) It suffices to check that $S_j^{*\beta_j+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}|_E = 0$ for $j = 1, \ldots, d$. We will verify this only for $j = d$. The verification for the other coordinates is the same. Let $f \in E$. Clearly, for $f = e_{\mathsf{root}}$, we have $S_d^{*\beta_d+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f = 0$. Let $F \in \mathscr{P}$ be such that $F \neq \emptyset$ and let $f \in \mathcal{L}_{u,F}$ for $u \in \Omega_F \subseteq \Phi_F$ (see (4.3) and (4.6)). If $d \notin F$ then once again $S_d^{*\beta_d+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f = 0$ since $f \in l^2(\mathsf{sib}_F(u))$ is supported on a subset of $V_1 \times \cdots \times V_{d-1} \times \{\mathsf{root}_d\}$. Hence we may assume that $d \in F$. Since $f \in \mathcal{L}_{u,F}$, by (4.10), $f = \sum_{v \in \mathsf{sib}_F(u)} f(v) e_v$ satisfies

$$\sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w) \lambda_w^{(i)} = 0, \quad i \in F \text{ and } v_G \in \mathsf{sib}_{F,G}(u).$$

However, since $\lambda_w^{(i)} = \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_i(w))}} \sqrt{\frac{\mathsf{d}_{w_i}}{|\mathsf{d}_w| + a - 1}}$ is constant on $w \in \mathsf{sib}_i(v_G|u_i)$, we obtain

$$\sum_{w \in \mathsf{sib}_i(v_G|u_i)} f(w) = 0, \quad i \in F \text{ and } v_G \in \mathsf{sib}_{F,G}(u). \tag{5.17}$$

Let $\alpha = \beta - \beta_d \epsilon_d$. One may now argue as in (i) to see that

$$\begin{aligned}
S_d^{*\beta_d+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f &= S_d^* \sum_{v \in \mathsf{sib}_F(u)} f(v) S_d^{*\beta_d} S_d^{\beta_d} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v \\
&= \sum_{v \in \mathsf{sib}_F(u)} \sqrt{\prod_{j=1}^d \frac{(\mathsf{d}_{v_j}+\alpha_j)!}{\mathsf{d}_{v_j}!}} \frac{f(v)}{\sqrt{(|\mathsf{d}_v|+a)(|\mathsf{d}_v|+a+1)\cdots(|\mathsf{d}_v|+a+|\alpha|-1)}} \\
&\quad \times \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \prod_{j=1}^d \prod_{l=0}^{\alpha_j-1} \frac{S_d^*(S_d^{*\beta_d} S_d^{\beta_d} e_w)}{\sqrt{\mathrm{card}(\mathsf{sib}_j(\mathsf{par}_j^{\langle l \rangle} \mathsf{par}_{j-1}^{\langle \alpha_{j-1} \rangle} \cdots \mathsf{par}_1^{\langle \alpha_1 \rangle}(w)))}} \\
&\overset{(5.16)}{=} \sum_{v \in \mathsf{sib}_F(u)} f(v) \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\| \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \gamma(w,\alpha)\ S_d^*(S_d^{*\beta_d} S_d^{\beta_d} e_w), \qquad (5.18)
\end{aligned}$$

where $\gamma(w,\alpha)$ is given by

$$\gamma(w,\alpha) := \prod_{j=1}^d \prod_{l=0}^{\alpha_j-1} \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_j(\mathsf{par}_j^{\langle l \rangle} \mathsf{par}_{j-1}^{\langle \alpha_{j-1} \rangle} \cdots \mathsf{par}_1^{\langle \alpha_1 \rangle}(w)))}}. \qquad (5.19)$$

However, by (5.15) and $\mathsf{d}_{w_d} = \mathsf{d}_{v_d}$,

$$\begin{aligned}
S_d^{*\beta_d} S_d^{\beta_d} e_w &= \frac{(\mathsf{d}_{w_d}+\beta_d)!}{\mathsf{d}_{w_d}!} \frac{(|\mathsf{d}_w|+a-1)!}{(|\mathsf{d}_w|+\beta_d+a-1)!} e_w \\
&= \frac{(\mathsf{d}_{v_d}+\beta_d)!}{\mathsf{d}_{v_d}!} \frac{(|\mathsf{d}_v|+|\alpha|+a-1)!}{(|\mathsf{d}_v|+|\alpha|+\beta_d+a-1)!} e_w.
\end{aligned}$$

This combined with (5.18) yields

$$\begin{aligned}
S_d^{*\beta_d+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f = \sum_{v \in \mathsf{sib}_F(u)} f(v) \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\| \frac{(\mathsf{d}_{v_d}+\beta_d)!}{\mathsf{d}_{v_d}!} \frac{(|\mathsf{d}_v|+|\alpha|+a-1)!}{(|\mathsf{d}_v|+|\alpha|+\beta_d+a-1)!} \\
\times \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \gamma(w,\alpha) S_d^* e_w.
\end{aligned}$$

Since $w_d = v_d$ and $\mathsf{d}_w = \mathsf{d}_v + \alpha$, we have

$$\begin{aligned}
S_d^* e_w &= \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_d(w))}} \sqrt{\frac{\mathsf{d}_{w_d}}{|\mathsf{d}_w|+a-1}}\, e_{\mathsf{par}_d(w)} \\
&= \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_d(v))}} \sqrt{\frac{\mathsf{d}_{v_d}}{|\mathsf{d}_v|+|\alpha|+a-1}}\, e_{\mathsf{par}_d(w)}.
\end{aligned}$$

This gives

$$\begin{aligned}
S_d^{*\beta_d+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f = \sum_{v \in \mathsf{sib}_F(u)} f(v) \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\| \frac{(\mathsf{d}_{v_d}+\beta_d)!}{\mathsf{d}_{v_d}!} \frac{(|\mathsf{d}_v|+|\alpha|+a-1)!}{(|\mathsf{d}_v|+|\alpha|+\beta_d+a-1)!} \\
\times \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_d(v))}} \sqrt{\frac{\mathsf{d}_{v_d}}{|\mathsf{d}_v|+|\alpha|+a-1}} \sum_{w \in \mathsf{Chi}^{\ll \alpha \gg}(v)} \gamma(w,\alpha) e_{\mathsf{par}_d(w)}.
\end{aligned}$$

It is clear from the definition of depth and siblings that the expression $\Gamma(v,\alpha)$ below is independent of $v \in \mathsf{sib}_F(u)$:

$$\Gamma(v,\alpha) := \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} e_v\| \frac{(\mathsf{d}_{v_d}+\beta_d)!}{\mathsf{d}_{v_d}!} \frac{(|\mathsf{d}_v|+|\alpha|+a-1)!}{(|\mathsf{d}_v|+|\alpha|+\beta_d+a-1)!} \times \frac{1}{\sqrt{\mathrm{card}(\mathsf{sib}_d(v))}} \sqrt{\frac{\mathsf{d}_{v_d}}{|\mathsf{d}_v|+|\alpha|+a-1}}.$$

Further, since $\alpha_d = 0$, we conclude from (5.19) that $\gamma(w,\alpha)$ is independent of $w_d = v_d$. Also, since $v = v_G|v_d$ for $G = F \setminus \{d\}$ and $v \in \mathsf{sib}_F(u)$, it follows that

$$\begin{aligned}
\frac{1}{\Gamma(u,\alpha)} S_d^{*\beta_d+1} S^{\beta}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f &= \sum_{v\in \mathsf{sib}_F(u)} f(v) \sum_{w\in \mathsf{Chi}^{\ll\alpha\gg}(v)} \gamma(w,\alpha) e_{\mathsf{par}_d(w)} \\
&= \sum_{v_G|v_d\in \mathsf{sib}_F(u)} \Big(\sum_{w\in \mathsf{Chi}^{\ll\alpha\gg}(v)} f(v_G|v_d)\gamma(w,\alpha) e_{\mathsf{par}_d(w)} \Big) \\
&= \sum_{v_G\in \mathsf{sib}_{F,G}(u)} \sum_{v_d\in \mathsf{sib}(u_d)} \Big(\sum_{w\in \mathsf{Chi}^{\ll\alpha\gg}(v)} f(v_G|v_d)\gamma(w,\alpha) e_{\mathsf{par}_d(w)} \Big) \\
&= \sum_{v_G\in \mathsf{sib}_{F,G}(u)} \sum_{w\in \mathsf{Chi}^{\ll\alpha\gg}(v)} \Big(\sum_{v_d\in \mathsf{sib}(u_d)} f(v_G|v_d)\gamma(w,\alpha) e_{\mathsf{par}_d(w)} \Big) \\
&= \sum_{v_G\in \mathsf{sib}_{F,G}(u)} \sum_{w\in \mathsf{Chi}^{\ll\alpha\gg}(v)} \Big(\sum_{v_d\in \mathsf{sib}(u_d)} f(v_G|v_d) \Big) \gamma(w,\alpha) e_{\mathsf{par}_d(w)},
\end{aligned}$$

where the sum in the inner bracket is 0 in view of (5.17). ∎

We are now in a position to complete the proof of Theorem 5.2.6.

Proof of Theorem 5.2.6. We divide the proof into several steps.

STEP I. In this step, we prove that $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ can be modeled as the multiplication tuple $\mathscr{M}_z$ on a Hilbert space of E-valued formal power series. Note that by Theorem 4.0.1 and Lemma 5.2.7(ii), we have

$$l^2(V) = \bigoplus_{\alpha\in\mathbb{N}^d} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} E.$$

Thus for any $f \in l^2(V)$, there exists a multisequence $\{f_\alpha\}_{\alpha\in\mathbb{N}^d}$ in E such that

$$f = \sum_{\alpha\in\mathbb{N}^d} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha.$$

Also, since $S_1, \ldots, S_d$ are injective (Corollary 3.3.4), the multisequence $\{f_\alpha\}_{\alpha\in\mathbb{N}^d}$ with the above property is unique. This unique representation allows us to form the inner product space $\mathscr{H}_{a,d}$ of E-valued formal power series by

$$\mathscr{H}_{a,d} := \Big\{ F(z) = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha : f_\alpha \in E \ (\alpha\in\mathbb{N}^d),\ \sum_{\alpha\in\mathbb{N}^d} \|S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha\|^2 < \infty \Big\}$$

endowed with the inner product

$$\langle F(z), G(z)\rangle := \sum_{\alpha\in\mathbb{N}^d} \langle S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha, S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} g_\alpha\rangle,$$

where $G(z) = \sum_{\alpha\in\mathbb{N}^d} g_\alpha z^\alpha$. Since $S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ is bounded below on E (Lemma 5.2.7(i)), there exists $M_\alpha > 0$ such that $\|F\|^2 \geq \sum_{\alpha\in\mathbb{N}^d} M_\alpha \|f_\alpha\|^2_{l^2(V)}$ for all $F \in \mathscr{H}_{a,d}$. Hence $\mathscr{H}_{a,d}$

is a Hilbert space. We now define unitary $U : l^2(V) \to \mathscr{H}_{a,d}$ by $U(f) = F$. Further, if $\mathscr{M}_{z,a}$ denotes the d-tuple of (densely defined) multiplication operators $\mathscr{M}_{z_1}, \ldots, \mathscr{M}_{z_d}$ in $\mathscr{H}_{a,d}$ then for $j = 1, \ldots, d$,

$$US_j(S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha) = US^{\alpha+\epsilon_j}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha = f_\alpha z^{\alpha+\epsilon_j} = \mathscr{M}_{z_j} f_\alpha z^\alpha = \mathscr{M}_{z_j} U(S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha).$$

Note that $US_j = \mathscr{M}_{z_j} U$ holds on a dense set. Since S_j is bounded, it follows that $\mathscr{M}_{z_j}$ is bounded for every $j = 1, \ldots, d$.

STEP II. In this step, we check that $\mathscr{H}_{a,d}$ is a reproducing kernel Hilbert space associated with the reproducing kernel

$$\kappa_{\mathscr{H}_{a,d}}(z, w) = \sum_{\alpha \in \mathbb{N}^d} D_\alpha z^\alpha \overline{w}^\alpha \quad (z, w \in \Omega),$$

where D_α is the inverse of $S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}|_E$ on E as ensured by Lemma 5.2.7(i), and Ω denotes the domain of convergence of $\kappa_{\mathscr{H}_{a,d}}$ (possibly $\{0\}$). Note that for $F(z) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha z^\alpha \in \mathscr{H}_{a,d}$, $g \in E$ and $w \in \Omega$,

$$\begin{aligned}\langle F, \kappa_{\mathscr{H}_{a,d}}(\cdot, w) g\rangle &= \sum_{\alpha \in \mathbb{N}^d} \langle S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} f_\alpha, S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} (D_\alpha g) \overline{w}^\alpha \rangle = \sum_{\alpha \in \mathbb{N}^d} \langle f_\alpha w^\alpha, S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} D_\alpha g\rangle \\ &= \langle F(w), g\rangle_E,\end{aligned}$$

where we have used the fact that $S^{*\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} S^{\alpha}_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} D_\alpha = I|_E$ for every $\alpha \in \mathbb{N}^d$. This completes the verification of Step II.

STEP III. We note that D_α is the diagonal operator on E with diagonal entries $a(a+1)\cdots(a+|\alpha|-1)/\alpha!$ (corresponding to e_{root}) and

$$\left(\prod_{j=1}^{d} \frac{\mathsf{d}_{u_j}!}{(\mathsf{d}_{u_j} + \alpha_j)!}\right)(|\mathsf{d}_u| + a)(|\mathsf{d}_u| + a + 1)\cdots(|\mathsf{d}_u| + a + |\alpha| - 1)$$

(corresponding to the component of E from $l^2(\mathsf{sib}_F(u))$). This is immediate from (5.16) and the definition of D_α.

STEP IV. We verify that the domain of convergence of $\kappa_{\mathscr{H}_{a,d}}$ equals the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. Indeed, by the preceding two steps, $\kappa_{\mathscr{H}_{a,d}}$ takes the form

$$\kappa_{\mathscr{H}_{a,d}}(z, w) = \sum_{\alpha \in \mathbb{N}^d} \frac{a(a+1)\cdots(a+|\alpha|-1)}{\alpha!} z^\alpha \overline{w}^\alpha P_{[e_{\mathsf{root}}]} + \sum_{\substack{F \in \mathscr{P} \\ F \neq \emptyset}} \sum_{u \in \Omega_F} \kappa_{u,F}(z, w),$$

where $\kappa_{u,F}(z, w)$ is given by (5.13). Since the first series in the expression for $\kappa_{\mathscr{H}_{a,d}}(z, w)$ is precisely $P_{[e_{\mathsf{root}}]}/(1 - \langle z, w\rangle)^a$, it suffices to check that the domain of convergence of $\kappa_{u,F}(\cdot, w)$ equals $\mathbb{B}^d$ for every $w \in \mathbb{B}^d$. However, the coefficients in this series are obtained (modulo some scalars) by adding the constant d-tuple d_u to the coefficients of $1/(1 - \langle z, w\rangle)^a$, and hence the domain of convergence is $\mathbb{B}^d$. ∎

REMARK 5.2.8. In case $\mathscr{T}_j = \mathscr{T}_{1,0}$ for $j = 1, \ldots, d$, then the reproducing kernel spaces $\mathscr{H}_{a,d}$ are precisely the spaces appearing in [24, (1.11)] (see Table 1 below).

COROLLARY 5.2.9. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index and let $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ be as in Example* 5.2.5. *Then the point spectrum of $S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ contains the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$.*

Table 1. Tree analogs of reproducing kernels $\kappa_{\mathcal{H}_{a,d}}(z,w)$

Kernel $\kappa_{\mathscr{H}_{a,d}}(z,w)$	$\mathscr{T}_1 \times \mathscr{T}_2$
$\frac{1}{(1-\langle z,w\rangle)^a} P_{[e_{(0,0)}]}$	$\mathscr{T}_{1,0} \times \mathscr{T}_{1,0}$
$\frac{1}{(1-\langle z,w\rangle)^a} P_{[e_{(0,0)}]} + \sum_{\alpha\in\mathbb{N}^d} \frac{1}{(\alpha+\epsilon_1)!}\prod_{j=0}^{\lvert\alpha\rvert-1}(a+1+j)z^\alpha\overline{w}^\alpha\, P_{\mathcal{L}_{(1,0),\{1\}}}$	$\mathscr{T}_{2,0} \times \mathscr{T}_{1,0}$
$\frac{1}{(1-\langle z,w\rangle)^a} P_{[e_{(0,0)}]} + \sum_{\alpha\in\mathbb{N}^d} \frac{1}{(\alpha+\epsilon_1)!}\prod_{j=0}^{\lvert\alpha\rvert-1}(a+1+j)z^\alpha\overline{w}^\alpha\, P_{\mathcal{L}_{(1,0),\{1\}}}$ $+\sum_{\alpha\in\mathbb{N}^d} \frac{1}{(\alpha+\epsilon_2)!}\prod_{j=0}^{\lvert\alpha\rvert-1}(a+1+j)z^\alpha\overline{w}^\alpha\, P_{\mathcal{L}_{(0,1),\{2\}}}$ $+\sum_{\alpha\in\mathbb{N}^d} \frac{1}{(\alpha+\epsilon_1+\epsilon_2)!}\prod_{j=0}^{\lvert\alpha\rvert-1}(a+2+j)z^\alpha\overline{w}^\alpha\, P_{\mathcal{L}_{(1,1),\{1,2\}}}$	$\mathscr{T}_{2,0} \times \mathscr{T}_{2,0}$

Proof. By Theorem 5.2.6, $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is unitarily equivalent to the multiplication d-tuple $\mathscr{M}_z = (\mathscr{M}_{z_1}, \ldots, \mathscr{M}_{z_d})$ acting on the reproducing kernel Hilbert space $\mathscr{H}_{a,d}$ of E-valued holomorphic functions defined on the unit ball $\mathbb{B}^d$. The desired conclusion now follows from the fact that $\mathscr{M}_{z_j}^*(\kappa_{\mathscr{H}_{a,d}}(\cdot,w)f) = \overline{w}_j\kappa_{\mathscr{H}_{a,d}}(\cdot,w)f$ for any $f\in E$, $w\in\mathbb{B}^d$ and $j=1,\ldots,d$, where $\kappa_{\mathscr{H}_{a,d}}$ denotes the reproducing kernel associated with $\mathscr{H}_{a,d}$. ■

We now turn our attention to the classification of spherically balanced multishifts. The notion of spherically balanced multishifts is closely related to that of spherical Cauchy dual tuple. Before we make this precise, note that by (1.3), $S_i^{\mathfrak{s}} = S_i\big(\sum_{i=1}^d S_i^*S_i\big)^{-1}$. It follows that the spherical Cauchy dual $S_{\boldsymbol\lambda}^{\mathfrak{s}} = (S_1^{\mathfrak{s}}, \ldots, S_d^{\mathfrak{s}})$ of a joint left invertible multishift $S_{\boldsymbol\lambda}$ is given by

$$S_i^{\mathfrak{s}} e_v = \Big(\sum_{j=1}^d \|S_j e_v\|^2\Big)^{-1} \sum_{w\in\mathsf{Chi}_i(v)} \lambda_w^{(i)} e_w \quad \text{for all } v\in V,\ i=1,\ldots,d.$$

Note that $S_{\boldsymbol\lambda}^{\mathfrak{s}}$ is also a multishift on $\mathscr{T}$ with weights

$$\lambda_w^{(i)}\Big(\sum_{j=1}^d \|S_j e_{\mathsf{par}_i(w)}\|^2\Big)^{-1}, \qquad w\in V^\circ,\ i=1,\ldots,d.$$

In the next proposition, we show that every joint left invertible spherically balanced multishift admits a polar decomposition in the following sense.

Proposition 5.2.10. *Let $\mathscr{T}=(V,\mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1,\ldots,\mathscr{T}_d$. Let $S_{\boldsymbol\lambda}=(S_1,\ldots,S_d)$ be a joint left invertible multishift on $\mathscr{T}$ and let $S_{\boldsymbol\lambda}^{\mathfrak{s}}$ denote the spherical Cauchy dual of $S_{\boldsymbol\lambda}$. Then the following statements are equivalent:*

(i) *$S_{\boldsymbol\lambda}^{\mathfrak{s}}$ is commuting.*

(ii) *For every $v\in V^\circ$, $\mathfrak{C}$ is constant on $\mathsf{Par}(v)$, where $\mathfrak{C}$ is as defined in (5.12).*

(iii) *There exists a joint isometry multishift $T_{\boldsymbol\lambda}=(T_1,\ldots,T_d)$ and a diagonal, positive, invertible bounded linear operator D_c on $l^2(V)$ such that*

$$S_j = T_j D_c, \qquad j=1,\ldots,d.$$

In this case, the above decomposition is unique.

Proof. We do not include the verification of the uniqueness part as it is similar to that of Proposition 5.1.1.

(i)⇔(ii): Fix $v\in V$. By the discussion prior to the proposition, we have

$$S_i^{\mathfrak{s}} e_v = \mathfrak{C}(v)^{-1} \sum_{w\in\mathsf{Chi}_i(v)} \lambda_w^{(i)} e_w.$$

Therefore,

$$S^{\mathfrak{s}}_j S^{\mathfrak{s}}_i e_v = \mathfrak{C}(v)^{-1} \sum_{w \in \mathsf{Chi}_i(v)} \lambda^{(i)}_w S^{\mathfrak{s}}_j e_w = \mathfrak{C}(v)^{-1} \sum_{w \in \mathsf{Chi}_i(v)} \lambda^{(i)}_w \mathfrak{C}(w)^{-1} \sum_{u \in \mathsf{Chi}_j(w)} \lambda^{(j)}_u e_u$$
$$= \mathfrak{C}(v)^{-1} \sum_{u \in \mathsf{Chi}_j \mathsf{Chi}_i(v)} \lambda^{(i)}_{\mathsf{par}_j(u)} \lambda^{(j)}_u \mathfrak{C}(\mathsf{par}_j(u))^{-1} e_u.$$

Similarly,

$$S^{\mathfrak{s}}_i S^{\mathfrak{s}}_j e_v = \mathfrak{C}(v)^{-1} \sum_{u \in \mathsf{Chi}_i \mathsf{Chi}_j(v)} \lambda^{(j)}_{\mathsf{par}_i(u)} \lambda^{(i)}_u \mathfrak{C}(\mathsf{par}_i(u))^{-1} e_u.$$

Since $S_{\boldsymbol\lambda}$ is commuting, by Proposition 3.1.7(i), $S^{\mathfrak{s}}_i S^{\mathfrak{s}}_j e_v = S^{\mathfrak{s}}_j S^{\mathfrak{s}}_i e_v$ for all $i, j = 1, \ldots, d$ if and only if

$$\mathfrak{C}(\mathsf{par}_i(u)) = \mathfrak{C}(\mathsf{par}_j(u)) \quad \text{for all } u \in \mathsf{Chi}_i\mathsf{Chi}_j(v) \text{ and } i, j = 1, \ldots, d.$$

This yields the desired equivalence of (i) and (ii).

(ii)⇔(iii): We introduce a multishift $T_{\boldsymbol\lambda} = (T_1, \ldots, T_d)$ on $\mathscr{T}$ with weights

$$\lambda^{(i)}_w / \sqrt{\mathfrak{C}(v)}, \quad w \in \mathsf{Chi}_i(v),\ v \in V,\ \text{and } i = 1, \ldots, d.$$

Note that for $v \in V$,

$$\sum_{j=1}^d \|T_j e_v\|^2 = \mathfrak{C}(v)^{-1} \sum_{j=1}^d \|S_j e_v\|^2 = 1.$$

One may argue as in the previous paragraph to see that $T_{\boldsymbol\lambda}$ is commuting if and only (ii) holds. We now define a diagonal operator D_c on $l^2(V)$ as follows:

$$D_c e_v := \sqrt{\mathfrak{C}(v)}\ e_v \quad \text{for any } v \in V. \tag{5.20}$$

Since $S_{\boldsymbol\lambda}$ is joint left invertible, by Proposition 3.1.7(viii), D_c is a diagonal, positive, invertible bounded linear operator. If (ii) holds then by the above argument $S_{\boldsymbol\lambda}$ has the decomposition given in (iii). Conversely, if (iii) holds then by the uniqueness of the decomposition, $T_{\boldsymbol\lambda}$ and D_c must be of the form as defined above. The desired conclusion in (ii) now follows from the commutativity of $T_{\boldsymbol\lambda}$. ∎

Remark 5.2.11. The spherical Cauchy dual $S^{\mathfrak{s}}_{\mathbf{w}}$ of a joint left invertible classical multishift $S_{\mathbf{w}}$ is the d-variable weighted shift given by

$$S^{\mathfrak{s}}_j e_\alpha = \frac{w^{(j)}_\alpha}{\delta_{\alpha, S_w}} e_{\alpha+\epsilon_j} \quad (1 \le j \le d),$$

where

$$\delta_{\alpha, S_{\mathbf{w}}} := \sum_{j=1}^d (w^{(j)}_\alpha)^2 \quad (\alpha \in \mathbb{N}^d).$$

It is easily seen that that $S^{\mathfrak{s}}_{\mathbf{w}}$ is commuting if and only if $\delta_{\alpha+\epsilon_j, S_{\mathbf{w}}} = \delta_{\alpha+\epsilon_k, S_{\mathbf{w}}}$ for all $1 \le j, k \le d$ [32, Section 6]. In this case, the function $\mathfrak{C}$ turns out to be a function of $|\alpha|$, $\alpha \in \mathbb{N}^d$ [33, Lemma 3.1]. This may also be deduced from the result above once we note that for any integer $t \ge 1$, one can order $\mathcal{G}_t$ as $\{u_1, \ldots, u_{\mathrm{card}(\mathcal{G}_t)}\}$ such that $\mathsf{Par}(u_i) \cap \mathsf{Par}(u_{i+1}) \neq \emptyset$ for every $i = 1, \ldots, \mathrm{card}(\mathcal{G}_t) - 1$.

We refer to $T_{\boldsymbol\lambda}$ and D_c as the *joint isometry part* and the *diagonal part* of the multishift $S_{\boldsymbol\lambda}$ respectively.

Corollary 5.2.12. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol\lambda}$ be a joint left invertible multishift on $\mathscr{T}$. If $S_{\boldsymbol\lambda}$ is spherically balanced, then for any $\beta \in \mathbb{N}^d$ and any $v \in V$, we have*

(i) $T^{\beta}_{\boldsymbol\lambda} e_v = \big(\prod_{p=0}^{|\beta|-1} \frac{1}{\sqrt{\mathfrak{C}_{|\mathsf{d}_v|+p}}}\big) S^{\beta}_{\boldsymbol\lambda} e_v$,

(ii) $\|T^{\beta}_{\boldsymbol\lambda} e_v\|^2 = \big(\prod_{p=0}^{|\beta|-1} \frac{1}{\mathfrak{C}_{|\mathsf{d}_v|+p}}\big) \|S^{\beta}_{\boldsymbol\lambda} e_v\|^2$,

where $\mathfrak{C}_t$ denotes the constant value of $\mathfrak{C}$ on the generation $\mathcal{G}_t$, and $T_{\boldsymbol\lambda}$ is the joint isometry part of $S_{\boldsymbol\lambda}$.

Proof. Assume that $S_{\boldsymbol\lambda}$ is spherically balanced. Note that the weights of $T_{\boldsymbol\lambda}$ take the form

$$\lambda^{(i)}_w / \sqrt{\mathfrak{C}_{|\mathsf{d}_v|}}, \qquad w \in \mathsf{Chi}_i(v),\ v \in V, \text{ and } i = 1, \ldots, d.$$

To see (i), let $v \in V$. Using induction on $k \in \mathbb{N}$, one can verify that for $i = 1, \ldots, d$,

$$T^k_i e_v = \prod_{p=0}^{k-1} \frac{1}{\sqrt{\mathfrak{C}_{|\mathsf{d}_v|+p}}} S^k_i e_v.$$

Further, in a similar fashion one can get

$$T^l_j T^k_i e_v = \prod_{p=0}^{k+l-1} \frac{1}{\sqrt{\mathfrak{C}_{|\mathsf{d}_v|+p}}} S^l_j S^k_i e_v.$$

Continuing, we obtain (i). Finally, (ii) is immediate from (i). ■

Note that part (ii) above gives a precise relation between the moments of $S_{\boldsymbol\lambda}$ and that of $T_{\boldsymbol\lambda}$. The multiplicative factor $\prod_{p=0}^{|\beta|-1} \frac{1}{\mathfrak{C}_{|\mathsf{d}_v|+p}}$ appearing in (ii) suggests introducing a classical unilateral weighted shift S_θ on some Hilbert space $H^2(\gamma)$ of formal power series, so that $\|T^{\beta}_{\boldsymbol\lambda} e_v\| = \|S^{|\beta|}_\theta f_0\| \, \|S^{\beta}_{\boldsymbol\lambda} e_v\|$ for some orthonormal basis $\{f_k\}_{k \in \mathbb{N}}$ of $H^2(\gamma)$. Unfortunately, in this formulation, the tree like structure of $\mathscr{T}$ does not reflect in the construction of S_θ on $H^2(\gamma)$. However, there is an alternative way to construct a shift on a directed tree arising naturally from $\mathscr{T}$ which at the same time gives the above relation between moments of $S_{\boldsymbol\lambda}$ and of $T_{\boldsymbol\lambda}$.

Definition 5.2.13. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Consider the component $\mathscr{T}^{\otimes}_{\mathsf{root}} = (V^{\otimes}, \mathcal{F})$ of the tensor product $\mathscr{T}^{\otimes}$ of $\mathscr{T}_1, \ldots, \mathscr{T}_d$, which contains root (see Theorem 2.2.3). For a bounded sequence $\{c_t\}_{t \in \mathbb{N}}$ of positive real numbers, consider the (one-variable) weighted shift S_θ on the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$ with weights given by

$$\theta_{\mathfrak{w}} := \sqrt{c_{\mathsf{d}_{\mathfrak{w}}-1}} / \sqrt{\mathrm{card}(\mathsf{sib}(\mathfrak{w}))} \qquad (\mathfrak{w} \in V^{\otimes} \setminus \mathsf{root}), \tag{5.21}$$

where $\mathsf{d}_{\mathfrak{w}}$ is the depth of $\mathfrak{w}$ in $\mathscr{T}^{\otimes}_{\mathsf{root}}$ and $\mathsf{sib}(\mathfrak{w})$ is the set of siblings of $\mathfrak{w}$ in the directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}}$. We refer to the weighted shift S_θ on $\mathscr{T}^{\otimes}_{\mathsf{root}}$ as the *balanced weighted shift associated with* $\{c_t\}_{t \in \mathbb{N}}$.

Example 5.2.14. If $\mathscr{T}_1 = \mathscr{T}_{1,0} = \mathscr{T}_2$, then as seen in Example 2.2.6, $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is (isomorphic to) $\mathscr{T}_{1,0}$, and hence S_θ is (unitarily equivalent to) the classical unilateral shift on $l^2(\mathbb{N})$.

If $\mathscr{T}_1 = \mathscr{T}_{2,0}, \mathscr{T}_2 = \mathscr{T}_{1,0}$, then as seen in Example 2.2.7, $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is $\mathscr{T}_{2,0}$, and hence S_θ is a weighted shift on the rooted directed tree $\mathscr{T}_{2,0}$ (see Figure 5.1).

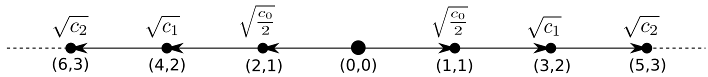

Fig. 5.1. The weights of S_θ on $\mathscr{T}^{\otimes}_{\mathsf{root}} = \mathscr{T}_{2,0}$

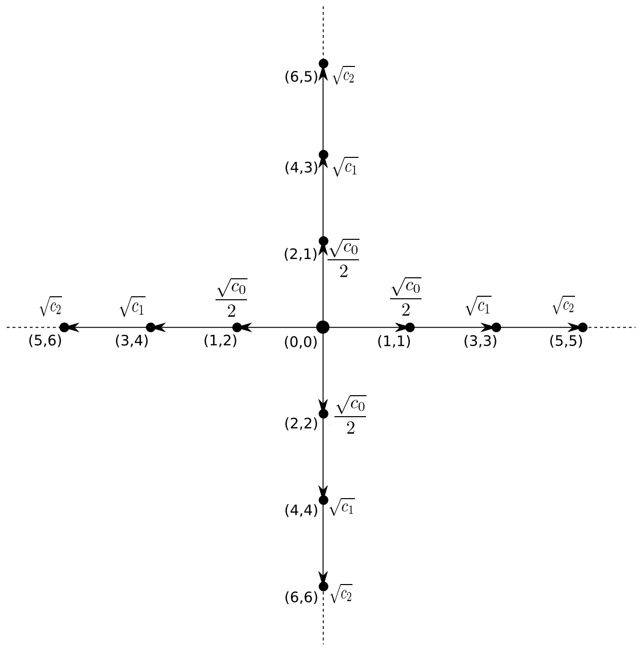

Fig. 5.2. The weights of S_θ on $\mathscr{T}^{\otimes}_{\mathsf{root}} = \mathscr{T}_{4,0}$

Finally, if $\mathscr{T}_1 = \mathscr{T}_{2,0} = \mathscr{T}_2$ then as seen in Example 2.2.8, $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is $\mathscr{T}_{4,0}$, and hence S_θ is a weighted shift on the rooted directed tree $\mathscr{T}_{4,0}$ (see Figure 5.2).

Before we present the main result of this section, recall that a finite positive Borel measure μ supported in the unit sphere $\partial\mathbb{B}^d$ in $\mathbb{C}^d$ is said to be $\mathbb{T}^d$-*invariant* if for any Borel measurable subset Δ of $\partial\mathbb{B}^d$,

$$\mu(\zeta \cdot \Delta) = \mu(\Delta) \quad \text{for all } \zeta \in \mathbb{T}^d,$$

where $\zeta \cdot \Delta = \{\zeta \cdot z : z \in \Delta\}$ with $\zeta \cdot z$ denoting the dot product of ζ and z. A *Reinhardt measure* is a $\mathbb{T}^d$-invariant probability measure on the unit sphere.

The following generalizes [33, Theorem 1.10] and [77, Theorem 1.3] (the case in which $\mathscr{T} = \mathscr{T}^d_{1,0}$, as described in Example 2.1.5, with $v = \mathsf{root}$).

THEOREM 5.2.15. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $\mathscr{T}^{\otimes}_{\mathsf{root}} = (V^{\otimes}, \mathcal{F})$ be the component of $\mathscr{T}^{\otimes}$ containing* root. *For $v \in V$, choose $\mathfrak{v} \in V^{\otimes}$ so that $|\mathsf{d}_v| = \mathsf{d}_{\mathfrak{v}}$ (see the last part of Theorem 2.2.3). If $S_{\boldsymbol{\lambda}}$ is a joint left invertible multishift on $\mathscr{T}$, then the following statements are equivalent:*

(i) *The multishift $S_{\boldsymbol{\lambda}}$ is spherically balanced.*

(ii) *For every $v \in V$, there exists a Reinhardt measure μ_v supported in the unit sphere $\partial\mathbb{B}^d$ and a balanced weighted shift S_θ on $\mathscr{T}^{\otimes}_{\mathsf{root}}$ associated with a bounded sequence $\{c_t\}_{t\in\mathbb{N}}$ such that*

$$\|f_k\|^2_{l^2(V)} = \int_{\partial\mathbb{B}^d} \|f_{\theta,k}(z)\|^2_{l^2(V^\otimes)}\, d\mu_v(z), \tag{5.22}$$

where

$$f_k = \sum_{\substack{\beta\in\mathbb{N}^d\\ |\beta|\le k}} a_\beta S^\beta_{\boldsymbol{\lambda}} e_v \in l^2(V),$$

$$f_{\theta,k}(z) = \sum_{\substack{\beta\in\mathbb{N}^d\\ |\beta|\le k}} a_\beta z^\beta S^{|\beta|}_\theta e_{\mathfrak{v}} \in l^2(V^\otimes)\ (k\in\mathbb{N}, z\in\mathbb{C}^d).$$

If any of the above equivalent statements holds, then:

(a) *If $\{f_k\}_{k\in\mathbb{N}}$ converges to f in $l^2(V)$, then $\{f_{\theta,k}(z)\}_{k\in\mathbb{N}}$ converges to some $f_\theta(z)$ in $l^2(V^\otimes)$ for μ_v-a.e. $z\in\partial\mathbb{B}^d$, and*

$$\|f\|^2_{l^2(V)} = \int_{\partial\mathbb{B}^d} \|f_\theta(z)\|^2_{l^2(V^\otimes)}\, d\mu_v(z). \tag{5.23}$$

(b) *If $Q^n_T(I)$ is as defined in (1.1), then for $n\in\mathbb{N}$,*

$$\langle Q^n_{S_{\boldsymbol{\lambda}}}(I)e_v, e_v\rangle = \|S^n_\theta e_{\mathfrak{v}}\|^2, \tag{5.24}$$

$$\|Q^n_{S_{\boldsymbol{\lambda}}}(I)\| = \|S^n_\theta\|^2. \tag{5.25}$$

Proof. Suppose that $S_{\boldsymbol{\lambda}}$ is a joint left invertible multishift on $\mathscr{T}$. To see (i)⇒(ii), assume that $S_{\boldsymbol{\lambda}}$ is spherically balanced and let $c_t := \mathfrak{C}_t$, the constant value of $\mathfrak{C}$ on the generation $\mathcal{G}_t$ for $t\in\mathbb{N}$. Consider the balanced weighted shift S_θ on $\mathscr{T}^{\otimes}_{\mathsf{root}}$ associated with $\{c_t\}_{t\in\mathbb{N}}$. We first prove the formula

$$\|S^k_\theta e_{\mathfrak{v}}\|^2 = \prod_{p=0}^{k-1} c_{\mathsf{d}_{\mathfrak{v}}+p} \qquad (k\ge 1,\ \mathfrak{v}\in V^\otimes) \tag{5.26}$$

by induction on $k\ge 1$. If $k=1$, then

$$\|S_\theta e_{\mathfrak{v}}\|^2 = \sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \theta^2_{\mathfrak{u}} \overset{(5.21)}{=} \sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \frac{c_{\mathsf{d}_{\mathfrak{u}}-1}}{\operatorname{card}(\mathsf{sib}(\mathfrak{u}))} = c_{\mathsf{d}_{\mathfrak{v}}} \sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \frac{1}{\operatorname{card}(\mathsf{sib}(\mathfrak{u}))} = c_{\mathsf{d}_{\mathfrak{v}}}.$$

Assume the formula holds for some $k\ge 1$. By the induction hypothesis, we obtain

$$\|S^{k+1}_\theta e_{\mathfrak{v}}\|^2 = \Big\|\sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \theta_{\mathfrak{u}} S^k_\theta e_{\mathfrak{u}}\Big\|^2 = \sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \theta^2_{\mathfrak{u}} \|S^k_\theta e_{\mathfrak{u}}\|^2 = \sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \frac{c_{\mathsf{d}_{\mathfrak{u}}-1}}{\operatorname{card}(\mathsf{sib}(\mathfrak{u}))}\Big(\prod_{p=0}^{k-1} c_{\mathsf{d}_{\mathfrak{u}}+p}\Big)$$

$$= c_{\mathsf{d}_{\mathfrak{v}}}\Big(\prod_{p=0}^{k-1} c_{\mathsf{d}_{\mathfrak{v}}+1+p}\Big) \sum_{\mathfrak{u}\in\mathsf{Chi}(\mathfrak{v})} \frac{1}{\operatorname{card}(\mathsf{sib}(\mathfrak{u}))} = \prod_{p=0}^{k} c_{\mathsf{d}_{\mathfrak{v}}+p}.$$

This completes the induction. We now verify that there exists a Reinhardt measure μ_v supported in the unit sphere $\partial\mathbb{B}^d$ such that

$$\|S^\beta_{\boldsymbol{\lambda}} e_v\|^2_{l^2(V)} = \|S^{|\beta|}_\theta e_{\mathfrak{v}}\|^2_{l^2(V^\otimes)} \int_{\partial\mathbb{B}^d} |z^\beta|^2\, d\mu_v, \qquad \beta\in\mathbb{N}^d. \tag{5.27}$$

In view of Corollary 5.2.12(ii), it suffices to find a Reinhardt measure μ_v supported in the unit sphere $\partial\mathbb{B}^d$ such that $\|T_{\boldsymbol\lambda}^{\beta}e_v\|^2 = \int_{\partial\mathbb{B}^d}|z^\beta|^2 d\mu_v$, where $T_{\boldsymbol\lambda}$ is the joint isometry part of $S_{\boldsymbol\lambda}$. Consider the $T_{\boldsymbol\lambda}$-invariant subspace $\mathcal{M} := \bigvee\{T_{\boldsymbol\lambda}^{\beta}e_v : \beta \in \mathbb{N}^d\}$ of $l^2(V)$. By Proposition 3.1.7(ix), $\langle T_{\boldsymbol\lambda}^{\beta}e_v, T_{\boldsymbol\lambda}^{\gamma}e_v\rangle = 0$ if $\beta \neq \gamma$. It follows that $T_{\boldsymbol\lambda}|_{\mathcal{M}}$ is a joint isometry classical multishift (up to unitary equivalence). The existence of the desired measure is now a consequence of the well-known fact that any joint isometry is joint subnormal [19, Proposition 2] (see [33, proof of Theorem 1.10] for more details).

We now check the integral representation appearing in (5.22). Note that $\langle S_{\boldsymbol\lambda}^{\beta}e_v, S_{\boldsymbol\lambda}^{\gamma}e_v\rangle = 0$ if $\beta \neq \gamma$ and $\langle S_\theta^k e_{\mathfrak{v}}, S_\theta^l e_{\mathfrak{v}}\rangle = 0$ if $k \neq l$ (see Proposition 3.1.7(ix)). It follows that

$$\begin{aligned}\|f_k\|^2_{l^2(V)} &= \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}} |a_\beta|^2\|S_{\boldsymbol\lambda}^{\beta}e_v\|^2_{l^2(V)}\\ &\overset{(5.27)}{=} \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}} |a_\beta|^2\|S_\theta^{|\beta|}e_{\mathfrak{v}}\|^2_{l^2(V^{\otimes})}\int_{\partial\mathbb{B}^d}|z^\beta|^2\,d\mu_v. \end{aligned}\tag{5.28}$$

Since μ_v is a Reinhardt measure, by [33, Lemma 2.3] the monomials $\{z^\alpha\}_{\alpha\in\mathbb{N}^d}$ are orthogonal in $L^2(\partial\mathbb{B}^d, \mu_v)$. Thus

$$\begin{aligned}\int_{\partial\mathbb{B}^d}\|f_{\theta,k}(z)\|^2_{l^2(V^{\otimes})}d\mu_v(z) &= \int_{\partial\mathbb{B}^d}\sum_{\substack{\beta,\gamma\in\mathbb{N}^d\\|\beta|,|\gamma|\le k}} a_\beta\overline{a}_\gamma\langle S_\theta^{|\beta|}e_{\mathfrak{v}}, S_\theta^{|\gamma|}e_{\mathfrak{v}}\rangle z^\beta\overline{z}^\gamma d\mu_v(z)\\ &= \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}}|a_\beta|^2\|S_\theta^{|\beta|}e_{\mathfrak{v}}\|^2_{l^2(V^{\otimes})}\int_{\partial\mathbb{B}^d}|z^\beta|^2 d\mu_v(z) \overset{(5.28)}{=} \|f_k\|^2_{l^2(V)}.\end{aligned}$$

This proves that (i) implies (ii). To see that (ii) implies (i), let $f_k = S_j e_v$ and $f_{\theta,k} = z_j S_\theta e_{\mathfrak{v}}$ in 5.22 and sum over $j = 1, \ldots, d$ to see that

$$\sum_{j=1}^d \|S_j e_v\|^2_{l^2(V)} = \|S_\theta e_{\mathfrak{v}}\|^2_{l^2(V^{\otimes})}\sum_{j=1}^d\int_{\partial\mathbb{B}^d}|z_j|^2 d\mu_v = \|S_\theta e_{\mathfrak{v}}\|^2_{l^2(V^{\otimes})} = c_{\mathsf{d}_{\mathfrak{v}}} = c_{|\mathsf{d}_v|},$$

which is constant on $\mathcal{G}_{|\mathsf{d}_v|}$.

In the remaining part of the proof, we assume that $S_{\boldsymbol\lambda}$ is spherically balanced.

(a) Note that $\|f_k\|_{l^2(V)} \uparrow \|f\|_{l^2(V)}$ and $\|f_{\theta,k}(z)\|_{l^2(V^{\otimes})} \uparrow g(z)$ (possibly in the extended real line) as $k \to \infty$, where

$$g(z) := \Big(\sum_{\beta\in\mathbb{N}^d}|a_\beta|^2|z^\beta|^2\|S_\theta^{|\beta|}e_{\mathfrak{v}}\|^2\Big)^{1/2} \quad (z \in \partial\mathbb{B}^d).$$

Applying the monotone convergence theorem to (5.22), we obtain

$$\|f\|^2_{l^2(V)} = \int_{\partial\mathbb{B}^d} g(z)^2\,d\mu_v(z).$$

Since the left hand side is a finite positive number, $0 \le g(z) < \infty$ except for z in a set of μ_v measure 0. It follows that $f_\theta(z) := \sum_{\beta\in\mathbb{N}^d} a_\beta z^\beta S_\theta^{|\beta|}e_{\mathfrak{v}} \in l^2(V^{\otimes})$ except for z in a set of μ_v measure 0 and $g(z) = \|f_\theta(z)\|_{l^2(V^{\otimes})}$. This also yields (5.23).

(b) Note that

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \frac{n!}{\alpha!}\|S_{\boldsymbol\lambda}^\alpha e_v\|^2 \overset{(5.27)}{=} \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \frac{n!}{\alpha!}\|S_\theta^{|\alpha|} e_{\mathfrak{v}}\|^2_{l^2(V^\otimes)} \int_{\partial\mathbb{B}^d} |z^\alpha|^2 d\mu_v$$
$$= \|S_\theta^n e_{\mathfrak{v}}\|^2_{l^2(V^\otimes)} \int_{\partial\mathbb{B}^d} \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \frac{n!}{\alpha!}|z^\alpha|^2 d\mu_v,$$

which is the same as $\|S_\theta^n e_{\mathfrak{v}}\|^2_{l^2(V^\otimes)}$ since

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=n}} \frac{n!}{\alpha!}|z^\alpha|^2 = \|z\|_2^{2n} \qquad (z\in\mathbb{C}^d)$$

and μ is a probability measure supported in the unit sphere. It follows that $\langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle = \|S_\theta^n e_{\mathfrak{v}}\|^2$. Note further that

$$\|Q^n_{S_{\boldsymbol\lambda}}(I)\| = \sup_{v\in V}\langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle = \sup_{\mathfrak{v}\in V^\otimes}\|S_\theta^n e_{\mathfrak{v}}\|^2 = \|S_\theta^n\|^2.$$

This completes the verification of (b). ■

For convenience, we refer to the balanced weighted shift S_θ on $\mathscr{T}^\otimes_{\mathsf{root}}$ as the *shift associated with the multishift $S_{\boldsymbol\lambda}$ on $\mathscr{T}$* (cf. [36, Definition 2.3]).

Here we discuss some consequences of the preceding theorem. The first one is a local spherical analog of von Neumann's inequality (cf. [77, Proposition 2.5] and [73, Theorem 7.6]).

Corollary 5.2.16. *Let $\mathscr{T} = (V,\mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1,\ldots,\mathscr{T}_d$. Let $S_{\boldsymbol\lambda}$ be a joint left invertible, spherically balanced multishift on $\mathscr{T}$. If $S_{\boldsymbol\lambda}$ is a joint contraction, then for any positive integer k and finite sequence $\{a_\beta : \beta\in\mathbb{N}^d,\ |\beta|\le k\}$, we have*

$$\sup_{v\in V}\Big\|\sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}} a_\beta S_{\boldsymbol\lambda}^\beta e_v\Big\| \le \sup_{z\in\mathbb{B}^d}\Big|\sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}} a_\beta z^\beta\Big|.$$

Proof. Suppose that $S_{\boldsymbol\lambda}$ is a joint contraction. Let S_θ be the balanced weighted shift associated with $S_{\boldsymbol\lambda}$ and fix $v\in V$. Note that by (5.25), S_θ is a contraction. On the other hand, by the preceding theorem, there exists a Reinhardt measure μ_v supported in $\partial\mathbb{B}^d$ such that

$$\Big\|\sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}} a_\beta S_{\boldsymbol\lambda}^\beta e_v\Big\|^2_{l^2(V)} \overset{(5.22)}{=} \int_{\partial\mathbb{B}^d}\Big\|\sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le k}} a_\beta z^\beta S_\theta^{|\beta|} e_{\mathfrak{v}}\Big\|^2_{l^2(V^\otimes)} d\mu_v(z)$$
$$= \int_{\partial\mathbb{B}^d}\Big\|\sum_{l=0}^k \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|=l}} a_\beta z^\beta S_\theta^l e_{\mathfrak{v}}\Big\|^2_{l^2(V^\otimes)} d\mu_v(z)$$
$$= \int_{\partial\mathbb{B}^d}\sum_{l=0}^k \Big|\sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|=l}} a_\beta z^\beta\Big|^2 \|S_\theta^l e_{\mathfrak{v}}\|^2_{l^2(V^\otimes)}\, d\mu_v(z)$$

$$\leq \int_{\partial\mathbb{B}^d} \sum_{l=0}^{k} \Big| \sum_{\substack{\beta\in\mathbb{N}^d \\ |\beta|=l}} a_\beta z^\beta \Big|^2 d\mu_v(z) = \int_{\partial\mathbb{B}^d} \Big| \sum_{\substack{\beta\in\mathbb{N}^d \\ |\beta|\leq k}} a_\beta z^\beta \Big|^2 d\mu_v(z)$$
$$\leq \sup_{z\in\partial\mathbb{B}^d} \Big| \sum_{\substack{\beta\in\mathbb{N}^d \\ |\beta|\leq k}} a_\beta z^\beta \Big|,$$

where we have used the orthogonality of monomials in $L^2(\partial\mathbb{B}^d, \mu_v)$. After taking the supremum over $v \in V$, the desired conclusion follows from the maximum modulus principle in several complex variables [85, p. 5]. ∎

COROLLARY 5.2.17. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a joint left invertible, spherically balanced multishift on $\mathscr{T}$ and let S_θ be the weighted shift on the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}} = (V^{\otimes}, \mathcal{F})$ associated with $S_{\boldsymbol\lambda}$. Let $\mathfrak{C}_t$ be the constant value of $\sum_{j=1}^d \|S_j e_v\|^2$ on the generation $\mathcal{G}_t$ of $\mathscr{T}$. Then:*

(i) $r(S_{\boldsymbol\lambda}) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} (\prod_{p=0}^{n-1} \mathfrak{C}_{k+p})^{1/(2n)} = r(S_\theta)$, *where $r(T)$ denotes the spectral radius of any commuting d-tuple T. In particular, the Taylor spectrum of $S_{\boldsymbol\lambda}$ is a Reinhardt set containing* 0 *and contained in the closed ball centered at the origin and of radius $r(S_\theta)$.*

(ii) $m_\infty(S_{\boldsymbol\lambda}) = \sup_{n\geq 1} \inf_{k\in\mathbb{N}} (\prod_{p=0}^{n-1} \mathfrak{C}_{k+p})^{1/(2n)} = m_\infty(S_\theta)$. *In particular, the left spectrum $\sigma_l(S_{\boldsymbol\lambda})$ of $S_{\boldsymbol\lambda}$ is contained in the closed ball shell centered at the origin with inner radius $m_\infty(S_\theta)$ and outer radius $r(S_\theta)$.*

Proof. The first part follows from (5.25), (5.26), and the spectral radius formula (1.5) for the Taylor spectrum:

$$r(S_{\boldsymbol\lambda}) = \lim_{n\to\infty} \|Q^n_{S_{\boldsymbol\lambda}}(I)\|^{1/(2n)} = \lim_{n\to\infty} \|S^n_\theta\|^{1/n} = r(S_\theta).$$

To see (ii), note that by (1.6),

$$m_\infty(S_{\boldsymbol\lambda}) \leq \sup_{n\geq 1} \inf_{v\in V} \langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle^{1/(2n)}.$$

Let $M_n := \inf_{v\in V} \langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle^{1/(2n)}$ for $n \geq 1$. Then for any $f = \sum_{v\in V} f(v)e_v \in l^2(V)$ of unit norm, by Proposition 3.1.7(vi)&(x),

$$\langle Q^n_{S_{\boldsymbol\lambda}}(I)f, f\rangle^{1/(2n)} = \Big(\sum_{v\in V} |f(v)|^2 \langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle\Big)^{1/(2n)} \geq M_n,$$

and hence $m_\infty(S_{\boldsymbol\lambda}) = \sup_{n\geq 1} \inf_{v\in V} \langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle^{1/(2n)}$. A similar observation holds for S_θ. The desired conclusion in (ii) may now be drawn from (5.24) and (5.26). ∎

EXAMPLE 5.2.18. Consider the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ as discussed in Example 5.2.5. Recall that $\mathfrak{c}_a = \{c_{a,t}\}_{t\in\mathbb{N}}$ is given by

$$c_{a,t} = \frac{t+d}{t+a} \quad (t \in \mathbb{N}).$$

Thus weights of $S_{\boldsymbol\lambda_{\mathfrak{c}_a}} = (S_1, \ldots, S_d)$ are given by

$$\lambda^{(j)}_w = \frac{1}{\sqrt{\mathrm{card}(\mathsf{Chi}_j(v))}} \sqrt{\frac{\mathsf{d}_{v_j} + 1}{|\mathsf{d}_v| + a}} \quad \text{for } w \in \mathsf{Chi}_j(v) \text{ and } j = 1, \ldots, d. \tag{5.29}$$

By the preceding corollary,

$$r(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \Big(\prod_{p=0}^{n-1} c_{a,k+p}\Big)^{1/(2n)} = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \Big(\prod_{p=0}^{n-1} \frac{k+p+d}{k+p+a}\Big)^{1/(2n)}.$$

We follow the argument in [36, Lemma 3.9] to see that $r(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) = 1$. Let $F(n,k) = \prod_{p=0}^{n-1} \frac{k+p+d}{k+p+a}$ ($k \in \mathbb{N}, n \geq 1$), and note that $F(n,k)$ is increasing (resp. decreasing) in k if and only if $a \geq d$ (resp. $a \leq d$). Thus the following possibilities occur:

$$\prod_{p=0}^{n-1} \Big(\frac{p+d}{p+a}\Big)^{1/(2n)} = F(n,0)^{1/(2n)} \leq F(n,k)^{1/(2n)} \leq 1 \quad \text{or}$$
$$1 \leq F(n,k)^{1/(2n)} \leq F(n,0)^{1/(2n)} = \prod_{p=0}^{n-1} \Big(\frac{p+d}{p+a}\Big)^{1/(2n)}.$$

In either case, $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \big(\prod_{p=0}^{n-1} \frac{k+p+d}{k+p+a}\big)^{1/(2n)} = 1$. This shows that $r(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) = 1$. One can argue similarly to see that $m_\infty(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) = 1$. In particular,

$$\sigma(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) \subseteq \mathrm{cl}(\mathbb{B}^d), \qquad \sigma_l(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) \subseteq \partial\mathbb{B}^d. \tag{5.30}$$

Finally, by Corollary 5.2.9, we must have $\sigma(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) = \mathrm{cl}(\mathbb{B}^d)$.

We conclude this section with a brief discussion on the essential spectrum of the multishift $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$. Recall first that a commuting d-tuple T is *essentially normal* if $[T_i^*, T_j] = T_i^*T_j - T_jT_i^*$ is compact for every $i, j = 1, \ldots, d$.

PROPOSITION 5.2.19. *Let $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ be a multishift as in Example* 5.2.5. *If $\mathscr{T}$ is of finite joint branching index $k_{\mathscr{T}}$, then $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} = (S_1, \ldots, S_d)$ is essentially normal.*

Proof. Assume that $\mathscr{T}$ is of finite joint branching index $k_{\mathscr{T}}$. By the Putnam–Fuglede Theorem [38] (applied to the image of S_j under the Calkin map), the compactness of the commutators $[S_j^*, S_j]$ ($1 \leq j \leq d$) implies that of the cross-commutators $[S_j^*, S_k]$ ($1 \leq j, k \leq d, j \neq k$). Thus it suffices to check that $[S_j^*, S_j]$ is compact for every $j = 1, \ldots, d$. For fixed $j = 1, \ldots, d$, define

$$W_j := \{v \in V : \mathrm{card}(\mathsf{Chi}_j(v)) = 1 \text{ and } \mathrm{card}(\mathsf{sib}_j(v)) = 1\}.$$

Note that $[S_j^*, S_j]$ decomposes into $A_j \oplus B_j$ on $l^2(V) = l^2(W_j) \oplus l^2(V \setminus W_j)$, where A_j, B_j are block diagonal operators given by

$$A_j e_v = ((\lambda_w^{(j)})^2 - (\lambda_v^{(j)})^2) e_v \quad (v \in W_j, \mathsf{Chi}_j(v) = \{w\}),$$
$$B_j e_v = \sum_{w\in\mathsf{Chi}_j(v)} (\lambda_w^{(j)})^2 e_v - \sum_{u\in\mathsf{sib}_j(v)} \lambda_v^{(j)}\lambda_u^{(j)} e_u \quad (v \in V \setminus W_j).$$

By (5.29), A_j is the diagonal operator with diagonal entries

$$(\lambda_w^{(j)})^2 - (\lambda_v^{(j)})^2 = \frac{|\mathsf{d}_v| - \mathsf{d}_{v_j} + a - 1}{(|\mathsf{d}_v| + a)(|\mathsf{d}_v| + a - 1)},$$

which tends to 0 as $|\mathsf{d}_v| \to \infty$. This shows that A_j is compact. To see that B_j is compact, note first that

$$\begin{aligned} B_j e_v &\overset{(5.29)}{=} \sum_{u \in \mathsf{Chi}_j(v)} \frac{1}{\mathrm{card}(\mathsf{Chi}_j(v))} \frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+a}\, e_v \\ &\quad - \sum_{w \in \mathsf{sib}_j(v)} \frac{1}{\mathrm{card}(\mathsf{sib}_j(v))} \frac{\mathsf{d}_{v_j}}{|\mathsf{d}_v|+a-1}\, e_w \\ &= \frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+a}\, e_v - \frac{1}{\mathrm{card}(\mathsf{sib}_j(v))} \frac{\mathsf{d}_{v_j}}{|\mathsf{d}_v|+a-1} \sum_{w \in \mathsf{sib}_j(v)} e_w. \end{aligned} \tag{5.31}$$

We next decompose $V \setminus W_j$ as $\bigsqcup_{v \in \Omega} \mathsf{sib}_j(v)$, where Ω is formed by picking up only one element from every $\mathsf{sib}_j(v)$ for $j = 1, \ldots, d$. The existence of Ω is ensured by the axiom of choice. Note that $l^2(\mathsf{sib}_j(v))$ is reducing for B_j for every $v \in \Omega$. This immediately yields the decomposition

$$B_j = \bigoplus_{v \in \Omega} B_{jv} \quad \text{on } l^2(V \setminus W_j) = \bigoplus_{v \in \Omega} l^2(\mathsf{sib}_j(v)),$$

where B_{jv} is a finite rank operator (since $\mathscr{T}_j$ is locally finite) for $j = 1, \ldots, d$. It now suffices to check that $\|B_{jv}\| \to 0$ as $|\mathsf{d}_v| \to \infty$ (see Remark 3.4.1). Before proceeding to this end, observe that

$$\sup_{v \in V \setminus W_j} \mathrm{card}(\mathsf{sib}_j(v)) \le M_j := \mathrm{card}(\mathsf{Chi}^{\langle k_{\mathscr{T}_j} \rangle}(\mathsf{root}_j)) < \infty, \tag{5.32}$$

$$\sup_{v \in V \setminus W_j} \mathsf{d}_{v_j} \le k_{\mathscr{T}_j}. \tag{5.33}$$

Let $f = \sum_{u \in \mathsf{sib}_j(v)} f(u) e_u \in l^2(\mathsf{sib}_j(v))$ and $\Upsilon_v := \sum_{u \in \mathsf{sib}_j(v)} f(u)$. Note that

$$\begin{aligned} B_{jv} f &\overset{(5.31)}{=} \sum_{u \in \mathsf{sib}_j(v)} f(u) \Big(\frac{\mathsf{d}_{u_j}+1}{|\mathsf{d}_u|+a}\, e_u - \frac{1}{\mathrm{card}(\mathsf{sib}_j(u))} \frac{\mathsf{d}_{u_j}}{|\mathsf{d}_u|+a-1} \sum_{w \in \mathsf{sib}_j(u)} e_w \Big) \\ &= \frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+a} \sum_{u \in \mathsf{sib}_j(v)} f(u) e_u - \frac{\Upsilon_v}{\mathrm{card}(\mathsf{sib}_j(v))} \frac{\mathsf{d}_{v_j}}{|\mathsf{d}_v|+a-1} \Big(\sum_{w \in \mathsf{sib}_j(v)} e_w \Big) \\ &= \sum_{u \in \mathsf{sib}_j(v)} \beta_u e_u, \end{aligned}$$

where β_u is given by

$$\beta_u = \frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+a} f(u) - \frac{\Upsilon_v}{\mathrm{card}(\mathsf{sib}_j(v))} \frac{\mathsf{d}_{v_j}}{|\mathsf{d}_v|+a-1}.$$

Since $|\Upsilon_v| \le \|f\| M_j$, by (5.33) and the Cauchy–Schwarz inequality,

$$\begin{aligned} |\beta_u| &\le \frac{(k_{\mathscr{T}_j}+1)|f(u)|}{|\mathsf{d}_v|+a} + \frac{|\Upsilon_v|}{\mathrm{card}(\mathsf{sib}_j(v))} \frac{k_{\mathscr{T}_j}}{|\mathsf{d}_v|+a-1} \\ &\le \frac{k_{\mathscr{T}_j}+1}{|\mathsf{d}_v|+a-1} \Big(1 + \frac{M_j}{\mathrm{card}(\mathsf{sib}_j(v))} \Big) \|f\|. \end{aligned}$$

It follows from (5.32) that

$$\|B_{jv} f\|^2 = \sum_{u \in \mathsf{sib}_j(v)} |\beta_u|^2 \le \frac{(k_{\mathscr{T}_j}+1)^2 (1+M_j)^3}{(|\mathsf{d}_v|+a-1)^2} \|f\|^2.$$

This shows that $\|B_{jv}\| \to 0$ as $|\mathsf{d}_v| \to \infty$. ∎

The conclusion of the preceding proposition does not hold true if we relax the assumption of finite joint branching index.

EXAMPLE 5.2.20. Consider the *n-ary tree* $\mathscr{T}^{(n)}$ given by

$$V^{(n)} = \{v_{k,l} : k \in \mathbb{N},\, l = 1, \ldots, 2^k\}, \quad \mathsf{Chi}(v_{k,l}) = \{v_{k+1,j} : n(l-1)+1 \leq j \leq nl\}.$$

Let $\mathscr{T} = (V, \mathcal{E})$ denote the directed product of $\mathscr{T}^{(n)}$ with itself. Note that for $v_{k,l}, v_{p,q} \in V^{(n)}$,

$$\mathsf{Chi}_1((v_{k,l}, v_{p,q})) = \{(v_{k+1,j}, v_{p,q}) \in V : n(l-1)+1 \leq j \leq nl\},$$
$$\mathsf{Chi}_2((v_{k,l}, v_{p,q})) = \{(v_{k,l}, v_{p+1,j}) \in V : n(q-1)+1 \leq j \leq nq\},$$

so that $\mathrm{card}(\mathsf{Chi}_j((v_{k,l}, v_{p,q}))) = n$ for $j = 1, 2$. Let $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ be as in Example 5.2.5 with $d = 2$. Note that the system $\boldsymbol{\lambda}$ is given by

$$\lambda_w^{(1)} = \frac{1}{\sqrt{n}}\sqrt{\frac{k+1}{k+p+a}}, \quad w \in \mathsf{Chi}_1((v_{k,l}, v_{p,q})),$$

$$\lambda_w^{(2)} = \frac{1}{\sqrt{n}}\sqrt{\frac{p+1}{k+p+a}}, \quad w \in \mathsf{Chi}_2((v_{k,l}, v_{p,q})).$$

We claim that $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ on $\mathscr{T}^{(n)}$ is essentially normal if and only if $n = 1$. In case $n = 1$, $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ are classical multishifts. The essential normality in this case is well-known (see, for example, [36]). To see the converse, assume that $n \geq 2$. Let $B_j := [S_j^*, S_j]$ for $j = 1, 2$. It suffices to check that $\|B_1 e_{(v_{k,l}, v_{p,q})}\| \nrightarrow 0$ as $k = p \to \infty$. For $v = (v_{k,l}, v_{p,q})$, note that

$$\begin{aligned}\langle B_1 e_v, e_v\rangle &= \sum_{u \in \mathsf{Chi}_1(v)} \frac{1}{n}\frac{k+1}{k+p+a} - \sum_{w \in \mathsf{sib}_j(v)} \frac{1}{n}\frac{k}{k+p+a-1}\langle e_w, e_v\rangle \\ &= \frac{k+1}{k+p+a} - \frac{1}{n}\frac{k}{k+p+a-1},\end{aligned}$$

which converges to $\frac{1}{2}(1 - 1/n)$ as $k = p \to \infty$.

COROLLARY 5.2.21. *Let $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ be the multishift as in Example* 5.2.5. *Assume that $\mathscr{T}$ is of finite joint branching index $k_{\mathscr{T}}$. Then*

$$\sigma_e(S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}) \subseteq \partial\mathbb{B}^d.$$

Proof. It may be concluded from [36, proof of Lemma 3.5] that for any essentially normal d-tuple T, the essential spectrum $\sigma_e(T)$ is contained in

$$\{w \in \mathbb{C}^d : \|w\|_2^2 \in \sigma_e(Q_T(I))\},$$

where $Q_T(\cdot)$ is as defined in (1.1). In view of the last result, it now suffices to check that $\sigma_e(Q_{S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}}(I))$ is equal to $\{1\}$. However, $Q_{S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}}(I)$ is the diagonal operator with diagonal entries $(|\mathsf{d}_v| + d)/(|\mathsf{d}_v| + a)$ (repeated $\mathrm{card}(\mathsf{Chi}^{\langle\langle \mathsf{d}_v \rangle\rangle}(\mathsf{root}))$ times) for $v \in V$. Since the only limit point of these eigenvalues of $Q_{S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}}(I)$ is 1, the essential spectrum of $Q_{S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}}(I)$ must be $\{1\}$ [38]. ■

REMARK 5.2.22. Since the point spectrum of $S_{\boldsymbol{\lambda}}$ is empty (Corollary 3.3.4), in dimension $d = 2$, the dimension of the cohomology group at the middle stage in the Koszul complex of $S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}} - \omega$ is constant for every $\omega \in \mathbb{B}^d$ (see (8)).

5.3. Joint subnormal multishifts. We begin this section with a simple characterization of joint subnormal multishifts in terms of complete monotonicity of their moments.

PROPOSITION 5.3.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda}$ be a toral contractive multishift on $\mathscr{T}$. Then $S_{\boldsymbol\lambda}$ is joint subnormal if and only if for every $v \in V$, the multisequence $\{\|S_{\boldsymbol\lambda}^{\alpha} e_v\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone.*

Proof. As recorded earlier, by [17, Theorem 4.4], a toral contractive d-tuple T on a complex Hilbert space $\mathcal{H}$ is joint subnormal if and only if for every $h \in \mathcal{H}$, the multisequence $\{\|T^{\alpha}h\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone. Since $\{S_{\boldsymbol\lambda}^{\alpha} e_v\}_{v\in V}$ is mutually orthogonal (Proposition 3.1.7(x)), for $f = \sum_{v\in V} f(v)e_v$,

$$\|S_{\boldsymbol\lambda}^{\alpha} f\|^2 = \sum_{v\in V} |f(v)|^2 \|S_{\boldsymbol\lambda}^{\alpha} e_v\|^2.$$

By the general theory [25, Chapter 4], we conclude that $S_{\boldsymbol\lambda}$ is joint subnormal if and only if for every $v \in V$, $\{\|S_{\boldsymbol\lambda}^{\alpha} e_v\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone. ∎

Although the preceding result characterizes all joint subnormal contractive multishifts on $\mathscr{T}$, the necessary and sufficient conditions include information about moments at all vertices. On the other hand, information about the moment at a single vertex (namely, $\{\|S_{\boldsymbol\lambda}^{\alpha} e_{\mathsf{root}}\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone) is sufficient to ensure joint subnormality in the context of classical multishifts. Thus a natural question arises whether joint subnormality of $S_{\boldsymbol\lambda}$ can be recovered from complete monotonicity at finitely many vertices. This question has an affirmative answer in case each $\mathscr{T}_j$ is locally finite with finite branching index.

THEOREM 5.3.2. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda}$ be a toral contractive multishift on $\mathscr{T}$. Let*

$$W := \bigcup_{\substack{\alpha\in\mathbb{N}^d \\ \alpha\leq k_{\mathscr{T}}}} \mathsf{Chi}^{\langle\!\langle\alpha\rangle\!\rangle}(\mathsf{root})$$

and let $\tilde{W} := W_1 \times \cdots \times W_d$, where

$$W_j := \mathsf{Chi}(V_{\prec}^{(j)}) \cup \{\mathsf{root}_j\}, \qquad j = 1, \ldots, d.$$

Then the following statements are equivalent:

(i) *$S_{\boldsymbol\lambda}$ is joint subnormal.*
(ii) *For every $v \in W$, $\{\|S_{\boldsymbol\lambda}^{\alpha} e_v\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone.*
(iii) *For every $v \in \tilde{W}$, $\{\|S_{\boldsymbol\lambda}^{\alpha} e_v\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone.*

Proof. The implication (i)⇒(ii) is clear from the previous result while (ii)⇒(iii) is obvious in view of the inclusion $\tilde{W} \subseteq W$. Let us check (ii)⇒(i). Assume that for every $v \in W$, the multisequence $\{\|S_{\boldsymbol\lambda}^{\alpha} e_v\|^2\}_{\alpha\in\mathbb{N}^d}$ is completely monotone. Fix $v \in V \setminus W$. We contend that there exist $w \in W$, $\tilde{\alpha} \in \mathbb{N}^d$ and $\gamma \in \mathbb{C}$ such that

$$e_v = \gamma S_{\boldsymbol\lambda}^{\tilde{\alpha}} e_w. \tag{5.34}$$

Note that there exists a subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, d\}$ such that $\mathsf{d}_{v_{i_j}} > k_{\mathscr{T}_{i_j}}$ for every $j = 1, \ldots, k$. Let $l_j = \mathsf{d}_{v_{i_j}} - k_{\mathscr{T}_{i_j}}$ $(j = 1, \ldots, k)$ and set

$$w := \mathsf{par}_{i_1}^{\langle l_1\rangle} \cdots \mathsf{par}_{i_k}^{\langle l_k\rangle}(v).$$

Then $\mathsf{d}_{w_j} \leq k_{\mathscr{T}_j}$ for every $j = 1, \ldots, d$, and hence $w \in W$. Now set

$$\tilde{\alpha} := l_1\epsilon_{i_1} + \cdots + l_k\epsilon_{i_k}.$$

Since $\mathsf{d}_{w_{i_j}} = k_{\mathscr{T}_{i_j}}$ for $j = 1, \ldots, k$, we must have $\gamma S_{\boldsymbol{\lambda}}^{\tilde{\alpha}} e_w = e_v$ for some scalar $\gamma \in \mathbb{C}$. This completes the verification of (5.34). It follows that for any $\beta \in \mathbb{N}^d$,

$$\sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq n}} (-1)^{|\alpha|} \binom{n}{\alpha} \|S_{\boldsymbol{\lambda}}^{\alpha+\beta} e_v\|^2 = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq n}} (-1)^{|\alpha|} \binom{n}{\alpha} |\gamma|^2 \|S_{\boldsymbol{\lambda}}^{\alpha+\beta+\tilde{\alpha}} e_w\|^2 \geq 0,$$

since $\{\|S_{\boldsymbol{\lambda}}^{\alpha} e_w\|^2\}_{\alpha \in \mathbb{N}^d}$ is assumed to be completely monotone. Now apply the preceding proposition to complete the verification of (ii)⇒(i).

Finally, we check the implication that (iii)⇒(ii). Let $v \in W \setminus \tilde{W}$. Define

$$\mathscr{F}_v := \{w \in \tilde{W} : \mathsf{Chi}^{\ll \alpha^{(w)} \gg}(w) \text{ contains } v \text{ for some } \alpha^{(w)} \in \mathbb{N}^d\}.$$

Then $\mathscr{F}_v$ is nonempty as $\mathsf{root} \in \mathscr{F}_v$. Now consider the set

$$\mathscr{G}_v := \{w \in \mathscr{F}_v : |\mathsf{d}_w| \geq |\mathsf{d}_u| \text{ for all } u \in \mathscr{F}_v\}.$$

We claim that for all $w \in \mathscr{G}_v$, there exists $\alpha^{(w)} = (\alpha_1^{(w)}, \ldots, \alpha_d^{(w)}) \in \mathbb{N}^d$ such that $\mathsf{Chi}^{\ll \alpha^{(w)} \gg}(w) = \{v\}$. If possible, suppose there are distinct vertices $v, v' \in \mathsf{Chi}^{\ll \alpha^{(w)} \gg}(w)$ for some $w \in \mathscr{G}_v$. Without loss of generality, we may assume that $v_1 \neq v_1'$. As $v_1, v_1' \in \mathsf{Chi}^{\langle \alpha_1^{(w)} \rangle}(w_1)$, there exists an integer k, $1 \leq k \leq \alpha_1^{(w)}$, such that $u_1 := \mathsf{par}^{\langle k \rangle}(v_1) \in V_{\prec}^{(1)}$. Let $\dot{u}_1 \in \mathsf{Chi}(u_1)$ be such that $v_1 \in \mathsf{Chi}^{\langle k-1 \rangle}(\dot{u}_1)$. Note that $\dot{u}_1 \in W_1$. Consider $w' := (\dot{u}_1, w_2, \ldots, w_d)$. Then $w' \in \tilde{W}$ and $v \in \mathsf{Chi}^{\ll \beta \gg}(w')$, where $\beta = (k-1, \alpha_2^{(w)}, \ldots, \alpha_d^{(w)})$. Thus $w' \in \mathscr{F}_v$, and hence $|\mathsf{d}_w| \geq |\mathsf{d}_{w'}|$. On the other hand, $|\mathsf{d}_{w'}| = |\mathsf{d}_w| + \alpha_1^{(w)} - k + 1 > |\mathsf{d}_w|$, which is a contradiction. This proves the claim that $\mathsf{Chi}^{\ll \alpha^{(w)} \gg}(w) = \{v\}$ for all $w \in \mathscr{G}_v$. It is now easy to see that for every $v \in W \setminus \tilde{W}$, there exist $w \in \tilde{W}$, $\tilde{\alpha} \in \mathbb{N}^d$ and $\gamma \in \mathbb{C}$ such that $e_v = \gamma S_{\boldsymbol{\lambda}}^{\tilde{\alpha}} e_w$. This immediately gives the complete monotonicity of $\{\|S_{\boldsymbol{\lambda}}^{\alpha} e_v\|^2\}_{\alpha \in \mathbb{N}^d}$. ∎

REMARK 5.3.3. If $\mathscr{T}$ is locally finite with finite joint branching index, then W (and hence $\tilde{W}$) is finite.

It is well-known that there is a class of tuples antithetical to joint subnormal tuples commonly known as (*toral* or *joint*) *completely hyperexpansive tuples* (refer to [21] and [32] for definitions and basic properties). A characterization similar to one given above can be obtained for toral completely hyperexpansive multishifts as well, where, as expected, the moments being completely monotone is replaced by moments being completely alternating (refer to [25] for the definition of completely alternating functions). Similarly, the class of joint q-isometries, as introduced and studied in [58], can be characterized within the class of multishifts.

The class of joint subnormal multishifts within the class of spherically balanced multishifts admits a handy characterization (cf. [36, Theorem 5.3(1)]).

PROPOSITION 5.3.4. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol{\lambda}} = (S_1, \ldots, S_d)$ be a joint left invertible, spherically balanced multishift on $\mathscr{T}$ and let S_θ be the weighted shift on the rooted directed tree*

$\mathscr{T}^{\otimes}_{\mathsf{root}} = (V^{\otimes}, \mathcal{F})$ *associated with* $S_{\boldsymbol\lambda}$. *If* $S_{\boldsymbol\lambda}$ *is a joint contraction, then the following statements are equivalent:*

(i) $S_{\boldsymbol\lambda}$ *is joint subnormal.*
(ii) $\{1, \mathfrak{C}_0, \mathfrak{C}_0\mathfrak{C}_1, \mathfrak{C}_0\mathfrak{C}_1\mathfrak{C}_2, \ldots\}$ *is completely monotone, where* $\mathfrak{C}_t$ *denotes the constant value of* $\sum_{j=1}^d \|S_j e_v\|^2$ *on the generation* $\mathcal{G}_t$ *of* $\mathscr{T}$.
(iii) S_θ *is subnormal.*

Proof. Assume that $S_{\boldsymbol\lambda}$ is a joint contraction. We have proved (see (5.24)) that

$$\langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle = \|S^n_\theta e_{\mathfrak{v}}\|^2 \quad (n \in \mathbb{N}),$$

where $Q^n_T(\cdot)$ is as given by (1.1). By [20, Theorem 5.2], $S_{\boldsymbol\lambda}$ is joint subnormal if and only if $\{\langle Q^n_{S_{\boldsymbol\lambda}}(I)e_v, e_v\rangle\}_{n\in\mathbb{N}}$ is completely monotone for every $v \in V$, and hence by the formula above, this is equivalent to the complete monotonicity of $\{\|S^n_\theta e_{\mathfrak{v}}\|^2\}_{n\in\mathbb{N}}$ for every $\mathfrak{v} \in V^{\otimes}$. This yields the equivalence of (i) and (iii). The equivalence of (ii) and (iii) is immediate from (5.26). ∎

Let us illustrate the previous result with the help of the family of multishifts discussed in Example 5.2.5.

Example 5.3.5. Let $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ be as in Example 5.2.5. Note that $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is a joint contraction if and only if $d \le a$. Assume that a is an integer such that $d \le a$. By the preceding proposition, $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is joint subnormal if and only if $\{\prod_{p=0}^n c_{a,p}\}_{n\in\mathbb{N}}$ is completely monotone, where

$$c_{a,t} = \frac{t+d}{t+a} \quad (t \in \mathbb{N}).$$

Let us verify the last statement. Recall that the product of completely monotone sequences $(\{\frac{i}{i+n}\}_{n\in\mathbb{N}}, i = d, \ldots, d+k)$ is completely monotone. Since for $k = a - d \in \mathbb{N}$,

$$\prod_{p=0}^n c_{a,p} = \begin{cases} 1 & \text{if } k = 0, \\ \frac{d(d+1)\cdots(d+k-1)}{(d+n+1)(d+n+2)\cdots(d+k+n)} & \text{otherwise,} \end{cases}$$

$\{\prod_{p=0}^n c_{a,p}\}_{n\in\mathbb{N}}$ is completely monotone.

5.4. Joint hyponormal multishifts. In this short section, we discuss the class of joint hyponormal multishifts.

Proposition 5.4.1. *Let* $\mathscr{T} = (V, \mathcal{E})$ *be the directed Cartesian product of rooted directed trees* $\mathscr{T}_1, \ldots, \mathscr{T}_d$. *Let* $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ *be a multishift on* $\mathscr{T}$. *Then* $S_{\boldsymbol\lambda}$ *is joint hyponormal if and only if for every* $t \in \mathbb{N}$ *and every* $f_1, \ldots, f_d \in l^2(V)$ *supported on* $\mathcal{G}_t$,

$$\sum_{i,j=1}^d \langle [S_j^*, S_i]f_j, f_i\rangle \ge 0.$$

Proof. Note that $\langle [S_j^*, S_i]e_v, e_w\rangle = 0$ $(i, j = 1, \ldots, d)$ for any $v, w \in V$ such that $|\mathsf{d}_v| \ne |\mathsf{d}_w|$ (see Lemma 2.1.10(vi)). It follows that for $f_t, g_s \in l^2(V)$ with supports on $\mathcal{G}_t$ and $\mathcal{G}_s$ respectively with $s \ne t$,

$$\langle [S_j^*, S_i]f_t, g_s\rangle = 0 \quad \text{for every } i, j = 1, \ldots, d. \tag{5.35}$$

For $j = 1, \ldots, d$, let $f_j \in l^2(V)$ and write $f_j = \sum_{t\in\mathbb{N}} f_{j,t}$, where $f_{j,t}$ is supported on $\mathcal{G}_t$. Then

$$\sum_{i,j=1}^{d} \langle [S_j^*, S_i] f_j, f_i\rangle = \sum_{t,s\in\mathbb{N}} \sum_{i,j=1}^{d} \langle [S_j^*, S_i] f_{j,t}, f_{i,s}\rangle \overset{(5.35)}{=} \sum_{t\in\mathbb{N}} \sum_{i,j=1}^{d} \langle [S_j^*, S_i] f_{j,t}, f_{i,t}\rangle.$$

The desired equivalence is now immediate. ∎

In the case of spherically balanced multishifts, the preceding characterization can be made more explicit (cf. [36, Theorem 5.3(5)]).

THEOREM 5.4.2. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a joint left invertible, spherically balanced multishift on $\mathscr{T}$ and let S_θ be the weighted shift on the rooted directed tree $\mathscr{T}^{\otimes}_{\mathsf{root}} = (V^{\otimes}, \mathcal{F})$ associated with $S_{\boldsymbol\lambda}$. Then the following statements are equivalent:*

- (i) *$S_{\boldsymbol\lambda}$ is joint hyponormal.*
- (ii) *$\{\mathfrak{C}_t\}_{t\in\mathbb{N}}$ is increasing, where $\mathfrak{C}_t$ is the constant value of $\sum_{j=1}^{d} \|S_j e_v\|^2$ on the generation $\mathcal{G}_t$ of $\mathscr{T}$.*
- (iii) *S_θ is hyponormal.*

Proof. We first verify (i)⇒(ii). Recall from [34, Lemma 4.10] that any joint hyponormal d-tuple T satisfies the inequality

$$Q_T^2(I) \geq Q_T(I)^2,$$

where $Q_T^n(\cdot)$ is as given in (1.1). In particular, for any $v \in V$,

$$\langle Q^2_{S_{\boldsymbol\lambda}}(I) e_v, e_v\rangle \geq \|Q_{S_{\boldsymbol\lambda}}(I) e_v\|^2.$$

However, by the polar decomposition obtained in Proposition 5.2.10,

$$S_j = T_j D_c, \quad j = 1, \ldots, d, \tag{5.36}$$

where $T_{\boldsymbol\lambda} = (T_1, \ldots, T_d)$ and D_c are the joint isometry part and the diagonal part of $S_{\boldsymbol\lambda}$ respectively. Since $T_{\boldsymbol\lambda}$ is a joint isometry, $\sum_{j=1}^{d} T_j^* T_j = I$, and hence by (5.36),

$$Q_{S_{\boldsymbol\lambda}}(I) = \sum_{j=1}^{d} D_c T_j^* T_j D_c = D_c^2.$$

It is now easy to see that

$$\begin{aligned} \mathfrak{C}_{|\mathsf{d}_v|}\mathfrak{C}_{|\mathsf{d}_v|+1} &= \langle Q_{S_{\boldsymbol\lambda}}(D_c^2) e_v, e_v\rangle = \langle Q^2_{S_{\boldsymbol\lambda}}(I) e_v, e_v\rangle \\ &\geq \|Q_{S_{\boldsymbol\lambda}}(I) e_v\|^2 = \|D_c^2 e_v\|^2 = \mathfrak{C}^2_{|\mathsf{d}_v|}, \end{aligned}$$

which implies that $\{\mathfrak{C}_t\}_{t\in\mathbb{N}}$ is increasing.

We next verify (ii)⇒(i). In view of Proposition 5.4.1, it suffices to check that for every $t \in \mathbb{N}$ and every $f_1, \ldots, f_d \in l^2(V)$ supported on $\mathcal{G}_t$,

$$\sum_{i,j=1}^{d} \langle [S_j^*, S_i] f_j, f_i\rangle \geq 0.$$

So let $f_1, \ldots, f_d \in l^2(V)$ be supported on $\mathcal{G}_t$ for some $t \in \mathbb{N}$. A routine verification using (5.36) shows that

$$[S_j^*, S_i] e_v = \mathfrak{C}_{|\mathsf{d}_v|}[T_j^*, T_i] e_v + (\mathfrak{C}_{|\mathsf{d}_v|} - \mathfrak{C}_{|\mathsf{d}_v|-1}) T_i T_j^* e_v \quad \text{for any } v \in V.$$

Hence

$$\sum_{i,j=1}^{d} \langle [S_j^*, S_i] f_j, f_i\rangle = \mathfrak{C}_t \sum_{i,j=1}^{d} \langle [T_j^*, T_i] f_j, f_i\rangle + (\mathfrak{C}_t - \mathfrak{C}_{t-1}) \sum_{i,j=1}^{d} \langle T_i T_j^* f_j, f_i\rangle, \tag{5.37}$$

where we have used the convention that $\mathfrak{C}_{-1} = 0$. However,

$$\sum_{i,j=1}^{d} \langle [T_j^*, T_i] f_j, f_i\rangle \geq 0$$

since $T_{\boldsymbol\lambda}$, being joint subnormal, is joint hyponormal, and

$$\sum_{i,j=1}^{d} \langle T_i T_j^* f_j, f_i\rangle = \Big\| \sum_{j=1}^{d} T_j^* f_j \Big\|^2.$$

Since $\{\mathfrak{C}_t\}_{t\in\mathbb{N}}$ is increasing, we conclude from (5.37) that

$$\sum_{i,j=1}^{d} \langle [S_j^*, S_i] f_j, f_i\rangle \geq 0.$$

We finally check the equivalence of (ii) and (iii). In view of [67, Theorem 5.1.2], it suffices to check that

$$\sum_{\mathfrak{w}\in \mathsf{Chi}(\mathfrak{v})} \frac{\theta_{\mathfrak{w}}^2}{\|S_\theta e_{\mathfrak{w}}\|^2} \leq 1 \quad \text{for every } \mathfrak{v} \in V^{\otimes}$$

if and only if $\{\mathfrak{C}_t\}_{t\in\mathbb{N}}$ is increasing. Since $\|S_\theta e_{\mathfrak{w}}\|^2 = \mathfrak{C}_{\mathsf{d}_{\mathfrak{w}}}$ and $\theta_{\mathfrak{w}} = \frac{\sqrt{\mathfrak{C}_{\mathsf{d}_{\mathfrak{w}}-1}}}{\sqrt{\mathrm{card}(\mathsf{sib}(\mathfrak{w}))}}$ ($\mathfrak{w} \in V^{\otimes} \setminus \mathsf{root}$) (see (5.26) and (5.21)), we obtain

$$\begin{aligned}\sum_{\mathfrak{w}\in \mathsf{Chi}(\mathfrak{v})} \frac{\theta_{\mathfrak{w}}^2}{\|S_\theta e_{\mathfrak{w}}\|^2} &= \sum_{\mathfrak{w}\in \mathsf{Chi}(\mathfrak{v})} \frac{1}{\mathrm{card}(\mathsf{sib}(\mathfrak{w}))} \frac{\mathfrak{C}_{\mathsf{d}_{\mathfrak{w}}-1}}{\mathfrak{C}_{\mathsf{d}_{\mathfrak{w}}}} \\ &= \frac{\mathfrak{C}_{\mathsf{d}_{\mathfrak{v}}}}{\mathfrak{C}_{\mathsf{d}_{\mathfrak{v}}+1}} \sum_{\mathfrak{w}\in \mathsf{Chi}(\mathfrak{v})} \frac{1}{\mathrm{card}(\mathsf{sib}(\mathfrak{w}))} = \frac{\mathfrak{C}_{\mathsf{d}_{\mathfrak{v}}}}{\mathfrak{C}_{\mathsf{d}_{\mathfrak{v}}+1}}.\end{aligned}$$

The equivalence of (ii) and (iii) is now clear. ■

Remark 5.4.3. Assume that $S_{\boldsymbol\lambda}$ is a joint hyponormal multishift. It is well-known that the spectral radius and norm of a hyponormal operator coincide [40]. One may now conclude from Corollary 5.2.17(i) and (5.25) that the spectral radius of $S_{\boldsymbol\lambda}$ equals

$$\|S_1^* S_1 + \cdots + S_d^* S_d\|^{1/2}.$$

Example 5.4.4. Let $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ be as in Example 5.2.5. Note that $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is a joint hyponormal if and only if $\{c_{a,t}\}_{t\in\mathbb{N}}$ is increasing, where

$$c_{a,t} = \frac{t+d}{t+a} \quad (t \in \mathbb{N}).$$

This holds if and only if $d \leq a$. In view of Example 5.3.5, we have the following equivalent statements (cf. [9, Lemma 3.3]):

1. $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is joint subnormal.
2. $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is joint hyponormal.
3. $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ is joint contraction.

Afterword. Needless to say, the work presented in this paper provides a framework to unify the theories of classical multishifts and weighted shifts on rooted directed trees. This framework also enables one to pose and peruse a diverse range of problems. More importantly, this framework allows the rich interplay of graph theory, complex function theory, and operator theory. One of the important outcomes of these investigations is perhaps the tree analogs $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ of extensively studied classical multishifts like Szegö, Bergman, and Drury–Arveson d-shifts. On the one hand, the tree analogs of these d-shifts share many important properties of their classical counterparts; e.g. $S_{\boldsymbol{\lambda}_{\mathfrak{c}_d}}$ (Szegö d-shift) is a joint isometry, $S_{\boldsymbol{\lambda}_{\mathfrak{c}_{d+1}}}$ (Bergman d-shift) is joint subnormal, and $S_{\boldsymbol{\lambda}_{\mathfrak{c}_1}}$ (Drury–Arveson d-shift) is a row contraction. On the other hand, due to abundance of directed tree structures, various intricacies may arise. For instance, unlike their classical counterparts, for a suitable choice of $\mathscr{T}$, the defect operator $\sum_{k=0}^{a}(-1)^k\binom{a}{k}Q^k_{S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}}(I)$ fails to be an orthogonal projection. Further, in the matrix decomposition of $S^*_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$, nondiagonal tuples of infinite rank operators appear naturally. We believe that the class of multishifts $S_{\boldsymbol{\lambda}_{\mathfrak{c}_a}}$ warrants further attention as they may play the role of building blocks in the classification of $\mathcal{G}$-homogeneous tuples associated with the action of various linear groups $\mathcal{G} \subseteq GL_d(\mathbb{C})$.

Appendix

We are grateful to V. M. Sholapurkar for kindly providing a multivariable analog of the identity given in (4.2) (along with proof).

LEMMA 1. *Let n be a positive integer and let $X = (x_1, \dots, x_d), Y = (y_1, \dots, y_d)$ be d-tuples such that the variables x_i, y_i $(i = 1, \dots, d)$ belong to a unital complex algebra. Then*

$$1 - \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| = n}} \binom{|\alpha|}{\alpha} X^\alpha Y^\alpha = \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| \leq n-1}} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta,$$

where $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1! \cdots \alpha_m!}$ $(\alpha \in \mathbb{N}^d)$.

Proof. We use induction on $n \in \mathbb{N}$. For $n = 1$, both sides of the identity reduce to $1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d$ and hence the result holds. Suppose the result holds for some $n \geq 1$. We now prove the identity for $n+1$. Starting with the right hand side, we split the sum

$$A := \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| \leq n-1}} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta$$

as $A_1 + A_2$, where

$$A_1 := \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| \leq n-1}} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta,$$

$$A_2 := \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = n}} \binom{|\beta|}{\beta} X^\beta (1 - x_1 y_1 - x_2 y_2 - \cdots - x_d y_d) Y^\beta.$$

Now by induction hypothesis, $A_1 = 1 - \sum_{|\beta|=n} \binom{n}{\beta} X^\beta Y^\beta$. Observe also that

$$A_2 = \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = n}} \binom{n}{\beta} X^\beta Y^\beta - \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = n}} \binom{n}{\beta} \Big[\sum_{i=1}^{d} X^{\beta + \epsilon_i} Y^{\beta + \epsilon_i} \Big].$$

Thus

$$\begin{aligned} A_1 + A_2 &= 1 - \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = n}} \binom{n}{\beta} \Big[\sum_{i=1}^{d} X^{\beta+\epsilon_i} Y^{\beta+\epsilon_i} \Big] \\ &= 1 - \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = n+1}} \Big[\sum_{i=1}^{d} \binom{n}{\beta - \epsilon_i} \Big] X^{\beta} Y^{\beta} \\ &= 1 - \sum_{\substack{\beta \in \mathbb{N}^d \\ |\beta| = n+1}} \binom{n+1}{\beta} X^{\beta} Y^{\beta}. \end{aligned}$$

This is the left hand side of the identity with n replaced by $n+1$. ■

Let us discuss some of the difficulties in the deduction of appropriate analogs of Shimorin's formula (4.1) in several variables. The very first difficulty which arises is the appropriate notion of Cauchy dual S' in several variables, where $S = (S_1, \ldots, S_d)$ is the given d-tuple of bounded linear operators on $\mathcal{H}$. To see what a correct choice would be, note that (4.2) may be rewritten as

$$I - T^n T'^{*n} = \sum_{k=0}^{n-1} T^k P_E T'^{*k},$$

where T is a left invertible operator on $\mathcal{H}$ and $P_E = I - TT'^*$ is the orthogonal projection onto the kernel of T^*. Hence, in view of the preceding lemma, the choice of Cauchy dual in several variables should necessarily ensure that $I - \sum_{i=1}^{d} S_i S_i'^*$ is an orthogonal projection. Examples show that none of the notions of Cauchy dual (toral and spherical) can be applied with success in this context. Keeping this aside and assuming this condition for a moment, what we obtain is the following formula:

$$\bigcap_{|\alpha|=n} \ker S'^{*\alpha} \subseteq \bigvee \{ S^{\beta} f : f \in \ker S^*,\ |\beta| \le n-1 \},$$

where it is not clear whether equality holds. For a single operator, we obtain equality, which can then be used to derive the duality formula (4.1) crucial in obtaining the wandering subspace property for T.

References

[1] G. Adams and P. McGuire, *Analytic tridiagonal reproducing kernels*, J. London Math. Soc. 64 (2001), 722–738.

[2] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203–217.

[3] J. Agler, G. Knese, and J. McCarthy, *Algebraic pairs of isometries*, J. Operator Theory 67 (2012), 215–236.

[4] D. Alpay, A. Dijksma et D. Volok, *Évaluation ponctuelle et espace de Hardy: le cas multiéchelle*, C. R. Math. Acad. Sci. Paris 340 (2005), 415–420.

[5] D. Alpay and H. T. Kaptanoğlu, *Toeplitz operators on Arveson and Dirichlet spaces*, Integral Equations Operator Theory 58 (2007), 1–33.

[6] D. Alpay and D. Volok, *Point evaluation and Hardy space on a homogeneous tree*, Integral Equations Operator Theory 53 (2005), 1–22.

[7] A. Anand, S. Chavan, Z. Jabłoński, and J. Stochel, *A solution to the Cauchy dual subnormality problem for 2-isometries*, arXiv:1702.01264v3 (2017).

[8] T. Andô, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged) 24 (1963), 88–90.

[9] J. Arazy and G. Zhang, *Homogeneous multiplication operators on bounded symmetric domains*, J. Funct. Anal. 202 (2003), 44–66.

[10] N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures for analytic Besov spaces*, Rev. Mat. Iberoamer. 18 (2002), 443–510.

[11] N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures and interpolating sequences for Besov spaces on complex balls*, Mem. Amer. Math. Soc. 182 (2006), no. 859, vi+163 pp.

[12] N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures for the Drury–Arveson Hardy space and other Besov–Sobolev spaces on complex balls*, Adv. Math. 218 (2008), 1107–1180.

[13] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337–404.

[14] W. Arveson, *Subalgebras of C^*-algebras. III. Multivariable operator theory*, Acta Math. 181 (1998), 159–228.

[15] W. Arveson, *The Dirac operator of a commuting d-tuple*, J. Funct. Anal. 189 (2002), 53–79.

[16] W. Arveson, D. Hadwin, T. Hoover, and E. Kymala, *Circular operators*, Indiana Univ. Math. J. 33 (1984), 583–595.

[17] A. Athavale, *Holomorphic kernels and commuting operators*, Trans. Amer. Math. Soc. 304 (1987), 101–110.

[18] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. 103 (1988), 417–423.

[19] A. Athavale, *On the intertwining of joint isometries*, J. Operator Theory 23 (1990), 339–350.

[20] A. Athavale, *Model theory on the unit ball*, J. Operator Theory 27 (1992), 347–358.

[21] A. Athavale and V. M. Sholapurkar, *Completely hyperexpansive operator tuples*, Positivity 3 (1999), 245–257.

[22] B. Bagchi and G. Misra, *Homogeneous tuples of multiplication operators on twisted Bergman spaces*, J. Funct. Anal. 136 (1996), 171–213.

[23] B. Bagchi and G. Misra, *The homogeneous shifts*, J. Funct. Anal. 204 (2003), 293–319.

[24] J. Ball and V. Bolotnikov, *Canonical transfer-function realization for Schur multipliers on the Drury–Arveson space and models for commuting row contractions*, Indiana Univ. Math. J. 61 (2012), 665–716.

[25] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups*, Springer, Berlin, 1984.

[26] P. Budzyński, P. Dymek, and M. Ptak, *Analytic structure of weighted shifts on directed trees*, Math. Nachr. 290 (2017), 1612–1629.

[27] P. Budzyński, Z. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded subnormal weighted shifts on directed trees*, J. Math. Anal. Appl. 394 (2012), 819–834.

[28] P. Budzyński, Z. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded subnormal weighted shifts on directed trees. II*, J. Math. Anal. Appl. 398 (2013), 600–608.

[29] P. Budzyński, Z. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded subnormal composition operators in L^2-spaces*, J. Funct. Anal. 269 (2015), 2110–2164.

[30] P. Budzyński, Z. Jabłoński, I. B. Jung, and J. Stochel, *A subnormal weighted shift on a directed tree whose nth power has trivial domain*, J. Math. Anal. Appl. 435 (2016), 302–314.

[31] P. Budzyński, Z. Jabłoński, I. B. Jung, and J. Stochel, *Subnormality of unbounded composition operators over one-circuit directed graphs: exotic examples*, Adv. Math. 310 (2017), 484–556.

[32] S. Chavan and R. Curto, *Operators Cauchy dual to 2-hyperexpansive operators: the multivariable case*, Integral Equations Operator Theory 73 (2012), 481–516.

[33] S. Chavan and S. Kumar, *Spherically balanced Hilbert spaces of formal power series in several variables–I*, J. Operator Theory 72 (2014), 405–428.

[34] S. Chavan and V. M. Sholapurkar, *Rigidity theorems for spherical hyperexpansions*, Complex Anal. Oper. Theory 7 (2013), 1545–1568.

[35] S. Chavan and S. Trivedi, *An analytic model for left-invertible weighted shifts on directed trees*, J. London Math. Soc. 94 (2016), 253–279.

[36] S. Chavan and D. Yakubovich, *Spherical tuples of Hilbert space operators*, Indiana Univ. Math. J. 64 (2015), 577–612.

[37] M. Chō and W. Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J. 21 (1992), 251–258.

[38] J. Conway, *A Course in Functional Analysis*, Springer, New York, 1985.

[39] J. Conway, *Towards a functional calculus for subnormal tuples: the minimal normal extension*, Trans. Amer. Math. Soc. 326 (1991), 543–567.

[40] J. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monogr. 36, Amer. Math. Soc., Providence, RI, 1991.

[41] M. Cowen and R. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), 187–261.

[42] R. Curto, *On the connectedness of invertible n-tuples*, Indiana Univ. Math. J. 29 (1980), 393–406.

[43] R. Curto, *Fredholm and invertible n-tuples of operators. The deformation problem*, Trans. Amer. Math. Soc. 266 (1981), 129–159.

[44] R. Curto, *Applications of several complex variables to multiparameter spectral theory*, in: Surveys of Some Recent Results in Operator Theory, Vol. II, Pitman Res. Notes Math. Ser. 192, Longman Sci. Tech., Harlow, 1988, 25–90.

[45] R. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, in: Proc. Sympos. Pure Math. 51, Amer. Math. Soc., 1990, 69–91.

[46] R. Curto and W. Lee, *Joint hyponormality of Toeplitz pairs*, Mem. Amer. Math. Soc. 150 (2001), no. 712, x+65 pp.

[47] R. Curto, P. Muhly, and J. Xia, *Hyponormal pairs of commuting operators*, in: Contributions to Operator Theory and its Applications (Mesa, AZ, 1987), Oper. Theory Adv. Appl. 35, Birkhäuser, Basel, 1988, 1–22.

[48] R. Curto and M. Putinar, *Existence of nonsubnormal polynomially hyponormal operators*, Bull. Amer. Math. Soc. 25 (1991), 373–378.

[49] R. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen–Douglas theory*, Amer. J. Math. 106 (1984), 447–488.

[50] R. Curto and N. Salinas, *Spectral properties of cyclic subnormal m-tuples*, Amer. J. Math. 107 (1985), 113–138.

[51] R. Curto and J. Yoon, *Spectral pictures of 2-variable weighted shifts*, C. R. Math. Acad. Sci. Paris 343 (2006), 579–584.

[52] M. Didas and J. Eschmeier, *Inner functions and spherical isometries*, Proc. Amer. Math. Soc. 139 (2011), 2877–2889.

[53] R. Diestel, *Graph Theory*, Grad. Texts in Math. 173, Springer, Heidelberg, 2010.

[54] R. Douglas and J. Eschmeier, *Spectral inclusion theorems*, in: Mathematical Methods in Systems, Optimization, and Control, Oper. Theory Adv. Appl. 222, Birkhäuser/Springer Basel AG, Basel, 2012, 113–128.

[55] J. Eschmeier and M. Putinar, *Some remarks on spherical isometries*, in: Systems, Approximation, Singular Integral Operators, and Related Topics (Bordeaux, 2000), Oper. Theory Adv. Appl. 129, Birkhäuser, Basel, 2001, 271–291.

[56] J. Feigenbaum, *Directed cartesian-product graphs have unique factorizations that can be computed in polynomial time*, Discrete Appl. Math. 15 (1986), 105–110.

[57] R. Gellar, *Circularly symmetric normal and subnormal operators*, J. Anal. Math. 32 (1977), 93–117.

[58] J. Gleason and S. Richter, *m-Isometric commuting tuples of operators on a Hilbert space*, Integral Equations Operator Theory 56 (2006), 181–196.

[59] J. Gleason, S. Richter, and C. Sundberg, *On the index of invariant subspaces in spaces of analytic functions of several complex variables*, J. Reine Angew. Math. 587 (2005), 49–76.

[60] K. Guo, J. Hu, and X. Xu, *Toeplitz algebras, subnormal tuples and rigidity on reproducing $\mathbb{C}[z_1, \dots, z_d]$-modules*, J. Funct. Anal. 210 (2004), 214–247.

[61] P. Halmos, *Naive Set Theory*, Undergrad. Texts in Math., Springer, New York, 1974.

[62] F. Harary and C. Trauth, *Connectedness of products of two directed graphs*, SIAM J. Appl. Math. 14 (1966), 250–254.

[63] M. Hartz, *Von Neumann's inequality for commuting weighted shifts*, Indiana Univ. Math. J. 66 (2017), 1065–1079.

[64] W. He, *Boundary representations on co-invariant subspaces of Bergman space*, Proc. Amer. Math. Soc. 138 (2010), 615–622.

[65] D. Herrero, *On multicyclic operators*, Integral Equations Operator Theory 1 (1978), 57–102.

[66] W. Imrich and S. Klavžar, *Product Graphs Structure and Recognition*, Wiley-Interscience, New York, 2000.

[67] Z. Jabłoński, I. B. Jung, and J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. 216 (2012), no. 1017, viii+106 pp.

[68] Z. Jabłoński, I. B. Jung, and J. Stochel, *A non-hyponormal operator generating Stieltjes moment sequences*, J. Funct. Anal. 262 (2012), 3946–3980.

[69] Z. Jabłoński, I. B. Jung, and J. Stochel, *Normal extensions escape from the class of weighted shifts on directed trees*, Complex Anal. Oper. Theory 7 (2013), 409–419.

[70] Z. Jabłoński, I. B. Jung, and J. Stochel, *A hyponormal weighted shift on a directed tree whose square has trivial domain*, Proc. Amer. Math. Soc. 142 (2014), 3109–3116.

[71] N. P. Jewell and A. R. Lubin, *Commuting weighted shifts and analytic function theory in several variables*, J. Operator Theory 1 (1979), 207–223.

[72] J. Kaad, *Joint torsion of several commuting operators*, Adv. Math. 229 (2012), 442–486.

[73] H. T. Kaptanoğlu, *Aspects of multivariable operator theory on weighted symmetric Fock spaces*, Comm. Contemp. Math. 16 (2014), 1350034, 49 pp.

[74] E. Katsoulis and D. Kribs, *Applications of the Wold decomposition to the study of row contractions associated with directed graphs*, Trans. Amer. Math. Soc. 357 (2005), 3739–3755.

[75] J. Kelley, *General Topology*, Grad. Texts in Math. 27, Springer, New York, 1975.

[76] D. Kribs, R. Levene and S. Power, *Commutants of weighted shift directed graph operator algebras*, Proc. Amer. Math. Soc. 145 (2017), 3465–3480.

[77] S. Kumar, *Spherically balanced Hilbert spaces of formal power series in several variables–II*, Complex Anal. Oper. Theory 10 (2016), 505–526.

[78] A. Lambert, *Unitary equivalence and reducibility of invertibly weighted shifts*, Bull. Austral. Math. Soc. 5 (1971), 157–173.

[79] W. Majdak and J. B. Stochel, *Weighted shifts on directed semi-trees: An application to creation operators on Segal–Bargmann spaces*, Complex Anal. Oper. Theory 10 (2016), 1427–1452.

[80] M. McAndrew, *On the product of directed graphs*, Proc. Amer. Math. Soc. 14 (1963), 600–606.

[81] W. Mlak, *Notes on circular operators. I–IV*, Univ. Iagel. Acta Math. 29 (1992), 145–152, 153–162, 163–170, 171–175.

[82] V. Müller and A. Sołtysiak, *Spectral radius formula for commuting Hilbert space operators*, Studia Math. 103 (1992), 329–333.

[83] V. Müller and F. H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. 117 (1993), 979–989.

[84] S. Richter and C. Sundberg, *Joint extensions in families of contractive commuting operator tuples*, J. Funct. Anal. 258 (2010), 3319–3346.

[85] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer, 2008.

[86] G. Sabidussi, *Graph multiplication*, Math. Z. 72 (1960), 446–457.

[87] R. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions, Theory and Applications*, 2nd ed., de Gruyter Stud. Math. 37, de Gruyter, Berlin, 2012.

[88] A. Shields, *Weighted shift operators and analytic function theory*, in: Topics in Operator Theory, Math. Surveys Monogr. 13, Amer. Math. Soc., Providence, RI, 1974, 49–128.

[89] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. 531 (2001), 147–189.

[90] J. Stochel, *Characterizations of subnormal operators*, Studia Math. 97 (1991), 227–238.

[91] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland and Akadémiai Kiadó, Amsterdam–Budapest, 1970.

[92] P. Weichsel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc. 13 (1962), 47–52.

[93] F. Zhang (ed.), *The Schur Complement and its Applications*, Numerical Methods Algorithms 4, Springer, New York, 2005.

Index of notation