INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

DISSERTATIONES MATHEMATICAE

528

NURULLA AZAMOV

Spectral flow and resonance index

WARSZAWA 2017

Nurulla Azamov
College of Science and Engineering
Flinders University, South Road, Tonsley
Adelaide, South Australia 5042
Australia
E-mail: nurulla.azamov@flinders.edu.au

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound in Poland by HermanDruK, Warszawa
Nakład 200 egz.

Abstracted/Indexed in: Mathematical Reviews, Zentralblatt MATH, Science Citation Index Expanded, Journal Citation Reports/Science Edition, Google Science, Scopus, EBSCO Discovery Service.

Available online at **http://journals.impan.pl**

doi: 10.4064/dm756-6-2017

ISSN 0012-3862

Contents

Abstract

It has been shown recently that spectral flow admits a natural integer-valued extension to essential spectrum. This extension admits four different interpretations; two of them are singular spectral shift function and total resonance index. In this work we study the resonance index outside the essential spectrum.

Among results of this paper are the following:

(1) Investigation of the root space of the compact operator $(H_0 + sV - \lambda)^{-1}V$ corresponding to an eigenvalue $(s - r_\lambda)^{-1}$, where H_0 is a self-adjoint operator and $r_\lambda \in \mathbb{R}$ is such that $H_0 + r_\lambda V$ belongs to the set

$$\mathcal{R}(\lambda) = \{H_0 + V \colon V = V^* \text{ is } H_0\text{-compact and } \lambda \in \sigma_d(H_0 + V)\}.$$

(2) (a) Criteria for a perturbation V to be tangent to the set $\mathcal{R}(\lambda)$ at a point H.
(b) Criteria for the order of tangency of a perturbation V to the set $\mathcal{R}(\lambda)$.

(3) Direct proof of the equality "total resonance index = intersection number".
(4) Direct proof of the equality "total resonance index = total Fredholm index".
(5) Total resonance index satisfies the Robbin–Salamon axioms for spectral flow.
(6) Direct proof of the equality "total resonance index = spectral shift function" at a point λ not in the essential spectrum $\sigma_{\text{ess}}(H_0)$.

This analysis gives a finer information about the behaviour of discrete spectrum compared to spectral flow.

Many results of this paper are non-trivial even in finite dimensions, in which case they can be and were tested in numerical experiments.

Acknowledgements. I thank Tom Daniels for a scrupulous reading of this paper which resulted in countless improvements including a countable subset of fixed articles. I also thank the referee whose constructive criticism and numerous suggestions led to a significant improvement of the presentation of this material.

2010 *Mathematics Subject Classification*: Primary 47A55, 47A53, 47A10; Secondary 47A75.
Key words and phrases: spectral flow, intersection number, resonance index, Fredholm index, spectral shift function, Robbin–Salamon axioms.
Received 28 July 2016; revised 21 June 2017.
Published online 3 January 2018.

1. Introduction

1.1. Introduction. The spectral flow of a continuous path $\{H_r\colon r \in [0,1]\}$ of self-adjoint operators through a point λ which does not belong to the common essential spectrum σ_{ess} of the operators H_r is naively understood as the number of eigenvalues of H_r which cross λ from left to right minus the number of eigenvalues of H_r which cross λ from right to left as the variable r moves from 0 to 1 [APS]. This naive definition was given a rigorous basis in [RoSa]. An alternative definition of spectral flow as the total Fredholm index of the path $\{H_r\}$ was suggested in [Ph, Ph2], [BCPRSW, Section 4]. This definition has some advantages compared to the intersection number, for example it can be applied to study spectral flow in von Neumann algebras, where the notion of intersection number does not make sense. Further, the total Fredholm index can be interpreted as an integral of a one-form defined on some real affine space of self-adjoint operators [Ge, CP, CP2, ACS, BCPRSW]; the possibility of such interpretation was first suggested by I. M. Singer in 1974. The spectral flow can also be interpreted as Maslov index [RoSa]. Finally, outside the essential spectrum the spectral flow is equal to the spectral shift function [L, Kr] (see e.g. [ACDS, ACS, Pu]).

These definitions of spectral flow are applicable only for numbers λ outside the common essential spectrum of a path of self-adjoint operators H_r, with the exception of the spectral shift function. The spectral shift function is not integer-valued inside the essential spectrum, and therefore it cannot be considered as a proper analogue of spectral flow for essential spectrum. In [Az2, Az3, Az4] an analogue of the Lebesgue decomposition $m = m^{(a)} + m^{(s)}$ of a measure m into its absolutely continuous and singular parts was suggested for the spectral shift function ξ. It was shown that the singular part $\xi^{(s)}$ of ξ, which can be correctly defined by

$$\xi^{(s)}(\lambda) = \frac{d}{d\lambda}\int_0^1 \operatorname{Tr}(VE_\lambda^{H_r^{(s)}})\,dr, \qquad \text{a.e. } \lambda, \tag{1.1.1}$$

where $H_r^{(s)}$ is the singular part of $H_r = H_0 + rV$, is a function which takes integer values for a.e. value of the spectral parameter λ, including those in σ_{ess}, and which coincides with ξ, and thus with the spectral flow, outside the essential spectrum.

Apparently, it is difficult to work directly with the definition (1.1.1) of the singular spectral shift function $\xi^{(s)}$. In [Az5] (see also [Az6, Section 6]) it was found that for trace class perturbations the singular spectral shift function can be interpreted as *total resonance index* (TRI),

$$\xi^{(s)}(\lambda) = \sum_{r_\lambda \in [0,1]} \operatorname{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V), \tag{1.1.2}$$

where $\mathrm{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$ is the so-called resonance index of the triple $(\lambda; H_{r_\lambda}, V)$, and where the sum is taken over the real resonance points r_λ of $(\lambda; H_0, V)$ which belong to $[0, 1]$. One of the ways to define the resonance points and the resonance index is as follows. Let

$$\sigma_z^1(s), \sigma_z^2(s), \dots$$

be the list of eigenvalues of the compact operator

$$(H_0 + sV - z)^{-1}V, \qquad s \in \mathbb{R}.$$

It is not difficult to show that for each of these eigenvalues there exists a number r_z^j such that

$$\sigma_z^j(s) = (s - r_z^j)^{-1}.$$

The numbers r_z^j are the *resonance points* of the triple $(z; H_0, V)$. A real number r_λ is a real resonance point of $(\lambda; H_0, V)$ if at least one of r_z^j approaches r_λ as $z = \lambda + iy \to \lambda + i0$. The *resonance index* of $(\lambda; H_{r_\lambda}, V)$ is the integer

$$N_+ - N_-,$$

where N_+ (respectively, N_-) is the number of resonance points which approach r_λ in the upper complex half-plane (respectively, lower complex half-plane). The resonance points r_z^j in this definition should be counted according to their algebraic multiplicities, which are transferred from the corresponding eigenvalues $\sigma_z^j(s)$.

If λ does not belong to the essential spectrum, then $\xi^{(s)}(\lambda) = \xi(\lambda)$, as the definition (1.1.1) of singular SSF turns into a well-known Birman–Solomyak [BS] formula for SSF:

$$\xi(\lambda) = \frac{d}{d\lambda} \int_0^1 \mathrm{Tr}(V E_\lambda^{H_r})\, dr, \qquad \text{a.e. } \lambda.$$

Thus, outside the essential spectrum the formula (1.1.2) turns into

$$\text{spectral flow through } \lambda = \sum_{r_\lambda \in [0,1]} \mathrm{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V).$$

Once this formula is obtained, one may choose to forget its origin and consider the right hand side as a new definition of spectral flow. Apart from the fact that, unlike other definitions of spectral flow, this definition makes perfect sense inside the essential spectrum, it has two other advantages. Firstly, it requires minimum assumptions in order to be defined, in particular it includes as special cases the spectral flow for operators with compact resolvent, and the spectral shift function for relatively trace class perturbations. Secondly, it is defined in the language of complex analysis and it can be investigated using tools of complex analysis. It is well known that proofs based on complex analysis are as a rule considerably simpler (and also more beautiful, but this depends on one's taste) and therefore more natural. As a historical example, Franz Rellich's perturbation theory of isolated eigenvalues was essentially simplified by introduction of Riesz idempotents.

In this paper we study spectral flow from the point of view of resonance index. In particular, we show that the total resonance index satisfies the Robbin–Salamon axioms for spectral flow, and give direct proofs of equality of the total resonance index with other classical definitions of spectral flow, such as intersection number and the total Fredholm index.

1.2. Basic assumption. Below we collect the basic assumptions, notation and terminology which will be used throughout this paper.

Assumption 1.2.1.

(1) H_0 is a self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$ with dense domain $\mathcal{D}$.

(2) $\mathcal{A}_0$ is a real vector space of self-adjoint operators V which are relatively compact with respect to H_0. The latter means by definition that the domain of V contains $\mathcal{D}$, and the product

$$R_z(H_0)V := (H_0 - z)^{-1}V$$

is compact for some and thus for any complex number z which does not belong to the spectrum of H_0. Elements of $\mathcal{A}_0$ will also be called *directions.*

(3) $\mathcal{A}$ is the real affine space $H_0 + \mathcal{A}_0$ of self-adjoint operators. Elements of $\mathcal{A}$ will also be called *points.*

(4) It follows from (2) and the second resolvent identity that all directions from $\mathcal{A}_0$ are relatively compact with respect to any point from $\mathcal{A}$. Therefore, by Weyl's theorem (see e.g. [RS]) all points from $\mathcal{A}$ have the same essential spectrum. We denote this common essential spectrum by σ_{ess} and refer to it as the *essential spectrum* of the affine space $\mathcal{A}$.

(5) There exists at least one real number λ which does not belong to σ_{ess}. Most of the time, λ will be fixed.

(6) We let $H_s = H_0 + sV$ and denote by r_λ a real number such that λ is an eigenvalue of

$$H_{r_\lambda} = H_0 + r_\lambda V$$

of multiplicity $m \geq 1$. Most of the time we will work with one operator H_{r_λ}, and the notation r_λ and m will be used strictly in this sense only. We also assume that for some real s the number λ is not an eigenvalue of H_s.

(7) Item (6) implies that the eigenvalue λ of multiplicity m moves away from λ as the coupling variable s is varied from r_λ. The resulting m (counting multiplicities) eigenvalue functions of s are denoted by $\lambda_\nu(s)$, $\nu = 1, \ldots, m$, and the corresponding eigenvectors, which are defined only up to scaling, by $\varphi_\nu(s)$, $\nu = 1, \ldots, m$.

These are the only assumptions which we shall make in this paper. They are quite generic in Hilbert space perturbation theory. As a special case they include the case of self-adjoint operators H_0 with compact resolvent and a vector space of bounded self-adjoint perturbations V, which is the main setting of spectral flow theory in differential geometry and global analysis.

We shall consistently use the words "point" and "direction" instead of "self-adjoint operator from $\mathcal{A}$" and "self-adjoint perturbation operator from $\mathcal{A}_0$". Elements of a vector space associated with an affine space are usually called vectors, but we shall not use this word to avoid confusion.

The set of all points H from $\mathcal{A}$ for which λ is an eigenvalue is called the *resonance set* and denoted by $\mathcal{R}(\lambda)$. Elements of $\mathcal{R}(\lambda)$ will be called λ-*resonant operators* or λ-*resonant points.* Since the real number λ will be fixed for most of the time, λ-resonant points will

often be called resonant points. We say that a resonance point H is *simple* if λ is an eigenvalue of H of multiplicity 1. An analytic path $H(s)$ will be said to be a *resonant path* if $H(s) \in \mathcal{R}(\lambda)$ for all s. Otherwise we say that $H(s)$ is a *regular path*. A regular path $H(s)$ may have resonant points on it, but the set of such points is discrete. A direction V at a resonant point H will be said to be *regular* if the straight line $H + sV$ is a regular path. In my previous papers, whether published or not, regular directions were called regularising.

The terminology "λ-resonant operator" was used in [Az6] in a study of spectral flow inside σ_{ess}, but for a real number λ outside the essential spectrum the definition of the λ-resonant operator reduces to the one given above.

A function defined on a finite-dimensional real affine space is called *analytic* if it is given by a real analytic function in some and thus in any affine system of coordinates. A subset R of a finite-dimensional real affine space is called an *analytic set* if R is the set of zeros of one or several real analytic functions. A subset R of any real affine space is called an *analytic set* if the intersection of R with any finite-dimensional affine subspace is an analytic set. By an affine space from now on we will always mean a real affine space. Finite subsets of an affine space and an affine space itself are analytic sets. Intersections and finite unions of analytic sets are also analytic. The resonance set $\mathcal{R}(\lambda)$ is an analytic set; a proof of this assertion follows verbatim that of [Az4, Theorem 4.2.5].

1.3. Preliminaries. The *second resolvent identity*

$$R_z(H_0 + V) - R_z(H_0) = -R_z(H_0 + V)VR_z(H_0) = -R_z(H_0)VR_z(H_0 + V) \tag{1.3.1}$$

holds for any pair of closed operators H_0 and V provided $H_0 + V$ is well-defined and z belongs to the resolvent sets of both H_0 and $H_0 + V$. This identity can be rewritten as follows:

$$R_z(H_0 + V) = R_z(H_0)(1 + VR_z(H_0))^{-1} = (1 + R_z(H_0)V)^{-1}R_z(H_0). \tag{1.3.2}$$

We shall extensively use the notation

$$A_z(s) = R_z(H_s)V \quad \text{and} \quad B_z(s) = VR_z(H_s), \tag{1.3.3}$$

where

$$H_s = H_0 + sV.$$

Since V is relatively compact with respect to all H_s, by definition of relative compactness the domain of V contains the range $\mathcal{D}$ of $R_z(H_s)$. Thus, the domain of $B_z(s) = VR_z(H_s)$ is the whole Hilbert space. The same cannot be said about $R_z(H_s)V$ in case V is unbounded, so by $A_z(s)$ we shall mean the closure of $R_z(H_s)V$.

By Assumption 1.2.1, the operators $A_z(s)$ and $B_z(s)$ are compact. The second resolvent identity (1.3.2) implies that for $s, r \in \mathbb{C}$,

$$A_z(s) = (1 + (s - r)A_z(r))^{-1}A_z(r) \tag{1.3.4}$$

and a similar equality holds for $B_z(s)$. This shows that $A_z(s)$ is a meromorphic function of s.

We start with a recap of some material of [Az6, Section 3].

Let $H_0 \in \mathcal{A}$, $V \in \mathcal{A}_0$ and let $H_s = H_0 + sV$, where $s \in \mathbb{C}$. Let $z = \lambda + iy \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}$. A point r_z is a *resonance point* of the triple $(z; H_0, V)$ if any one of the following equivalent conditions hold:

(1) r_z is a pole of the meromorphic function $\mathbb{C} \ni s \mapsto A_z(s)$.
(2) r_z is a pole of the meromorphic function $\mathbb{C} \ni s \mapsto B_z(s)$.
(3) The operator $1 + (r_z - s)A_z(s)$ has a non-zero kernel for some $s \in \mathbb{C}$. The kernel
$$\Upsilon_z^1(r_z; H_0, V)$$
of this operator does not depend on s.
(4) The operator $1 + (r_z - s)B_z(s)$ has a non-zero kernel. The kernel
$$\Psi_z^1(r_z; H_0, V)$$
of this operator also does not depend on s.
(5) The number z is an eigenvalue of the operator
$$H_{r_z} := H_0 + r_z V.$$
The corresponding eigenspace is $\Upsilon_z^1(r_z; H_0, V)$.

In the definition of $\Upsilon_z^1(r_z; H_0, V)$ the operator $A_z(s)$ is the closure of $R_z(H_s)V$, but it can be shown that $\Upsilon_z^1(r_z; H_0, V)$ is a subspace of $\mathrm{dom}(V)$, and $V\Upsilon_z^1(r_z; H_0, V) = \Psi_z^1(r_z; H_0, V)$.

With every resonance point r_z of the triple $(z; H_0, V)$, one can associate an idempotent operator $P_z(r_z)$, the vector space $\Upsilon_z(r_z) := \mathrm{im}\, P_z(r_z)$ and a nilpotent operator $\mathbf{A}_z(r_z)$, which is reduced by the vector space $\Upsilon_z(r_z)$. They can be defined by [Az6, (3.2.6)]
$$P_z(r_z) = \frac{1}{2\pi i} \oint_{C_{r_z}} A_z(s)\, ds \tag{1.3.5}$$
and [Az6, (3.3.1)]
$$\mathbf{A}_z(r_z) = \frac{1}{2\pi i} \oint_{C_{r_z}} (s - r_z)A_z(s)\, ds, \tag{1.3.6}$$
where C_{r_z} is a contour encircling r_z and no other resonance points. Similarly one defines an idempotent operator $Q_z(r_z)$ and a nilpotent operator $\mathbf{B}_z(r_z)$, by replacing $A_z(s)$ in (1.3.5) and (1.3.6) by $B_z(s)$. For these operators we have [Az6, (3.3.12) and (3.3.11)]
$$\mathbf{B}_z(r_z)V = V\mathbf{A}_z(r_z) \quad \text{and} \quad (\mathbf{A}_z(r_z))^* = \mathbf{B}_{\bar{z}}(\bar{r}_z). \tag{1.3.7}$$
The Laurent expansion of the meromorphic function $A_z(s)$ at a resonance point r_z has the following form [Az6, (3.3.16)]:
$$A_z(s) = \tilde{A}_z(s) + (s - r_z)^{-1}P_z(r_z) + (s - r_z)^{-2}\mathbf{A}_z(r_z) + \cdots + (s - r_z)^{-d}\mathbf{A}_z^{d-1}(r_z), \tag{1.3.8}$$
where d is the order of the resonance point r_z (see below) and $\tilde{A}_z(s)$ is the holomorphic part of the Laurent series. The idempotent operators $P_z(r_z)$ and $Q_z(r_z)$ have the following properties. If $r_z^{(1)}$ and $r_z^{(2)}$ are two different resonance points corresponding to z, then [Az6, (3.2.8)]
$$P_z(r_z^{(1)})P_z(r_z^{(2)}) = 0. \tag{1.3.9}$$

Further, [Az6, (3.2.5)]

$$P_z^*(r_z) = Q_{\bar z}(\bar r_z), \tag{1.3.10}$$

and [Az6, (3.2.10)]

$$VP_z(r_z) = Q_z(r_z)V. \tag{1.3.11}$$

Proofs of these equalities are quite elementary.

The operator $Q_{\bar z}(\bar r_z)VP_z(r_z)$ is a finite-rank self-adjoint operator which in [Az6] is called the *resonance matrix*. Since $P_\lambda(r_\lambda)$ is an idempotent, from (1.3.11) we have

$$VP_\lambda(r_\lambda) = Q_\lambda(r_\lambda)VP_\lambda(r_\lambda), \tag{1.3.12}$$

and so, since λ and r_λ are real and $\lambda \notin \sigma_{\mathrm{ess}}$, taking into account (1.3.10), we conclude that

$$VP_\lambda(r_\lambda) \text{ is self-adjoint and has finite rank.} \tag{1.3.13}$$

This operator plays an important role in this paper.

The vector space $\Upsilon_z(r_z)$ consists of all vectors χ such that for some positive integer k,

$$(1 + (r_z - s)A_z(s))^k\chi = 0, \tag{1.3.14}$$

where s is any number which is not a pole of $A_z(s)$. This definition does not depend on the choice of s. A vector from $\Upsilon_z(r_z)$ will be called the *resonance vector*. The equality (1.3.14) will be called the *resonance equation* of order k. The smallest integer k such that (1.3.14) holds for some (and thus for any) s will be called the *order* of χ. The vector space of resonance vectors of order $\le k$ is denoted by $\Upsilon_z^k(r_z)$. The operator $\mathbf{A}_z(r_z)$ maps $\Upsilon_z^k(r_z)$ onto $\Upsilon_z^{k-1}(r_z)$, that is, $\mathbf{A}_z(r_z)$ lowers the order of a resonance vector by 1. A resonance vector χ has *depth* at least k if $\chi \in \operatorname{im}\mathbf{A}_z^k(r_z)$, and has depth k if it has depth at least k but not of at least $k+1$.

The dimensions of $\Upsilon_z(r_z)$ and $\Upsilon_z^1(r_z)$ are denoted by N and m respectively. The vector space $\Upsilon_z^1(r_z)$ is the eigenspace of $H_{r_z} = H_0 + r_zV$ corresponding to the eigenvalue z, and we also denote it by $\mathcal{V}_z(r_z)$ or $\mathcal{V}_z$ if there is no danger of confusion.

The smallest positive integer d such that $\Upsilon_z(r_z) = \Upsilon_z^d(r_z)$ will be called the *order* of the perturbation V at H_{r_z} and the *order* of the resonance point r_z. A regular direction V is said to be *simple* at a λ-resonance point H if V has order 1.

Jordan decomposition of the nilpotent operator $\mathbf{A}_z(r_z)$ consists of m Jordan blocks; we use lower case Greek letters ν and μ to enumerate them. We denote the size of the νth block by d_ν and assume that $d_1 \ge \cdots \ge d_m$. A basis

$$\chi_\nu^{(j)}, \qquad \nu = 1, \dots, m,\ j = 0, 1, \dots, d_\nu - 1,$$

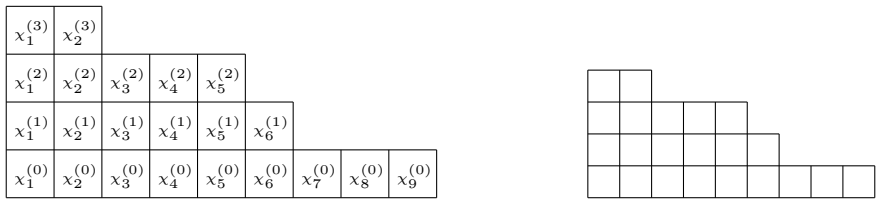
of $\Upsilon_z(r_z)$ is a *Jordan basis* if

$$\mathbf{A}_z(r_z)\chi_\nu^{(j)} = \chi_\nu^{(j-1)},$$

where it is assumed that $\chi_\nu^{(-1)} = 0$. In particular,

$$\ker(\mathbf{A}_z(r_z)) \cap \Upsilon_z(r_z) = \Upsilon_z^1(r_z) = \mathcal{V}_z.$$

A Jordan basis can be depicted by either of the two Young diagrams

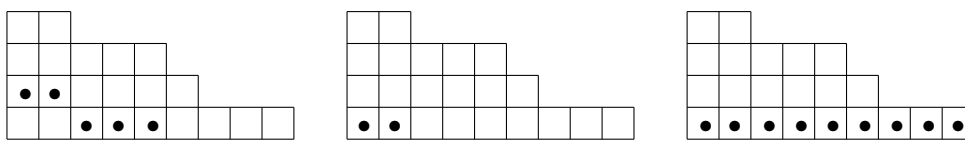


In such a diagram each square represents a resonance vector from a Jordan basis, and the height of the square is the order of the vector. The number of squares is N, the width of the diagram is m, and its height is d. Each Jordan basis defines a direct sum decomposition of the resonance vector space $\Upsilon_z(r_z)$, the νth summand of which we denote by $\Upsilon_z^{[\nu]}(r_z)$. Thus,

$$\Upsilon_z(r_z) = \Upsilon_z^{[1]}(r_z) \dotplus \cdots \dotplus \Upsilon_z^{[m]}(r_z).$$

The height of the νth column in the Young diagram is $d_\nu = \dim \Upsilon_z^{[\nu]}(r_z)$.

For example, the first two of the following three Young diagrams show those elements of a Jordan basis which have depths 2 and 3, respectively, while the third diagram shows vectors of order 1, which form a basis of the eigenspace $\Upsilon_z^1(r_z) = \mathcal{V}_z$.



The vector space $\Upsilon_z^1(z; H_0, V)$ depends only on the operator $H_{r_z} = H_0 + r_z V$ and does not depend on V, but the vector spaces $\Upsilon_z^k(r_z; H_0, V)$, $k \geq 2$, depend on both H_{r_z} and V [Az6, Section 3].

A complex number r_z is a resonance point iff

$$\sigma_z(s) = (s - r_z)^{-1}$$

is an eigenvalue of $A_z(s)$. The eigenvalue $\sigma_z(s)$ has algebraic multiplicity N and geometric multiplicity m.

Though z can be any complex number outside essential spectrum, we are mainly interested in the case where z and the corresponding resonance point r_z are real numbers. In this case, if a real number λ is shifted to $\lambda + iy$ with small $y > 0$, then the real eigenvalue $\sigma_\lambda(s)$ of $A_\lambda(s)$ splits into N_+ and N_- (where $N_\pm \geq 0$, $N_+ + N_- \geq 1$) eigenvalues in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, and all shifted eigenvalues are non-real. The *resonance index* of the triple $(\lambda; H_{r_\lambda}, V)$ is by definition the difference

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = N_+ - N_-. \tag{1.3.15}$$

The objects such as $P_z(r_z)$, $\Upsilon_z(r_z)$, etc., depend on H_0 and V too, but since for the most part the operators H_0 and V are fixed, usually we do not indicate this dependence. If necessary we write $P_z(r_z; H_0, V)$, etc., or $P_z(H_{r_z}, V)$, etc., where $H_{r_z} = H_0 + r_z V$. The notation such as $P_z(H_{r_z}, V)$ is not ambiguous, since $P_z(H_{r_z}, V)$ depends on H_{r_z} and V but not on H_0.

1.4. Description of results

1.4.1. Section 2. Let H_{r_λ} be a resonance point, V a regular direction and

$$H_s = H_{r_\lambda} + (s - r_\lambda)V.$$

We use the following notation: $\mathcal{V}_\lambda$ is the eigenspace of H_{r_λ} corresponding to the eigenvalue λ,

$$\hat{\mathcal{H}} = \mathcal{V}_\lambda^\perp,$$

and $\hat{P}$ is the orthogonal projection onto $\hat{\mathcal{H}}$. Further, we let

$$\hat{H}_s = \hat{P}H_s\hat{P}, \quad \hat{V} = \hat{P}V\hat{P}, \quad v = \hat{P}V\hat{P}^\perp,$$

and $R_\lambda(\hat{H}_s)$ is the resolvent of $\hat{H}_s$ on $\hat{\mathcal{H}} = \mathcal{V}_\lambda^\perp$ and zero on $\mathcal{V}_\lambda$. In analogy with (1.3.3), we let

$$\hat{A}_\lambda(s) = R_\lambda(\hat{H}_s)\hat{V}.$$

The product $R_\lambda(\hat{H}_{r_\lambda})V$ will be encountered quite often, and, as we shall see later, its negative can be interpreted as a near inverse of $\mathbf{A}_\lambda(r_\lambda)$. For this reason we shall introduce a special notation for this product:

$$S_\lambda = R_\lambda(\hat{H}_{r_\lambda})V.$$

THEOREM 1.4.1 (Theorem 2.1.4). *Let $k \geq 2$. If $\hat{\varphi} \in \hat{\mathcal{H}}$ is a resonance vector of order k then $\hat{\varphi}$ belongs to the linear subspace*

$$\operatorname{im} R_\lambda(\hat{H}_{r_\lambda})v \dotplus \operatorname{im} \hat{A}_\lambda(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v \dotplus \cdots \dotplus \operatorname{im} \hat{A}_\lambda^{k-2}(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})v.$$

The following theorem provides a criterion for a resonance vector to have depth at least 1. This criterion is used several times in the remaining sections.

THEOREM 1.4.2 (Theorem 2.3.1). *For a resonance vector χ the following three assertions are equivalent:*

(1) $V\chi \perp \mathcal{V}_\lambda$.
(2) χ *has depth at least* 1.
(3) $\mathbf{A}_\lambda(r_\lambda)S_\lambda\chi = -\chi$.

In particular, if $V\chi \perp \mathcal{V}_\lambda$, then $S_\lambda\chi$ is a resonance vector.

The operator S_λ satisfies the following equality.

THEOREM 1.4.3 (Theorem 2.4.4). *Under Assumption* 1.2.1*, for all $j = 1, \ldots, d$,*

$$-S_\lambda\mathbf{A}_\lambda^j(r_\lambda) = \hat{P}\mathbf{A}_\lambda^{j-1}(r_\lambda).$$

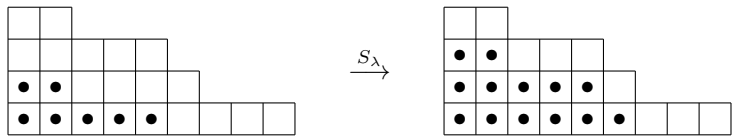
The last two theorems show that $-S_\lambda$ behaves to a certain extent as the inverse of the nilpotent operator $\mathbf{A}_\lambda(r_\lambda)$, in particular S_λ increases the order of a resonance vector by 1, if there is room for that. Since $\mathbf{A}_\lambda(r_\lambda)$ decreases order by 1 and increases depth by 1, this raises a natural question of whether S_λ decreases depth by 1. This assertion is not proved, but Theorem 2.6.1 provides a criterion for this property.

THEOREM 1.4.4 (Theorem 2.6.1). *Under Assumption* 1.2.1*, the following assertions are equivalent:*

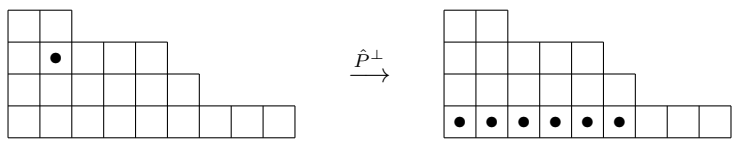
(1) *For all $j = 1, \ldots, d-1$,* $\operatorname{im}(S_\lambda\mathbf{A}_\lambda^j) \subset \operatorname{im}(\mathbf{A}_\lambda^{j-1})$.
(2) *For all $j = 1, \ldots, d-1$,* $\operatorname{im}(\hat{P}^\perp\mathbf{A}_\lambda^{j-1}) \subset \operatorname{im}(\mathbf{A}_\lambda^{j-1})$.

That is, the operator S_λ decreases the depth of resonance vectors by no more than 1 if and only if the orthogonal projection $\hat{P}^\perp$ onto the eigenspace $\mathcal{V}_\lambda = \Upsilon^1_\lambda(r_\lambda)$ does not decrease depth.

Item (1) of Theorem 1.4.4 can be illustrated by the following picture:



Item (2) can be illustrated by the following example: the projection of a vector of depth 1, shown in the left Young diagram, into $\mathcal{V}_\lambda$ is a linear combination of eigenvectors of depth at least 1, shown in the right Young diagram.



1.4.2. Section 3. The eigenvalue λ of multiplicity m of the self-adjoint operator H_{r_λ} splits into m analytic eigenvalue functions $\lambda_\nu(s)$, $\nu = 1, \dots, m$, of the operator $H_s = H_{r_\lambda} + (s - r_\lambda)V$, so that $\lambda_\nu(r_\lambda) = \lambda$ for all $\nu = 1, \dots, m$. The corresponding eigenvector function is denoted by $\varphi_\nu(s)$. The eigenvalue functions $\lambda_\nu(s)$ are not necessarily distinct, and if they are not, we list them according to their multiplicities, but in any case the analytic eigenvector functions $\varphi_\nu(s)$ can be chosen to be pairwise orthogonal: for any s and any $\nu \neq \mu$, $\langle \varphi_\nu(s), \varphi_\mu(s) \rangle = 0$. We assume such a choice throughout this paper.

THEOREM 1.4.5 (Theorem 3.1.7). *Under Assumption* 1.2.1, *let* $k \geq 2$ *and* $\varphi(s)$ *be an analytic path of eigenvectors of* $H_s = H_0 + sV$. *Then the following assertions are equivalent:*

(i) *The vectors*
$$V\varphi(r_\lambda), V\varphi'(r_\lambda), \dots, V\varphi^{(k-2)}(r_\lambda)$$
are orthogonal to the eigenspace $\mathcal{V}_\lambda$.

(ii) *The vectors*
$$V\varphi(r_\lambda), V\varphi'(r_\lambda), \dots, V\varphi^{(k-2)}(r_\lambda)$$
are orthogonal to the vector $\varphi(r_\lambda)$.

(iii) *The equalities*
$$\lambda'(r_\lambda) = 0, \dots, \lambda^{(k-1)}(r_\lambda) = 0$$
hold, where $\lambda(s)$ *is an analytic path of eigenvalues of* H_s *which corresponds to* $\varphi(s)$.

(iv) *For all* $j = 1, \dots, k-1$,
$$(H_{r_\lambda} - \lambda)\varphi^{(j)}(r_\lambda) = -jV\varphi^{(j-1)}(r_\lambda).$$

(v) *For all* $j = 1, \dots, k-1$,
$$\mathbf{A}_\lambda(r_\lambda)\varphi^{(j)}(r_\lambda) = j\varphi^{(j-1)}(r_\lambda).$$

(vi) $\varphi(r_\lambda)$ *is an eigenvector of depth at least* $k-1$.

An eigenpath $\varphi(s)$ will be said to have *order* at least k if it has any of these properties. This definition is correct in the sense that an eigenpath $\varphi(s)$ has order k if and only if an eigenpath $a(s)\varphi(s)$ has order k for any analytic function $a(s)$ such that $a(r_\lambda) \neq 0$. We shall also say that an eigenvalue function $\lambda(s)$ has *order* k if the corresponding eigenpath has order k.

According to item (iii) of Theorem 1.4.5, an eigenvalue function of order 2 makes a U-turn at λ, an eigenvalue function of order 3 makes a double U-turn at λ, etc. Schematically, for example, eigenvalue functions of orders 2, 3 and 4 will be depicted using the following figures:

THEOREM 1.4.6 (Lemma 3.1.5). *Under Assumption* 1.2.1, *if an eigenpath* $\varphi(s)$ *of* $H_0 + sV$ *has order at least* k, *then the vectors*

$$\varphi(r_\lambda), \varphi'(r_\lambda), \dots, \varphi^{(k-1)}(r_\lambda)$$

are resonance vectors of orders respectively $1, 2, \dots, k$.

1.4.3. Section 4. It is interesting to find out conditions under which a straight line of operators $H_s = H_{r_\lambda} + (s - r_\lambda)V$ is tangent to the resonance set at a resonance point H_{r_λ}. We say that a direction V is *tangent* to the resonance set $\mathcal{R}(\lambda)$ at H_{r_λ} *to order at least* k, if there exists a resonant path $\{H(s)\} \subset \mathcal{R}(\lambda)$ such that for some (necessarily real) numbers $c_2, \dots, c_{k-1}$,

$$H(s) = H_{r_\lambda} + (s - r_\lambda)V + \sum_{j=2}^{k-1} c_j (s - r_\lambda)^j V + O((s - r_\lambda)^k), \qquad s \to r_\lambda. \tag{1.4.1}$$

In this case we also say that the path $H(s)$ is tangent to V at $H(r_\lambda) = H_{r_\lambda}$ to order at least k. The *order of tangency* of a direction V to $\mathcal{R}(\lambda)$ is the largest positive integer k such that (1.4.1) holds for some resonance path $H(s)$. We say that a direction V is *tangent* at H_{r_λ} if V is tangent to order at least 2. If a direction V is tangent only to order 1 at $H_{r_\lambda} \in \mathcal{R}(\lambda)$, then V is *transversal* at H_{r_λ}.

THEOREM 1.4.7 (Theorem 4.1.2). *Under Assumption* 1.2.1, *let* $k \geq 1$, *let* H_{r_λ} *be a resonance point and* V *be a regular direction at* H_{r_λ}. *If* $H(s)$ *is a resonant path tangent to* V *at* H_{r_λ} *to order at least* k *and if* $\chi(s)$ *is a corresponding analytic eigenpath, then*

(i) *the vectors* $\chi(r_\lambda), \chi'(r_\lambda), \dots, \chi^{(k-1)}(r_\lambda)$ *have orders respectively* $1, 2, \dots, k$,
(ii) *the direction* V *has order at least* k,
(iii) *for any* $j = 1, \dots, k$,

$$\mathbf{A}_\lambda(r_\lambda)\chi^{(k-1)}(r_\lambda) = (k-1)\chi^{(k-2)}(r_\lambda) + \sum_{j=2}^{k-1} j! \binom{k-1}{j} c_j \chi^{(k-1-j)}(r_\lambda),$$

where the numbers $c_2, \dots, c_k$ *are as in* (1.4.1), *and*
(iv) *the eigenvector* $\chi(r_\lambda)$ *has depth at least* $k - 1$.

If a resonance path $H(s)$ is tangent to V to order k at a resonant point H_{r_λ}, then changing the parameter s if necessary we can always make the operators $H''(r_\lambda), \dots, H^{(k-1)}(r_\lambda)$

equal to zero, so that

$$H(s) = H_{r_\lambda} + (s - r_\lambda)V + O((s - r_\lambda)^k).$$

A path of this form will be called *standard*.

According to Theorem 4.1.2, with a resonant path tangent to order k we can associate a set of resonance vectors $\chi_0, \ldots, \chi_{k-1}$ of respective orders $1, \ldots, k$, namely, the first k coefficients

$$\chi_j = \frac{1}{j!}\chi^{(j)}(r_\lambda), \quad j = 0, 1, \ldots, k-1,$$

of the Taylor expansion of a resonance eigenpath $\chi(s)$.

PROPOSITION 1.4.8 (Proposition 4.1.4). *Under Assumption* 1.2.1*, let V be a regular direction at H_{r_λ} and let $H(s)$ be a resonant path tangent to V at H_{r_λ} to order k. The path $H(s)$ is standard if and only if for all $j = 1, \ldots, k-1$,*

$$\mathbf{A}_\lambda(r_\lambda)\chi_j(r_\lambda) = \chi_{j-1}(r_\lambda),$$

where $\chi(s)$ is a corresponding analytic path of eigenvectors.

Given a direction V of order d it is possible to exhibit a resonant path $H(s)$ which is tangent to V to order d. Namely, let χ be an eigenvector of $H(r_\lambda)$ corresponding to the eigenvalue λ, and let $W = \langle \chi, \cdot \rangle \chi$. We consider the intersection of the two-dimensional real affine plane

$$\alpha = H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W$$

with $\mathcal{R}(\lambda)$. A sufficiently small neighbourhood of H_{r_λ} in $\alpha \cap \mathcal{R}(\lambda)$ consists of a unique simple curve (Theorem 4.2.1). We denote this analytic curve by γ_χ. The curve γ_χ can be normalised so that $\gamma_\chi(r_\lambda) = \chi$ (Theorem 4.2.3).

THEOREM 1.4.9 (Theorem 4.3.1). *If χ has depth at least $k-1$ then*

(1) *the analytic curve γ_χ is tangent to V to order k,*
(2) *for any analytic parametrisation $\gamma(s)$ of γ_χ, and any corresponding analytic eigenpath $\chi(s)$, the vectors*

$$\chi(r_\lambda), \chi'(r_\lambda), \ldots, \chi^{(k-1)}(r_\lambda)$$

have orders respectively $1, 2, \ldots, k$, and
(3) *if the parametrisation $\gamma(s)$ of γ_χ is standard then for all $j = 1, \ldots, k-1$,*

$$\mathbf{A}_\lambda(r_\lambda)\chi^{(j)}(r_\lambda) = j\chi^{(j-1)}(r_\lambda).$$

Theorems 4.1.2 and 4.3.1 provide the following geometric interpretation of the order of a direction V.

THEOREM 1.4.10 (Theorem 4.3.2). *The order of tangency of a regular direction V at a resonance point H_{r_λ} to the resonance set is equal to the order of V.*

1.4.4. Section 5. Assume that r_λ is a resonance point of algebraic multiplicity N and geometric multiplicity m. A resonance point r_z depends analytically on z outside the essential spectrum, and as λ varies, the resonance point r_λ splits into up to N resonance points r_z^j. Theorem 5.1.1 asserts that all these resonance points r_z^j have order 1 in some deleted neighbourhood of λ. When z makes one round about λ, these N resonance points undergo a permutation. Theorem 5.2.3 asserts that this permutation is a product

of m disjoint cycles of lengths $d_1, \dots, d_m$. We denote these cycles by

$$r_\nu^{(j)}(z), \quad \nu = 1, \dots, m, \;\; j = 0, \dots, d_\nu - 1.$$

Functions of each cycle $r_\nu^{(\cdot)}(z)$ represent branches of a multivalued holomorphic function. The idempotents $P_z(r_\nu^{(j)}(z))$ which correspond to these resonance points are also multivalued, but the sum

$$P_z^{[\nu]} = \sum_{j=0}^{d_\nu - 1} P_z(r_\nu^{(j)}(z))$$

is single-valued in a neighbourhood of λ. Theorem 5.3.1(1) asserts that this sum admits analytic continuation to the point λ. Thus, the limit operator $P_\lambda^{[\nu]}(r_\lambda)$ is defined. Similarly, one can define operators $Q_z^{[\nu]}$, or they can also be defined by $Q_z^{[\nu]} = (P_{\bar z}^{[\nu]})^*$. The operators $P_z^{[\nu]}$, including the case of $z = \lambda$, have the following properties:

$$P_z^{[\nu]} P_z^{[\mu]} = \delta_{\nu\mu} P_z^{[\nu]}, \quad \mathbf{A}_z(r_\lambda) P_z^{[\nu]} = P_z^{[\nu]} \mathbf{A}_z(r_\lambda), \quad P_z(r_\lambda) = \sum_{\nu=1}^{m} P_z^{[\nu]},$$

and

$$V P_z^{[\nu]} = Q_z^{[\nu]} V.$$

Here $P_z(r_\lambda)$ (respectively, $\mathbf{A}_z(r_\lambda)$) is the sum of the idempotents $P_z(r_\nu^{(j)}(z))$ (respectively, nilpotent operators $\mathbf{A}_z(r_\nu^{(j)}(z))$) over all resonance points $r_\nu^{(j)}(z)$ of the group of r_λ. In particular, the image $\Upsilon_\lambda^{[\nu]}$ of $P_\lambda^{[\nu]}$ reduces $\mathbf{A}_\lambda(r_\lambda)$. This reduction is denoted by $\mathbf{A}_\lambda^{[\nu]}$. The vector space $\Upsilon_\lambda^{[\nu]}$ has dimension d_ν and the vectors

$$\varphi(r_\lambda), \varphi'(r_\lambda), \dots, \varphi^{(d_\nu - 1)}(r_\lambda)$$

form its basis (Theorem 5.3.1(2)). The Jordan cell decomposition of the restriction of this operator to $\Upsilon_\lambda^{[\nu]}$ consists of only one Jordan cell (Theorem 5.3.1(5)).

Proposition 5.5.2 asserts that in the Puiseux series (5.5.2)

$$r_\nu^{(j)}(z) = \sum_{k=0}^{\infty} r_{k/d_\nu} \varepsilon_{d_\nu}^{kj} (z - \lambda)^{k/d_\nu}, \quad j = 0, \dots, d_\nu - 1,$$

of the function $r_\nu^{(\cdot)}(z)$ the coefficients r_{k/d_ν} are real and $r_{1/d_\nu} \neq 0$.

Theorem 5.2.1 and Proposition 5.2.2 assert that there is a natural one-to-one correspondence between cycles $r_\nu^{(\cdot)}(z)$ and eigenvalue functions $\lambda_\nu(\cdot)$, and therefore with eigenpaths $\varphi_\nu(\cdot)$. Namely, restriction of one of the functions $r_\nu^{(0)}(z)$ of a cycle $r_\nu^{(\cdot)}(z)$ to at least one of the half-intervals $[\lambda, \lambda + \varepsilon)$ or $(\lambda - \varepsilon, \lambda]$ takes real values, and it is the inverse of $\lambda_\nu(s)$ in some left or right neighbourhood of r_λ. Such a function $r_\nu^{(0)}(z)$ is unique in the sense that there is no other branch of $r_\nu^{(\cdot)}(z)$ which takes real values when restricted to the same half-interval as $r_\nu^{(0)}(z)$. Theorem 5.2.1 provides more properties of this one-to-one correspondence.

Theorem 5.2.3 provides further information about this correspondence. It asserts that for each $\nu = 1, \dots, m$ the following numbers are equal (d_ν is their common value):

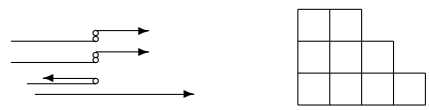
(1) the order of the eigenpath $\varphi_\nu(s)$ (see Theorem 1.4.5),
(2) the size of the cycle $r_\nu^{(\cdot)}(z)$,
(3) the size of the νth Jordan cell of $R_\lambda(H_s)V$ corresponding to eigenvalue $(s - r_\lambda)^{-1}$,

and that the set of vectors

$$\varphi_\nu(r_\lambda), \varphi'_\nu(r_\lambda), \dots, \varphi_\nu^{(d_\nu-1)}(r_\lambda) \tag{1.4.2}$$

is a basis of the range $\Upsilon_\lambda^{[\nu]}(r_\lambda)$ of the idempotent $P_\lambda^{[\nu]}$. In particular, the set of vectors (1.4.2), $\nu = 1, \dots, m$, is a Jordan basis of $\Upsilon_\lambda(r_\lambda)$.

For example, for $m = 4$ and $d_1 = d_2 = 3$, $d_3 = 2$, $d_4 = 1$, the following figures demonstrate the correspondence between items (1) and (3):



The Puiseux series of the idempotent $P_z(r_\nu^{(j)}(z))$ has the form (Proposition 5.6.4)

$$P_z(r_\nu^{(j)}(z)) = \tilde{P}_\nu^{(j)}(z) + \sum_{l=0}^{d_\nu-1} e^{-2\pi i l j/d_\nu} (z-\lambda)^{-l/d_\nu} P_{-l/d_\nu},$$

where $\tilde{P}_\nu^{(j)}(z)$ is continuous at $z = \lambda$. The informative part of this formula is the upper summation limit $d_\nu - 1$. Further, for each $\nu = 1, \dots, m$ and for all $k \geq 0$ (Proposition 5.6.5),

$$\lim_{z\to\lambda} \sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda)^k P_z(r_\nu^{(j)}(z)) = P_\lambda^{[\nu]}(r_\lambda) \mathbf{A}_\lambda^k(r_\lambda).$$

The following theorem provides a relationship between different eigenpaths $\varphi_\nu(s)$.

THEOREM 1.4.11 (Theorem 3.1.12, Corollary 5.3.2). *Let $\varphi_\nu(s)$ and $\varphi_\mu(s)$ be two different eigenpaths of H_s, $s \in \mathbb{R}$. For all $j = 0, 1, \dots, d_\nu - 1$ and all $k = 0, 1, \dots, d_\mu - 1$,*

$$\langle \varphi_\nu^{(j)}(r_\lambda), V \varphi_\mu^{(k)}(r_\lambda) \rangle = 0.$$

This is a stronger version of Theorem 3.1.12 in that it shows that the numbers d_ν add up to N.

Proposition 5.6.6 gives an expression for the restriction of a power of the operator $\mathbf{A}_\lambda(r_\lambda)$ to the vector space $\Upsilon_\lambda^{[\nu]}(r_\lambda)$ via Puiseux coefficients of $r_\nu^{(j)}(z)$ and $P_z(r_\nu^{(j)}(z))$: for all $\nu = 1, \dots, m$ and all $k = 1, \dots, d_\nu - 1$,

$$P_\lambda^{[\nu]}(r_\lambda) \mathbf{A}_\lambda^k(r_\lambda) = d_\nu \sum_{l=k}^{d_\nu-1} \Big(\sum_{m_1+\dots+m_k=l} r_{m_1/d_\nu} \dots r_{m_k/d_\nu} \Big) P_{-l/d_\nu},$$

where in the sum $m_1, \dots, m_k \geq 1$.

Inside the essential spectrum the geometric meaning of the total resonance index is obscure, but outside the essential spectrum it has a clear geometric interpretation, which is allowed by the fact that r_λ depends on λ analytically. Assume that r_0 is a resonance point corresponding to λ_0, that is, λ_0 is an eigenvalue of $H_0 + r_0 V$. If r_λ is an increasing function of λ in a neighbourhood I of λ_0, and therefore if $\lambda(r)$ is an increasing function of r, then intuitively the contribution of $\lambda(r)$ to the spectral flow through λ_0 is $+1$. Since r_λ is an increasing function, the derivative $\frac{d}{d\lambda} r_\lambda$ is positive on I. Therefore, according to the geometric interpretation of the derivative of a holomorphic function, when $z = \lambda_0$ is perturbed to $z = \lambda_0 + iy$ with small $y > 0$, the real value r_0 of the function r_z rotates

towards the half-plane $\mathbb{C}_+$. This gives a contribution of $+1$ to the total resonance index. Similarly, a decreasing function r_λ contributes -1 to TRI.

A more interesting situation occurs if at some value r_0 of the coupling constant the derivative $\lambda'(r_0)$ vanishes. Geometrically, this means that the eigenvalue $\lambda(r)$ of H_r stops at λ_0 when the coupling constant attains r_0 and may either turn back or go through λ_0, thus contributing -1, 0, or $+1$ to the spectral flow through λ_0. Since $\lambda'(r_0)$ vanishes, the inverse function $r(z)$ is not single-valued in a neighbourhood of $z = \lambda_0$. Accordingly, when λ_0 is shifted to $\lambda_0 + iy$ with $y > 0$, the resonance point r_0 splits into two or more resonance points. About one half d_+ of those resonance points goes towards $\mathbb{C}_+$ and another half d_- goes towards $\mathbb{C}_-$, resulting in the contribution of $d_+ - d_-$ to the total resonance index. The overall number $d = d_+ + d_-$ of resonance points equals the smallest positive number d such that

$$\left.\frac{d^d\lambda}{dr^d}\right|_{r=r_0} \neq 0.$$

Moreover, those d resonance points undergo a cyclic permutation when z makes one round about λ_0. Combining this with the fact that r_z cannot be real for non-real z, one can already infer that $d_+ - d_-$ is necessarily equal to the contribution of $\lambda(r)$ to the spectral flow. If λ has geometric multiplicity $m > 1$, then this argument applies to each of the m eigenvalue functions $\lambda_\nu(s)$. This formal observation is made precise in Theorems 5.7.1 and 5.8.1.

THEOREM 1.4.12 (Theorem 5.7.1). *For each $\nu = 1, \dots, m$ and for all small enough $\varepsilon > 0$ and $y > 0$, the signs of the following real numbers coincide:*

(1) $\lambda_\nu(r_\lambda + \varepsilon) - \lambda_\nu(r_\lambda)$,
(2) $\langle \varphi_\nu(r_\lambda), V\varphi_\nu^{(d_\nu - 1)}(r_\lambda)\rangle$,
(3) $\operatorname{Im} r_\nu^{(0)}(z + iy)$ *for all z from some small enough left or right neighbourhood of λ on which the branch $r_\nu^{(0)}(z)$ takes real values larger than r_λ (such an interval and such a branch exist and are unique).*

This sign will be called the *sign* of a cycle ν and denoted $\operatorname{sign}(\nu)$. Let

$$b_\nu = \begin{cases} 0 & \text{if } d_\nu \text{ is even,} \\ 1 & \text{if } d_\nu \text{ is odd.} \end{cases}$$

The *intersection number* through λ for a resonance point r_λ can be defined by

$$\sum_{\nu=1}^{m} b_\nu \operatorname{sign}(\nu), \tag{1.4.3}$$

since, according to Theorems 1.4.12(1) and 1.4.5(iii), each eigenvalue function $\lambda_\nu(s)$ which reaches λ at $s = r_\lambda$ contributes $b_\nu \operatorname{sign}(\nu)$ to the spectral flow through λ, where the case $b_\nu = 0$ corresponds to eigenvalues making a U-turn at λ.

THEOREM 1.4.13 (Theorem 5.8.1). *The sum of the intersection numbers* (1.4.3) *of the resonance points of a path H_r, $r \in [0, 1]$, through λ is equal to the total resonance index.*

Theorem 1.4.11 allows us to prove the following two theorems. The first one is an explicit formula for the idempotent $P_\lambda(r_\lambda)$ in terms of resonance vectors of a Jordan

basis. The second asserts that the resonance index equals the signature of the so-called resonance matrix. The latter holds for λ inside the essential spectrum too [Az6, Section 9], but here we provide a new and simpler proof for $\lambda \notin \sigma_{\text{ess}}$.

THEOREM 1.4.14 (Theorem 5.9.1). *The idempotent operator $P_\lambda(r_\lambda)$ can be written as*

$$P_\lambda(r_\lambda) = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \sum_{k=0}^{d_\mu - 1} \sum_{j=0}^{d_\nu - 1} \frac{1}{k!j!} \alpha_{\mu\nu}^{kj} \langle V\varphi_\mu^{(k)}(r_\lambda), \cdot \rangle \varphi_\nu^{(j)}(r_\lambda),$$

where the $N \times N$ matrix α is a direct sum of self-adjoint skew-upper-triangular Hankel matrices of sizes $d_1, \dots, d_m$.

For the next theorem, see (1.3.15) for the definition of $\operatorname{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V)$ and (1.3.12) for a discussion of the finite-rank self-adjoint operator $VP_\lambda(r_\lambda)$, which we call the *resonance matrix*.

THEOREM 1.4.15 (Theorem 5.10.3). *We have the equality*

$$\operatorname{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V) = \operatorname{sign}(VP_\lambda(r_\lambda)).$$

Here for a finite-rank self-adjoint operator A,

$$\operatorname{sign}(A) = \operatorname{rank}(A_+) - \operatorname{rank}(A_-)$$

is the *signature* of A.

Theorem 1.4.15 is non-trivial even in finite dimensions, in which case it can be and was tested in numerical experiments using MATLAB.

1.4.5. Section 6. Here we study the dependence of the resonance index $\operatorname{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V)$ on the direction V. This requires some topology in $\mathcal{A}$. We postulate that the topology of $\mathcal{A}$ has the following properties: for some non-real complex number z and for some $H_0 \in \mathcal{A}$, (1) $VR_z(H_0)$ continuously depends on V, and (2) $V_1 R_z(H_0) V_2$ is compact and jointly continuously depends on V_1 and V_2. These properties hold if $\mathcal{A}_0$ consists of bounded operators and the topology of $\mathcal{A}_0$ is the uniform topology.

The definition of the topology of $\mathcal{A}_0$ ensures that the resonance points $r_z(H_0, V)$ depend continuously on H_0 and V.

The sets of regular and simple directions are open in the norm of $\mathcal{A}_0$ (Lemma 6.1.4). The restrictions of the mappings

$$\mathcal{A}_0 \ni V \mapsto P_\lambda(H_{r_\lambda}, V) \quad \text{and} \quad \mathcal{A}_0 \ni V \mapsto VP_\lambda(H_{r_\lambda}, V)$$

to the open set of simple directions are continuous in the norm of $\mathcal{A}_0$ (Lemma 6.2.1). Further, the mapping

$$\mathcal{A}_0 \ni V \mapsto \operatorname{ind}_{\text{res}}(\lambda; H_{r_\lambda}, V)$$

is locally constant on the set of simple directions (Theorem 6.3.1).

For regular directions these assertions are not true, but the following theorem holds. Here for simplicity we assume that $r_\lambda = 0$.

THEOREM 1.4.16 (Theorems 6.3.3 and 6.3.4). *Let V be a regular direction at a resonance point H_0. Let W be a small (in the norm of $\mathcal{A}_0$) perturbation of V, and let H_0' be a small perturbation of H_0. Let $r_\lambda^1(H_0', W), r_\lambda^2(H_0', W), \dots$ be resonance points of the triple*

$(\lambda; H_0', W)$ *which belong to the group of the resonance point* $s = 0$ *of the triple* $(\lambda; H_0, V)$. *Then* $\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V)$ *is equal to*

$$\sum_j \operatorname{ind}_{\mathrm{res}}(\lambda; H'_{r_\lambda^j}, W),$$

where the sum is over real resonance points of the group of $s = 0$, *and* $H_r' = H_0' + rW$.

This theorem allows us to prove homotopy stability of the total resonance index.

THEOREM 1.4.17 (Theorem 6.3.5). *Let* H_0 *and* H_1 *be operators from* $\mathcal{A}$ *which are not resonant at* $\lambda \notin \sigma_{\mathrm{ess}}$. *Then there exist neighbourhoods* $\mathcal{U}_0$ *and* $\mathcal{U}_1$ *of* H_0 *and* H_1 *respectively such that for all* $H_0' \in \mathcal{U}_0$ *and all* $H_1' \in \mathcal{U}_1$,

$$\sum_{r\in[0,1]} \operatorname{ind}_{\mathrm{res}}(\lambda; H_r, V) = \sum_{r\in[0,1]} \operatorname{ind}_{\mathrm{res}}(\lambda; H_r', V'),$$

where $V' = H_1' - H_0'$ *and* $H_r' = H_0' + rV'$.

Theorem 6.4.1 asserts that the total resonance index satisfies the Robbin–Salamon axioms for the spectral flow. Since the Robbin–Salamon axioms uniquely identify the spectral flow (Theorem 6.4.2), this proves the equality of the TRI and the spectral flow.

In Subsection 6.5 we give proofs of some well-known properties of the resonance set $\mathcal{R}(\lambda)$, such as: $\operatorname{codim}\mathcal{R}(\lambda) = 1$, $\mathcal{R}(\lambda)$ has no cusps, and any plane section of $\mathcal{R}(\lambda)$ in a neighbourhood of a resonance point H_{r_λ} of multiplicity m consists of no more than m simple curves.

1.4.6. Section 7. In this section we also assume for convenience that the resonance point r_λ is 0. In Subsection 7.1 we observe that the finite-rank self-adjoint operator

$$VP_\lambda(H_0, V),$$

which is called the *resonance matrix* of the triple $(\lambda; H_0, V)$, preserves many properties of the initial direction V: if V is regular then so is VP_λ (Theorem 7.1.1) further, the operators P_λ and $\mathbf{A}_\lambda$ are the same for the triples $(\lambda; H_0, V)$ and $(\lambda; H_0, VP_\lambda)$ (Theorem 7.1.3), and the resonance matrices of the directions V and VP_λ are equal (Theorem 7.1.4). In particular, the resonance indices of $(\lambda; H_0, V)$ and $(\lambda; H_0, VP_\lambda)$ coincide (Theorem 7.1.5):

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V) = \operatorname{ind}_{\mathrm{res}}(\lambda; H_0, VP_\lambda).$$

All these assertions in essence follow from the following observation (Theorem 7.1.2):

$$(H_0 + sVP_\lambda - \lambda)^{-1}VP_\lambda = (H_0 + sV - \lambda)^{-1}VP_\lambda.$$

Further, the operators V and VP_λ are *plane homotopic*, that is, a direction V can be deformed to a direction VP_λ in the affine plane generated by these directions without crossing the resonance set (Theorem 7.1.7).

In Subsection 7.2 we give a direct proof of the equality of TRI and the total Fredholm index (Theorem 7.2.1).

1.4.7. Section 8. In this sction we give a direct proof of the equality of the total resonance index and the spectral shift function for $\lambda \notin \sigma_{\mathrm{ess}}$. This is a special case of equality of TRI and the singular spectral shift function [Az6, Section 6], [Az5]. It also follows from the fact that the spectral shift function satisfies the Robbin–Salamon axioms.

The proof highlights a key idea of the proof of the more general result given in [Az6, Section 6], [Az5].

2. 2×2 matrix representations

2.1. 2×2 representation of $A_z(s)$. Recall that we are working in the setting of Assumption 1.2.1. Let H_{r_λ} be a resonance point of multiplicity m. Accordingly, let

$$\mathcal{H} = \hat{\mathcal{H}} \oplus \mathcal{V}_\lambda \tag{2.1.1}$$

be the orthogonal decomposition of the Hilbert space $\mathcal{H}$ on which H_{r_λ} acts into the sum of the m-dimensional eigenspace

$$\mathcal{V}_\lambda = \{\chi \in \mathcal{H} \colon H_{r_\lambda}\chi = \lambda\chi\}$$

and its orthogonal complement which we denote $\hat{\mathcal{H}}$. Operators acting on the Hilbert space (2.1.1) can be written as 2×2 matrices. The operator H_{r_λ} has the matrix representation

$$H_{r_\lambda} = \begin{pmatrix} \hat{H}_{r_\lambda} & 0 \\ 0 & \lambda I_m \end{pmatrix}. \tag{2.1.2}$$

The operator V has the form

$$V = \begin{pmatrix} \hat{V} & v \\ v^* & \mathfrak{a} \end{pmatrix}, \tag{2.1.3}$$

where $\hat{V}$ is a self-adjoint operator in $\hat{\mathcal{H}}$, v is an operator from $\mathcal{V}_\lambda$ to $\hat{\mathcal{H}}$ and $\mathfrak{a}$ is a self-adjoint operator on $\mathcal{V}_\lambda$. In [RoSa, p. 14] the operator $\mathfrak{a}$ is called the *crossing operator*.

We agree to identify an element $\hat{\chi}$ of the Hilbert space $\hat{\mathcal{H}}$ with the element $\left(\begin{smallmatrix} \hat{\chi} \\ 0 \end{smallmatrix}\right)$ of $\mathcal{H}$. Analogously, an operator $v \colon \mathcal{V}_\lambda \to \hat{\mathcal{H}}$ will also be considered as an operator from $\mathcal{H}$ to $\mathcal{H}$. This remark applies to other operators such as $\hat{H}_s$, v^* and $\mathfrak{a}$.

The identity operator on a Hilbert space $\mathcal{K}$ is denoted by $1_{\mathcal{K}}$, but if there is no ambiguity we may write 1 for $1_{\mathcal{K}}$. The same remark applies to scalar operators.

We denote by $\hat{P}$ the orthogonal projection from $\mathcal{H}$ onto $\hat{\mathcal{H}}$. Given two vectors f and g from $\mathcal{H}$, we can form the vector $\hat{P}f + \hat{P}^\perp g$, which can also be written as

$$\begin{pmatrix} \hat{P}f \\ \hat{P}^\perp g \end{pmatrix} \in \hat{\mathcal{H}} \oplus \mathcal{V}_\lambda.$$

But to make formulas less cumbersome, we agree to write a vector $\hat{P}f + \hat{P}^\perp g$ as

$$\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \hat{P}f \\ \hat{P}^\perp g \end{pmatrix} \in \hat{\mathcal{H}} \oplus \mathcal{V}_\lambda.$$

That is, we assume a default application of $\hat{P}$ to the first component and of $\hat{P}^\perp$ to the second component of a two-column, if those components are not already in $\hat{\mathcal{H}}$ and $\mathcal{V}_\lambda$ respectively.

So, we write $\left(\begin{smallmatrix} \chi \\ 0 \end{smallmatrix}\right)$ for $\hat{P}\chi$ and $\left(\begin{smallmatrix} 0 \\ \chi \end{smallmatrix}\right)$ for $\hat{P}^\perp\chi$; with these agreements χ and $\left(\begin{smallmatrix} \chi \\ \chi \end{smallmatrix}\right)$ are two ways to write the same vector. Hopefully this should not create confusion.

Similarly, the number 1 is treated as the identity operator on $\mathcal{H}$, or $\hat{\mathcal{H}}$ or $\hat{\mathcal{H}}^{\perp}$, depending on the context. The components of the operator V in its 2×2 representation (2.1.3) can be defined by

$$\hat{V}=\hat{P}V\hat{P},\quad v=\hat{P}V\hat{P}^{\perp}\quad\text{and}\quad \mathfrak{a}=\hat{P}^{\perp}V\hat{P}^{\perp}.$$

The 2×2 representation of the operator

$$H_s=H_{r_\lambda}+(s-r_\lambda)V$$

is given by

$$H_s=\begin{pmatrix}\hat{H}_s & (s-r_\lambda)v\\ (s-r_\lambda)v^* & \lambda+(s-r_\lambda)\mathfrak{a}\end{pmatrix},$$

where $\hat{H}_s=\hat{H}_{r_\lambda}+(s-r_\lambda)\hat{V}$. According to our agreements above, we can rewrite this formula as

$$H_s=\hat{H}_s+\lambda\hat{P}^{\perp}+(s-r_\lambda)(v+v^*)+(s-r_\lambda)\mathfrak{a}.$$

We shall often use both ways of writing this kind of formulas, but mostly we prefer the matrix version.

In this subsection we consider 2×2 matrix representations of some relevant operators, such as $A_\lambda(s)$, $P_\lambda(r_\lambda)$, etc., with respect to the decomposition (2.1.1) of the Hilbert space $\mathcal{H}$. One of the outcomes of these lengthy and somewhat tedious calculations is Theorem 2.3.1, which will be essentially used a few times in subsequent sections.

The starting point is the following well-known fact of linear algebra.

Lemma 2.1.1. *Assume that A is an invertible operator (with bounded inverse). A block operator*

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

has bounded inverse if and only if the operator $D-CA^{-1}B$ has bounded inverse. In this case,

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}=\begin{pmatrix}A^{-1}+A^{-1}B\mathcal{D}CA^{-1} & -A^{-1}B\mathcal{D}\\ -\mathcal{D}CA^{-1} & \mathcal{D}\end{pmatrix},$$

where

$$\mathcal{D}=(D-CA^{-1}B)^{-1}.$$

We apply this lemma to the inverse of the 2×2 representation of H_s-z. In this case the inverse A^{-1} of the $(1,1)$-entry is the sliced resolvent $R_z(\hat{H}_s)$, and so

$$\begin{aligned}(H_s-z)^{-1}&=\begin{pmatrix}\hat{H}_s-z & (s-r_\lambda)v\\ (s-r_\lambda)v^* & \lambda-z+(s-r_\lambda)\mathfrak{a}\end{pmatrix}^{-1}\\ &=\begin{pmatrix}R_z(\hat{H}_s)+(s-r_\lambda)^2R_z(\hat{H}_s)v\mathcal{D}_z(s)v^*R_z(\hat{H}_s) & (r_\lambda-s)R_z(\hat{H}_s)v\mathcal{D}_z(s)\\ (r_\lambda-s)\mathcal{D}_z(s)v^*R_z(\hat{H}_s) & \mathcal{D}_z(s)\end{pmatrix},\end{aligned}\tag{2.1.4}$$

where

$$\mathcal{D}_z(s)=\big(\lambda-z+(s-r_\lambda)\mathfrak{a}-(s-r_\lambda)^2v^*R_z(\hat{H}_s)v\big)^{-1}.\tag{2.1.5}$$

We use the notation

$$\hat{A}_z(s)=R_z(\hat{H}_s)\hat{V}.$$

We have

$$A_z(s) := (H_s - z)^{-1}V$$
$$= \begin{pmatrix} R_z(\hat{H}_s) + (s - r_\lambda)^2 R_z(\hat{H}_s) v \mathcal{D}_z(s) v^* R_z(\hat{H}_s) & (r_\lambda - s) R_z(\hat{H}_s) v \mathcal{D}_z(s) \\ (r_\lambda - s)\mathcal{D}_z(s) v^* R_z(\hat{H}_s) & \mathcal{D}_z(s) \end{pmatrix} \begin{pmatrix} \hat{V} & v \\ v^* & \mathfrak{a} \end{pmatrix}$$
$$= \begin{pmatrix} \hat{A}_z(s) + (r_\lambda - s) R_z(\hat{H}_s) v \mathcal{D}_z(s) v^* \mathcal{F}_z(s) & R_z(\hat{H}_s) v (1 + (r_\lambda - s)[\ldots]) \\ \mathcal{D}_z(s) v^* \mathcal{F}_z(s) & [\ldots] \end{pmatrix}, \tag{2.1.6}$$

with

$$[\ldots] = \mathcal{D}_z(s)[\mathfrak{a} + (r_\lambda - s) v^* R_z(\hat{H}_s) v]$$

and

$$\mathcal{F}_z(s) = 1_{\hat{\mathcal{H}}} + (r_\lambda - s)\hat{A}_z(s), \tag{2.1.7}$$

where $1_{\hat{\mathcal{H}}}$ is the identity operator on $\hat{\mathcal{H}}$. The operator $\mathcal{F}_z(s)$ is invertible and

$$\mathcal{F}_z^{-1}(s) = 1_{\hat{\mathcal{H}}} + (s - r_\lambda)\hat{A}_z(r_\lambda). \tag{2.1.8}$$

The second resolvent identity implies that

$$\mathcal{F}_z(s)\hat{A}_z(r_\lambda) = \hat{A}_z(s). \tag{2.1.9}$$

From now on we consider the case of $z = \lambda$.

For a regular direction V we have, from (2.1.5),

$$\mathcal{D}_\lambda(s) = \big((s - r_\lambda)\mathfrak{a} - (s - r_\lambda)^2 v^* R_\lambda(\hat{H}_s) v\big)^{-1}. \tag{2.1.10}$$

Lemma 2.1.2. *A direction V given by* (2.1.3) *is regular at a resonance point H_{r_λ} given by* (2.1.2) *if and only if the matrix*

$$\mathfrak{a} + (r_\lambda - s) v^* R_\lambda(\hat{H}_s) v$$

is defined and invertible for some real s.

Proof. This follows from Lemma 2.1.1, the 2×2 representation (2.1.4) of $(H_s - \lambda)^{-1}$, and from the definition (2.1.10) of $\mathcal{D}_\lambda(s)$. ∎

Corollary 2.1.3. *If a direction V given by* (2.1.3) *is regular at H_{r_λ} then for any non-zero eigenvector χ of H_{r_λ} at least one of the two vectors $\mathfrak{a}\chi$, $v\chi$ is non-zero. That is, if V is regular then for any non-zero eigenvector χ the vector $V\chi$ is also non-zero.*

This necessary condition of regularity of V is not sufficient. A simple three-dimensional example can be found in [Az6, §14.6.1].

Using the 2×2 representation (2.1.6) of $A_\lambda(s)$ and the equality (2.1.10), a direct calculation gives

$$A_\lambda(s) = \begin{pmatrix} \hat{A}_\lambda(s) + (r_\lambda - s) R_\lambda(\hat{H}_s) v \mathcal{D}_\lambda(s) v^* \mathcal{F}_\lambda(s) & 0 \\ \mathcal{D}_\lambda(s) v^* \mathcal{F}_\lambda(s) & (s - r_\lambda)^{-1} \end{pmatrix} \tag{2.1.11}$$

and

$$1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s) = \begin{pmatrix} [1_{\hat{\mathcal{H}}} + (s - r_\lambda)^2 R_\lambda(\hat{H}_s) v \mathcal{D}_\lambda(s) v^*] \mathcal{F}_\lambda(s) & 0 \\ (r_\lambda - s)\mathcal{D}_\lambda(s) v^* \mathcal{F}_\lambda(s) & 0 \end{pmatrix}. \tag{2.1.12}$$

The function $A_\lambda(s)$ is not holomorphic at $s = r_\lambda$. As can be seen from (2.1.11), apart from the $(2,2)$-entry of $A_\lambda(s)$, the only factor which fails holomorphicity is $\mathcal{D}_\lambda(s)$; the other terms are holomorphic at $s = r_\lambda$.

One can note that a vector $\hat\varphi \in \hat{\mathcal{H}}$ has order 2 if and only if

$$[1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s)]\begin{pmatrix}\hat\varphi\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \dots\end{pmatrix},$$

where the dots denote a non-zero vector. By (2.1.12), this equality is equivalent to

$$[1_{\hat{\mathcal{H}}} + (s - r_\lambda)^2 R_\lambda(\hat H_s) v\mathcal{D}_\lambda(s)v^*]\mathcal{F}_\lambda(s)\hat\varphi = 0.$$

It follows that $\hat\varphi$ is a vector of order 2 if and only if

$$\begin{aligned}\hat\varphi &= -(s - r_\lambda)^2\mathcal{F}_\lambda^{-1}(s)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat\varphi\\ &= -(s - r_\lambda)^2 R_\lambda(\hat H_{r_\lambda})v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat\varphi.\end{aligned}$$

Thus, for a vector $\hat\varphi$ from $\hat{\mathcal{H}}$,

$$\hat\varphi \in \Upsilon_\lambda^2 \;\Leftrightarrow\; \hat\varphi = -(s - r_\lambda)^2 R_\lambda(\hat H_{r_\lambda})v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat\varphi. \tag{2.1.13}$$

In particular, any resonance vector $\hat\varphi \in \hat{\mathcal{H}}$ of order 2 belongs to the image of $R_\lambda(\hat H_{r_\lambda})v$. The following theorem generalises this to vectors of arbitrary order.

THEOREM 2.1.4. *Let $k \geq 2$. A resonance vector $\hat\varphi$ of order k from $\hat{\mathcal{H}}$ belongs to the linear span of the images of the operators*

$$R_\lambda(\hat H_{r_\lambda})v,\ \hat A_\lambda(r_\lambda)R_\lambda(\hat H_{r_\lambda})v, \dots, \hat A_\lambda^{k-2}(r_\lambda)R_\lambda(\hat H_{r_\lambda})v.$$

Proof. The induction base with $k = 2$ follows from (2.1.13). Assume that the claim holds for a vector of order less than k and let $\hat\varphi_k \in \hat{\mathcal{H}}$ be a vector of order k. Since the operator $1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s)$ decreases the order of a resonance vector by 1, the vector $\hat\varphi_{k-1}(s) = (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))\hat\varphi_k$ has order $k-1$. Using the 2×2 representation (2.1.12) of $1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s)$, we have

$$[1_{\hat{\mathcal{H}}} + (s - r_\lambda)^2 R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*]\mathcal{F}_\lambda(s)\hat\varphi_k = \hat\varphi_{k-1}(s).$$

This can be rewritten as

$$\hat\varphi_k + (s - r_\lambda)^2\mathcal{F}_\lambda^{-1}(s)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat\varphi_k = \mathcal{F}_\lambda^{-1}(s)\hat\varphi_{k-1}(s),$$

which by (2.1.8) is equivalent to

$$\hat\varphi_k = -(s - r_\lambda)^2 R_\lambda(\hat H_{r_\lambda})v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)\hat\varphi_k + \big(1 + (s - r_\lambda)\hat A_\lambda(r_\lambda)\big)\hat\varphi_{k-1}(s).$$

Using induction assumption proves the claim. ∎

2.2. The operator S_λ. In what follows, the product of $R_\lambda(\hat H_{r_\lambda})$ and V will be encountered very often; for this reason, we shall write

$$S_\lambda = R_\lambda(\hat H_{r_\lambda})V = \begin{pmatrix}\hat A_\lambda(r_\lambda) & R_\lambda(\hat H_{r_\lambda})v\\ 0 & 0\end{pmatrix}. \tag{2.2.1}$$

Restriction of the power S_λ^k to $\hat{\mathcal{H}}$ coincides with $\hat A_\lambda^k(r_\lambda)$, while its restriction to $\mathcal{V}_\lambda$ coincides with $\hat A_\lambda^{k-1}(r_\lambda)R_\lambda(\hat H_{r_\lambda})v$.

LEMMA 2.2.1. *For any* $\hat{f} \in \hat{\mathcal{H}}$,

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda \hat{f} = A_\lambda(s)\hat{f} + \begin{pmatrix} (s - r_\lambda)R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*\hat{f} \\ -\mathcal{D}_\lambda(s)v^*\hat{f} \end{pmatrix}.$$

Proof. Since $\hat{f} \in \hat{\mathcal{H}}$, we have $S_\lambda \hat{f} = \hat{A}_\lambda(r_\lambda)\hat{f}$. Hence,

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda \hat{f} = (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))\hat{A}_\lambda(r_\lambda)\hat{f}.$$

From (2.1.8) we have

$$\hat{A}_\lambda(r_\lambda) = (s - r_\lambda)^{-1}(\mathcal{F}_\lambda^{-1}(s) - 1_{\hat{\mathcal{H}}}).$$

Combining the last two equalities and using the 2×2 representation (2.1.12) of the operator $1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s)$, we get

$$\begin{aligned}(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda \hat{f} &= (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))\hat{A}_\lambda(r_\lambda)\hat{f} \\ &= (s - r_\lambda)^{-1}(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))(\mathcal{F}_\lambda^{-1}(s) - 1)\hat{f} \\ &= -(s - r_\lambda)^{-1}(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))\hat{f} \\ &\quad + (s - r_\lambda)^{-1}\begin{pmatrix} [1_{\hat{\mathcal{H}}} + (s - r_\lambda)^2 R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*]\hat{f} \\ (r_\lambda - s)\mathcal{D}_\lambda(s)v^*\hat{f} \end{pmatrix} \\ &= A_\lambda(s)\hat{f} + \begin{pmatrix} (s - r_\lambda)R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*\hat{f} \\ -\mathcal{D}_\lambda(s)v^*\hat{f} \end{pmatrix}. \blacksquare\end{aligned}$$

THEOREM 2.2.2. *If* $\chi \in \mathcal{H}$ *and* $V\chi$ *is orthogonal to the eigenspace* $\mathcal{V}_\lambda$, *then*

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda \chi = A_\lambda(s)\chi. \tag{2.2.2}$$

Proof. (A) The premise $V\chi \perp \mathcal{V}_\lambda$ means that the second component of $V\chi$ in the direct sum $\mathcal{H} = \hat{\mathcal{H}} \oplus \mathcal{V}_\lambda$ is zero, that is, using (2.1.3),

$$v^*\chi + \mathfrak{a}\chi = 0. \tag{2.2.3}$$

(B) Here we prove that if $V\chi \perp \mathcal{V}_\lambda$, then

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})v\chi = \begin{pmatrix} (r_\lambda - s)R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*\chi \\ (s - r_\lambda)^{-1}\chi + \mathcal{D}_\lambda(s)v^*\chi \end{pmatrix}. \tag{2.2.4}$$

Using (2.1.12), we have

$$\begin{aligned}(E) &:= (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat{H}_{r_\lambda})v\chi \\ &= \begin{pmatrix} [1_{\hat{\mathcal{H}}} + (s - r_\lambda)^2 R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*]\mathcal{F}_\lambda(s)R_\lambda(\hat{H}_{r_\lambda})v\chi \\ (r_\lambda - s)\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)R_\lambda(\hat{H}_{r_\lambda})v\chi \end{pmatrix} \\ &= \begin{pmatrix} [1_{\hat{\mathcal{H}}} + (s - r_\lambda)^2 R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*]R_\lambda(\hat{H}_s)v\chi \\ (r_\lambda - s)\mathcal{D}_\lambda(s)v^*R_\lambda(\hat{H}_s)v\chi \end{pmatrix},\end{aligned} \tag{2.2.5}$$

where in the second equality the second resolvent identity (2.1.9) is used. The second component of this vector can be transformed as follows:

$$\begin{aligned}(r_\lambda - s)\mathcal{D}_\lambda(s)v^*R_\lambda(\hat{H}_s)v\chi &= \mathcal{D}_\lambda(s)[\mathfrak{a}\chi + (r_\lambda - s)v^*R_\lambda(\hat{H}_s)v\chi] - \mathcal{D}_\lambda(s)\mathfrak{a}\chi \\ &= (s - r_\lambda)^{-1}\hat{P}^\perp\chi - \mathcal{D}_\lambda(s)\mathfrak{a}\chi,\end{aligned}$$

where the second equality follows from the definition (2.1.10) of $\mathcal{D}_\lambda(s)$. Combining this with (2.2.3) gives

$$(r_\lambda - s)\mathcal{D}_\lambda(s)v^*R_\lambda(\hat H_s)v\chi = (s-r_\lambda)^{-1}\hat P^\perp\chi + \mathcal{D}_\lambda(s)v^*\chi.$$

Substituting the right hand side into (2.2.5) yields

$$\begin{aligned}(E) &= \begin{pmatrix} R_\lambda(\hat H_s)v[\chi + (s-r_\lambda)^2\mathcal{D}_\lambda(s)v^*R_\lambda(\hat H_s)v\chi] \\ (s-r_\lambda)^{-1}\chi + \mathcal{D}_\lambda(s)v^*\chi\end{pmatrix}\\ &= \begin{pmatrix} R_\lambda(\hat H_s)v[\chi + (r_\lambda - s)((s-r_\lambda)^{-1}\hat P^\perp\chi + \mathcal{D}_\lambda(s)v^*\chi)] \\ (s-r_\lambda)^{-1}\chi + \mathcal{D}_\lambda(s)v^*\chi\end{pmatrix}\\ &= \begin{pmatrix} (r_\lambda - s)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\chi \\ (s-r_\lambda)^{-1}\chi + \mathcal{D}_\lambda(s)v^*\chi\end{pmatrix},\end{aligned} \tag{2.2.6}$$

where in the last line we have used $v\hat P^\perp = v$.

(C) By Lemma 2.2.1 applied to the vector $\hat P\chi = \hat\chi$,

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat H_{r_\lambda})\hat V\chi = A_\lambda(s)\hat\chi + \begin{pmatrix}(s-r_\lambda)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\hat\chi \\ -\mathcal{D}_\lambda(s)v^*\hat\chi\end{pmatrix}. \tag{2.2.7}$$

Using this and (2.2.4) gives

$$\begin{aligned}&(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda\chi\\ &\quad= (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat H_{r_\lambda})V\chi\\ &\quad= (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat H_{r_\lambda})\hat V\chi + (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat H_{r_\lambda})v\chi\\ &\quad= A_\lambda(s)\hat\chi + \begin{pmatrix}(s-r_\lambda)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\hat\chi \\ -\mathcal{D}_\lambda(s)v^*\hat\chi\end{pmatrix} + (1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))R_\lambda(\hat H_{r_\lambda})v\chi\\ &\quad= A_\lambda(s)\hat\chi + \begin{pmatrix}(s-r_\lambda)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\hat\chi \\ -\mathcal{D}_\lambda(s)v^*\hat\chi\end{pmatrix} + \begin{pmatrix}(r_\lambda - s)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)v^*\chi \\ (s-r_\lambda)^{-1}\chi + \mathcal{D}_\lambda(s)v^*\chi\end{pmatrix}.\end{aligned} \tag{2.2.8}$$

Since $v^*\chi = v^*\hat\chi$, we get

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = A_\lambda(s)\hat\chi + (s-r_\lambda)^{-1}\hat P^\perp\chi.$$

Since $\hat P^\perp\chi$ is an eigenvector of H_{r_λ}, we have $A_\lambda(s)\hat P^\perp\chi = (s-r_\lambda)^{-1}\hat P^\perp\chi$. Hence,

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = A_\lambda(s)\hat\chi + A_\lambda(s)\hat P^\perp\chi = A_\lambda(s)\chi. \ \blacksquare$$

This argument shows that in general (without the assumption $V\chi \perp \mathcal{V}_\lambda$) we have

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = A_\lambda(s)\chi + \begin{pmatrix}(s-r_\lambda)R_\lambda(\hat H_s)v\mathcal{D}_\lambda(s)(v^* + \mathfrak{a})\chi \\ -\mathcal{D}_\lambda(s)(v^* + \mathfrak{a})\chi\end{pmatrix}. \tag{2.2.9}$$

2.3. Depth of resonance vectors and depth criteria. Recall that a resonance vector χ has *depth at least* k if

$$\chi \in \operatorname{im}(\mathbf{A}_\lambda^k).$$

A resonance vector has depth k if it has depth at least k but not at least $k+1$.

The following theorem provides a criterion for a resonance vector to have depth ≥ 1. This result will be used quite a few times in subsequent sections.

THEOREM 2.3.1. *Let χ be a resonance vector. The following assertions are equivalent:*

(i) *The vector $V\chi$ is orthogonal to $\mathcal{V}_\lambda$.*
(ii) *The depth of χ is at least* 1.
(iii) $\mathbf{A}_\lambda(r_\lambda)S_\lambda\chi = -\chi$.

Proof. (i)⇒(iii). By (1.3.8), the meromorphic operator-valued function $A_\lambda(s)$ has the following Laurent expansion at $s = r_\lambda$:

$$A_\lambda(s) = \tilde{A}_\lambda(s) + (s - r_\lambda)^{-1}P_\lambda + \sum_{j=1}^{d-1}(s - r_\lambda)^{-j-1}\mathbf{A}_\lambda^j. \tag{2.3.1}$$

From this we obtain

$$1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s) = (r_\lambda - s)\tilde{A}_\lambda(s) + (1 - P_\lambda) - \sum_{j=1}^{d-1}(s - r_\lambda)^{-j}\mathbf{A}_\lambda^j.$$

Applying the latter series to the vector $S_\lambda\chi$ and retaining on the right hand side only the term $(s - r_\lambda)^{-1}$ gives

$$(1_{\mathcal{H}} + (r_\lambda - s)A_\lambda(s))S_\lambda\chi = \cdots - (s - r_\lambda)^{-1}\mathbf{A}_\lambda S_\lambda\chi + \cdots .$$

Since χ is a resonance vector, so that $P_\lambda\chi = \chi$, applying the operator (2.3.1) to χ also gives

$$A_\lambda(s)\chi = \cdots + (s - r_\lambda)^{-1}\chi + \cdots .$$

Since $V\chi \perp \mathcal{V}_\lambda$, according to Theorem 2.2.2 the last two Laurent expansions are equal, and therefore $\mathbf{A}_\lambda S_\lambda\chi = -\chi$.

(iii)⇒(ii). This immediately follows from the definition of depth.

(ii)⇒(i). Let χ be a resonance vector of depth at least 1. By definition of depth, there exists a resonance vector χ' such that $\mathbf{A}_\lambda\chi' = \chi$. Hence, for any eigenvector $\varphi \in \mathcal{V}_\lambda$,

$$\langle V\chi, \varphi\rangle = \langle V\mathbf{A}_\lambda\chi', \varphi\rangle = \langle V\chi', \mathbf{A}_\lambda\varphi\rangle = 0,$$

where the last equality holds since $\mathbf{A}_\lambda(r_\lambda)$ eliminates any eigenvector. ∎

COROLLARY 2.3.2. *We have*

$$(v^* + \mathfrak{a})\mathbf{A}_\lambda = 0.$$

Proof. Let $f \in \mathcal{H}$ and $\chi = \mathbf{A}_\lambda f$. Then χ is a resonance vector of depth at least 1. So, by Theorem 2.3.1, the vector $V\chi$ is orthogonal to $\mathcal{V}_\lambda$. In terms of the decomposition $\hat{\mathcal{H}} \oplus \mathcal{V}_\lambda$ this means that the $\mathcal{V}_\lambda$ component of $V\chi$ is zero. But this component is $(v^* + \mathfrak{a})\mathbf{A}_\lambda f$. ∎

COROLLARY 2.3.3. *The kernel of the crossing operator $\mathfrak{a}$ consists of eigenvectors of depth at least* 1, *that is,*

$$\ker(\mathfrak{a}) \subset \operatorname{im}(\mathbf{A}_\lambda),$$

where by $\ker(\mathfrak{a})$ *we mean* $\ker(\mathfrak{a}) \cap \mathcal{V}_\lambda$.

Proof. Let $\chi \in \mathcal{V}_\lambda$ be an eigenvector of H_{r_λ} such that $\mathfrak{a}\chi = 0$. Then

$$V\chi = \begin{pmatrix} \hat{V} & v \\ v^* & \mathfrak{a} \end{pmatrix}\begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} v\chi \\ \mathfrak{a}\chi \end{pmatrix} = \begin{pmatrix} v\chi \\ 0 \end{pmatrix}.$$

Hence $V\chi \perp \mathcal{V}_\lambda$, and so, by Theorem 2.3.1, χ has depth at least 1. ∎

QUESTION. Are the items of this theorem also equivalent to this one: (iv) $S_\lambda\chi$ is a resonance vector? The implication (i)⇒(iv) follows from Theorem 2.2.2 and (1.3.14), so the question is whether (iv)⇒(iii).

2.4. 2×2 representations of $P_\lambda(r_\lambda)$ and $\mathbf{A}_\lambda(r_\lambda)$. In this subsection we obtain 2×2 matrix representations of the operators $P_\lambda(r_\lambda)$ and $\mathbf{A}^j_\lambda(r_\lambda)$, $j=1,2,\dots$. These formulas are used to prove that

$$-S_\lambda\mathbf{A}^j_\lambda=\hat P\mathbf{A}^{j-1}_\lambda,$$

which, in addition to Theorem 2.3.1, is yet another indication that $-S_\lambda$ is a kind of "near inverse" of the nilpotent operator $\mathbf{A}_\lambda$. Since $\mathbf{A}_\lambda$ increases the depth of resonance vectors by 1, another property one would expect from the "near inverse" $-S_\lambda$ is that it decreases the depth of vectors by no more than 1. I have spent quite some time trying to prove this conjecture, but all I could prove (Theorem 2.6.1) is that this property of S_λ is equivalent to the following statement: the orthogonal projection onto the eigenspace $\mathcal{V}_\lambda$ does not decrease the depth of resonance vectors.

The results of subsequent sections do not rely on the remaining part of this section.

The meromorphic operator-valued function $\mathcal{D}_\lambda(s)$ defined in (2.1.10) acts on the finite-dimensional Hilbert space $\mathcal{V}_\lambda$. Since the function

$$\mathcal{D}^{-1}_\lambda(s)=(s-r_\lambda)\mathfrak{a}-(s-r_\lambda)^2v^*R_\lambda(\hat H_s)v$$

is holomorphic at $s=r_\lambda$, by the analytic Fredholm alternative, the Laurent series of $\mathcal{D}_\lambda(s)$ at the pole r_λ has finitely many terms with negative powers.

LEMMA 2.4.1. *The meromorphic function $\mathcal{D}_\lambda(s)$ has a pole of order d at $s=r_\lambda$.*

Proof. Since by (2.3.1) the operator $A_\lambda(s)$ has a pole of order d at $s=r_\lambda$, it follows from the 2×2 representation (2.1.11) that $\mathcal{D}_\lambda(s)$ has a pole of order at least d at $s=r_\lambda$, since the other factors in this 2×2 representation are holomorphic at $s=r_\lambda$. That the order of this pole is not greater than d can be observed from the second resolvent identity

$$R_\lambda(H_s)=R_\lambda(H_{s_0})-(s-s_0)R_\lambda(H_s)VR_\lambda(H_{s_0})$$

and the fact that $\mathcal{D}_\lambda(s)$ is the $(2,2)$-entry of $R_\lambda(H_s)$, since the right hand side of the resolvent identity above has a pole of order at most d. ∎

This lemma shows that the Laurent expansion of $\mathcal{D}_\lambda(s)$ can be written as

$$\mathcal{D}_\lambda(s)=\sum_{j=-d+1}^{\infty}D_{-j}(s-r_\lambda)^{j-1}. \tag{2.4.1}$$

The additional factor $(s-r_\lambda)^{-1}$ is introduced for convenience; also, since we shall be working mainly with coefficients of negative powers of $s-r_\lambda$, we denote the coefficient of $(s-r_\lambda)^{j-1}$ by D_{-j}. Since the meromorphic function $\mathcal{D}_\lambda(s)$ depends only on the triple $(\lambda;H_{r_\lambda},V)$, the operators $D_1,\dots,D_{d-1}$ are also invariants of this triple. Further, since $\mathcal{D}_\lambda(s)$ is self-adjoint for real values of s, the operators D_j are self-adjoint.

LEMMA 2.4.2. *For any $j = d-1, d-2, \dots, 2, 1$ and -1,*

$$D_j\mathfrak{a} = \sum_{k=1}^{d-1-j} D_{j+k} v^* (-\hat{A}_\lambda(r_\lambda))^k R_\lambda(\hat{H}_{r_\lambda}) v,$$

and (for $j = 0$)

$$D_0\mathfrak{a} = 1_{\mathcal{V}_\lambda} + \sum_{k=1}^{d-1} D_k v^* (-\hat{A}_\lambda(r_\lambda))^k R_\lambda(\hat{H}_{r_\lambda}) v.$$

Proof. By definition of $\mathcal{D}_\lambda(s)$, we have

$$1_{\mathcal{V}_\lambda} = (s - r_\lambda)\mathcal{D}_\lambda(s)(\mathfrak{a} + (r_\lambda - s)v^* R_\lambda(\hat{H}_s)v).$$

Replacing $\mathcal{D}_\lambda(s)$ by its Laurent expansion (2.4.1) and the resolvent $R_\lambda(\hat{H}_s)$ by its Neumann expansion gives

$$1_{\mathcal{V}_\lambda} = \sum_{k=-d+1}^{\infty} D_{-k}(s - r_\lambda)^k \Big(\mathfrak{a} + \sum_{j=0}^{\infty} (-1)^{j+1}(s - r_\lambda)^{j+1} v^* \hat{A}_\lambda^j(r_\lambda) R_\lambda(\hat{H}_{r_\lambda}) v\Big).$$

Comparing powers of $s - r_\lambda$ on both sides gives the required equalities. ∎

In the next theorem we obtain 2×2 representations of the operators $\mathbf{A}_\lambda^j(r_\lambda)$, $j = 1, \dots, d-1$, $P_\lambda(r_\lambda)$ and $\tilde{A}_\lambda(r_\lambda)$. To simplify the resulting formulas we use the notation

$$Y_j := R_\lambda(\hat{H}_{r_\lambda}) v D_j v^*$$

and

$$\{\hat{A}_\lambda^l(r_\lambda), Y_j\} = \hat{A}_\lambda^l(r_\lambda) Y_j + \hat{A}_\lambda^{l-1}(r_\lambda) Y_j \hat{A}_\lambda(r_\lambda) + \hat{A}_\lambda^{l-2}(r_\lambda) Y_j \hat{A}_\lambda^2(r_\lambda) + \dots + Y_j \hat{A}_\lambda^l(r_\lambda).$$

In particular,

$$\{\hat{A}_\lambda^0(r_\lambda), Y_j\} = Y_j.$$

We also observe that

$$\hat{A}_\lambda(r_\lambda)\{\hat{A}_\lambda^l(r_\lambda), Y_j\} = \{\hat{A}_\lambda^{l+1}(r_\lambda), Y_j\} - Y_j \hat{A}_\lambda^{l+1}(r_\lambda). \tag{2.4.2}$$

THEOREM 2.4.3. *The 2×2 representations of operators $\mathbf{A}_\lambda^j(r_\lambda)$, $j = 1, \dots, d-1$, $P_\lambda(r_\lambda)$ and $\tilde{A}_\lambda(r_\lambda)$ are given by*

$$\mathbf{A}_\lambda^j = \begin{pmatrix} \sum_{k=1}^{d-j-1} (-1)^k \{\hat{A}_\lambda^{k-1}(r_\lambda), Y_{j+k}\} & 0 \\ \sum_{k=0}^{d-j-1} (-1)^k D_{j+k} v^* \hat{A}_\lambda^k(r_\lambda) & 0 \end{pmatrix}, \tag{2.4.3}$$

$$P_\lambda = \begin{pmatrix} \sum_{k=1}^{d-1} (-1)^k \{\hat{A}_\lambda^{k-1}(r_\lambda), Y_k\} & 0 \\ \sum_{k=0}^{d-1} (-1)^k D_k v^* \hat{A}_\lambda^k(r_\lambda) & 1 \end{pmatrix}, \tag{2.4.4}$$

$$\tilde{A}_\lambda(r_\lambda) = \begin{pmatrix} \hat{A}_\lambda(r_\lambda) + \sum_{k=0}^{d-1} (-1)^{k+1} \{\hat{A}_\lambda^k(r_\lambda), Y_k\} & 0 \\ \sum_{k=0}^{d} (-1)^k D_{k-1} v^* \hat{A}_\lambda^k(r_\lambda) & 0 \end{pmatrix}, \tag{2.4.5}$$

where $\tilde{A}_\lambda(s)$ is the holomorphic part of the Laurent expansion of $A_\lambda(s)$ at $s = r_\lambda$ (see (2.3.1)).

In particular,

$$\mathbf{A}_\lambda^{d-1} = \begin{pmatrix} 0 & 0 \\ D_{d-1}v^* & 0 \end{pmatrix}, \quad \mathbf{A}_\lambda^{d-2} = \begin{pmatrix} -Y_{d-1} & 0 \\ D_{d-2}v^* - D_{d-1}v^*\hat{A}_\lambda(r_\lambda) & 0 \end{pmatrix}.$$

These formulas generalise those given in [Az6, §14.4] considering that in this setting the eigenvalue λ is degenerate. Note that the $(2,2)$-entry of P_λ is 1.

Proof of Theorem 2.4.3. The operators P_λ and $\mathbf{A}_\lambda^j$ are coefficients of the Laurent series (2.3.1) of the meromorphic function $A_\lambda(s)$. Combining this with the 2×2 representation (2.1.11) of $A_\lambda(s)$, the Laurent series (2.4.1) for $\mathcal{D}_\lambda(s)$ and the Neumann series

$$R_\lambda(\hat{H}_s) = \sum_{m=0}^{\infty}(-1)^m(s-r_\lambda)^m\hat{A}_\lambda^m(r_\lambda)R_\lambda(\hat{H}_{r_\lambda}), \qquad \mathcal{F}_\lambda(s) = \sum_{k=0}^{\infty}(-1)^k(s-r_\lambda)^k\hat{A}_\lambda^k(r_\lambda),$$

one can calculate 2×2 representations of these operators. For example, the $(1,1)$-entry of $\mathbf{A}_\lambda^j$ is the coefficient of $(s-r_\lambda)^{-j-1}$ in the $(1,1)$-entry of $A_\lambda(s)$, which is

$$\begin{aligned}
(E) &:= \hat{A}_\lambda(s) + (r_\lambda - s)R_\lambda(\hat{H}_s)v\mathcal{D}_\lambda(s)v^*\mathcal{F}_\lambda(s)\\
&= \hat{A}_\lambda(s) + (r_\lambda - s)R_\lambda(\hat{H}_s)v\\
&\quad\times\Big(\sum_{n=-d+1}^{\infty} D_{-n}(s-r_\lambda)^{n-1}\Big)v^*\Big(\sum_{k=0}^{\infty}(-1)^k(s-r_\lambda)^k\hat{A}_\lambda^k(r_\lambda)\Big)\\
&= \hat{A}_\lambda(s) - \sum_{m=0}^{\infty}\sum_{n=-d+1}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}(s-r_\lambda)^{m+n+k}\hat{A}_\lambda^m(r_\lambda)R_\lambda(\hat{H}_{r_\lambda})vD_{-n}v^*\hat{A}_\lambda^k(r_\lambda)\\
&= \hat{A}_\lambda(s) - \sum_{m=0}^{\infty}\sum_{n=-d+1}^{\infty}\sum_{k=0}^{\infty}(-1)^{m+k}(s-r_\lambda)^{m+n+k}\hat{A}_\lambda^m(r_\lambda)Y_{-n}\hat{A}_\lambda^k(r_\lambda)\\
&= \hat{A}_\lambda(s) - \sum_{l=0}^{\infty}\sum_{n=-d+1}^{\infty}(-1)^l(s-r_\lambda)^{n+l}\sum_{\substack{m,k\geq 0\\ m+k=l}}\hat{A}_\lambda^m(r_\lambda)Y_{-n}\hat{A}_\lambda^k(r_\lambda)\\
&= \hat{A}_\lambda(s) - \sum_{l=0}^{\infty}\sum_{n=-d+1}^{\infty}(-1)^l(s-r_\lambda)^{n+l}\{\hat{A}_\lambda^l(r_\lambda), Y_{-n}\}.
\end{aligned}$$

Now we collect in $(\dots)$ the positive powers of $s-r_\lambda$, which are not of interest for us (more precisely, $A = (\dots) + B$ means that A and B are equal up to a function of s holomorphic in a neighbourhood of $s = r_\lambda$):

$$\begin{aligned}
(E) &= (\dots) + \hat{A}_\lambda(r_\lambda) - \sum_{l=0}^{d-1}\sum_{n=-d+1}^{-l}(-1)^l(s-r_\lambda)^{n+l}\{\hat{A}_\lambda^l(r_\lambda), Y_{-n}\}\\
&= (\dots) + \hat{A}_\lambda(r_\lambda) + \sum_{l=0}^{d-1}\sum_{n=l}^{d-1}(-1)^{l+1}(s-r_\lambda)^{l-n}\{\hat{A}_\lambda^l(r_\lambda), Y_n\}.
\end{aligned}$$

Replacing n by $l+j+1$ gives

$$(E) = (\dots) + \hat{A}_\lambda(r_\lambda) + \sum_{l=0}^{d-1}\sum_{j=-1}^{d-l-2}(s-r_\lambda)^{-j-1}(-1)^{l+1}\{\hat{A}_\lambda^l(r_\lambda), Y_{l+j+1}\}$$
$$= (\dots) + \hat{A}_\lambda(r_\lambda) + \sum_{j=-1}^{d-2}(s-r_\lambda)^{-j-1}\sum_{l=0}^{d-j-2}(-1)^{l+1}\{\hat{A}_\lambda^l(r_\lambda), Y_{l+j+1}\}.$$

The coefficient of $(s-r_\lambda)^{-j-1}$ here is (after the change $l=k-1$)

$$\sum_{k=1}^{d-j-1}(-1)^k\{\hat{A}_\lambda^{k-1}(r_\lambda), Y_{k+j}\},$$

as required.

Other entries of P_λ and $\mathbf{A}_\lambda^j$ are found by similar calculations. ∎

THEOREM 2.4.4. *In the 2×2 representation, for all $j=1,\dots,d$, the only non-zero entry of the operator $-S_\lambda\mathbf{A}_\lambda^j$ is the $(1,1)$-entry, which is equal to the $(1,1)$-entry of $\mathbf{A}_\lambda^{j-1}$. In other words,*

$$-S_\lambda\mathbf{A}_\lambda^j = \hat{P}\mathbf{A}_\lambda^{j-1}. \tag{2.4.6}$$

Proof. This is a direct calculation based on the 2×2 representation (2.4.3) of the operator $\mathbf{A}_\lambda^j$. For $j\geq 1$, we have

$$S_\lambda\mathbf{A}_\lambda^j(r_\lambda) = \begin{pmatrix}\hat{A}_\lambda(r_\lambda) & R_\lambda(\hat{H}_{r_\lambda})v\\ 0 & 0\end{pmatrix}\begin{pmatrix}\sum_{k=1}^{d-j-1}(-1)^k\{\hat{A}_\lambda^{k-1}(r_\lambda), Y_{j+k}\} & 0\\ \sum_{k=0}^{d-j-1}(-1)^k D_{j+k}v^*\hat{A}_\lambda^k(r_\lambda) & 0\end{pmatrix}.$$

The only non-zero entry of this product is the $(1,1)$-entry, which is equal to

$$(E) := \sum_{k=1}^{d-j-1}(-1)^k\hat{A}_\lambda(r_\lambda)\{\hat{A}_\lambda^{k-1}(r_\lambda), Y_{j+k}\} + \sum_{k=0}^{d-j-1}(-1)^k Y_{j+k}\hat{A}_\lambda^k(r_\lambda).$$

Using (2.4.2), we can rewrite this as follows:

$$\begin{aligned}(E) &= \sum_{k=1}^{d-j-1}(-1)^k\big(\{\hat{A}_\lambda^k(r_\lambda), Y_{j+k}\} - Y_{j+k}\hat{A}_\lambda^k(r_\lambda)\big) + \sum_{k=0}^{d-j-1}(-1)^k Y_{j+k}\hat{A}_\lambda^k(r_\lambda)\\ &= \sum_{k=1}^{d-j-1}(-1)^k\{\hat{A}_\lambda^k(r_\lambda), Y_{j+k}\} - \sum_{k=1}^{d-j-1}(-1)^k Y_{j+k}\hat{A}_\lambda^k(r_\lambda) + \sum_{k=0}^{d-j-1}(-1)^k Y_{j+k}\hat{A}_\lambda^k(r_\lambda)\\ &= \sum_{k=1}^{d-j-1}(-1)^k\{\hat{A}_\lambda^k(r_\lambda), Y_{j+k}\} + Y_j = \sum_{k=0}^{d-j-1}(-1)^k\{\hat{A}_\lambda^k(r_\lambda), Y_{j+k}\}\\ &= -\sum_{k=1}^{d-j}(-1)^k\{\hat{A}_\lambda^{k-1}(r_\lambda), Y_{j-1+k}\}.\end{aligned}$$

Since, according to (2.4.3), this is the negative of $(1,1)$-entry of $\mathbf{A}_\lambda^{j-1}$, we are done. ∎

Subtracting $\mathbf{A}_\lambda^{j-1}$ from both sides of (2.4.6) and replacing $j-1$ by j in the resulting formula gives, for each $j = 0, 1, \dots, d-1$,

$$(1_{\mathcal{H}} + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j = \hat{P}^\perp \mathbf{A}_\lambda^j.$$

(In this equality, $1_{\mathcal{H}}$ can be replaced by P_λ, since the latter gets absorbed by $\mathbf{A}_\lambda^j$). Since $\hat{P}^\perp$ is the orthogonal projection onto the eigenspace $\mathcal{V}_\lambda$, combining this equality with (2.4.3) gives, for $j = 1, 2, \dots,$

$$(1_{\mathcal{H}} + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j = \sum_{k=0}^{d-1-j} (-1)^k D_{j+k} v^* \hat{A}_\lambda^k(r_\lambda). \tag{2.4.7}$$

If $j = 0$, then we have to use (2.4.4), which has $(2,2)$-entry 1. This entry results in the additional summand $\hat{P}^\perp$. Hence,

$$(1_{\mathcal{H}} + S_\lambda \mathbf{A}_\lambda)P_\lambda = \sum_{k=0}^{d-1} (-1)^k D_k v^* \hat{A}_\lambda^k(r_\lambda) + \hat{P}^\perp. \tag{2.4.8}$$

THEOREM 2.4.5. *For any $j = 1, \dots, d-1$,*

$$D_j v^* = (1_{\mathcal{H}} + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j (1_{\mathcal{H}} + \mathbf{A}_\lambda \hat{A}_\lambda(r_\lambda)).$$

Also,

$$D_0 v^* = -\hat{P}^\perp + (1_{\mathcal{H}} + S_\lambda \mathbf{A}_\lambda)P_\lambda (1_{\mathcal{H}} + \mathbf{A}_\lambda \hat{A}_\lambda(r_\lambda)).$$

Proof. Since $\mathbf{A}^d = 0$, the equality (2.4.7), taken with $j = d-1$, gives

$$D_{d-1} v^* = \mathbf{A}_\lambda^{d-1}.$$

Further we proceed by induction from $j = d-1$ to $j = 1$. Moving all summands of the right side of (2.4.7), except the first one, to the left side and then using the induction assumption we obtain (we write 1 for $1_{\mathcal{H}}$)

$$\begin{aligned}
D_j v^* &= (1 + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j + \sum_{k=1}^{\infty} (-1)^{k+1} D_{j+k} v^* \hat{A}_\lambda^k(r_\lambda) \\
&= (1 + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j + \sum_{k=1}^{\infty} (-1)^{k+1} (1 + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^{j+k} (1 + \mathbf{A}_\lambda \hat{A}_\lambda(r_\lambda)) \hat{A}_\lambda^k(r_\lambda) \\
&= (1 + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j \Big(1 + \sum_{k=1}^{\infty} (-1)^{k+1} \mathbf{A}_\lambda^k (1 + \mathbf{A}_\lambda \hat{A}_\lambda(r_\lambda)) \hat{A}_\lambda^k(r_\lambda)\Big) \\
&= (1 + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j \Big(1 + \sum_{k=1}^{\infty} (-1)^{k+1} \mathbf{A}_\lambda^k \hat{A}_\lambda^k(r_\lambda) + \sum_{k=1}^{\infty} (-1)^{k+1} \mathbf{A}_\lambda^{k+1} \hat{A}_\lambda^{k+1}(r_\lambda)\Big) \\
&= (1 + S_\lambda \mathbf{A}_\lambda)\mathbf{A}_\lambda^j (1 + \mathbf{A}_\lambda \hat{A}_\lambda(r_\lambda)).
\end{aligned}$$

The appearance of the additional summand $-\hat{P}^\perp$ in the case of $j = 0$ was explained before the statement of the theorem. We also remark that the infinite sums above are in fact finite, but for simplicity the upper summation indices are replaced by infinity. ∎

Since $\mathfrak{a} S_\lambda = 0$, this theorem implies the equality

$$\mathfrak{a} D_j v^* = \mathfrak{a} \mathbf{A}_\lambda^j (1 + \mathbf{A}_\lambda \hat{A}_\lambda(r_\lambda)), \quad j = 1, \dots, d-1. \tag{2.4.9}$$

Using the 2×2 representations of operators $P_\lambda, \mathbf{A}_\lambda, \tilde{A}_\lambda(r_\lambda)$ and Lemma 2.4.2, straightforward but somewhat lengthy calculations prove that

$$P_\lambda + \mathbf{A}_\lambda S_\lambda = \sum_{l=0}^{d-1} (-1)^l S_\lambda^l D_l(v^* + \mathfrak{a}),$$

$$\tilde{A}_\lambda(r_\lambda) + P_\lambda S_\lambda = S_\lambda + \sum_{l=-1}^{d-1} (-1)^{l+1} S_\lambda^{l+1} D_l(v^* + \mathfrak{a}),$$

where S_λ^0 is the identity operator on the whole Hilbert space $\mathcal{H}$. Since these formulas are not used further, their proofs are omitted. As $V\chi \perp \mathcal{V}_\lambda$ if and only if $(v^* + \mathfrak{a})\chi = 0$, Theorem 2.3.1 immediately follows from the first of these formulas.

2.5. Resonance points with property B. In this subsection we prove the following theorem. We say that a resonance point r_λ has *property B* if one (and hence all) of the equivalent conditions of this theorem hold.

THEOREM 2.5.1. *The following assertions are equivalent:*

(1) $\mathfrak{a}D_j = 0$ *for all* $j = 1, \dots, d-1$.
(2) $\mathfrak{a}D_j v^* = 0$ *for all* $j = 1, \dots, d-1$.
(3) $\mathfrak{a}\mathbf{A}_\lambda = 0$.
(4) $v^*\mathbf{A}_\lambda = 0$.
(5) *The function* $(s - r_\lambda)\mathcal{D}_\lambda(s)\mathfrak{a}$ *is holomorphic at* $s = r_\lambda$.
(6) *The function* $(s - r_\lambda)v\mathcal{D}_\lambda(s)\mathfrak{a}$ *is holomorphic at* $s = r_\lambda$.
(7) $\operatorname{im}(\hat{P}^\perp \mathbf{A}_\lambda) \subset \operatorname{im}(\mathbf{A}_\lambda)$.

Proof. We prove the following equivalences:

$$(1)\Leftrightarrow(2),\ (2)\Leftrightarrow(3),\ (3)\Leftrightarrow(4),\ (1)\Leftrightarrow(5),\ (2)\Leftrightarrow(6),\ (3)\Leftrightarrow(7).$$

(1)⇒(2) is obvious.

(2)⇒(1). We shall prove that $vD_j\mathfrak{a} = 0$ implies $D_j\mathfrak{a} = 0$.

The equality $D_{d-1}\mathfrak{a} = 0$ follows from Lemma 2.4.2. Lemma 2.4.2 also implies that $\mathfrak{a}D_{d-2}\mathfrak{a} = 0$. So, if $D_{d-2}\mathfrak{a} \neq 0$ then by Corollary 2.1.3 we would have $vD_{d-2}\mathfrak{a} \neq 0$, which contradicts the premise. Hence, $D_{d-2}\mathfrak{a} = 0$.

The equalities $D_{d-1}\mathfrak{a} = 0$, $D_{d-2}\mathfrak{a} = 0$ and Lemma 2.4.2 imply that $\mathfrak{a}D_{d-3}\mathfrak{a} = 0$. So, if $D_{d-3}\mathfrak{a} \neq 0$ then by Corollary 2.1.3 we would have $vD_{d-3}\mathfrak{a} \neq 0$, which contradicts the premise. Hence, $D_{d-3}\mathfrak{a} = 0$. And so on.

(3)⇒(2) follows immediately from (2.4.9).

(2)⇒(3). Using (2.4.9) with $j = d-1$ we infer that

$$0 = \mathfrak{a}D_{d-1}v^* = \mathfrak{a}\mathbf{A}_\lambda^{d-1}.$$

Hence, using (2.4.9) with $j = d-2$ we infer that

$$0 = \mathfrak{a}D_{d-2}v^* = \mathfrak{a}\mathbf{A}_\lambda^{d-2}.$$

And so on:

$$0 = \mathfrak{a}D_1 v^* = \mathfrak{a}\mathbf{A}_\lambda^1.$$

(3)⇔(4). This follows from the formula $(v^* + \mathfrak{a})\mathbf{A}_\lambda = 0$ of Corollary 2.3.2.

The equivalences (1)⇔(5), (2)⇔(6) obviously follow from the Laurent expansion (2.4.1) of $\mathcal{D}_\lambda(s)$.

(7)⇒(3). Since $\mathfrak{a} = \mathfrak{a}\hat{P}^\perp$, we have

$$\mathfrak{a}\,\mathrm{im}(\mathbf{A}_\lambda) = \mathfrak{a}\,\mathrm{im}(\hat{P}^\perp\mathbf{A}_\lambda).$$

By the premise, we have

$$\mathrm{im}(\hat{P}^\perp\mathbf{A}_\lambda) \subset \mathrm{im}(\mathbf{A}_\lambda).$$

Combining this with the obvious inclusion

$$\mathrm{im}(\hat{P}^\perp\mathbf{A}_\lambda) \subset \mathcal{V}_\lambda$$

gives

$$\mathfrak{a}\,\mathrm{im}(\mathbf{A}_\lambda) = \mathfrak{a}\,\mathrm{im}(\hat{P}^\perp\mathbf{A}_\lambda) \subset \mathfrak{a}(\mathrm{im}(\mathbf{A}_\lambda) \cap \mathcal{V}_\lambda).$$

An element χ of $\mathrm{im}(\mathbf{A}_\lambda)\cap\mathcal{V}_\lambda$ is an eigenvector of depth at least 1. Hence, by Theorem 2.3.1, $V\chi \perp \mathcal{V}_\lambda$, that is, the second component of $V\chi$, which is $\mathfrak{a}\chi$, is zero.

(3)⇒(7). Since $\mathfrak{a} = \mathfrak{a}\hat{P}^\perp$, we have $\mathfrak{a}\hat{P}^\perp\mathbf{A}_\lambda = 0$. Since by Corollary 2.3.3 the kernel of $\mathfrak{a}$ consists of vectors of depth at least 1, we are done. ∎

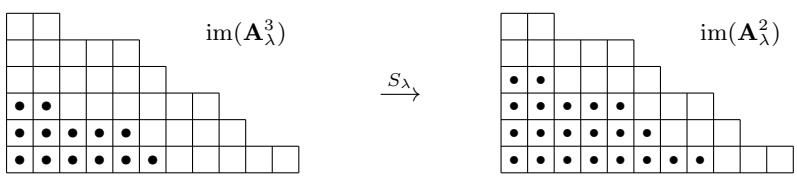
2.6. Resonance points with property *A*. Theorem 2.3.1 and formula (2.4.6) indicate that the operators $\mathbf{A}_\lambda$ and $-S_\lambda$ restricted to the vector space $\Upsilon_\lambda(r_\lambda)$ behave to a certain extent as inverses of each other. In particular, $\mathbf{A}_\lambda$ decreases the order of a resonance vector χ by 1, while S_λ increases the order of χ by 1, provided there is some room for increasing the order. Another property of $\mathbf{A}_\lambda$ is that it increases the depth of a resonance vector by 1 (this is, in fact, the definition of depth). It is therefore reasonable to inquire whether S_λ decreases the depth of a resonance vector by 1. We say that a resonance point has *property A* if it has this property. We conjecture that all resonance points have property *A*. In this subsection we give three conditions which are equivalent to property *A*.

THEOREM 2.6.1. *For a real resonance point r_λ which does not belong to the essential spectrum, the following assertions are equivalent:*

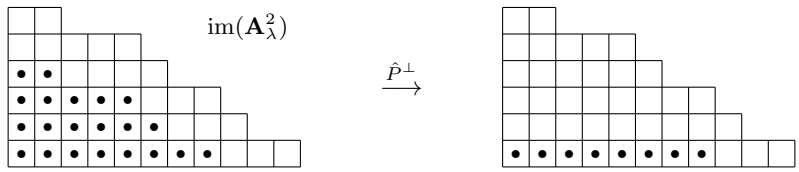
(1) *For all* $j = 1, \dots, d-1$, $\mathrm{im}(D_j v^*) \subset \mathrm{im}(\mathbf{A}_\lambda^j)$.
(2) *For all* $j = 1, \dots, d-1$, $\mathrm{im}(S_\lambda\mathbf{A}_\lambda^j) \subset \mathrm{im}(\mathbf{A}_\lambda^{j-1})$.
(3) *For all* $j = 1, \dots, d-1$, $\mathrm{im}(\hat{P}\mathbf{A}_\lambda^{j-1}) \subset \mathrm{im}(\mathbf{A}_\lambda^{j-1})$.
(4) *For all* $j = 1, \dots, d-1$, $\mathrm{im}(\hat{P}^\perp\mathbf{A}_\lambda^{j-1}) \subset \mathrm{im}(\mathbf{A}_\lambda^{j-1})$.

Proof. The equivalence (1)⇔(2) follows from (2.4.7) applied for $j = d-1, d-2, \dots$; (2)⇔(3) follows immediately from (2.4.6); and (3)⇔(4) is obvious. ∎

We note that item (2) of this theorem asserts that S_λ decreases the depth of resonance vectors by no more than 1, as demonstrated in the following picture:



Item (4) asserts that the orthogonal projection onto the eigenspace $\mathcal{V}_\lambda$ does not decrease the depth of resonance vectors:



Precisely, these diagrams should be interpreted as follows: the linear span of resonance vectors in the left Young diagram is mapped into the linear span of resonance vectors in the right Young diagram.

Property A implies property B, since item (7) of Theorem 2.5.1 is a special case of item (4) of Theorem 2.6.1. Property A holds trivially in two cases: if $m = 1$ or $d \leq 2$. Also, if $d \leq 3$ then property B trivially implies property A.

Conjecture 1.

(i) *Property B implies property A.*
(ii) *Property B holds.*
(iii) *Property A holds.*

3. On eigenpaths $\varphi_\nu(s)$ of $H_0 + sV$

3.1. Order of an eigenpath. Let H_{r_λ} be a λ-resonant operator. Eigenvectors of H_{r_λ} corresponding to the eigenvalue λ form a vector space $\mathcal{V}_\lambda$, whose dimension we denote by m. Given a regular direction V, among the eigenvectors of H_{r_λ} one can distinguish vectors which are also eigenvectors of the crossing operator $\mathfrak{a} = \hat{P}^\perp V \hat{P}^\perp$.

We denote by $\lambda_\nu(s)$, $\nu = 1, \ldots, m$, the eigenvalue functions of H_s, and by $\varphi_\nu(s)$ the corresponding eigenvector functions of H_s.

Proposition 3.1.1. *If $\varphi_\nu(s)$ is an eigenpath of H_s then $\varphi_\nu(r_\lambda)$ is also an eigenvector of the crossing operator $\mathfrak{a} = \hat{P}^\perp V \hat{P}^\perp$. Moreover, the corresponding eigenvalue is $\lambda'_\nu(r_\lambda)$.*

Proof. Differentiating $H_s\varphi_\nu(s) = \lambda_\nu(s)\varphi_\nu(s)$ and letting $s = r_\lambda$ we obtain

$$V\varphi_\nu(r_\lambda) + H_{r_\lambda}\varphi'_\nu(r_\lambda) = \lambda'_\nu(r_\lambda)\varphi_\nu(r_\lambda) + \lambda_\nu(r_\lambda)\varphi'_\nu(r_\lambda).$$

Applying $\hat{P}^\perp$ to both sides gives $\hat{P}^\perp V\varphi_\nu(r_\lambda) = \lambda'_\nu(r_\lambda)\varphi_\nu(r_\lambda)$, as required. ∎

Definition 3.1.2. Let $\varphi(s)$ be an analytic path of eigenvectors of H_s and let k be a positive integer. We say that $\varphi(s)$ has *order at least k* if the vectors

$$V\varphi(r_\lambda), V\varphi'(r_\lambda), \ldots, V\varphi^{(k-2)}(r_\lambda) \tag{3.1.1}$$

are orthogonal to the eigenspace $\mathcal{V}_\lambda$.

We also say that $\varphi(s)$ has *order k* if in addition $V\varphi^{(k-1)}(r_\lambda)$ is not orthogonal to $\mathcal{V}_\lambda$.

Since for $k = 1$ the set of vectors (3.1.1) is empty, the order of every eigenpath is at least 1. As we shall see later, the largest order of eigenpaths is equal to the order of the direction V.

LEMMA 3.1.3. *The definition of the order of an eigenpath is correct in the sense that it does not depend on a normalisation of $\varphi(s)$. That is, if $a(s)$ is an analytic function such that $a(r_\lambda) \neq 0$, then for the eigenpath $\psi(s) = a(s)\varphi(s)$ the vectors*

$$V\psi(r_\lambda), V\psi'(r_\lambda), \dots, V\psi^{(k-2)}(r_\lambda)$$

are orthogonal to $\mathcal{V}_\lambda$ if and only if so are the vectors (3.1.1).

Proof. This immediately follows from the Leibniz rule

$$\psi^{(k)}(s) = \sum_{j=0}^{k} \binom{k}{j} a^{(j)}(s)\varphi^{(k-j)}(s). \ \blacksquare \tag{3.1.2}$$

If $(H_0 + sV)\varphi(s) = \lambda(s)\varphi(s)$, then $\lambda'(r_\lambda) = \langle \varphi(r_\lambda), V\varphi(r_\lambda)\rangle$, which is a well-known fact in perturbation theory (see e.g. [LL, §38]). In particular, if $\lambda'(r_\lambda) = 0$, then vector $V\varphi(r_\lambda)$ is orthogonal to $\varphi(r_\lambda)$. The following lemma is a generalisation of this statement.

LEMMA 3.1.4. *Let $k \geq 2$ and let $\varphi(s)$ be an analytic path of eigenvectors of the path*

$$H_s = H_{r_\lambda} + (s - r_\lambda)V.$$

The following assertions are equivalent:

(i) *The path $\varphi(s)$ has order at least k.*
(ii) *The vectors $V\varphi(r_\lambda), V\varphi'(r_\lambda), \dots, V\varphi^{(k-2)}(r_\lambda)$ are orthogonal to $\varphi(r_\lambda)$.*
(iii) *The equalities $\lambda'(r_\lambda) = 0, \dots, \lambda^{(k-1)}(r_\lambda) = 0$ hold, where $\lambda(s)$ is an analytic path of eigenvalues of H_s which corresponds to $\varphi(s)$.*
(iv) *For all $j = 1, \dots, k-1$, $(H_{r_\lambda} - \lambda)\varphi^{(j)}(r_\lambda) = -jV\varphi^{(j-1)}(r_\lambda)$.*

Proof. Since $\varphi(r_\lambda)$ is an eigenvector, (i) plainly implies (ii). The implication (iv)⇒(i) is also obvious. We shall prove (ii)⇒(iii) and (iii)⇒(iv).

(ii)⇒(iii). Differentiating the eigenvalue equation $H_s\varphi(s) = \lambda(s)\varphi(s)$ $k-1$ times gives

$$(k-1)V\varphi^{(k-2)}(s) + H_s\varphi^{(k-1)}(s) = \sum_{j=0}^{k-1} \binom{k-1}{j} \lambda^{(j)}(s)\varphi^{(k-1-j)}(s). \tag{3.1.3}$$

Here we let $s = r_\lambda$ and take the scalar product of both sides with $\varphi(r_\lambda)$. This leads to cancellation of the second summand of the left hand side with the first summand of the right hand side. Hence,

$$(k-1)\langle \varphi(r_\lambda), V\varphi^{(k-2)}(r_\lambda)\rangle = \sum_{j=1}^{k-1} \binom{k-1}{j} \lambda^{(j)}(r_\lambda)\langle \varphi(r_\lambda), \varphi^{(k-1-j)}(r_\lambda)\rangle. \tag{3.1.4}$$

If $k = 2$ then $\langle \varphi(r_\lambda), V\varphi(r_\lambda)\rangle = \lambda'(r_\lambda)\langle \varphi(r_\lambda), \varphi(r_\lambda)\rangle$. This implies the assertion for $k = 2$. Assume that the claim holds for $k < n$. Then from (3.1.4) with $k = n$, using the induction

assumption, we get

$$(n-1)\langle\varphi(r_\lambda), V\varphi^{(n-2)}(r_\lambda)\rangle = \lambda^{(n-1)}(r_\lambda)\langle\varphi(r_\lambda),\varphi(r_\lambda)\rangle.$$

Since $\langle\varphi(r_\lambda), V\varphi^{(n-2)}(r_\lambda)\rangle = 0$ by the premise, this gives $\lambda^{(n-1)}(r_\lambda) = 0$.

(iii)⇒(iv). Letting $s = r_\lambda$ in (3.1.3) gives

$$(k-1)V\varphi^{(k-2)}(r_\lambda) + H_{r_\lambda}\varphi^{(k-1)}(r_\lambda) = \sum_{j=0}^{k-1}\binom{k-1}{j}\lambda^{(j)}(r_\lambda)\varphi^{(k-1-j)}(r_\lambda).$$

By the premise, the right hand side simplifies to $\lambda(r_\lambda)\varphi^{(k-1)}(r_\lambda) = \lambda\varphi^{(k-1)}(r_\lambda)$. Hence,

$$(H_{r_\lambda} - \lambda)\varphi^{(k-1)}(r_\lambda) = -(k-1)V\varphi^{(k-2)}(r_\lambda). \ \blacksquare$$

This proof also shows that if an eigenpath $\varphi(s)$ has order k, then

$$\langle\varphi(r_\lambda), V\varphi^{(k-1)}(r_\lambda)\rangle = \frac{1}{k}\lambda^{(k)}(r_\lambda)\langle\varphi(r_\lambda),\varphi(r_\lambda)\rangle. \tag{3.1.5}$$

In particular, in this case the number $\langle\varphi(r_\lambda), V\varphi^{(k-1)}(r_\lambda)\rangle$ is non-zero and real.

LEMMA 3.1.5. *Let $k \geq 2$ and let $\varphi(s)$ be an analytic path of eigenvectors of the path $H_s = H_0 + sV$. If $\varphi(s)$ has order at least k, then*

(1) *$\varphi^{(j)}(r_\lambda)$ is a resonance vector of order $j+1$ for all $j = 0, 1, \ldots, k-1$, and*
(2) *for all $j = 1, \ldots, k-1$,*

$$\mathbf{A}_\lambda(r_\lambda)\varphi^{(j)}(r_\lambda) = j\varphi^{(j-1)}(r_\lambda); \tag{3.1.6}$$

in particular, $\varphi(r_\lambda)$ is an eigenvector of depth at least $k-1$.

Proof. We use induction on k.

(A) Let $k = 2$. Differentiation of the eigenvalue equation gives

$$V\varphi(s) + H_s\varphi'(s) = \lambda'(s)\varphi(s) + \lambda(s)\varphi'(s).$$

Since $k = 2$, by the premise and Lemma 3.1.4(iii), the first summand on the right vanishes at $s = r_\lambda$. Hence,

$$V\varphi(r_\lambda) + H_{r_\lambda}\varphi'(r_\lambda) = \lambda(r_\lambda)\varphi'(r_\lambda),$$

which can be rewritten as

$$(H_{r_\lambda} - \lambda)\varphi'(r_\lambda) = -V\varphi(r_\lambda).$$

Applying $R_\lambda(\hat{H}_{r_\lambda})$ to both sides gives

$$\varphi'(r_\lambda) = -S_\lambda\varphi(r_\lambda) + \text{order 1 vector}.$$

Combining this with the premise $V\varphi(r_\lambda) \perp \mathcal{V}_\lambda$ and Theorem 2.3.1 implies that $\varphi'(r_\lambda)$ is a resonance vector of order 2. Theorem 2.3.1 also implies (3.1.6) for $j = 2$.

(B) Assume that the claim holds for $k < n$ and let $\varphi(s)$ be an eigenpath of order $\geq n$. By Lemma 3.1.4(iv), we have

$$(H_{r_\lambda} - \lambda)\varphi^{(n-1)}(r_\lambda) = -(n-1)V\varphi^{(n-2)}(r_\lambda).$$

Applying the sliced resolvent $R_\lambda(\hat{H}_{r_\lambda})$ to both sides gives

$$\frac{1}{n-1}\varphi^{(n-1)}(r_\lambda) = -S_\lambda\varphi^{(n-2)}(r_\lambda) + \text{order 1 vector}.$$

By the induction assumption the order of the vector $\varphi^{(n-2)}(r_\lambda)$ is $n-1$, and since $\varphi(s)$ has order $\geq n$, we have $V\varphi^{(n-2)} \perp \mathcal{V}_\lambda$. Hence, Theorem 2.3.1 implies that $\varphi^{(n-1)}$ is a vector of order n and that (3.1.6) holds. ∎

LEMMA 3.1.6. *Let $k \geq 2$. If $\varphi(s)$ is an analytic path of eigenvectors of the path $H_s = H_0 + sV$ such that $\varphi(r_\lambda)$ has depth at least $k-1$, then $\varphi(s)$ has order at least k.*

Proof. If $k = 2$ then the assertion follows from the equivalence for $\varphi(r_\lambda)$ to have depth at least 1 and to be V-orthogonal to $\mathcal{V}_\lambda$ (see Theorem 2.3.1). Assume that the claim holds for $k < n$, and let $\varphi(r_\lambda)$ be of depth at least $n-1$. For the eigenvalue function $\lambda(s)$ which corresponds to $\varphi(s)$, by the induction assumption and Lemma 3.1.4(iii) we have $\lambda'(r_\lambda) = \cdots = \lambda^{(n-2)}(r_\lambda) = 0$. Combining this with (3.1.4), we obtain

$$(n-1)\langle \varphi(r_\lambda), V\varphi^{(n-2)}(r_\lambda)\rangle = \lambda^{(n-1)}(r_\lambda)\langle \varphi(r_\lambda), \varphi(r_\lambda)\rangle.$$

Since $\varphi(r_\lambda)$ has depth at least $n-1$, there exists a vector f such that $\mathbf{A}_\lambda^{n-1} f = \varphi(r_\lambda)$. Hence,

$$\begin{aligned}\lambda^{(n-1)}(r_\lambda)\langle \varphi(r_\lambda), \varphi(r_\lambda)\rangle &= (n-1)\langle \mathbf{A}_\lambda^{n-1} f, V\varphi^{(n-2)}(r_\lambda)\rangle \\ &= (n-1)\langle f, V\mathbf{A}_\lambda^{n-1}\varphi^{(n-2)}(r_\lambda)\rangle = 0,\end{aligned}$$

where the last equality follows from the induction assumption, according to which and Lemma 3.1.5 the vector $\varphi^{(n-2)}(r_\lambda)$ has order $n-1$ and therefore $\mathbf{A}_\lambda^{n-1}\varphi^{(n-2)}(r_\lambda) = 0$. This gives $\lambda^{(n-1)}(r_\lambda) = 0$. Hence, by Lemma 3.1.4 the proof is complete. ∎

Since the eigenvalue λ has geometric multiplicity m, there are m eigenpaths $\varphi_\nu(s)$, $\nu = 1, \ldots, m$; we denote their orders by $\tilde{d}_\nu$.

We summarise Lemmas 3.1.4–3.1.6 in the following theorem.

THEOREM 3.1.7. *Under Assumption* 1.2.1*, for each $\nu = 1, \ldots, m$ the following assertions are equivalent:*

(i) *The eigenpath $\varphi_\nu(s)$ has order $\tilde{d}_\nu$.*

(ii) *The vectors*

$$V\varphi_\nu(r_\lambda), V\varphi_\nu'(r_\lambda), \ldots, V\varphi_\nu^{(\tilde{d}_\nu-2)}(r_\lambda)$$

are orthogonal to $\varphi_\nu(r_\lambda)$, but the vector $V\varphi_\nu^{(\tilde{d}_\nu-1)}(r_\lambda)$ is not.

(iii) *The equalities*

$$\lambda_\nu'(r_\lambda) = 0, \quad \lambda_\nu''(r_\lambda) = 0, \quad \ldots, \quad \lambda_\nu^{(\tilde{d}_\nu-1)}(r_\lambda) = 0$$

hold but $\lambda_\nu^{(\tilde{d}_\nu)}(r_\lambda) \neq 0$, where $\lambda_\nu(s)$ is an analytic path of eigenvalues of H_s which corresponds to $\varphi_\nu(s)$.

(iv) *For each $j = 1, \ldots, \tilde{d}_\nu - 1$ the equalitly*

$$(H_{r_\lambda} - \lambda)\varphi_\nu^{(j)}(r_\lambda) = -jV\varphi_\nu^{(j-1)}(r_\lambda)$$

holds, but it fails for $j = \tilde{d}_\nu$.

(v) *For each $j = 1, \ldots, \tilde{d}_\nu - 1$ the equalitly*

$$\mathbf{A}_\lambda(r_\lambda)\varphi_\nu^{(j)}(r_\lambda) = j\varphi_\nu^{(j-1)}(r_\lambda)$$

holds, but it fails for $j = \tilde{d}_\nu$.

(vi) *$\varphi_\nu(r_\lambda)$ is an eigenvector of depth $\tilde{d}_\nu - 1$.*

We shall refer to the positive integer $\tilde{d}_\nu$ as the *order* of the eigenvalue function $\lambda_\nu(s)$ too. Thus, the order of $\lambda_\nu(s)$ is the smallest positive integer $\tilde{d}_\nu$ such that $\lambda_\nu^{(\tilde{d}_\nu)}(s) \neq 0$, or

$$\lambda_\nu(s) = \lambda + \varepsilon_\nu (s - r_\lambda)^{\tilde{d}_\nu} + O((s - r_\lambda)^{\tilde{d}_\nu + 1}), \qquad s \to r_\lambda, \tag{3.1.7}$$

where $\varepsilon_\nu \neq 0$.

Remark 3.1.8. The formula (3.1.6) is independent of the choice of normalisation of the eigenpath $\varphi(s)$. That is, if $\psi(s) = a(s)\varphi(s)$ is another eigenpath, where $a(s)$ is an analytic function with $a(r_\lambda) \neq 0$, then $\mathbf{A}_\lambda \psi^{(j)}(r_\lambda) = j\psi^{(j-1)}(r_\lambda)$. This equality also follows from (3.1.6) and (3.1.2).

Remark 3.1.9. In this paper we use $\varphi^{(k-1)}$ to denote a vector of order k. This is consistent with the usage of the bracketed superscript (k) for the kth order derivative. In [Az6] the notation $\varphi^{(k)}$ was used to denote a vector of order k.

Assume that H_{r_λ} is a λ-resonance point of multiplicity m. Let V be a regular direction and let $H_s = H_{r_\lambda} + (s - r_\lambda)V$. Let $\lambda_\nu(s)$, $\nu = 1, \dots, m$, be eigenvalue functions of H_s, listed counting multiplicities, and let $\varphi_\nu(s)$, $1, \dots, m$, be corresponding eigenvector functions. If the eigenvalue λ of H_{r_λ} is *splitting*, that is, the functions $\lambda_1(s), \dots, \lambda_m(s)$ are distinct, then for $\mu \neq \nu$ the vectors $\varphi_\nu(s)$ and $\varphi_\mu(s)$ are orthogonal as eigenvectors of a self-adjoint operator H_s corresponding to different eigenvalues $\lambda_\nu(s)$ and $\lambda_\mu(s)$. Even if the eigenvalue λ of H_{r_λ} is not splitting, it is always possible to choose the eigenvector functions $\varphi_\nu(s)$ to be pairwise orthogonal. Thus, we can and do assume that for any s,

$$\langle \varphi_\nu(s), \varphi_\mu(s) \rangle = 0. \tag{3.1.8}$$

In particular, $\langle \varphi_\nu(r_\lambda), \varphi_\mu(r_\lambda) \rangle = 0$. Hence, a regular direction V induces a natural orthogonal decomposition of the eigenspace. This decomposition has an additional property given by the following lemma.

Lemma 3.1.10. *In the setting above, if* $\mu \neq \nu$ *then* $\langle V\varphi_\mu(r_\lambda), \varphi_\nu(r_\lambda) \rangle = 0$.

Proof. Taking the derivative of (3.1.8) we get

$$\langle \varphi_\nu'(s), \varphi_\mu(s) \rangle + \langle \varphi_\nu(s), \varphi_\mu'(s) \rangle = 0. \tag{3.1.9}$$

Further, for any s from some neighbourhood of r_λ we have

$$0 = \langle \lambda_\nu(s)\varphi_\nu(s), \varphi_\mu(s) \rangle = \langle H_s \varphi_\nu(s), \varphi_\mu(s) \rangle.$$

Hence,

$$0 = \frac{d}{ds} \langle H_s \varphi_\nu(s), \varphi_\mu(s) \rangle = \langle H_s' \varphi_\nu(s), \varphi_\mu(s) \rangle + \langle H_s \varphi_\nu'(s), \varphi_\mu(s) \rangle + \langle H_s \varphi_\nu(s), \varphi_\mu'(s) \rangle.$$

With $s = r_\lambda$ this gives

$$\begin{aligned} 0 &= \langle V\varphi_\nu(r_\lambda), \varphi_\mu(r_\lambda) \rangle + \langle H_{r_\lambda} \varphi_\nu'(r_\lambda), \varphi_\mu(r_\lambda) \rangle + \langle H_{r_\lambda} \varphi_\nu(r_\lambda), \varphi_\mu'(r_\lambda) \rangle \\ &= \langle V\varphi_\nu(r_\lambda), \varphi_\mu(r_\lambda) \rangle + \lambda \langle \varphi_\nu'(r_\lambda), \varphi_\mu(r_\lambda) \rangle + \lambda \langle \varphi_\nu(r_\lambda), \varphi_\mu'(r_\lambda) \rangle = \langle V\varphi_\nu(r_\lambda), \varphi_\mu(r_\lambda) \rangle, \end{aligned}$$

where the last equality follows from (3.1.9). ∎

The following lemma is a special case of the next theorem, but nevertheless we give it here separately as a warm up, since its proof uses the same idea but with less obscure calculations.

LEMMA 3.1.11. *Let $\varphi_\mu(s)$ and $\varphi_\nu(s)$ be (orthogonal) eigenpaths of H_0+sV with corresponding eigenvalue functions $\lambda_\mu(s)$ and $\lambda_\nu(s)$. If $\varphi_\mu(s)$ has order at least k, then for all $j=0,1,\dots,k-1$, $\langle V\varphi_\mu^{(j)}(r_\lambda),\varphi_\nu(r_\lambda)\rangle=0$.*

Proof. It is sufficient to prove $\langle V\varphi_\mu^{(k-1)}(r_\lambda),\varphi_\nu(r_\lambda)\rangle=0$. Since by (3.1.8) the function $s\mapsto\langle H_s\varphi_\mu(s),\varphi_\nu(s)\rangle$ is zero, for any non-negative integer k the Leibniz rule implies

$$\begin{aligned}0&=\frac{d^k}{ds^k}\langle H_s\varphi_\mu,\varphi_\nu\rangle\\&=\langle H_s\varphi_\mu^{(k)},\varphi_\nu\rangle+\sum_{j=0}^{k-1}\binom{k}{j}\langle H_s\varphi_\mu^{(j)},\varphi_\nu^{(k-j)}\rangle+\sum_{j=0}^{k-1}k\binom{k-1}{j}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(k-1-j)}\rangle.\end{aligned}\tag{3.1.10}$$

Here we have explicitly separated the first summand $\langle H_s\varphi_\mu^{(k)},\varphi_\nu\rangle$ since it will be treated differently. After this special treatment, this summand will be returned into the sum.

In this equality we replace s by r_λ, but nevertheless we write φ_μ instead of $\varphi_\mu(r_\lambda)$, etc. We have $H_{r_\lambda}\varphi_\mu=\lambda\varphi_\mu$. By Lemma 3.1.4, the premise asserts that the path $\varphi_\mu(s)$ has order at least k. Hence, Lemma 3.1.4(iv) implies that for all $j=0,1,\dots,k-1$,

$$H_{r_\lambda}\varphi_\mu^{(j)}=\lambda\varphi_\mu^{(j)}-jV\varphi_\mu^{(j-1)}.$$

Using these equalities, from (3.1.10) with $s=r_\lambda$ we get

$$\begin{aligned}0&=\langle H_{r_\lambda}\varphi_\mu^{(k)},\varphi_\nu\rangle+\sum_{j=0}^{k-1}\binom{k}{j}\langle H_{r_\lambda}\varphi_\mu^{(j)},\varphi_\nu^{(k-j)}\rangle+\sum_{j=0}^{k-1}k\binom{k-1}{j}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(k-1-j)}\rangle\\&=\lambda\langle\varphi_\mu^{(k)},\varphi_\nu\rangle+\lambda\sum_{j=0}^{k-1}\binom{k}{j}\langle\varphi_\mu^{(j)},\varphi_\nu^{(k-j)}\rangle\\&\quad-\sum_{j=0}^{k-1}j\binom{k}{j}\langle V\varphi_\mu^{(j-1)},\varphi_\nu^{(k-j)}\rangle+\sum_{j=0}^{k-1}k\binom{k-1}{j}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(k-1-j)}\rangle\\&=\lambda\sum_{j=0}^{k}\binom{k}{j}\langle\varphi_\mu^{(j)},\varphi_\nu^{(k-j)}\rangle-\sum_{j=1}^{k-1}\frac{k!}{(j-1)!(k-j)!}\langle V\varphi_\mu^{(j-1)},\varphi_\nu^{(k-j)}\rangle\\&\quad+\sum_{j=0}^{k-1}\frac{k!}{j!(k-1-j)!}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(k-1-j)}\rangle.\end{aligned}$$

The first summand of the last expression is equal to

$$\lambda\frac{d^k}{ds^k}\langle\varphi_\mu,\varphi_\nu\rangle\Big|_{s=r_\lambda},$$

which by (3.1.8) is zero. Hence,

$$\begin{aligned}0 &= -\sum_{j=0}^{k-2}\frac{k!}{j!(k-1-j)!}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(k-1-j)}\rangle+\sum_{j=0}^{k-1}\frac{k!}{j!(k-1-j)!}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(k-1-j)}\rangle\\ &= k\langle V\varphi_\mu^{(k-1)},\varphi_\nu\rangle.\ \blacksquare\end{aligned}$$

THEOREM 3.1.12. *If $\varphi_\mu(s)$ and $\varphi_\nu(s)$ are distinct eigenpaths with orders $\tilde d_\mu$ and $\tilde d_\nu$ respectively, then for any $j=0,1,\dots,\tilde d_\mu-1$ and $k=0,1,\dots,\tilde d_\nu-1$,*

$$\langle V\varphi_\mu^{(j)}(r_\lambda),\varphi_\nu^{(k)}(r_\lambda)\rangle=0.$$

Proof. As in Lemma 3.1.11, it is sufficient to prove $\langle V\varphi_\mu^{(\tilde d_\mu-1)},\varphi_\nu^{(\tilde d_\nu-1)}\rangle=0$.

Let $l=\tilde d_\mu+\tilde d_\nu-2$. Since the eigenpaths $\varphi_\mu(s)$ and $\varphi_\nu(s)$ are distinct, we have

$$\langle H_s\varphi_\mu(s),\varphi_\nu(s)\rangle=\lambda_\mu(s)\langle\varphi_\mu(s),\varphi_\nu(s)\rangle=0,$$

where $\lambda_\mu(s)$ is the eigenvalue function for $\varphi_\mu(s)$. The Leibniz rule gives

$$\begin{aligned}0&=\frac{d^{l+1}}{ds^{l+1}}\langle H_s\varphi_\mu,\varphi_\nu\rangle\\ &=\sum_{j=0}^{l+1}\binom{l+1}{j}\langle H_s\varphi_\mu^{(j)},\varphi_\nu^{(l+1-j)}\rangle+\sum_{j=0}^{l}(l+1)\binom{l}{j}\langle V\varphi_\mu^{(j)},\varphi_\nu^{(l-j)}\rangle.\end{aligned}\tag{3.1.11}$$

We transform the first summand as follows:

$$\begin{aligned}&\sum_{j=0}^{l+1}\binom{l+1}{j}\langle H_s\varphi_\mu^{(j)},\varphi_\nu^{(l+1-j)}\rangle\\ &\qquad=\sum_{j=\tilde d_\mu}^{l+1}\binom{l+1}{j}\langle\varphi_\mu^{(j)},H_s\varphi_\nu^{(l+1-j)}\rangle+\sum_{j=0}^{\tilde d_\mu-1}\binom{l+1}{j}\langle H_s\varphi_\mu^{(j)},\varphi_\nu^{(l+1-j)}\rangle.\end{aligned}$$

Here we let $s=r_\lambda$ and apply the equalities

$$H_{r_\lambda}\varphi_\nu^{(l+1-j)}=\lambda\varphi_\nu^{(l+1-j)}-(l+1-j)V\varphi_\nu^{(l-j)},\qquad H_{r_\lambda}\varphi_\mu^{(j)}=\lambda\varphi_\mu^{(j)}-jV\varphi_\mu^{(j-1)},$$

which follow from the premise that the eigenpaths $\varphi_\nu(s)$ and $\varphi_\mu(s)$ have orders $\tilde d_\nu$ and $\tilde d_\mu$ respectively, according to Theorem 3.1.7(iv). This gives

$$\begin{aligned}(E)&:=\sum_{j=0}^{l+1}\binom{l+1}{j}\langle H_{r_\lambda}\varphi_\mu^{(j)},\varphi_\nu^{(l+1-j)}\rangle\\ &=\sum_{j=\tilde d_\mu}^{l+1}\binom{l+1}{j}\lambda\langle\varphi_\mu^{(j)},\varphi_\nu^{(l+1-j)}\rangle+\sum_{j=0}^{\tilde d_\mu-1}\binom{l+1}{j}\lambda\langle\varphi_\mu^{(j)},\varphi_\nu^{(l+1-j)}\rangle\\ &\quad-\sum_{j=\tilde d_\mu}^{l+1}(l+1-j)\binom{l+1}{j}\langle\varphi_\mu^{(j)},V\varphi_\nu^{(l-j)}\rangle-\sum_{j=0}^{\tilde d_\mu-1}j\binom{l+1}{j}\langle V\varphi_\mu^{(j-1)},\varphi_\nu^{(l+1-j)}\rangle.\end{aligned}$$

We can rewrite this expression as follows:

$$\begin{aligned}(E) &= \lambda \sum_{j=0}^{l+1} \binom{l+1}{j} \langle \varphi_\mu^{(j)}, \varphi_\nu^{(l+1-j)} \rangle \\ &\quad - \sum_{j=\tilde{d}_\mu}^{l+1} \frac{(l+1)!}{j!(l-j)!} \langle \varphi_\mu^{(j)}, V\varphi_\nu^{(l-j)} \rangle - \sum_{j=1}^{\tilde{d}_\mu - 1} \frac{(l+1)!}{(j-1)!(l+1-j)!} \langle V\varphi_\mu^{(j-1)}, \varphi_\nu^{(l+1-j)} \rangle \\ &= 0 - \sum_{j=\tilde{d}_\mu}^{l+1} \frac{(l+1)!}{j!(l-j)!} \langle \varphi_\mu^{(j)}, V\varphi_\nu^{(l-j)} \rangle - \sum_{j=0}^{\tilde{d}_\mu - 2} \frac{(l+1)!}{j!(l-j)!} \langle V\varphi_\mu^{(j)}, \varphi_\nu^{(l-j)} \rangle.\end{aligned}$$

Here the first summand is zero for the same reason as in Lemma 3.1.11. Combining this equality with (3.1.11) gives $\langle V\varphi_\mu^{(\tilde{d}_\mu - 1)}, \varphi_\nu^{(\tilde{d}_\nu - 1)} \rangle = 0$. ∎

3.2. On the ground eigenvalue. If H_{r_λ} is bounded below then its smallest eigenvalue, if it exists, is called the *ground eigenvalue*. Here we make some observations about this eigenvalue.

In the following proposition we use the notation of Subsection 2.1.

PROPOSITION 3.2.1. *Let λ be a simple eigenvalue of H_{r_λ}. The order of the triple $(\lambda; H_{r_\lambda}, V)$ is d if and only if*

$$\mathfrak{a} = 0,\ \langle R_\lambda(\hat{H}_{r_\lambda})v, v\rangle = 0, \dots, \langle (R_\lambda(\hat{H}_{r_\lambda})V)^{d-3} R_\lambda(\hat{H}_{r_\lambda})v, v\rangle = 0$$

and

$$\langle (R_\lambda(\hat{H}_{r_\lambda})V)^{d-2} R_\lambda(\hat{H}_{r_\lambda})v, v\rangle \neq 0.$$

Proof. Since $\mathcal{V}_\lambda$ is one-dimensional, the matrix v is reduced to a vector $V\chi$, where χ is an eigenvector. Further, since λ is a simple eigenvalue, by Theorem 2.3.1 the order is equal to d if and only if

$$\chi \perp V\chi,\ S_\lambda\chi \perp V\chi,\ \dots, S_\lambda^{d-2}\chi \perp V\chi$$

and $S_\lambda^{d-1}\chi$ is not orthogonal to $V\chi$. It remains to note that

$$\langle (R_\lambda(\hat{H}_{r_\lambda})V)^j R_\lambda(\hat{H}_{r_\lambda})v, v\rangle = \langle S_\lambda^{j+1}\chi, V\chi\rangle.$$ ∎

THEOREM 3.2.2. *If λ is a non-degenerate ground eigenvalue, then the order of the corresponding resonance point is at most* 2.

Proof. Since λ is the smallest eigenvalue, the sliced resolvent $R_\lambda(\hat{H}_{r_\lambda})$ is a strictly positive operator. Hence, if the order d is greater than 2, then Proposition 3.2.1 gives $\langle R_\lambda(\hat{H}_{r_\lambda})v, v\rangle = 0$, and combining this with $R_\lambda(\hat{H}_{r_\lambda}) > 0$ gives $R_\lambda(\hat{H}_{r_\lambda})v = 0$, which is a contradiction. ∎

If the order of a resonance point is even, then the corresponding eigenvalue makes a U-turn. One may ask whether this is a left or a right U-turn. At the ground eigenvalue, the following theorem answers this question.

THEOREM 3.2.3. *If λ is a non-degenerate ground eigenvalue and the order of a direction V is equal to* 2, *then the corresponding U-turn is a left U-turn.*

Proof. Since $d = 2$, using Theorem 3.1.7 one infers that

$$0 \neq \langle \varphi(r_\lambda), V\varphi'(r_\lambda)\rangle = -\langle \varphi(r_\lambda), VS_\lambda\varphi(r_\lambda)\rangle = -\langle \chi, VR_\lambda(\hat{H}_{r_\lambda})V\chi\rangle.$$

Since λ is a ground eigenvalue, the operator $R_\lambda(\hat{H}_{r_\lambda})$ is positive. Combining this with the above display gives $\langle \varphi(r_\lambda), V\varphi'(r_\lambda)\rangle < 0$. It follows from this and (3.1.5) that $\lambda''(r_\lambda) < 0$. Hence, the U-turn is the left one. ∎

4. Characterisation of order k directions

Recall that a *resonant path* $H(s)$ is an analytic path of self-adjoint operators in an affine space $\mathcal{A}$ such that λ is an eigenvalue of all $H(s)$. For any resonant path $H(s)$ there exists (see [Ka]) an analytic path of eigenvectors $\chi(s)$, that is,

$$H(s)\chi(s) = \lambda\chi(s). \tag{4.0.1}$$

Such an analytic path of eigenvectors $\chi(s)$ will be a *resonant eigenpath*. We say that a resonant path $H(s)$ is *simple* if at least one point on that path is simple. If a resonant path is simple, then all its points except a discrete set are simple. Further, a simple resonant path has a unique, up to scaling, resonant eigenpath (see e.g. [Ka]).

4.1. Tangent directions have high orders. We say that a direction V is *tangent* to the resonance set $\mathcal{R}(\lambda)$ at H_{r_λ} *to order at least* k, if there exists a resonant path $H(s) \subset \mathcal{R}(\lambda)$ such that for some (necessarily real) numbers $c_2, \dots, c_{k-1}$,

$$H(s) = H_{r_\lambda} + (s - r_\lambda)V + \sum_{j=2}^{k-1} c_j(s - r_\lambda)^j V + O((s - r_\lambda)^k), \qquad s \to r_\lambda. \tag{4.1.1}$$

In this case we also say that the path $H(s)$ is tangent to V at $H(r_\lambda) = H_{r_\lambda}$ to order at least k. If k is the greatest positive integer with this property, then we say that V is *tangent* at H_{r_λ} *to order* k.

We say that a direction V is *tangent* at H_{r_λ} if it is tangent to order at least 2. If V is tangent only to order 1 at $H_{r_\lambda} \in \mathcal{R}(\lambda)$, then we say that V is *transversal* at H_{r_λ}.

Lemma 4.1.1. *Under Assumption* 1.2.1*, let H_{r_λ} be a resonant point and V a regular direction. If V is tangent to the resonance set $\mathcal{R}(\lambda)$ at H_{r_λ}, then the order of V is at least* 2.

Proof. As usual, we let $H_s = H_0 + sV$. Since V is tangent to $\mathcal{R}(\lambda)$ at H_{r_λ}, there exists a resonant path $\{H(s)\}$ such that $H(r_\lambda) = H_{r_\lambda}$ and $H'(r_\lambda) = V$. Let $\chi(s)$ be a corresponding eigenpath. Differentiating the eigenvalue equation (4.0.1) we get

$$H'(s)\chi(s) + H(s)\chi'(s) = \lambda\chi'(s).$$

Letting $s = r_\lambda$ gives $V\chi(r_\lambda) + H_{r_\lambda}\chi'(r_\lambda) = \lambda\chi'(r_\lambda)$. This can be rewritten as

$$(1 + (r_\lambda - s)R_\lambda(H_s)V)\chi'(r_\lambda) = -R_\lambda(H_s)V\chi(r_\lambda),$$

where s is any real number such that $H_s - \lambda$ has bounded inverse (such s exist since V is a regular direction). The eigenvalue equation $H_{r_\lambda}\chi(r_\lambda) = \lambda\chi(r_\lambda)$ is equivalent to

$(1+(r_\lambda-s)R_\lambda(H_s)V)\chi(r_\lambda)=0$. Combining this equality with the previous one gives

$$(1+(r_\lambda-s)R_\lambda(H_s)V)\chi'(r_\lambda)=(r_\lambda-s)^{-1}\chi(r_\lambda).$$

Therefore, $(1+(r_\lambda-s)R_\lambda(H_s)V)^2\chi'(r_\lambda)=0$. Hence, if V is tangent to $\mathcal{R}(\lambda)$, then $\chi'(r_\lambda)$ is a vector of order 2, and therefore V has order at least 2. ∎

THEOREM 4.1.2. *Under Assumption* 1.2.1, *let* $k\geq 1$, *let* H_{r_λ} *be a resonance point and* V *a regular direction at* H_{r_λ}. *If* $H(s)$ *is a resonant path tangent to* V *at* H_{r_λ} *to order at least* k *and if* $\chi(s)$ *is a corresponding analytic eigenpath, then*

(i) *the vectors* $\chi(r_\lambda),\chi'(r_\lambda),\dots,\chi^{(k-1)}(r_\lambda)$ *have orders respectively* $1,2,\dots,k$,
(ii) *the direction* V *has order at least* k,
(iii) *for any* $j=1,\dots,k$,

$$\mathbf{A}_\lambda(r_\lambda)\chi^{(j-1)}(r_\lambda)=(j-1)\chi^{(j-2)}(r_\lambda)+\sum_{i=2}^{j-1}i!\binom{j-1}{i}c_i\chi^{(j-1-i)}(r_\lambda),$$

where the numbers $c_2,\dots,c_k$ *are as in* (4.1.1), *and*
(iv) *the eigenvector* $\chi(r_\lambda)$ *has depth at least* $k-1$.

Proof. (i) We prove this item by induction on k.

The vector $\chi(r_\lambda)$ has order 1 since it is an eigenvector. That in the case of $k\geq 2$ the vector $\chi'(r_\lambda)$ has order 2 was proved in Lemma 4.1.1. Now, assuming that the assertion holds for $k=n-1$, we prove it for $k=n$; still, we write k instead of n.

Since a path $H(s)$ which is tangent to V at H_{r_λ} to order at least k is also tangent to order at least $k-1$, it follows from the induction assumption that the vectors

$$\chi(r_\lambda),\chi'(r_\lambda),\dots,\chi^{(k-2)}(r_\lambda)$$

have orders $1,2,\dots,k-1$ respectively. Differentiating $k-1$ times the eigenvalue equation (4.0.1) gives

$$\sum_{j=0}^{k-1}\binom{k-1}{j}H^{(j)}(s)\chi^{(k-1-j)}(s)=\lambda\chi^{(k-1)}(s). \tag{4.1.2}$$

Since $H(s)$ is tangent to V at H_{r_λ} to order k, the operators $H'(r_\lambda),\dots,H^{(k-1)}(r_\lambda)$ are co-linear to V. Therefore, it follows from the previous equality, taken with $s=r_\lambda$, that

$$H_{r_\lambda}\chi^{(k-1)}(r_\lambda)+(k-1)V\chi^{(k-2)}(r_\lambda)+\tilde{c}_2V\chi^{(k-3)}(r_\lambda)+\cdots+\tilde{c}_{k-1}V\chi(r_\lambda)=\lambda\chi^{(k-1)}(r_\lambda), \tag{4.1.3}$$

where

$$\tilde{c}_j=\binom{k-1}{j}j!c_j$$

and the numbers c_j are from (4.1.1). Adding $(s-r_\lambda)V\chi^{(k-1)}(r_\lambda)$ to both sides of (4.1.3) gives

$$(H_s-\lambda)\chi^{(k-1)}(r_\lambda)+(k-1)V\chi^{(k-2)}(r_\lambda)+\tilde{c}_2V\chi^{(k-3)}(r_\lambda)+\cdots+\tilde{c}_{k-1}V\chi(r_\lambda)$$
$$=(s-r_\lambda)V\chi^{(k-1)}(r_\lambda).$$

Since V is regular, there exists $s\in\mathbb{R}$ such that $R_\lambda(H_s)=(H_s-\lambda)^{-1}$ exists. Multiplying

both sides of the last displayed equality by $R_\lambda(H_s)$, we obtain

$$\chi^{(k-1)}(r_\lambda) + R_\lambda(H_s)V((k-1)\chi^{(k-2)}(r_\lambda) + \tilde{c}_2\chi^{(k-3)}(r_\lambda) + \cdots + \tilde{c}_{k-1}\chi(r_\lambda)) = (s - r_\lambda)R_\lambda(H_s)V\chi^{(k-1)}(r_\lambda), \quad (4.1.4)$$

which can be rewritten as

$$(1 + (r_\lambda - s)R_\lambda(H_s)V)\chi^{(k-1)}(r_\lambda) = -R_\lambda(H_s)V\big((k-1)\chi^{(k-2)}(r_\lambda) + \tilde{c}_2\chi^{(k-3)}(r_\lambda) + \cdots + \tilde{c}_{k-1}\chi(r_\lambda)\big).$$

By the induction assumption, the sum in brackets has order $k-1$. Since by the definition (1.3.14) of vectors of order k the operator $R_\lambda(H_s)V$ preserves the order of vectors, the right hand side above has order $k-1$ too. Since by the same definition (1.3.14) the operator $1 + (r_\lambda - s)R_\lambda(H_s)V$ decreases the order of vectors by 1, it follows that $\chi^{(k-1)}(r_\lambda)$ has order k.

(ii) This follows from (i).

(iii) Taking contour integrals of both sides of (4.1.4) over a contour enclosing the resonance point $s = r_\lambda$ and using (1.3.5) and (1.3.6) we obtain

$$\mathbf{A}_\lambda(r_\lambda)\chi^{(k-1)}(r_\lambda) = (k-1)\chi^{(k-2)}(r_\lambda) + \tilde{c}_2\chi^{(k-3)}(r_\lambda) + \cdots + \tilde{c}_{k-1}\chi(r_\lambda).$$

(iv) This follows immediately from (iii). ∎

If a resonant path $H(s)$ is tangent to V to order k at a resonant point H_{r_λ}, then changing the parameter s if necessary we can always make the operators $H''(r_\lambda), \dots, H^{(k-1)}(r_\lambda)$ equal to zero, so that the path $H(s)$ takes the form

$$H(s) = H_{r_\lambda} + (s - r_\lambda)V + O((s - r_\lambda)^k), \quad s \to r_\lambda. \quad (4.1.5)$$

For example, assuming that $r_\lambda = 0$, to eliminate c_2 one can replace s by $t - c_2t^2$. Once c_2 is eliminated, the change of variables $s = t - c_3t^3$ eliminates c_3, and so on.

Definition 4.1.3. A path of the form (4.1.5) will be called *standard*.

According to Theorem 4.1.2, with a resonant path tangent to order k we can associate a set of resonance vectors $\chi_0, \dots, \chi_{k-1}$ of respective orders $1, \dots, k$, namely the first k coefficients

$$\chi_j = \frac{1}{j!}\chi^{(j)}(r_\lambda), \quad j = 0, 1, \dots, \quad (4.1.6)$$

of the Taylor expansion of a resonant eigenpath $\chi(s)$.

Proposition 4.1.4. *Under Assumption* 1.2.1*, let V be a regular direction at H_{r_λ} and let $H(s)$ be a resonance path tangent to V at H_{r_λ} to order k. The path $H(s)$ is standard if and only if for all $j = 1, \dots, k-1$ we have*

$$\mathbf{A}_\lambda(r_\lambda)\chi_j = \chi_{j-1}, \quad (4.1.7)$$

where $\chi(s)$ is a corresponding analytic path of eigenvectors.

Proof. This follows from Theorem 4.1.2(iii). ∎

4.2. A resonance curve associated with an eigenvector. As usual, we assume that V is a regular direction at a resonance point H_{r_λ}. We choose another direction W and consider the real affine plane $\alpha = H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W$ in the affine space $\mathcal{A}$ determined

by the point H_{r_λ} and the directions V and W. It is possible that the intersection of α and the resonance set $\mathcal{R}(\lambda)$ in a neighbourhood of H_{r_λ} consists of only one point H_{r_λ}. To avoid this one can choose W to be transversal to the resonance set. Since, as we shall see later, the resonance set has co-dimension 1, this will ensure that the intersection of the plane with the resonance set is a curve.

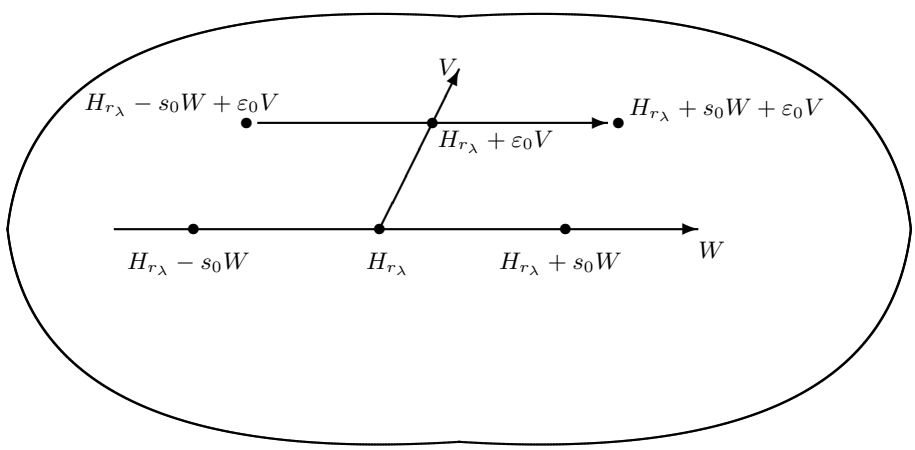
In this and the next subsection we assume that $\mathcal{A}_0$ contains rank-one self-adjoint operators.

THEOREM 4.2.1. *Under Assumption* 1.2.1*, in particular, for H_{r_λ} a resonance point and V a regular direction, let χ be an eigenvector of H_{r_λ} and $W = \langle \chi, \cdot \rangle \chi$. Then the intersection of the real affine plane*

$$\alpha := H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W$$

with a sufficiently small neighbourhood of the point H_{r_λ} in $\mathcal{R}(\lambda) \setminus \{H_{r_\lambda} + \mathbb{R}W\}$ consists of a single analytic curve γ. Moreover, this curve is simple.

Proof. First we show that any neighbourhood of H_{r_λ} has a resonance point in α which is not in $H_{r_\lambda} + \mathbb{R}W$. Assume the contrary. Then there exists a convex neighbourhood O of H_{r_λ} which does not have a resonance point outside the line $H_{r_\lambda} + \mathbb{R}W$. Fix small enough $s_0 > 0$ so that $H_{r_\lambda} \pm s_0 W$ are in O. The operators $H_{r_\lambda} + s_0 W$ and $H_{r_\lambda} - s_0 W$ have the eigenvalue λ of multiplicity $m - 1$ and the non-degenerate eigenvalues $\lambda + s_0$ and $\lambda - s_0$ respectively.



Since V is regular, for all small enough non-zero ε the operators $H_{r_\lambda} + \varepsilon V$ are non-resonant. We choose a small enough $\varepsilon_0 > 0$ so that all the operators

$$H_{r_\lambda} + sW + \varepsilon V, \quad (s, \varepsilon) \in [-s_0, s_0] \times [-\varepsilon_0, \varepsilon_0],$$

are in O. We also choose ε_0 small enough so that the perturbed eigenvalues $\lambda + s_0 + \cdots$ and $\lambda - s_0 + \cdots$ of the perturbed operators $H_{r_\lambda} \pm s_0 W + \varepsilon_0 V$ are on the same side of λ as the original non-perturbed eigenvalues $\lambda + s_0$ and $\lambda - s_0$ respectively. The other $m - 1$ eigenvalues of the operators

$$H_{r_\lambda} \pm s_0 W + \varepsilon V|_{\varepsilon=0}$$

will move away from λ as ε becomes non-zero, since by assumption there are no resonance

points in O outside the line $H_{r_\lambda} + \mathbb{R}W$. Thus, when we deform $H_{r_\lambda} - s_0 W + \varepsilon_0 V$ to $H_{r_\lambda} + s_0 W + \varepsilon_0 V$ by changing $-s_0$ to s_0, the number of eigenvalues on each side of λ changes. Hence, for some s between $-s_0$ and s_0 the operator $H_{r_\lambda} + sW + \varepsilon_0 V$ has λ as an eigenvalue. This operator is therefore resonant and belongs to O. This contradicts our assumption.

That there can only be one simple resonance curve in a neighbourhood of H_{r_λ} in the plane α follows from the non-negativity of W so that an eigenvalue of $H_{r_\lambda} + sW + \varepsilon_0 V$ can only move in the positive direction as s increases. The analyticity of this curve follows from that of the resonance set $\mathcal{R}(\lambda)$. ∎

Definition 4.2.2. The unique curve γ determined by Theorem 4.2.1 is denoted by γ_χ, or by $\gamma_\chi(\lambda; H_{r_\lambda}, V)$ if necessary.

Since the resonance curve γ_χ is simple, there exists only one (up to scaling) analytic eigenpath $\chi(s)$ corresponding to γ_χ. Therefore, to an eigenvector χ we can assign another eigenvector $\chi(r_\lambda)$. The following theorem shows that they are in fact the same.

Theorem 4.2.3. *Under Assumption* 1.2.1*, let H_{r_λ}, V and χ be as in Theorem* 4.2.1*. If $\chi(s)$ is an eigenpath corresponding to the curve of operators γ_χ, then the vectors $\chi(r_\lambda)$ and χ are co-linear.*

Proof. If for H_{r_λ} the eigenvalue λ is simple, then the assertion is trivial. We shall reduce the general case to the case of a simple eigenvalue λ by slightly perturbing H_{r_λ}.

Let $G_t = H_{r_\lambda} + tW$, with W as in Theorem 4.2.1. The operator G_t has an eigenvalue λ of multiplicity $m - 1$ and, for all small enough t, an eigenvalue $\lambda + t$ of multiplicity 1 corresponding to the eigenvector χ. Since $\lambda + t$ is a simple eigenvalue of G_t, the curve

$$\gamma_\chi(\lambda + t; G_t, V)$$

has a resonance eigenpath $\chi_t(s)$ which starts at χ (that is, $\chi_t(r_\lambda) = \text{const} \cdot \chi$). When t is deformed to zero, the curve $\gamma_\chi(\lambda + t; G_t, V)$ is analytically deformed to $\gamma_\chi(\lambda; H_{r_\lambda}, V)$. Thus, the resonance eigenpath $\chi_t(s)$ gets deformed to $\chi_0(s)$ in such a way that the base $\chi_t(r_\lambda)$ of this resonance eigenpath stays co-linear to χ for all t. Hence, in the final position $\chi_0(s)$ of this deformation the base of $\chi_0(s)$ will still be co-linear to χ. Finally, it remains to note that any two eigenpaths corresponding to γ_χ are co-linear, since γ_χ is a simple curve. Hence, $\chi(s)$ starts at χ too. ∎

4.3. High order directions are tangent. Now we are going to prove the converse of Theorem 4.1.2: if V is a direction of order k at H_{r_λ}, then V is tangent to $\mathcal{R}(\lambda)$ at H_{r_λ} to order k.

Theorem 4.3.1. *Under Assumption* 1.2.1*, let k be an integer greater than* 1*. If χ_0 is an eigenvector of depth at least $k - 1$ for the triple $(\lambda; H_{r_\lambda}, V)$, then*

(1) *the direction V is tangent to the resonance curve γ_{χ_0} to order at least k,*
(2) *in the Taylor expansion,*

$$\chi(s) = \sum_{j=0}^{\infty} (s - r_\lambda)^j \chi_j, \tag{4.3.1}$$

of a resonance eigenpath $\chi(s)$ corresponding to γ_{χ_0}, the vectors $\chi_0, \chi_1, \dots, \chi_{k-1}$ have orders respectively $1, 2, \dots, k$,

(3) *for any resonance eigenpath* (4.3.1) *corresponding to γ_{χ_0} and for all $j = 1, \dots, k-1$ the vector $\mathbf{A}_\lambda \chi_j$ is a linear combination of $\chi_0, \chi_1, \dots, \chi_{j-1}$; moreover, if the parametrisation of γ_{χ_0} is standard then $\mathbf{A}_\lambda \chi_j = \chi_{j-1}$, and*

(4) *the vectors $\chi_0, \chi_1, \dots, \chi_{k-2}$ have depths at least 1 and are V-orthogonal to $\mathcal{V}_\lambda$.*

Proof. Let $H(s)$ be a parametrisation of the resonance curve γ_{χ_0}. Since γ_{χ_0} is the intersection of the resonance set and the plane $H_{r_\lambda} + \mathbb{R}V + \mathbb{R}W$, where $W = \langle \chi_0, \cdot \rangle \chi_0$, the Taylor expansion of the path $H(s)$ has the form

$$H(s) = H_{r_\lambda} + \sum_{j=1}^{\infty} (s - r_\lambda)^j (\alpha_j V + \beta_j W). \tag{4.3.2}$$

By Theorem 4.2.3, a resonance eigenpath $\chi(s)$ corresponding to this path has Taylor series

$$\chi(s) = \chi_0 + (s - r_\lambda)\chi_1 + (s - r_\lambda)^2 \chi_2 + \cdots,$$

which starts at χ_0, that is, $\chi(r_\lambda) = \chi_0$. Comparing the coefficients of $s - r_\lambda$ on both sides of the eigenvalue equation (4.0.1) gives

$$(H_{r_\lambda} - \lambda)\chi_1 = -(\alpha_1 V + \beta_1 W)\chi_0.$$

The vector $(H_{r_\lambda} - \lambda)\chi_1$ is orthogonal to the eigenspace $\mathcal{V}_\lambda$ and in particular to χ_0. Hence,

$$\langle \chi_0, (\alpha_1 V + \beta_1 W)\chi_0 \rangle = 0.$$

Since by the premise χ_0 has depth at least 1, it follows from Theorem 2.3.1 that $\langle \chi_0, V\chi_0 \rangle = 0$. Combining this equality with the previous one implies that $\beta_1 = 0$, and therefore $H(s)$ is tangent to V at H_{r_λ} (to order at least 2). So, $(H_{r_\lambda} - \lambda)\chi_1 = -\alpha_1 V \chi_0$. Consequently,

$$\chi_1 = -\alpha_1 S_\lambda \chi_0 + \text{order 1 vector}.$$

Since χ_0 has depth at least 1, by Theorem 2.3.1 this implies $\mathbf{A}_\lambda \chi_1 = \alpha_1 \chi_0$. In particular, χ_1 is a vector of order 2. Further, if the parametrisation of $\gamma_{\chi_0}(s)$ is standard, then $\alpha_1 = 1$.

This proves the theorem in the case of $k = 2$. We proceed by induction on k. So, assume that the claim holds for $k \le n$ and let χ_0 be an eigenvector of depth $\ge n$. Differentiating n times the eigenvalue equation $H(s)\chi(s) = \lambda\chi(s)$ we obtain

$$\sum_{j=0}^{n} \binom{n}{j} H^{(j)}(s)\chi^{(n-j)}(s) = \lambda \chi^{(n)}(s).$$

Letting $s = r_\lambda$ and replacing $\chi^{(j)}(r_\lambda)/j!$ by χ_j gives

$$H_{r_\lambda}\chi_n + \sum_{j=1}^{n} (\alpha_j V + \beta_j W)\chi_{n-j} = \lambda \chi_n.$$

By the induction assumption,

$$\beta_1 = \cdots = \beta_{n-1} = 0. \tag{4.3.3}$$

Hence,

$$(H_{r_\lambda} - \lambda)\chi_n + \sum_{j=1}^{n-1} \alpha_j V\chi_{n-j} + (\alpha_n V + \beta_n W)\chi_0 = 0. \tag{4.3.4}$$

Since χ_0 has depth at least n, for some vector g we have $\chi_0 = \mathbf{A}_\lambda^n g$. Since, by the induction assumption, χ_j, $j = 0, 1, \dots, n-1$, is a vector of order $j+1$, it follows that for all $j = 0, 1, \dots, n-1$,

$$\langle \chi_0, V\chi_j \rangle = \langle \mathbf{A}_\lambda^n g, V\chi_j \rangle = \langle g, V\mathbf{A}_\lambda^n \chi_j \rangle = 0.$$

Hence, it follows from (4.3.4) by taking the scalar product of the left hand side and χ_0 that

$$\beta_n = 0 \tag{4.3.5}$$

and so

$$(H_{r_\lambda} - \lambda)\chi_n + \sum_{j=1}^{n} \alpha_j V\chi_{n-j} = 0.$$

It follows from (4.3.2), (4.3.3) and (4.3.5) that V is tangent to γ_{χ_0} to order at least $n+1$. Further, $(H_{r_\lambda} - \lambda)\chi_n$ is orthogonal to $\mathcal{V}_\lambda$, and $V\chi_0, \dots, V\chi_{n-2}$ are orthogonal to $\mathcal{V}_\lambda$ by the induction assumption (since $\chi_0, \dots, \chi_{n-2}$ have depth at least 1). Hence, according to the last equality, so is $V\chi_{n-1}$. Also by the last equality,

$$\chi_n + \sum_{j=1}^{n} \alpha_j S_\lambda \chi_{n-j} = \text{order 1 vector}.$$

Hence, by Theorem 2.3.1,

$$\mathbf{A}_\lambda \chi_n = \sum_{j=1}^{n} \alpha_j \chi_{n-j}.$$

Since, by the induction assumption, the vectors $\chi_0, \dots, \chi_{n-2}$ have depth at least 1, the last equality implies that so does χ_{n-1}.

Further, if the parametrisation of $\gamma_{\chi_0}(s)$ is standard, then $\alpha_2 = \cdots = \alpha_n = 0$ and $\alpha_1 = 1$, and therefore $\mathbf{A}_\lambda \chi_n = \chi_{n-1}$. ∎

Theorem 4.3.1 combined with Theorem 4.1.2 proves the following.

THEOREM 4.3.2. *Under Assumption* 1.2.1*, a regular direction at any resonance point H_{r_λ} is tangent to order at least k if and only if the order of the direction is at least k.*

Theorem 4.3.2 has the following corollary.

THEOREM 4.3.3. *Under Assumption* 1.2.1*, a regular direction at a resonance point is simple if and only if it is transversal.*

The last two theorems give a geometric interpretation of the order of a regular direction in the case where λ is outside the essential spectrum.

5. Resonance points as functions of the spectral parameter

In this section we study resonance points r_z as functions of the spectral parameter z. To stress this, we will often write $r(z)$ instead of r_z.

5.1. Order of a resonance function $r_z = r(z)$. In this subsection we consider the following question. Every resonance point r_z corresponding to a complex number z from the complement of the essential spectrum has an order, which is a positive number. A natural question is how the order of r_z depends on z. In the following theorem we show that the order of r_z is equal to 1 for all values of z except a discrete set, provided that r_z admits analytic continuation to a gap in the essential spectrum in the real axis and that the analytic continuation has a real value at least at one point. We conjecture that this property holds in general without this assumption, but since we are interested in analytic continuation of r_λ, this hypothesis automatically holds in our case.

THEOREM 5.1.1. *Let r_λ be a real resonance point of the triple $(\lambda; H_0, V)$. Analytic continuation of r_λ as a function of the complex variable λ has order 1 except on a discrete set.*

Proof. Assume the contrary: there exists a non-empty open subset G of the domain of holomorphy of r_z such that for all $z \in G$ the resonance point r_z has order at least 2. Then by [Az6, Corollary 3.4.7] there exist holomorphic vector-functions $\chi_1(z)$ and $\chi_2(z)$ such that

$$(H_0 + r(z)V - z)\chi_1(z) = 0, \tag{5.1.1}$$

$$(H_0 + r(z)V - z)\chi_2(z) = -V\chi_1(z). \tag{5.1.2}$$

Differentiation of (5.1.1) with respect to z gives

$$(r'(z)V - 1)\chi_1(z) + (H_0 + r(z)V - z)\chi_1'(z) = 0. \tag{5.1.3}$$

Let $\varphi(\bar z)$ be an anti-resonance vector-function of order 1, that is,

$$(H_0 + \bar r(z)V - \bar z)\varphi(\bar z) = 0. \tag{5.1.4}$$

Since

$$\langle \varphi(\bar z), (H_0 + r(z)V - z)\chi_1'(z)\rangle = \langle (H_0 + \bar r(z)V - \bar z)\varphi(\bar z), \chi_1'(z)\rangle = 0,$$

taking the scalar product of $\varphi(\bar z)$ with both sides of (5.1.3) gives

$$\langle \varphi(\bar z), (r'(z)V - 1)\chi_1(z)\rangle = 0.$$

Further, (5.1.2) implies that

$$\begin{aligned} -\langle \varphi(\bar z), V\chi_1(z)\rangle &= \langle \varphi(\bar z), (H_0 + r(z)V - z)\chi_2(z)\rangle \\ &= \langle (H_0 + \bar r(z)V - \bar z)\varphi(\bar z), \chi_2(z)\rangle = 0. \end{aligned} \tag{5.1.5}$$

Combining this with the previous equality implies

$$\langle \varphi(\bar z), \chi_1(z)\rangle = 0.$$

Since $r(z)$ takes real values in some interval I of the real axis, we can take $\chi_1(z)$ to be the holomorphic extension of a first order vector-function in I, and we can take $\varphi(\bar z)$ to be the anti-holomorphic extension of the same function. Since the scalar product is anti-linear in the first argument, the scalar product of this pair of holomorphic and anti-holomorphic vector-functions will be holomorphic, and according to the last displayed formula it will vanish on I. This implies that both these functions are zero in I, and therefore everywhere. Since $\chi_1(z)$ is an eigenvector, this gives a contradiction. ∎

5.2. Cycles of resonance points. Let r_λ be a resonance point of geometric multiplicity m and algebraic multiplicity N. When λ is shifted to $z = \lambda + iy$ with small $y > 0$, the resonance point r_λ splits into $N = \dim \Upsilon_\lambda(r_\lambda)$ resonance points $r_z^{(j)}$, counting algebraic multiplicities. The resonance points $r_z^{(j)}$ are holomorphic functions of z in a neighbourhood of λ. When z makes one round around λ, these N holomorphic functions undergo a permutation. We shall show in this subsection that this permutation consists of m disjoint cycles of lengths $d_1, \dots, d_m$, where m is the number of Jordan cells of the compact operator $A_\lambda(s)$ corresponding to the eigenvalue $(s - r_\lambda)^{-1}$, and d_ν is the size of the νth cell. Until then we denote the lengths of those cycles by $\hat{d}_\nu$.

For convenience, in this subsection we shall often indicate dependence of a resonance point r_z on the spectral parameter z in the usual way as $r(z)$ instead of using subscript. If there is no danger of confusion, we may choose to drop the variable z from the notation altogether.

In Section 3 we studied eigenvalues $\lambda_\nu(r)$ of $H_r = H_0 + rV$ as functions of the coupling constant r. The latter was treated as a real variable. In this section we consider r as a function of the spectral parameter λ, but unlike Section 3, we shall treat both r and λ as complex variables. Since the spectral variable treated as a complex variable is denoted by z, the functions under study are $r_\nu(z)$, which are inverses of $\lambda_\nu(r)$. According to Theorem 3.1.7, the eigenvalue functions $\lambda_\nu(r)$ of order $\tilde{d}_\nu > 1$ satisfy $\lambda'_\nu(r_\lambda) = 0$. Hence, in general the corresponding inverse function $r_\nu(z)$ is a branching multivalued holomorphic function in a neighbourhood of $z = \lambda$.

THEOREM 5.2.1.

(a) *For each cycle $r_\nu^{(\cdot)}(z)$ of resonance points there exists $\varepsilon > 0$ and a resonance point $r_\nu^{(0)}(z)$ of this cycle which takes real values for all $z \in I$, where I is either $[\lambda, \lambda + \varepsilon)$ or $(\lambda - \varepsilon, \lambda]$.*
(b) *The number of real resonance points in a cycle for $z \in I$ is either 1 or 2.*
(c) *In the case there are two real resonance points r' and r'' in a cycle for $z \in I$, the numbers $r' - r_\lambda$ and $r'' - r_\lambda$ have different signs.*
(d) *The numbers of non-real resonance points in each cycle for $z \in I$ are the same in $\mathbb{C}_+$ and $\mathbb{C}_-$.*
(e) *In case there are two real resonance points r' and r'' in a cycle for $z \in I$, they shift to different half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$ as $z \in I$ is shifted to $z + iy$ with small $y > 0$.*

Proof. Part (a) follows from the fact that an isolated eigenvalue λ of $H_0 + r_\lambda V$ is *stable*, that is, an eigenvalue $\lambda_\nu(s)$ of H_s depends on s continuously.

Parts (b) and (c) follow from the formula (see (3.1.7))

$$\lambda_\nu(s) = \lambda + \varepsilon_\nu (s - r_\lambda)^{\tilde{d}_\nu} + O((s - r_\lambda)^{\tilde{d}_\nu + 1}), \qquad s \to r_\lambda,$$

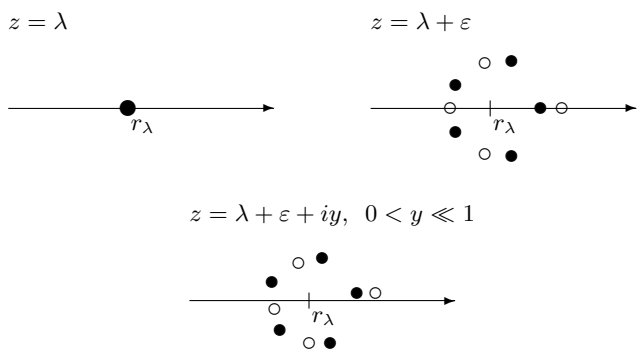
where $\tilde{d}_\nu$ is the order of $\lambda_\nu(s)$. Namely, with $\varepsilon > 0$ sufficiently small, if $\tilde{d}_\nu$ is odd then there is a unique real resonance point for all $z \in I$, where $I = [\lambda, \lambda + \varepsilon)$ or $I = (\lambda - \varepsilon, \lambda]$. In this case, I can be chosen to be either $[\lambda, \lambda + \varepsilon)$ or $(\lambda - \varepsilon, \lambda]$. If $\tilde{d}_\nu$ is even, then there are exactly two real resonance points for all $z \in I$ and they are located on different sides

of r_λ in the real axis of the coupling constant. In this case, if $\varepsilon_\nu > 0$ then I is $[\lambda, \lambda + \varepsilon)$, and if $\varepsilon_\nu < 0$ then I is $[\lambda - \varepsilon, \lambda)$.

(d) By Lemma 6.3.2 below, if r_z is a resonance point corresponding to z, then $\bar{r}_z$ is a resonance point corresponding to $\bar{z}$. Hence, the set of resonance points corresponding to a real value of λ is symmetric with respect to the real axis. We still need to show that if a resonance point $r_\nu^{(j)}(z)$ belongs to a cycle ν, then its conjugate also belongs to the same cycle, but this readily follows from the Schwarz reflection principle.

(e) By (d), for real $z \in I$ the set of resonance points in a cycle is symmetric with respect to the real axis. Combining this with the fact that for non-real z there can be no real resonance points, one can infer the claim. ∎

The following figure demonstrates this theorem by showing one of the possible scenarios of behaviour of resonance points of the group of r_λ, when λ is perturbed slightly first along the real axis and then off the real axis. In this figure there are two cycles: of length $\hat{d}_1 = 5$ (black dots) and $\hat{d}_2 = 4$ (white dots).



Proposition 5.2.2. *The number of cycles of the permutation of the N resonance points $r_z^{(j)}$ is equal to the geometric multiplicity m of the resonance point r_λ. More precisely, there is a natural one-to-one correspondence between cycles of resonance points $r_\nu^{(j)}(z)$ and the eigenvalue functions $\lambda_\nu(s)$ of the operator H_s, given by the following diagram:*

$$\varphi_\nu(s) \ \leftrightarrow \ \lambda_\nu(s) \ \leftrightarrow \ r_\nu^{(\cdot)}(z). \tag{5.2.1}$$

That is, with an eigenpath $\varphi_\nu(s)$ we associate an eigenvalue function $\lambda_\nu(s)$, and the inverse of this eigenvalue function is the multivalued holomorphic function $r_\nu^{(\cdot)}(z)$.

Proof. This immediately follows from Theorem 5.2.1(a)–(c). ∎

Now we shall prove that the order $\tilde{d}_\nu$ of the eigenpath $\varphi_\nu(s)$ is equal to the size $\hat{d}_\nu$ of a cycle corresponding to $\varphi_\nu(s)$.

By Theorem 3.1.7(i), (iii), the order $\tilde{d}_\nu$ of $\varphi_\nu(s)$ is determined by (3.1.7). Hence, the inverse of the function $\lambda_\nu(s)$ in a neighbourhood of $s = r_\lambda$ is a multivalued function $r_\nu^{(\cdot)}(z)$ with $\tilde{d}_\nu$ branches in a neighbourhood of $z = \lambda$, and these branches form a single cycle. This gives

$$\tilde{d}_\nu = \hat{d}_\nu.$$

Since the sum of the cycle sizes $\hat{d}_\nu$ of resonance points is N, so is the sum of the orders $\tilde{d}_\nu$ of the eigenpaths $\varphi_\nu(s)$, and therefore, by Theorem 3.1.7(v), the m sets of vectors

$$\frac{1}{j!}\varphi_\nu^{(j)}(r_\lambda), \qquad \nu = 1, \dots, m,\ j = 0, 1, \dots, \tilde{d}_\nu - 1,$$

form a Jordan basis for the nilpotent operator $\mathbf{A}_\lambda(r_\lambda)$.

We collect these assertions in the following theorem.

THEOREM 5.2.3. *For each $\nu = 1, \dots, m$, the following numbers are equal (assuming that they are arranged in decreasing order):*

(1) *the order $\tilde{d}_\nu$ of the eigenpath $\varphi_\nu(s)$,*
(2) *the size $\hat{d}_\nu$ of the cycle ν of resonance points of the group of r_λ,*
(3) *the size d_ν of the νth Jordan block of the nilpotent operator $\mathbf{A}_\lambda(r_\lambda)$.*

Moreover, the vectors

$$\frac{1}{j!}\varphi_\nu^{(j)}(r_\lambda), \qquad \nu = 1, \dots, m,\ j = 0, 1, \dots, d_\nu - 1,$$

form a Jordan basis of the nilpotent operator $\mathbf{A}_\lambda(r_\lambda)$.

5.3. Decomposition of $P_\lambda(r_\lambda)$. Let

$$r_\nu^{(0)}(z), \dots, r_\nu^{(d_\nu - 1)}(z)$$

be a cycle ν of resonance points. Since when z makes one round around λ, the elements of the cycle simply undergo a permutation, the function

$$P_z^{[\nu]} := \sum_{j=0}^{d_\nu - 1} P_z(r_\nu^{(j)}(z))$$

is single-valued in a neighbourhood of λ. Since $P_z(r_\nu^{(j)}(z))P_z(r_\nu^{(k)}(z)) = 0$ for different resonance points $r_\nu^{(j)}$ and $r_\nu^{(k)}$, the operator $P_z^{[\nu]}$ is an idempotent. In this subsection we shall prove some properties of this idempotent.

Let φ be a holomorphic function of one complex variable s with values in a Banach space (this is what we need). We recall that the *divided difference* $\varphi^{[j]}$ of order $j = 1, 2, \dots$ of φ is a function of $j + 1$ variables defined recursively by

$$\varphi^{[1]}(s_0, s_1) = \frac{\varphi(s_1) - \varphi(s_0)}{s_1 - s_0},$$

$$\varphi^{[j+1]}(s_0, s_1, \dots, s_{j+1}) = \frac{\varphi^{[j]}(s_1, \dots, s_{j+1}) - \varphi^{[j]}(s_0, s_1, \dots, s_j)}{s_{j+1} - s_0}.$$

The scalar version of the formula

$$\lim_{s_0, \dots, s_j \to s} \varphi^{[j]}(s_0, s_1, \dots, s_j) = \frac{1}{j!}\varphi^{(j)}(s) \tag{5.3.1}$$

is well-known and can be found in any textbook on numerical analysis; the vector version easily follows from the scalar version.

THEOREM 5.3.1. *For each $\nu = 1, \dots, m$, the following assertions hold:*

(1) *The function $P_z^{[\nu]}$ of z is holomorphic in a neighbourhood of λ.*

(2) *The range*

$$\Upsilon_\lambda^{[\nu]} := \operatorname{im} P_\lambda^{[\nu]}$$

of the idempotent $P_\lambda^{[\nu]}$ *(which exists by* (1)*) has dimension* d_ν *and basis*

$$\varphi_\nu(r_\lambda), \varphi_\nu'(r_\lambda), \dots, \varphi_\nu^{(d_\nu-1)}(r_\lambda). \tag{5.3.2}$$

(3) *The operators* $P_\lambda^{[\nu]}(r_\lambda)$ *and* $\mathbf{A}_\lambda(r_\lambda)$ *commute, and thus the vector space* $\Upsilon_\lambda^{[\nu]}$ *is invariant with respect to* $\mathbf{A}_\lambda(r_\lambda)$.

(4) *The dimension of* $\Upsilon_\lambda^{[\nu]} \cap \mathcal{V}_\lambda$ *is equal to* 1.

(5) *The Jordan cell decomposition of the restriction of* $\mathbf{A}_\lambda^{[\nu]}$ *to* $\Upsilon_\lambda^{[\nu]}$ *consists of only one Jordan cell.*

Proof. We split the proof into several parts.

(A) *The range of* $P_z^{[\nu]}$ *converges as* $z \to \lambda$ *to the linear span of the vectors*

$$\varphi_\nu(r_\lambda), \varphi_\nu'(r_\lambda), \dots, \varphi_\nu^{(d_\nu-1)}(r_\lambda).$$

By (5.3.1), the divided differences $\varphi_\nu^{[j]}$ for $j = 1, \dots, d_\nu - 1$, evaluated at the points

$$r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)$$

of the νth cycle, converge to $\frac{1}{j!}\varphi_\nu^{(j)}(r_\lambda)$ as $z \to \lambda$, since all $r_\nu^{(k)}(z)$ converge to r_λ. On the other hand, by

$$P_z^{[\nu]} \varphi_\nu(r_\nu^{(k)}(z)) = \varphi_\nu(r_\nu^{(k)}(z))$$

we have

$$P_z^{[\nu]} \varphi_\nu^{[j]}(r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)) = \varphi_\nu^{[j]}(r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)).$$

Since, by Theorem 5.2.3, the vectors $\varphi_\nu^{(0)}(r_\lambda), \dots, \varphi_\nu^{(d_\nu-1)}(r_\lambda)$ are linearly independent and the range of $P_z^{[\nu]}$ has dimension d_ν, the proof of part (A) is complete.

(B) *For each* $\nu = 1, \dots, m$, *the function* $P_z^{[\nu]}$ *of* z *is holomorphic in a neighbourhood of* λ. By [Ka, Theorem II.1.8], applied to the function $z \mapsto A_z(r_\lambda)$ in a neighbourhood of $z = \lambda$, the Laurent expansion of $P_z^{[\nu]}$ can only have finitely many terms with negative powers of $z - \lambda$. The sum $\sum_{\nu=1}^m P_z^{[\nu]}$ converges to $P_\lambda(r_\lambda)$ as $z \to \lambda$, and therefore is bounded in a neighbourhood of λ. Since $P_z^{[\nu]} P_z^{[\mu]} = \delta_{\nu\mu} P_z^{[\nu]}$, combining this with part (A) one infers that each $P_z^{[\nu]}$ is also bounded in a neighbourhood of λ.

(C) By (B), the operator $P_z^{[\nu]}$ has a limit as $z \to \lambda$, which we denote by

$$P_\lambda^{[\nu]}(r_\lambda) := \lim_{z \to \lambda} P_z^{[\nu]}. \tag{5.3.3}$$

We have

$$P_\lambda^{[\nu]} P_\lambda^{[\mu]} = \delta_{\nu\mu} P_\lambda^{[\nu]} \tag{5.3.4}$$

and

$$P_\lambda(r_\lambda) = \sum_{\nu=1}^m P_\lambda^{[\nu]}(r_\lambda).$$

(D) *The operators* $P_\lambda^{[\nu]}(r_\lambda)$ *and* $\mathbf{A}_\lambda(r_\lambda)$ *commute.* Since the limits of $P_z^{[\nu]}$ and $A_z(s)$ as $z \to \lambda$ exist, it is enough to take limits of both sides of the equality $P_z^{[\nu]} A_z(s) = A_z(s) P_z^{[\nu]}$, and then use Laurent expansion of $A_\lambda(s)$.

(E) *The dimension of* $\Upsilon_\lambda^{[\nu]} \cap \mathcal{V}_\lambda$ *is equal to* 1. By (3), the dimension of $\Upsilon_\lambda^{[\nu]} \cap \mathcal{V}_\lambda$ is at least 1. Hence, it is 1, since otherwise we get a contradiction with Proposition 5.2.2.

(F) *The Jordan cell decomposition of the restriction of* $\mathbf{A}_\lambda^{[\nu]}$ *to* $\Upsilon_\lambda^{[\nu]}$ *consists of only one Jordan cell.* This follows immediately from (3) and (4). ∎

We denote by $\mathbf{A}_\lambda^{[\nu]}(r_\lambda)$ the restriction of $\mathbf{A}_\lambda(r_\lambda)$ to $\Upsilon_\lambda^{[\nu]}$.

In a similar way, we introduce operators $Q_z^{[\nu]}$ by

$$Q_z^{[\nu]} := \sum_{j=0}^{d_\nu - 1} Q_z(r_\nu^{(j)}(z)).$$

Many properties of $Q_z^{[\nu]}$ are analogous to those of $P_z^{[\nu]}$.

It follows from (1.3.11) that $VP_z^{[\nu]} = Q_z^{[\nu]}V$. Letting $z \to \lambda$ in this equality, we obtain

$$VP_\lambda^{[\nu]} = Q_\lambda^{[\nu]}V. \tag{5.3.5}$$

Now, using results of this subsection, we give a second proof of Theorem 3.1.12.

COROLLARY 5.3.2. *Let* $\varphi_\nu(s)$ *and* $\varphi_\mu(s)$ *be different eigenpaths of* H_s, $s \in \mathbb{R}$. *For all* $j = 0, 1, \dots, d_\nu - 1$ *and all* $k = 0, 1, \dots, d_\mu - 1$,

$$\langle \varphi_\nu^{(j)}(r_\lambda), V\varphi_\mu^{(k)}(r_\lambda)\rangle = 0.$$

Proof. By Theorem 5.3.1(2), the vectors $\varphi_\nu(r_\lambda), \varphi_\nu'(r_\lambda), \dots, \varphi_\nu^{(d_\nu-1)}(r_\lambda)$ span $\Upsilon_\lambda^{[\nu]} = \operatorname{im} P_\lambda^{[\nu]}$. Hence, using (5.3.5) and (5.3.4), we have

$$\langle \varphi_\nu^{(j)}(r_\lambda), V\varphi_\mu^{(k)}(r_\lambda)\rangle = \langle \varphi_\nu^{(j)}(r_\lambda), VP_\lambda^{[\mu]}\varphi_\mu^{(k)}(r_\lambda)\rangle = \langle P_\lambda^{[\mu]}\varphi_\nu^{(j)}(r_\lambda), V\varphi_\mu^{(k)}(r_\lambda)\rangle = 0. \ ∎$$

Actually, this corollary is stronger than Theorem 3.1.12, since the latter does not assert that the sum of $\tilde{d}_\nu$ is N.

5.4. Two lemmas. Since by Theorem 5.1.1 the functions $r_\nu^{(j)}(z)$ have order 1, the operator-valued function $A_z(s)$ near the (real) point r_λ has the Laurent expansion

$$A_z(s) = \tilde{A}_z(s) + \sum_{\nu=1}^{m} \sum_{j=0}^{d_\nu - 1} \frac{P_z(r_\nu^{(j)}(z))}{s - r_\nu^{(j)}(z)}, \tag{5.4.1}$$

where $\tilde{A}_z(s)$ is a meromorphic function which has no poles in a neighbourhood of r_λ which includes all $r_\nu^{(j)}(z)$. The functions $P_z(r_\nu^{(j)}(z))/(s - r_\nu^{(j)}(z))$ of z (with s fixed), taken individually, are not single-valued in a neighbourhood of λ, unless $d_\nu = 1$, but each of the m sums

$$\sum_{j=0}^{d_\nu - 1} \frac{P_z(r_\nu^{(j)}(z))}{s - r_\nu^{(j)}(z)} \tag{5.4.2}$$

is a single-valued function of z in a neighbourhood of λ. We also have

$$P_z^{[\nu]}A_z(s) = \sum_{j=0}^{d_\nu - 1} \frac{P_z(r_\nu^{(j)}(z))}{s - r_\nu^{(j)}(z)}. \tag{5.4.3}$$

LEMMA 5.4.1. *As $z \to \lambda$, the fractional part*

$$P_z(r_\lambda)A_z(s) = \sum_{\nu=1}^{m}\sum_{j=0}^{d_\nu-1}\frac{P_z(r_\nu^{(j)}(z))}{s-r_\nu^{(j)}(z)}$$

of the Laurent expansion (5.4.1) *of the meromorphic function $A_z(s)$ at the N poles*

$$r_\nu^{(j)}(z), \quad \nu = 1,\dots,m,\ j = 0,\dots,d_\nu-1,$$

of the group of r_λ converges in the norm topology to the fractional part of the Laurent expansion of $A_\lambda(s)$ at the pole r_λ, which is

$$P_\lambda(r_\lambda)A_\lambda(s) = \frac{P_\lambda(r_\lambda)}{s-r_\lambda} + \frac{\mathbf{A}_\lambda(r_\lambda)}{(s-r_\lambda)^2} + \cdots + \frac{\mathbf{A}_\lambda^{d-1}(r_\lambda)}{(s-r_\lambda)^{d-1}}. \tag{5.4.4}$$

The convergence is uniform with respect to s on compact subsets of a deleted neighbourhood of r_λ.

Proof. The meromorphic function $A_z(s)$ converges in norm to $A_\lambda(s)$ as $z \to \lambda$ uniformly on compact subsets of a deleted neighbourhood of the pole r_λ in $\mathbb{C}$. Furthermore, $P_z(r_\lambda)$ converges to $P_\lambda(r_\lambda)$ in norm as $z \to \lambda$ (in fact, even in trace class norm; see [Az6, Lemma 5.2.2]). The claim follows. ∎

LEMMA 5.4.2. *For each $\nu = 1,\dots,m$,*

$$\lim_{z\to\lambda}\sum_{j=0}^{d_\nu-1}\frac{P_z(r_\nu^{(j)}(z))}{s-r_\nu^{(j)}(z)} = \frac{P_\lambda^{[\nu]}(r_\lambda)}{s-r_\lambda} + \frac{\mathbf{A}_\lambda^{[\nu]}(r_\lambda)}{(s-r_\lambda)^2} + \cdots + \frac{(\mathbf{A}_\lambda^{[\nu]}(r_\lambda))^{d_\nu-1}}{(s-r_\lambda)^{d_\nu-1}}, \tag{5.4.5}$$

where d_ν is the size of the cycle ν.

Proof. By Theorem 5.3.1(1),

$$\lim_{z\to\lambda} P_z^{[\nu]}A_z(s) = P_\lambda^{[\nu]}A_\lambda(s),$$

where the convergence is in norm and is uniform on compact subsets of a deleted neighbourhood of r_λ. This equality is exactly (5.4.5). ∎

5.5. Puiseux series for $r_\nu^{(j)}(z)$. For a positive integer d let

$$\varepsilon_d = e^{2\pi i/d}.$$

For an integer k and a positive integer d let

$$[d|k] = \begin{cases} 0 & \text{if } d \text{ does not divide } k,\\ 1 & \text{if } d \text{ divides } k.\end{cases}$$

For any integer k we have

$$\sum_{j=0}^{d-1}\varepsilon_d^{jk} = [d|k]\cdot d. \tag{5.5.1}$$

For each $\nu = 1,\dots,m$, the functions $r_\nu^{(0)}(z),\dots,r_\nu^{(d_\nu-1)}(z)$ are different branches of a multivalued holomorphic function with Puiseux series

$$r_\nu^{(j)}(z) = \sum_{k=0}^{\infty} r_{k/d_\nu}\varepsilon_{d_\nu}^{kj}(z-\lambda)^{k/d_\nu}, \quad j = 0,\dots,d_\nu-1. \tag{5.5.2}$$

The Puiseux series for $r_\nu^{(j)}(z)$ does not have the part with negative powers of $z-\lambda$, since this function is continuous at $z=\lambda$.

PROPOSITION 5.5.1. *The coefficients r_{k/d_ν} of the Puiseux series* (5.5.2) *are real.*

Proof. More precisely, these numbers can be chosen to be real. For $j=0$ we have

$$r_\nu^{(0)}(z)=\sum_{k=0}^{\infty} r_{k/d_\nu}(z-\lambda)^{k/d_\nu}.$$

Recall that λ is an isolated eigenvalue of the self-adjoint operator H_{r_λ}. Due to stability of isolated eigenvalues, one of the functions $r_\nu^{(j)}(z)$ must take real values for real values of z close to λ on the left or on the right. From this, one can infer that the numbers r_{k/d_ν} need to be real. ∎

PROPOSITION 5.5.2. *The first coefficient r_{1/d_ν} of the Puiseux series* (5.5.2) *for $r_\nu^{(j)}(z)$ is non-zero.*

Proof. Let r_{f_ν/d_ν} be the first non-zero coefficient of (5.5.2). Since $r_\nu^{(\cdot)}(z)$ is the inverse of $\lambda_\nu(s)$ (see diagram (5.2.1)), it follows from Theorem 3.1.7(iii) and (5.5.2) that

$$\lambda_\nu(s)=\lambda+\varepsilon_\nu(s-r_\lambda)^{d_\nu/f_\nu}+O((s-r_\lambda)^{d_\nu/f_\nu+1})$$

with non-zero ε_ν. Since $\lambda_\nu(s)$ is analytic at $s=r_\lambda$, f_ν divides d_ν. If $f_\nu>1$, then this would imply that the set of d_ν numbers $\{r_\nu^{(j)}(z)\colon j=0,\dots,d_\nu-1\}$ is not a cycle. Hence, the only possibility is $f_\nu=1$. ∎

5.6. Puiseux series for $P_z(r_\nu^{(j)}(z))$. In this subsection we shall study the Puiseux series of $P_z(r_\nu^{(j)}(z))$ at $z=\lambda$.

For each $\nu=1,\dots,m$, the operator-valued functions

$$P_z(r_\nu^{(0)}(z)),\dots,P_z(r_\nu^{(d_\nu-1)}(z))$$

are different branches of a multivalued holomorphic function with Puiseux series

$$P_z(r_\nu^{(j)}(z))=\tilde P_\nu^j(z)+\sum_{l=0}^{p}\varepsilon_{d_\nu}^{-lj}(z-\lambda)^{-l/d_\nu}P_{-l/d_\nu},\qquad j=0,\dots,d_\nu-1 \tag{5.6.1}$$

where $\tilde P_\nu^j(z)$ is the part with positive powers of $z-\lambda$. Since by the definition (1.3.5) the operator $P_z(r_\nu^{(j)}(z))$ is an eigenprojection of the compact operator $A_z(s)$ corresponding to a non-zero eigenvalue

$$(s-r_\nu^{(j)}(z))^{-1},$$

by [Ka, Theorem II.1.8], the part of the Puiseux expansion (5.6.1) with negative powers is finite.

Our first aim is to prove that the upper limit p in (5.6.1) is equal to $d_\nu-1$. For this we need two auxiliary lemmas.

LEMMA 5.6.1. *For each cycle of resonance points $r_\nu^{(\cdot)}(z)$,*

$$\sum_{j=0}^{d_\nu-1}\frac{P_z(r_\nu^{(j)}(z))}{s-r_\nu^{(j)}(z)}=\sum_{j=0}^{d_\nu-1}\frac{A_j^\nu(z)}{(s-r_\nu^{(0)}(z))\dots(s-r_\nu^{(j)}(z))}, \tag{5.6.2}$$

where for $j = 0, 1, \dots, d_\nu - 1$,

$$A_j^\nu(z) = \sum_{b=j}^{d_\nu - 1} (r_\nu^{(b)}(z) - r_\nu^{(0)}(z)) \dots (r_\nu^{(b)}(z) - r_\nu^{(j-1)}(z)) P_z(r_\nu^{(b)}(z)). \tag{5.6.3}$$

Proof. This lemma has general character as it holds for any set of numbers $r_\nu^{(j)}(z)$ and any bounded operators $P_z(r_\nu^{(j)}(z))$. For this reason we write r_j for $r_\nu^{(j)}(z)$, P_j for $P_z(r_\nu^{(j)}(z))$, d for d_ν, and A_j for $A_j^\nu(z)$.

We prove (5.6.2) by induction on d. For $d = 1$ the assertion is trivial. Assume that the assertion holds for smaller values of d. Since the operators A_j depend on d, we will write $A_j^{(d)}$.

By the induction assumption

$$\sum_{j=0}^{d-1} \frac{P_j}{s - r_j} = \sum_{j=0}^{d-2} \frac{A_j^{(d-1)}}{(s - r_0) \dots (s - r_j)} + \frac{P_{d-1}}{s - r_{d-1}}.$$

We need to find operators $A_j^{(d)}$ such that

$$\sum_{j=0}^{d-1} \frac{A_j^{(d)}}{(s - r_0) \dots (s - r_j)} = \sum_{j=0}^{d-2} \frac{A_j^{(d-1)}}{(s - r_0) \dots (s - r_j)} + \frac{P_{d-1}}{s - r_{d-1}}.$$

Multiplying both sides by $\prod_{k=0}^{d-1}(s - r_k)$ gives

$$\sum_{j=0}^{d-1} A_j^{(d)} \prod_{k=j+1}^{d-1} (s - r_k) = \sum_{j=0}^{d-2} A_j^{(d-1)} \prod_{k=j+1}^{d-1} (s - r_k) + \prod_{k=0}^{d-2} (s - r_k) P_{d-1}.$$

Replacing s by r_{d-1} gives

$$A_{d-1}^{(d)} = 0 + \prod_{k=0}^{d-2} (r_{d-1} - r_k) P_{d-1}.$$

Replacing s by r_{d-2} gives

$$A_{d-1}^{(d)} + A_{d-2}^{(d)}(r_{d-2} - r_{d-1}) = A_{d-2}^{(d-1)}(r_{d-2} - r_{d-1}) + 0.$$

Therefore,

$$\begin{aligned} A_{d-2}^{(d)} &= \frac{A_{d-1}^{(d)}}{r_{d-1} - r_{d-2}} + A_{d-2}^{(d-1)} = \prod_{k=0}^{d-3} (r_{d-1} - r_k) P_{d-1} + \sum_{b=d-2}^{d-2} (r_b - r_0) \dots (r_b - r_{d-3}) P_b \\ &= \prod_{k=0}^{d-3} (r_{d-1} - r_k) P_{d-1} + (r_{d-2} - r_0) \dots (r_{d-2} - r_{d-3}) P_{d-2} \\ &= \sum_{b=d-2}^{d-1} (r_b - r_0) \dots (r_b - r_{d-3}) P_b, \end{aligned} \tag{5.6.4}$$

where the second equality follows from the previous one and the induction assumption.

Continuing this way, we complete the proof. ∎

LEMMA 5.6.2. *Let $r^* \in \mathbb{C}$ and let $r_0^n, \dots, r_{d-1}^n$, $n = 1, 2, \dots$, be d sequences of complex numbers such that for all $j = 0, 1, \dots, d-1$,*

$$\lim_{n\to\infty} r_j^n = r^*.$$

Assume that $B_0, \dots, B_{d-1}$ are bounded operators and $A_0^n, \dots, A_{d-1}^n$, $n = 1, 2, \dots$, are d sequences of bounded operators such that for all $s \in \mathbb{C} \setminus \{r^*\}$,

$$\lim_{n\to\infty} \left(\frac{A_0^n}{s - r_0^n} + \frac{A_1^n}{(s - r_0^n)(s - r_1^n)} + \cdots + \frac{A_{d-1}^n}{(s - r_0^n)(s - r_1^n)\dots(s - r_{d-1}^n)} \right) = \frac{B_0}{s - r^*} + \frac{B_1}{(s - r^*)^2} + \cdots + \frac{B_{d-1}}{(s - r^*)^d},$$

where the limit is in norm and is uniform on compact subsets of $\mathbb{C} \setminus \{r^*\}$. Then for all $j = 0, 1, \dots, d-1$ the sequence A_j^n converges in norm to B_j as $n \to \infty$.

Proof. Multiplying the equality in the lemma by

$$\lim_{n\to\infty} (s - r_0^n) \dots (s - r_{d-1}^n) = (s - r^*)^d$$

gives for all $s \neq r^*$ the equality

$$\lim_{n\to\infty} (A_0^n (s - r_1^n) \dots (s - r_{d-1}^n) + \cdots + A_{d-2}^n (s - r_{d-1}^n) + A_{d-1}^n) = B_0 (s - r^*)^{d-1} + \cdots + B_{d-1}.$$

Since the convergence is uniform, we can differentiate the left hand side under the limit sign (see e.g. [Di, §IX.12]). Hence, applying the operator $\frac{d^d}{ds^d}$ to both sides of the last equality, we infer that $A_0^n \to B_0$. Applying $\frac{d^{d-1}}{ds^{d-1}}$, we infer that $A_1^n \to B_1$, and so on. ∎

LEMMA 5.6.3. *For each cycle ν and each $k = 0, 1, \dots, d_\nu - 1$,*

$$\lim_{z\to\lambda} A_k^\nu(z) = \mathbf{A}_\lambda^k(r_\lambda) P_\lambda^{[\nu]}(r_\lambda),$$

where the convergence is in norm.

Proof. Combining Lemmas 5.4.2 and 5.6.1, we get

$$\lim_{z\to\lambda} \sum_{j=0}^{d_\nu - 1} \frac{A_j^\nu(z)}{(s - r_\nu^{(0)}(z)) \dots (s - r_\nu^{(j)}(z))} = \frac{P_\lambda^{[\nu]}(r_\lambda)}{s - r_\lambda} + \frac{\mathbf{A}_\lambda^{[\nu]}(r_\lambda)}{(s - r_\lambda)^2} + \cdots + \frac{(\mathbf{A}_\lambda^{[\nu]}(r_\lambda))^{d_\nu - 1}}{(s - r_\lambda)^{d_\nu - 1}},$$

where the convergence is in norm and is uniform in s on compact subsets of a deleted neighbourhood of r_λ. Hence, Lemma 5.6.2 completes the proof. ∎

PROPOSITION 5.6.4. *The integer p from the Puiseux series* (5.6.1) *for $P_z(r_\nu^{(j)}(z))$ satisfies*

$$p = d_\nu - 1. \tag{5.6.5}$$

Proof. By Lemma 5.6.3, we have

$$\lim_{z\to\lambda} A_{d_\nu - 1}^\nu(z) = \mathbf{A}_\lambda^{d_\nu - 1}(r_\lambda) P_\lambda^{[\nu]}(r_\lambda) \neq 0. \tag{5.6.6}$$

Here the last inequality ($\neq 0$) follows from the fact that the operator $\mathbf{A}_\lambda^{d_\nu - 1}(r_\lambda)$ is reduced by the image of the idempotent $P_\lambda^{[\nu]}(r_\lambda)$ and that the reduction has only one Jordan cell

(see Theorem 5.3.1(5)). By (5.6.3), we have

$$A^{\nu}_{d_\nu-1}(z) = (r_\nu^{(d_\nu-1)}(z) - r_\nu^{(0)}(z)) \dots (r_\nu^{(d_\nu-1)}(z) - r_\nu^{(d_\nu-2)}(z)) P_z(r_\nu^{(d_\nu-1)}(z)).$$

Since by Proposition 5.5.2 the first Puiseux series coefficient r_{1/d_ν} for $r_\nu^{(j)}(z)$ is non-zero, the first non-zero fractional term of the Laurent expansion of the product

$$(r_\nu^{(d_\nu-1)}(z) - r_\nu^{(0)}(z)) \dots (r_\nu^{(d_\nu-1)}(z) - r_\nu^{(d_\nu-2)}(z))$$

is

$$(z-\lambda)^{(d_\nu-1)/d_\nu}.$$

The term with the smallest power in the Laurent expansion (5.6.1) of $P_z(r_\nu^{(d_\nu-1)}(z))$ is

$$(z-\lambda)^{-p/d_\nu}.$$

Hence, if $p < d_\nu - 1$, the limit in (5.6.6) would be zero, and if $p > d_\nu - 1$, the limit would diverge, and so $p = d_\nu - 1$. ∎

Proposition 5.6.5. *Let $\nu = 1, \dots, m$. For any $k \geq 0$ the function*

$$\sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda)^k P_z(r_\nu^{(j)}(z))$$

is analytic at $z = \lambda$. Moreover, the limit of this sum as $z \to \lambda$ is equal to $P_\lambda^{[\nu]} \mathbf{A}_\lambda^k(r_\lambda)$.

Proof. The function is symmetric with respect to $r_\nu^{(j)}(z)$, $j = 0, 1, \dots, d_\nu - 1$, and therefore it is single-valued in a neighbourhood of λ. By Proposition 5.6.4, this function also cannot have whole negative powers of $z-\lambda$ in its power series expansion at λ. Hence, it is analytic at λ. In particular, if $k = 0$, then its limit as $z \to \lambda$ is equal to $P_\lambda^{[\nu]}(r_\lambda)$.

Further, we have

$$\sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda) P_z(r_\nu^{(j)}(z)) = \sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\nu^{(0)}(z)) P_z(r_\nu^{(j)}(z)) + \sum_{j=0}^{d_\nu-1} (r_\nu^{(0)}(z) - r_\lambda) P_z(r_\nu^{(j)}(z)).$$

The first summand converges to $P_\lambda^{[\nu]} \mathbf{A}_\lambda(r_\lambda)$ by Lemma 5.6.3 and (5.6.3) (taken with $j = 1$). The second summand converges to zero, since by Theorem 5.3.1(1),

$$\sum_{j=0}^{d_\nu-1} P_z(r_\nu^{(j)}(z))$$

is analytic and $r_\nu^{(0)}$ is continuous at $z = \lambda$. Thus, the claim holds for $k = 1$. Using this, for any $k \geq 1$ we have

$$\lim_{z\to\lambda} \sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda)^k P_z(r_\nu^{(j)}(z)) = \lim_{z\to\lambda} \Big(\sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda) P_z(r_\nu^{(j)}(z)) \Big)^k = P_\lambda^{[\nu]} \mathbf{A}_\lambda^k(r_\lambda),$$

where the first equality follows from (1.3.9). ∎

Since $p = d_\nu - 1$, we have

$$\lim_{z\to\lambda} (z-\lambda) P_z(r_\nu^{(j)}(z)) = 0.$$

PROPOSITION 5.6.6. *For all $\nu = 1, \dots, m$ and all $k = 1, \dots, d_\nu - 1$,*

$$P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^k(r_\lambda) = d_\nu \cdot \sum_{l=k}^{d_\nu-1} \Big(\sum_{m_1+\cdots+m_k=l} r_{m_1/d_\nu} \dots r_{m_k/d_\nu} \Big) P_{-l/d_\nu}, \tag{5.6.7}$$

where in the sum $m_1, \dots, m_k \geq 1$.

Proof. We prove this only for $k = 2$. The general case follows by the same calculation.

By Proposition 5.6.5 we have

$$\begin{aligned} P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^2(r_\lambda) &= \lim_{z\to\lambda} \sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda)^2 P_z(r_\nu^{(j)}(z)) \\ &= \lim_{z\to\lambda} \sum_{j=0}^{d_\nu-1} \Big(\sum_{k=1}^{\infty} r_{k/d_\nu} \varepsilon_{d_\nu}^{kj} (z-\lambda)^{k/d_\nu} \Big) \\ &\quad \times \Big(\sum_{m=1}^{\infty} r_{m/d_\nu} \varepsilon_{d_\nu}^{mj} (z-\lambda)^{m/d_\nu} \Big) \Big(\sum_{l=1}^{d_\nu-1} \varepsilon_{d_\nu}^{-lj} (z-\lambda)^{-l/d_\nu} P_{-l/d_\nu} \Big) \\ &= \lim_{z\to\lambda} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{d_\nu-1} \sum_{j=0}^{d_\nu-1} \varepsilon_{d_\nu}^{(k+m-l)j} (z-\lambda)^{(k+m-l)/d_\nu} r_{k/d_\nu} r_{m/d_\nu} P_{-l/d_\nu}. \end{aligned}$$

In this sum, there are finitely many terms with negative powers of $z - \lambda$ and the sum of positive powers of $z - \lambda$ converges absolutely in some neighbourhood of λ. Hence, all the interchanges of summations, which have been performed so far and which are about to follow, are justified.

Let $x = k + m$. Then

$$\begin{aligned} P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^2(r_\lambda) &= \lim_{z\to\lambda} \sum_{x=2}^{\infty} \sum_{m=1}^{x-1} \sum_{l=1}^{d_\nu-1} \sum_{j=0}^{d_\nu-1} \varepsilon_{d_\nu}^{(x-l)j} (z-\lambda)^{(x-l)/d_\nu} r_{(x-m)/d_\nu} r_{m/d_\nu} P_{-l/d_\nu} \\ &= d_\nu \cdot \lim_{z\to\lambda} \sum_{x=2}^{\infty} \sum_{m=1}^{x-1} \sum_{l=1}^{d_\nu-1} [d_\nu | x-l] (z-\lambda)^{(x-l)/d_\nu} r_{(x-m)/d_\nu} r_{m/d_\nu} P_{-l/d_\nu}, \end{aligned}$$

where in the last equality we have used (5.5.1). The terms with $x > l$ disappear after taking the limit $z \to \lambda$. Since $x \geq 2$ and $1 \leq l \leq d_\nu - 1$, there are no non-zero terms with $x < l$ and $d_\nu \mid x - l$. Hence, the factor $[d_\nu | x - l]$ can be replaced by Kronecker's symbol δ_{xl}. This gives

$$\begin{aligned} P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^2(r_\lambda) &= d_\nu \cdot \lim_{z\to\lambda} \sum_{x=2}^{\infty} \sum_{l=1}^{d_\nu-1} \delta_{xl} (z-\lambda)^{(x-l)/d_\nu} \Big(\sum_{m=1}^{x-1} r_{(x-m)/d_\nu}\, r_{m/d_\nu} \Big) P_{-l/d_\nu} \\ &= d_\nu \cdot \sum_{l=2}^{d_\nu-1} \Big(\sum_{m=1}^{l-1} r_{(l-m)/d_\nu} r_{m/d_\nu} \Big) P_{-l/d_\nu}. \ \blacksquare \end{aligned}$$

Two special cases of (5.6.7) are the formulas

$$P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda(r_\lambda) = d_\nu \sum_{l=1}^{d_\nu-1} r_{l/d_\nu} P_{-l/d_\nu} \quad \text{and} \quad P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^{d_\nu-1}(r_\lambda) = d_\nu \cdot r_{1/d_\nu}^{d_\nu-1} P_{(1-d_\nu)/d_\nu}.$$

COROLLARY 5.6.7. *For any $j = 1, \dots, d_\nu - 1$ the operator $P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^j(r_\lambda)$ is a linear combination of the operators*

$$P_{-j/d_\nu}, \dots, P_{-(d_\nu-1)/d_\nu}$$

from the Puiseux expansion of $P_z(r_\nu^{(j)}(z))$ at $z = \lambda$.

We summarise the results of this section in the following theorem.

THEOREM 5.6.8. *Let r_λ be a real resonance point of the line $H_{r_\lambda} + (s - r_\lambda)V$, corresponding to a point λ outside essential spectrum. Let N and m be respectively the algebraic and geometric multiplicities of r_λ. Let $d_1, \dots, d_m$ be the sizes of Jordan cells of the compact operator $A_\lambda(s)$ corresponding to the eigenvalue $(s - r_\lambda)^{-1}$. Let $r_z^1, \dots, r_z^N$ be resonance points of the group of r_λ corresponding to $z \approx \lambda \in \mathbb{C}$. Then:*

(1) *As z makes one round about λ, the set of N resonance points undergoes a permutation which is the product of m disjoint cyclic permutations of the resonance points*

$$r_\nu^{(0)}(z), \dots, r_\nu^{(d_\nu-1)}(z), \qquad \nu = 1, \dots, m,$$

and, as the notation indicates, the sizes of these cyclic permutations are the same as the sizes $d_1, \dots, d_m$ of the Jordan cells.

(2) *For real z close to λ, there are either one or two* real *resonance points in each of these m cycles of resonance points. In case there are two real resonance points in a cycle, one of them is greater than r_λ and the other is smaller than r_λ; further, as z is shifted off the real axis, these two points shift off to different complex half-planes.*

(3) *The Puiseux series of $r_\nu^{(j)}(z)$ has the form*

$$r_\nu^{(j)}(z) = \sum_{k=0}^{\infty} r_{k/d_\nu} e^{2\pi i k j/d_\nu} (z - \lambda)^{k/d_\nu}$$

with $r_{1/d_\nu} \neq 0$ (where $r_0 = r_\lambda$) and all coefficients r_{k/d_ν} are real.

(4) *The Puiseux series of the idempotent $P_z(r_\nu^{(j)}(z))$ has the form*

$$P_z(r_\nu^{(j)}(z)) = \tilde{P}_\nu^{(j)}(z) + \sum_{l=0}^{d_\nu-1} e^{-2\pi i l j/d_\nu} (z - \lambda)^{-l/d_\nu} P_{-l/d_\nu},$$

where $\tilde{P}_\nu^{(j)}(z)$ is continuous at $z = \lambda$.

(5) *For each $\nu = 1, \dots, m$ and for all $k \geq 0$,*

$$\lim_{z \to \lambda} \sum_{j=0}^{d_\nu-1} (r_\nu^{(j)}(z) - r_\lambda)^k P_z(r_\nu^{(j)}(z)) = P_\lambda^{[\nu]}(r_\lambda)\mathbf{A}_\lambda^k(r_\lambda),$$

where $P_\lambda^{[\nu]}(r_\lambda)$ is an idempotent of rank d_ν which commutes with $\mathbf{A}_\lambda(r_\lambda)$ and which is given by

$$P_\lambda^{[\nu]}(r_\lambda) = \lim_{z \to \lambda} \sum_{j=0}^{d_\nu-1} P_z(r_\nu^{(j)}(z)).$$

In the rest of this subsection we give another proof of the equality

$$\mathbf{A}_\lambda(r_\lambda)\varphi_\nu^{(j)}(r_\lambda) = j\varphi_\nu^{(j-1)}(r_\lambda).$$

We denote by $M\varphi$ the function

$$(M\varphi)(s) = (s - r_\lambda)\varphi(s).$$

On the one hand, using Proposition 5.6.5 (with $k = 1$) and (5.3.1), we have

$$(E) := \lim_{z\to\lambda} \sum_{k=0}^{d_\nu - 1} (r_\nu^{(k)}(z) - r_\lambda) P_z(r_\nu^{(k)}(z))\varphi_\nu^{[j]}(r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)) = \frac{1}{j!}\mathbf{A}_\lambda^{[\nu]}(r_\lambda)\varphi_\nu^{(j)}(r_\lambda).$$

On the other hand, since

$$P_z(r_\nu^{(k)}(z))\varphi_\nu(r_\nu^{(l)}(z)) = \delta_{kl}\varphi_\nu(r_\nu^{(l)}(z)),$$

one can infer that

$$\sum_{k=0}^{d_\nu - 1} (r_\nu^{(k)}(z) - r_\lambda) P_z(r_\nu^{(k)}(z))\varphi_\nu^{[j]}(r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)) = (M\varphi_\nu)^{[j]}(r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)),$$

so that

$$(E) = \lim_{z\to\lambda} (M\varphi_\nu)^{[j]}(r_\nu^{(0)}(z), \dots, r_\nu^{(j)}(z)) = \frac{1}{j!}\frac{d^j}{ds^j}(M\varphi_\nu)(s)\Big|_{s=r_\lambda} = \frac{1}{(j-1)!}\varphi_\nu^{(j-1)}(r_\lambda),$$

where the second equality follows from (5.3.1). Hence, $\mathbf{A}_\lambda^{[\nu]}(r_\lambda)\varphi_\nu^{(j)}(r_\lambda) = j\varphi_\nu^{(j-1)}(r_\lambda)$.

5.7. Sign of a cycle. Previous results show that the decomposition

$$d_1 + \dots + d_m$$

of the algebraic multiplicity N of a real resonance point r_λ admits three interpretations: d_ν is the length of the νth cycle, the size of the νth Jordan cell, and the smallest positive integer such that $\lambda_\nu^{(d_\nu)}(r_\lambda) \neq 0$. In this subsection we show that with each of the cycles $\nu = 1, \dots, m$ one can associate a sign ± 1. This can be done in several equivalent ways.

THEOREM 5.7.1. *For each $\nu = 1, \dots, m$ and for all small enough $\varepsilon > 0$ and $y > 0$, the signs of the following real numbers coincide:*

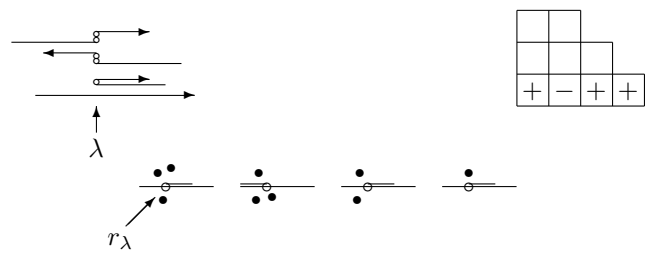
(1) $\lambda_\nu(r_\lambda + \varepsilon) - \lambda_\nu(r_\lambda)$,
(2) $\langle \varphi_\nu(r_\lambda), V\varphi_\nu^{(d_\nu - 1)}(r_\lambda)\rangle$,
(3) $\operatorname{Im} r_\nu^{(0)}(z + iy)$ *for all $z \in I$, where for some $\delta > 0$, I is one of the two intervals $(\lambda, \lambda + \delta)$ or $(\lambda - \delta, \lambda)$ on which the branch $r_\nu^{(0)}(z)$ takes real values larger than r_λ (such an interval and such a branch exist and are unique).*

Proof. That the signs of the first and second numbers coincide follows from (3.1.5). Equality of the signs of the first and third numbers can be inferred by considering the four cases: $\lambda^{(d_\nu)}(r_\lambda) > 0$ and $\lambda^{(d_\nu)}(r_\lambda) < 0$ for even and odd d_ν. This comparison is straightforward, and therefore the details are omitted. (Roughly speaking, since the derivative of, say, an increasing real-analytic function $I \to \mathbb{R}$ is positive, and since the derivative of the analytic extension of this function into the complex plane preserves the orientation of rotation of tangent vectors, if dz rotates into the upper half-plane (that is, if $y = \operatorname{Im} z$ becomes positive), then so does $dr_\nu(z)$.) ∎

This sign will be called the *sign of a cycle* ν.

If an eigenvalue $\lambda_\nu(s)$ crosses the threshold value λ from one side to the other, as s crosses r_λ in the positive direction, then the sign of the corresponding cycle ν is the contribution of the eigenvalue $\lambda_\nu(s)$ to the spectral flow through λ. There is a dichotomy in the way of assigning a sign to a cycle ν which corresponds to an eigenvalue $\lambda_\nu(s)$ making a U-turn at the threshold value λ. Theorem 5.7.1 provides one way of assigning the sign to such an eigenvalue, but since this sign does not contribute to the spectral flow, it is not essential which way to choose.

The following figure demonstrates the correspondence between the three different interpretations of the sign of a νth cycle in the case of $m = 4$, where the signs of the first, third, and fourth cycles are $+1$ and the sign of the second cycle is -1.



In the third interpretation the small bold interval is the interval I from Theorem 5.7.1(3). If the contribution of a cycle to the resonance index is $+1$ (respectively, -1), then I is a right interval (respectively, a left interval), but if the contribution is zero, it can be any.

5.8. Resonance index and intersection number. Let $\operatorname{sign}(\nu)$ be the sign of a cycle ν, and let

$$b_\nu = \begin{cases} 0 & \text{if } d_\nu \text{ is even,} \\ 1 & \text{if } d_\nu \text{ is odd.} \end{cases}$$

The intersection number through a resonance point r_λ is equal to

$$\sum_{\nu=1}^{m} b_\nu \operatorname{sign}(\nu). \tag{5.8.1}$$

Indeed, the value of b_ν determines whether or not the corresponding eigenvalue function $\lambda_\nu(s)$ makes a U-turn at $s = r_\lambda$, and if it does not, the value of $\operatorname{sign}(\nu)$ shows whether the eigenvalue $\lambda_\nu(s)$ crosses the threshold value λ in the positive or the negative direction. Whatever the sign of ν is, it does not contribute to the intersection number if $b_\nu = 0$. Further, Theorems 5.2.1, 5.2.3 and 5.7.1 imply that cycles with even d_ν do not contribute to the resonance index, while cycles with odd d_ν contribute $\operatorname{sign}(\nu)$. Hence, we have proved the following

THEOREM 5.8.1. *The sum of the intersection numbers* (5.8.1) *of the eigenvalues of a path* H_r, $r \in [0, 1]$, *through* λ *is equal to the total resonance index.*

Since each cycle ν contributes one of the three numbers ± 1 or 0 to the TRI, it follows that

$$|\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)| \le m. \tag{5.8.2}$$

This is the U-turn inequality which holds for a.e. λ inside essential spectrum too [Az6, Theorem 10.1.6].

5.9. A representation of $P_\lambda(r_\lambda)$. A matrix $A = (a_{ij})$ is said to be a *Hankel matrix* if its entries a_{ij} depend only on $i + j$, that is, if all skew-diagonals of A are constant. The theory of Hankel matrices is extremely rich and beautiful, but we shall use the words "Hankel matrix" in this subsection only as a terminology.

Let T be an operator of rank $N < \infty$. If $b_1, \dots, b_N$ is a basis of $\operatorname{im}(T)$ and if α is an invertible $N \times N$ matrix, then there exists a unique basis $a_1, \dots, a_N$ of $\operatorname{im}(T^*)$ such that

$$T = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} \langle a_i, \cdot \rangle b_j. \tag{5.9.1}$$

Also, if (a_i) and (b_i) are bases of $\operatorname{im}(T^*)$ and $\operatorname{im}(T)$ respectively, then there exists a unique invertible matrix α such that (5.9.1) holds. In the case of a finite-rank operator $P_\lambda(r_\lambda)$, there exists one natural Jordan basis

$$\frac{1}{j!}\varphi_\nu^{(j)}(r_\lambda), \qquad \nu = 1, \dots, m,\ j = 0, 1, \dots, d_\nu - 1,$$

of the vector space $\operatorname{im}(P_\lambda(r_\lambda)) = \Upsilon_\lambda(r_\lambda)$, provided by Theorems 5.2.3 and 5.3.1(2). Since by (1.3.11) and (1.3.10)(2), $\operatorname{im}(P_\lambda^*(r_\lambda)) = V \operatorname{im}(P_\lambda(r_\lambda))$, we also have a natural basis

$$\frac{1}{j!}V\varphi_\nu^{(j)}(r_\lambda), \qquad \nu = 1, \dots, m,\ j = 0, 1, \dots, d_\nu - 1,$$

of $\operatorname{im}(P_\lambda^*(r_\lambda)) = \Psi_\lambda(r_\lambda)$. Hence, there exists a unique invertible $N \times N$ matrix α such that

$$P_\lambda(r_\lambda) = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \sum_{k=0}^{d_\mu - 1} \sum_{j=0}^{d_\nu - 1} \frac{1}{k!j!} \alpha_{\mu\nu}^{kj} \langle V\varphi_\mu^{(k)}(r_\lambda), \cdot \rangle \varphi_\nu^{(j)}(r_\lambda). \tag{5.9.2}$$

Since $P_\lambda(r_\lambda)$ is an idempotent, we have $P_\lambda(r_\lambda)\varphi_\nu^{(j)}(r_\lambda) = \varphi_\nu^{(j)}(r_\lambda)$, that is,

$$\frac{1}{j'!}\varphi_{\nu'}^{(j')}(r_\lambda) = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \sum_{k=0}^{d_\mu - 1} \sum_{j=0}^{d_\nu - 1} \frac{1}{k!j!} \alpha_{\mu\nu}^{kj} \left\langle V\varphi_\mu^{(k)}(r_\lambda), \frac{1}{j'!}\varphi_{\nu'}^{(j')}(r_\lambda) \right\rangle \varphi_\nu^{(j)}(r_\lambda).$$

Introducing a matrix

$$b = (b_{\mu\nu}^{kj}) := \left(\frac{1}{k!j!} \langle V\varphi_\mu^{(k)}(r_\lambda), \varphi_\nu^{(j)}(r_\lambda) \rangle \right), \tag{5.9.3}$$

we can rewrite the last equality as

$$\frac{1}{j'!}\varphi_{\nu'}^{(j')}(r_\lambda) = \sum_{\nu,j} \Big(\sum_{\mu,k} \alpha_{\mu\nu}^{kj} b_{\mu\nu'}^{kj'} \Big) \frac{1}{j!}\varphi_\nu^{(j)}(r_\lambda).$$

This implies

$$\sum_{\mu,k} \alpha_{\mu\nu}^{kj} b_{\mu\nu'}^{kj'} = \delta_{\nu\nu'}\delta_{jj'}.$$

Therefore, the matrix α is the inverse of the transpose of b (but as we shall soon see, b is symmetric, so in fact $\alpha = b^{-1}$).

By Corollary 5.3.2, the matrix (5.9.3) has the following property:

$$b^{kj}_{\mu\nu} = \delta_{\mu\nu} b^{kj}_{\mu\nu}. \tag{5.9.4}$$

Hence, b is a direct sum of m matrices $b^{kj}_{\nu\nu}$ of size $d_\nu \times d_\nu$.

Further, by Theorems 3.1.7(v) and 5.2.3, while for $\mu \neq \nu$ the numbers (5.9.3) are zero, for $\mu = \nu$ they depend only on $k + j$:

$$b^{kj}_{\nu\nu} = b^{k-1,j+1}_{\nu\nu}, \tag{5.9.5}$$

and if $k + j \leq d_\nu - 2$, then they are zero. Indeed, from Theorem 3.1.7(v) and (1.3.7),

$$\begin{aligned} b^{kj}_{\nu\nu} &= \frac{1}{k!j!}\langle V\varphi^{(k)}_\nu(r_\lambda), \varphi^{(j)}_\nu(r_\lambda)\rangle = \frac{1}{k!(j+1)!}\langle V\varphi^{(k)}_\nu(r_\lambda), \mathbf{A}_\lambda(r_\lambda)\varphi^{(j+1)}_\nu(r_\lambda)\rangle \\ &= \frac{1}{k!(j+1)!}\langle \mathbf{B}_\lambda(r_\lambda)V\varphi^{(k)}_\nu(r_\lambda), \varphi^{(j+1)}_\nu(r_\lambda)\rangle = \frac{1}{k!(j+1)!}\langle V\mathbf{A}_\lambda(r_\lambda)\varphi^{(k)}_\nu(r_\lambda), \varphi^{(j+1)}_\nu(r_\lambda)\rangle \\ &= \frac{1}{(k-1)!(j+1)!}\langle V\varphi^{(k-1)}_\nu(r_\lambda), \varphi^{(j+1)}_\nu(r_\lambda)\rangle = b^{k-1,j+1}_{\nu\nu}, \end{aligned}$$

and if $k + j + 1 \leq d_\nu - 1$, then by the same argument

$$\begin{aligned} b^{kj}_{\nu\nu} &= b^{0,j+k}_{\nu\nu} = \frac{1}{(j+k)!}\langle V\varphi^{(0)}_\nu(r_\lambda), \varphi^{(j+k)}_\nu(r_\lambda)\rangle \\ &= \frac{1}{(j+k)!}\langle V\mathbf{A}_\lambda(r_\lambda)\varphi^{(0)}_\nu(r_\lambda), \varphi^{(j+k+1)}_\nu(r_\lambda)\rangle = 0, \end{aligned}$$

since $\mathbf{A}_\lambda(r_\lambda)\varphi^{(0)}_\nu(r_\lambda) = 0$. Hence, the matrix $(b^{kj}_{\mu\nu})$ is a direct sum of m Hankel matrices with zeros above the main skew-diagonal. It follows that the inverse α of the matrix $(b^{kj}_{\mu\nu})$ is a direct sum of Hankel matrices with zeros below the skew-diagonal.

Further, combining the definition (5.9.3) of b with the Hankel property (5.9.5) implies that the numbers $b^{kj}_{\mu\nu}$ are real. Hence, the matrix α is also real.

Thus, we have proved the following theorem.

THEOREM 5.9.1. *The idempotent operator $P_\lambda(r_\lambda)$ can be written in the form* (5.9.2)*, where the $N \times N$ matrix α is a direct sum of m real symmetric skew-upper-triangular Hankel matrices of sizes $d_1, \dots, d_m$.*

5.10. Signature of the resonance matrix $VP_\lambda(r_\lambda)$. For a finite-rank self-adjoint operator A we denote by $\operatorname{sign}(A)$ the signature $N_+ - N_-$ of A, where N_+ and N_- are, respectively, the number of positive and negative eigenvalues of A counted with multiplicities.

In this subsection we prove the equality

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V) = \operatorname{sign}(VP_\lambda(r_\lambda)). \tag{5.10.1}$$

This equality was proved in [Az6] in a more general setting of λ from the essential spectrum. Here we give a new proof, which easily follows from previous results. The proof is based on the following two well-known lemmas.

LEMMA 5.10.1. *The signature of a self-adjoint skew-upper-triangular $d \times d$ Hankel matrix is zero if d is even, and is equal to the sign of the skew-diagonal entry if d is odd.*

Proof. Let T be a self-adjoint skew-upper-triangular $d\times d$ Hankel matrix with entry α on the main skew-diagonal. If α is zero, then the assertion is obvious, so we assume $\alpha\neq 0$.

If all entries of T above the main skew-diagonal are zero, then T has eigenvalues α and $-\alpha$ of multiplicities $[(d+1)/2]$ and $[d/2]$ respectively, where $[x]$ is the integer part of a real number x. This implies that in this case $\operatorname{sign}(T)$ is equal to the sign of α if d is odd, and $\operatorname{sign}(T)=0$ if d is even.

In general, we continuously deform the matrix T to a skew-diagonal Hankel matrix with skew-diagonal entry α along the straight line. These matrices have constant non-zero determinant, and so are invertible. Hence, the signatures of all matrices along the deformation are constant, since eigenvalues do not cross zero. ∎

LEMMA 5.10.2. *If $b_1,\dots,b_d$ are linearly independent vectors in a Hilbert space and α is a $d\times d$ self-adjoint matrix, then the signature of the self-adjoint operator*

$$T=\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\langle b_i,\cdot\rangle b_j$$

is equal to the signature of α.

Proof. If $b_1,\dots,b_d$ is an orthonormal system of vectors, then the matrix of T in an orthonormal basis which extends $b_1,\dots,b_d$ is a block matrix of the form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 0\end{smallmatrix}\right)$, and therefore $\operatorname{sign}(T)=\operatorname{sign}(\alpha)$.

In general let $e_1,\dots,e_d$ be an orthonormal system of vectors such that

$$\operatorname{span}(e_1,\dots,e_d)=\operatorname{span}(b_1,\dots,b_d).$$

Let $C=(c_{ij})$ be a $d\times d$ invertible matrix such that $b_j=\sum_{l=1}^{d}\bar c_{lj}e_l$. Then

$$T=\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\langle b_i,\cdot\rangle b_j=\sum_{k=1}^{d}\sum_{l=1}^{d}\beta_{kl}\langle e_k,\cdot\rangle e_l,$$

where

$$\beta_{kl}=\sum_{i=1}^{d}\sum_{j=1}^{d}c_{ki}\alpha_{ij}\bar c_{lj},$$

so that $(\beta_{kl})=\beta=C\alpha C^*$. Since the system $e_1,\dots,e_d$ is orthonormal, we have $\operatorname{sign}(T)=\operatorname{sign}(\beta)$. Since also $\operatorname{sign}(C\alpha C^*)=\operatorname{sign}(\alpha)$, this gives $\operatorname{sign}(T)=\operatorname{sign}(\alpha)$. ∎

THEOREM 5.10.3. *The equality* (5.10.1) *holds.*

Proof. For each $\nu=1,\dots,m$, the signs of the main skew-diagonal entries of the $d_\nu\times d_\nu$ Hankel matrices (5.9.4) and of $(\alpha_{\mu\nu}^{kj})$ from (5.9.2) are equal. Hence, by Lemma 5.10.1 and Theorem 5.7.1 the resonance index on the left hand side of (5.10.1) is equal to the signature of the matrix $(\alpha_{\mu\nu}^{kj})$.

From (5.9.2) we have

$$VP_\lambda(r_\lambda)=\sum_{\mu,k}\sum_{\nu,j}\frac{1}{k!j!}\alpha_{\mu\nu}^{kj}\langle V\varphi_\mu^{(k)}(r_\lambda),\cdot\rangle V\varphi_\nu^{(j)}(r_\lambda).$$

By Lemma 5.10.2, the signature of this operator equals the signature of the matrix α. ∎

6. On stability of resonance index

In this section we study the behaviour of the resonance index $\mathrm{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$ as a function of the perturbation V.

6.1. Topology of the vector space of directions. In previous sections we worked with a fixed direction V, and for this reason there was no need in having a topology in the real vector space $\mathcal{A}_0$ of directions. Now we are going to consider stability of resonance index $\mathrm{ind}_{\mathrm{res}}(\lambda; H_{r_\lambda}, V)$ with respect to small perturbations of V, and therefore we need to discuss the topology of $\mathcal{A}_0$.

The approach taken is to impose conditions on the topology of the affine space $\mathcal{A}$ which allow us to prove stability results. These conditions hold trivially if $\mathcal{A}_0$ consists of bounded self-adjoint operators and if the topology of $\mathcal{A}_0$ is the norm topology or stronger.

Assumption 6.1.1. *The real vector space $\mathcal{A}_0$ is endowed with a norm $\|\cdot\|_{\mathcal{A}_0}$ such that for some non-real z and for some $H_0 \in \mathcal{A}$ the following three conditions hold:*

(VR) *the function $\mathcal{A}_0 \ni V \mapsto VR_z(H_0) \in \mathcal{B}(\mathcal{H})$ is continuous,*
(VRV1) *the product $V_1 R_z(H_0) V_2$ is compact,*
(VRV2) *the product $V_1 R_z(H_0) V_2$ is a continuous function of V_1 and V_2.*

Since Assumption 1.2.1(2) implies compactness of $V_1 \operatorname{Im} R_z(H_0) V_2$, condition (VRV1) is equivalent to compactness of $V_1 \operatorname{Re} R_z(H_0) V_2$.

The topology in $\mathcal{A}_0$ induces a topology in the affine space $\mathcal{A}$. We assume that $\mathcal{A}$ is endowed with this topology.

Lemma 6.1.2.

(i) *For any $V \in \mathcal{A}_0$ and any $H \in \mathcal{A}$,*

$$\lim_{y\to\infty} \|VR_{\lambda+iy}(H)\| = 0.$$

Further, if $\mathcal{A}_0$ is endowed with a norm which satisfies property (VR) *then:*

(ii) *The operator $VR_z(H)$ jointly continuously depends on $z \in \mathbb{C} \setminus \mathbb{R}$, $V \in \mathcal{A}_0$ and $H \in \mathcal{A}$.*
(iii) *The operator $R_z(H)$ jointly continuously depends on $z \in \mathbb{C} \setminus \mathbb{R}$ and $H \in \mathcal{A}$.*

Proof. (i) We have

$$VR_{\lambda+iy}(H) = VR_{\lambda+i}(H)\frac{H-\lambda-i}{H-\lambda-iy}.$$

Since the operator $VR_{\lambda+i}(H)$ is compact and $\frac{H-\lambda-i}{H-\lambda-iy}$ converges to zero in the $*$-strong topology as $y \to \infty$, [Ya, Lemma 6.1.3] completes the proof.

(ii) The first resolvent identity combined with (VR) implies that $VR_z(H_0)$ depends jointly continuously on z and V. Further, by the second resolvent identity we have

$$VR_z(H) = VR_z(H_0)\big(1 + (H - H_0)R_z(H_0)\big)^{-1}. \tag{6.1.1}$$

For non-real z the operator $1 + (H - H_0)R_z(H_0)$ is invertible. Hence, (6.1.1) shows that $VR_z(H)$ depends continuously on z, V and H. Item (iii) is proved by the same argument. ∎

Since $(s - r_z)^{-1}$ is an eigenvalue of $VR_z(H_s)$, by Lemma 6.1.2(ii), a resonance point $r_z = r_z(H_0, V)$ depends continuously on the pair (H_0, V).

LEMMA 6.1.3. *If $\mathcal{A}_0$ is endowed with a norm which satisfies Assumption* 6.1.1*, then the operator $V_1R_z(H)V_2$ is compact for any non-real z and any $H \in \mathcal{A}$, and it jointly continuously depends on $z \in \mathbb{C} \setminus \mathbb{R}$, $V_1, V_2 \in \mathcal{A}_0$ and $H \in \mathcal{A}$.*

Proof. By the first resolvent identity we have

$$V_1R_z(H_0)V_2 - V_1R_w(H_0)V_2 = (z - w)V_1R_z(H_0)\, R_w(H_0)V_2. \tag{6.1.2}$$

Since by Assumption 1.2.1 the right hand side is compact, the operator $V_1R_z(H_0)V_2$ is compact for any non-real value of z, provided it is compact for some value of z.

For $H = H_0 + V_3$ we have

$$\begin{aligned} V_1R_z(H)V_2 &= V_1R_z(H_0)(1 + V_3R_z(H_0))^{-1}V_2 = V_1R_z(H_0)\sum_{k\geq 0}(-1)^k(V_3R_z(H_0))^kV_2 \\ &= V_1R_z(H_0)V_2 + \sum_{k\geq 1}(-1)^k(V_3R_z(H_0))^{k-1}\cdot[V_3R_z(H_0)V_2]. \end{aligned} \tag{6.1.3}$$

For large $y = \operatorname{Im} z$, convergence of the series and compactness of this operator follow from Lemma 6.1.2(i) and (VRV1). The first summand and the product in the pair of square brackets are continuous by assumption (VRV2). Since for large enough y the geometric series converges uniformly, for such y the last series depends continuously on V_3 by assumption (VR). For other values of y the claim can now be inferred from (6.1.2). ∎

Usually we denote a resonance point of a triple $(\lambda; H_0, V)$, where H_0 is a λ-regular operator, by r_λ. But in this section for convenience we assume that $r_\lambda = 0$, so that the operator H_0 itself is λ-resonant.

LEMMA 6.1.4. *If $\mathcal{A}_0$ is endowed with a norm which satisfies Assumption* 6.1.1*, then the sets of all* (a) *regular and* (b) *simple directions at a resonance point H_0 are open in the norm of $\mathcal{A}_0$.*

Proof. (a) Let V be a regular direction. By continuity of the mapping $V \mapsto R_z(H)V$ (Lemma 6.1.2(ii)), there exists a neighbourhood O_V of V in $\mathcal{A}_0$ such that for all $W \in O_V$,

$$\|R_\lambda(H_0 + V)(V - W)\| < 1. \tag{6.1.4}$$

Hence, by the second resolvent identity, for all $W \in O_V$,

$$R_\lambda(H_0 + W) = \big(1 + R_\lambda(H_0 + V)(V - W)\big)^{-1}R_\lambda(H_0 + V),$$

where the inverse exists due to (6.1.4). It follows that all directions from O_V are regular.

(b) Here we use this characterisation of simple directions: a regular direction V at a resonance point H_0 is simple if and only if the algebraic multiplicity N of the eigenvalue s^{-1} of the compact operator $R_\lambda(H_0 + sV)V$ is equal to the geometric multiplicity m of that eigenvalue.

Since a simple direction is regular, by part (a) there exists a neighbourhood O_V of a simple direction V such that all directions W from O_V are regular. By Lemma 6.1.2(ii), the operator $R_\lambda(H_0 + W)W$ depends continuously on $W \in \mathcal{A}_0$ in some neighbourhood of V. Further, since V is simple, $R_\lambda(H_0 + sV)V$ has s^{-1} as an eigenvalue of algebraic and

geometric multiplicity m. Perturbation of V may not change the geometric multiplicity of the eigenvalue s^{-1}, since this number is the dimension of the eigenspace $\mathcal{V}_\lambda$ of H_0 and thus it depends only on H_0. Hence, a perturbation of V may not decrease the algebraic multiplicity of s^{-1}, but it may increase it.

The operator $R_\lambda(H_0+sW)W$ also has s^{-1} as an eigenvalue of geometric multiplicity m. Since $R_\lambda(H_0+sW)W$ is close to $R_\lambda(H_0+sV)V$ in the operator norm for all W close enough to V in the norm of $\mathcal{A}_0$, it follows that there exists a neighbourhood $\tilde{O}_V$ of V such that for all W from $\tilde{O}_V$ the algebraic multiplicity m of s^{-1} does not increase for $R_\lambda(H_0+sW)W$. Thus, all directions from $\tilde{O}_V$ are simple. ∎

6.2. Continuous dependence of P_λ and VP_λ on simple directions

LEMMA 6.2.1. *Let H_0 be a resonance point and let V be a regular direction of order* 1. *Then the idempotent $P_\lambda(H_0,V)$ depends on V continuously, that is, for any $\varepsilon>0$ there exists $\delta>0$ such that if W is a regular direction with $\|V-W\|_{\mathcal{A}_0}<\delta$ then*

$$\|P_\lambda(H_0,V)-P_\lambda(H_0,W)\|<\varepsilon. \tag{6.2.1}$$

Proof. Let m be the geometric multiplicity of the resonance point H_0. By definition of P_λ, we have

$$P_\lambda(H_0,V)=\frac{1}{2\pi i}\oint_C R_\lambda(H_0+sV)V\,ds,$$

where the contour C encloses only the resonance point $s=0$ of the path H_0+sV, and this resonance point has both geometric and algebraic multiplicity m, since V is a simple direction. By upper semicontinuity of the spectrum we can choose $\delta>0$ small enough so that, for all W with $\|V-W\|_{\mathcal{A}_0}<\delta$, inside the contour C there will only be one resonance point $s=0$ of the path H_0+sW, and it will have both geometric and algebraic multiplicity equal to m (indeed, the total algebraic multiplicity of all resonance points inside C is at least m, since H_0 has geometric multiplicity m and the total algebraic multiplicity is at most m due to upper semicontinuity of the spectrum). Now, compactness of the contour C and joint continuity of $R_\lambda(H_0+sV)W$ (Lemma 6.1.2(ii)) imply that there exists a possibly smaller $\delta>0$, if necessary, such that (6.2.1) holds as long as $\|V-W\|_{\mathcal{A}_0}<\delta$. ∎

For directions of order greater than 1 this proof does not work, since in this case the algebraic multiplicity of the resonance point $s=0$ is greater than m, and consequently, while for the perturbed path H_0+sW the point $s=0$ will still have geometric multiplicity m, other resonance points can appear inside the contour C which could have split from the resonance point $s=0$.

THEOREM 6.2.2. *Let H_0 be a resonance point and let V be a regular direction of order* 1. *Then the resonance matrix $VP_\lambda(H_0,V)$ depends on V continuously, that is, for any $\varepsilon>0$ there exists $\delta>0$ such that if W is a regular direction with $\|V-W\|_{\mathcal{A}_0}<\delta$, then*

$$\|VP_\lambda(H_0,V)-WP_\lambda(H_0,W)\|<\varepsilon.$$

Proof. The mapping $V\mapsto VR_\lambda(H_0+sV)V$ is continuous by Lemma 6.1.3. Since V is simple, it has a neighbourhood consisting of simple directions. Hence, inside a small

enough contour C enclosing the resonance point $s = 0$, for all W from the neighbourhood there will be no other resonance points of the triple $(\lambda; H_0, W)$. Hence, the formula

$$VP_\lambda(H_0, V) = \frac{1}{2\pi i} \oint_C VR_\lambda(H_0 + sV)V\, ds$$

completes the proof. ∎

6.3. Homotopy stability of total resonance index. We recall some definitions from previous sections. A point (that is, a self-adjoint operator) H_0 of the affine space $\mathcal{A}$ is *resonant* if a fixed real number λ which does not belong to the common essential spectrum σ_{ess} of operators from $\mathcal{A}$ is an eigenvalue of H_0. A resonance point H_0 is *simple* if the eigenvalue λ has multiplicity 1.

A regular direction V is *simple* at a resonance point H_0 if V is not tangent to $\mathcal{R}(\lambda)$ at H_0; by this we mean that V is not the tangent vector of any differentiable path in $\mathcal{R}(\lambda)$ which passes through H_0.

THEOREM 6.3.1. *Let H_0 be a resonance point and let V be a simple direction. The resonance index* $\operatorname{ind}_{\text{res}}(\lambda; H_0, V)$ *is stable under small perturbations of V within any finite-dimensional subspace of $\mathcal{A}_0$.*

Proof. Since the order of V is equal to 1, the rank N of the resonance matrix $VP_\lambda(H_0, V)$ is equal to m, and, by Theorem 4.3.3, the direction V is transversal. Hence, there exists a small enough convex neighbourhood of V in the finite-dimensional subspace such that all directions from that neighbourhood are also transversal, and therefore, by Theorem 4.3.3, have order 1. Hence, the ranks of the resonance matrices $WP_\lambda(H_0, W)$ for all directions W from the neighbourhood are equal to m. Since by Theorem 6.2.2 the resonance matrix depends continuously on V for simple directions V, it follows that the signature of the resonance matrix $VP_\lambda(H_0, V)$ is stable under small perturbations of V. ∎

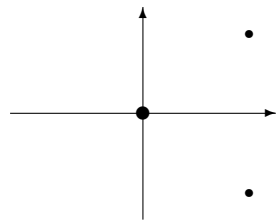
While the resonance index of a direction is stable if the direction is simple, in general this is not true. Geometrically, the reason is that a tangent direction may cross the resonance set however small a perturbation of that direction is. This leads to a sudden change of the intersection number of that direction. This connection of resonance index with intersection number of eigenvalues was discussed earlier. Analytically, the reason for the instability of the resonance index is that the resonance point may split into two or more resonance points as a non-simple direction V is perturbed. In other words, as a non-simple direction V is perturbed to a close direction W, near a resonance point $s = r_\lambda$ there may appear other resonance points, which may "take away" part of the resonance index. Thus, while the resonance index is not stable, the total resonance index is. In this subsection we prove the relevant theorems.

LEMMA 6.3.2 ([Az6, Corollary 3.1.5]). *If a non-real complex number r_λ^j is a resonance point of the triple $(\lambda; H_0, V)$, then so is the conjugate of r_λ^j, and moreover it has the same algebraic multiplicity.*

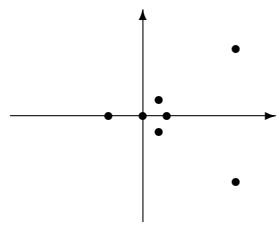
Let H_0 be a resonance point and let V be a regular direction. With every pair (H_0, V) we can associate the set of resonance points of the pair. If the direction V is slightly perturbed and if a resonance point is degenerate, then it can split. The resonance point

$s = 0$ itself will not move when V is changed, since H_0 is resonant, but some other resonance points may break away from $s = 0$ if V is not simple. The following figure shows one of the possible scenarios.

Resonance points of (H_0, V):



Resonance points of (H_0, W) where $W \approx V$:



THEOREM 6.3.3. *Let V be a regular direction at a resonance point H_0. Let W be a small perturbation of V, and let $r^1_\lambda(H_0, W), r^2_\lambda(H_0, W), \dots$ be resonance points of the triple $(\lambda; H_0, W)$ which belong to the group of the resonance point $s = 0$ of the triple $(\lambda; H_0, V)$, where $H_r = H_0 + rV$. Then*

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V) = \sum_j \operatorname{ind}_{\mathrm{res}}(\lambda; H_{r^j_\lambda}, W),$$

where the sum is taken over real resonance points of the group of $s = 0$.

Proof. The resonance index $\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V)$ is equal to the difference $N_+ - N_-$, where $N_\pm$ is the number of resonance points of the triple $(\lambda + iy; H_0, V)$ for small enough y which belong to the group of $s = 0$ and lie in $\mathbb{C}_\pm$. If V is deformed to W, then the resonance point $s = 0$ of (H_0, V) will in general split to some number of resonance points including the original resonance point $s = 0$; the algebraic multiplicity of this resonance point may decrease, but the geometric multiplicity will stay the same. We shall also refer to these resonance points of (H_0, W) as resonance points of the group of $s = 0$. Some of these resonance points can be real and some can be non-real.

If λ is perturbed slightly to $\lambda + iy$ with small positive y, then the non-real resonance points of (H_0, W) which belong to the group of $s = 0$ will stay in the same half-plane, and the real resonance points of the group of $s = 0$,

$$r^1_\lambda(H_0, W), r^2_\lambda(H_0, W), \dots,$$

will shift from the real axis, thus giving a sum of resonance indices for (H_0, W). We have to show that this sum is equal to $N_+ - N_-$.

Let M_+ (respectively, M_-) be the number of resonance points of (H_0, W) which belong to the group of $s = 0$ and which appear in the upper (respectively, lower) half-plane as λ is shifted to $\lambda + iy$, with small $y > 0$. The difference $M_+ - M_-$ is equal to the resonance

index of the triple $(\lambda; H_0, V)$, since we can deform V to W with $y > 0$ fixed, and as we do so, resonance points of V will get deformed to resonance points of W without crossing $\mathbb{R}$. Hence, equality of the total resonance index of (H_0, W) to the resonance index of (H_0, V) follows from Lemma 6.3.2 according to which the numbers of non-real resonance points of the group of $s = 0$ in both half-planes are the same. ∎

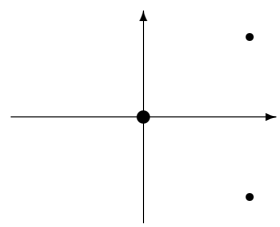
THEOREM 6.3.4. *Let V be a regular direction at a resonance point H_0. Let H_0' be a small perturbation of H_0, and let $r_\lambda^1(H_0', V), r_\lambda^2(H_0', V), \dots$ be resonance points of $(\lambda; H_0', V)$ which belong to the group of the resonance point $s = 0$ of $(\lambda; H_0', V)$, where $H_r' = H_0' + rV$. Then*

$$\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V) = \sum_j \operatorname{ind}_{\mathrm{res}}(\lambda; H'_{r_\lambda^j}, V),$$

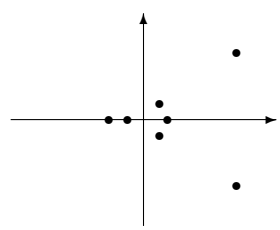
where the sum is over real resonance points of the group of $s = 0$.

The proof of this theorem follows almost verbatim the argument used for Theorem 6.3.3, and is therefore omitted. The only difference is that the resonance point $s = 0$ itself may not only split but also shift.

Resonance points of (H_0, V):



Resonance points of (H_0', V), where $H_0' \approx H_0$:



THEOREM 6.3.5. *Let H_0 and H_1 be operators from $\mathcal{A}$ which are not resonant at $\lambda \notin \sigma_{\mathrm{ess}}$. Then there exist neighbourhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of H_0 and H_1 respectively such that for all $H_0' \in \mathcal{U}_0$ and all $H_1' \in \mathcal{U}_1$,*

$$\sum_{r\in[0,1]} \operatorname{ind}_{\mathrm{res}}(\lambda; H_r, V) = \sum_{r\in[0,1]} \operatorname{ind}_{\mathrm{res}}(\lambda; H_r', V'),$$

where $V' = H_1' - H_0'$ and $H_r' = H_0' + rV'$.

Proof. This follows immediately from Theorems 6.3.3 and 6.3.4. ∎

6.4. Robbin–Salamon axioms for spectral flow and resonance index. In [RoSa] it was shown that for self-adjoint operators with compact resolvent the spectral flow of a continuous path of such operators can be uniquely characterised as a mapping which

satisfies five axioms: Homotopy, Constancy, Concatenation, Direct Sum, and Normalisation. In this subsection we show that the total resonance index satisfies these axioms, following closely [RoSa].

Assume that Assumptions 1.2.1 and 6.1.1 hold.

For any real numbers a and b, $a < b$, and any two operators H_a and H_b from the affine space $\mathcal{A}$ let $PC^1([a,b], H_a, H_b)$ be the set of all continuous piecewise C^1 paths $\{H_s\colon s \in [a,b]\}$ of operators from $\mathcal{A}$ such that (1) λ does not belong to the spectrum of H_a and H_b, (2) λ does not belong to the spectrum of the vertex points of the C^1 subpaths of H_s, and (3) all operators H_s, $s \in [a,b]$, belong to a finite-dimensional subspace of $\mathcal{A}$. The conditions (2) and (3) are not necessary, but they will allow us to avoid unnecessary technical complications.

By definition, the total resonance index of a path $\{H_s\colon s \in [a,b]\}$ from $PC^1[a,b]$ is the total resonance index of a continuous piecewise linear path K_s from $PC^1[a,b]$ which has the same end-points H_a and H_b and has $\sup_{s\in[a,b]} \|H_s - K_s\|_{\mathcal{A}_0}$ small enough. Using a standard compactness argument and the homotopy invariance of the total resonance index, one can show that this definition is correct.

Let μ be a mapping $PC^1([a,b], H_a, H_b) \to \mathbb{Z}$ which satisfies the following Robbin–Salamon axioms for spectral flow:

(i) (Homotopy) If paths f and g from $PC^1([a,b], H_a, H_b)$ are homotopic, then $\mu(f) = \mu(g)$.
(ii) (Constancy) If a path f from $PC^1([a,b], H_a, H_b)$ is constant, then $\mu(f) = 0$.
(iii) (Concatenation) If $f \in PC^1([a,b], H_a, H_b)$ and $g \in PC^1([b,c], H_b, H_c)$ and $f(b) = g(b)$, then $\mu(f \circledast g) = \mu(f) + \mu(g)$, where $\circledast$ denotes concatenation of paths.
(iv) (Direct Sum) If $f \in PC^1([a,b], H_a, H_b)$ and $g \in PC^1([a,b], \tilde{H}_a, \tilde{H}_b)$, where $\tilde{H}_a, \tilde{H}_b$ are operators from another affine space $\tilde{\mathcal{A}}$ satisfying the above conditions, then $\mu(f \oplus g) = \mu(f) + \mu(g)$.
(v) (Normalisation) Let $\mathcal{H} = \mathbb{C}$ be a one-dimensional Hilbert space, $a < \lambda < b$ and $f(t) = t$, $f \in PC^1([a,b], a, b)$. Then $\mu(f) = 1$.

THEOREM 6.4.1. *That total resonance index satisfies all five Robbin–Salamon axioms.*

Proof. All axioms except the Homotopy Axiom are trivially satisfied. The Homotopy Axiom is satisfied by Theorem 6.3.5.

Although the Normalisation Axiom is trivially satisfied, we shall check it explicitly. So, let $\mathcal{H} = \mathbb{C}$, $a < \lambda < b$, $H_s = s$. Then $V = 1$ and

$$A_{\lambda+iy}(s) = R_{\lambda+iy}(H_s)V = (s - \lambda - iy)^{-1}.$$

For $y > 0$ the only pole $s = \lambda + iy$ of this meromorphic function which belongs to the group of the pole $s = \lambda$ of $A_\lambda(s)$ is situated in the upper half-plane, and therefore $\operatorname{ind}_{\mathrm{res}}(\lambda; H_\lambda, V) = 1$. ∎

THEOREM 6.4.2. *A mapping μ which satisfies the Robbin–Salamon axioms exists and is unique.*

Proof. The original proof of this theorem from [RoSa] was given for operators with compact resolvent, but an inspection shows that with obvious minor changes it applies ver-

batim for the present case too. Nevertheless, for the readers' convenience here we outline this proof.

In [RoSa] the proof of the existence part is based on checking that the intersection number satisfies the Robbin–Salamon axioms. Here for the existence part we can also refer to Theorem 6.4.1.

For the uniqueness, in [RoSa] it is shown that any putative spectral flow mapping coincides with the intersection number. Again, here we show that any putative spectral flow mapping coincides with total resonance index.

(A) For every path $H(s)$ from $PC^1([a,b], H_a, H_b)$ there exist an integer n and a path $B(s)$ of self-adjoint $n \times n$ matrices such that $H \oplus B$ is homotopic to a constant path. Indeed, firstly, the Homotopy Axiom allows us to replace $H(s)$ by a path with only simple crossings (that is, resonance points with algebraic multiplicity $N = 1$). Further, the Concatenation Axiom allows us to reduce the problem to the case where $H(s)$ has only one simple crossing.

Without loss of generality we assume that $a = -1$, $b = 1$, $\lambda = 0$, and the crossing point is $s = 0$. Let χ be the eigenvector of H_0, that is, $H_0\chi = 0$. Let $B(s) = -s$, $B \in PC^1([-1,1], \mathbb{C})$, and let $\varphi(s)$, $s \in [-1,1]$, be an eigenpath of H_s, that is, $H_s\varphi(s) = 0$. Let

$$\tilde{H}_{s,t} = \begin{pmatrix} H_s & t\varphi(s) \\ t\varphi^*(s) & -s \end{pmatrix}.$$

Then $\tilde{H}_{s,0} = H_s \oplus B(s)$ and for $t > 0$ the operator $\tilde{H}_{s,t}$ is invertible for all $s \in [-1,1]$. Indeed, assume the contrary. Then, since $\lambda = 0$ does not belong to the essential spectrum, there exists a non-zero vector $\tilde{f} = f \oplus x \in \mathcal{H} \oplus \mathbb{C}$ such that $\tilde{H}_{s,t}\tilde{f} = 0$. It follows that $t\langle\varphi(s), f\rangle = sx$ and $H_s f = -tx\varphi(s)$. The latter equality implies that

$$-tx\langle\varphi(s), \varphi(s)\rangle = \langle H_s\varphi(s), f\rangle = 0,$$

and hence, $x = 0$. Combining this with the former equality gives $\varphi(s) \perp f$. Also, $H_s f = 0$, and therefore, since λ is a simple eigenvalue, f is co-linear to $\varphi(s)$. Hence, $f = 0$.

So, the path $H_s \oplus B(s)$ is homotopic to a path K_s without resonance points. Such a path can be continuously deformed to a constant path.

(B) Let $\tilde{\mu}$ be a putative spectral flow mapping. A piecewise linear path of self-adjoint matrices $B(s)$, $s \in [a,b]$, is homotopic to a path of diagonal matrices. Hence, by the Homotopy, Direct Sum and Normalisation Axioms both $\tilde{\mu}$ and the total resonance index of the path $B(s)$ are equal to

$$\tfrac{1}{2}\operatorname{sign} B(b) - \tfrac{1}{2}\operatorname{sign} B(a).$$

Now let $H(s)$ be any curve from $PC^1([a,b], H_a, H_b)$ and choose $B(s)$ as in part (A). Then it follows from the Homotopy and Constancy Axioms that $\tilde{\mu}(H \oplus B) = 0$. Hence, by the Direct Sum Axiom,

$$\tilde{\mu}(H) = -\tilde{\mu}(B) = -\mathrm{TRI}(B) = \mathrm{TRI}(H). \quad \blacksquare$$

The proof in [RoSa] does not use the Concatenation Axiom, which therefore follows from the other four axioms. In the proof above we used the Concatenation Axiom for simplicity, though it is not necessary.

Since spectral flow is deemed to be characterised by the Robbin–Salamon axioms, Theorem 6.4.1 shows that total resonance index and spectral flow are identical notions. Nevertheless, in Subsection 7.2 we give a direct proof of the equality

$$\text{total resonance index} = \text{total Fredholm index}.$$

6.5. Geometric properties of the resonance set. In this subsection we give proofs of some well-known geometric properties of the resonance set, with the aim to provide an intuitive interpretation of the spectral flow in terms of the resonance set.

THEOREM 6.5.1. *Under Assumption* 1.2.1*, the resonance set* $\mathcal{R}(\lambda)$ *has co-dimension* 1.

Proof. If the co-dimension of $\mathcal{R}(\lambda)$ is ≥ 2, then there exists a two-dimensional affine plane in $\mathcal{A}$ which intersects the resonance set transversally at a λ-resonant point H_0. A point $H_0 + V$ on a small circle neighbourhood of H_0 in this plane can be deformed to $H_0 - V$ along the circle. By Theorem 6.3.1, all deformations are simple and have constant resonance index. This contradicts the equality $\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, -V) = -\operatorname{ind}_{\mathrm{res}}(\lambda; H_0, V)$. ∎

At the same time, for λ inside the essential spectrum the spectral flow is not path-independent (see [Az4, §8.3]). This indicates that the resonance set may have co-dimension greater than 1 for λ inside the essential spectrum.

Theorem 6.5.1 implies that the intersection of the resonance set $\mathcal{R}(\lambda)$ with any k-dimensional affine space which passes through a resonance point H_0 and which is parallel to a simple direction V has dimension $k - 1$.

According to Theorem 6.5.1, the resonance set $\mathcal{R}(\lambda)$ divides a small enough neighbourhood of any simple resonance point H_0 into two parts. The operators in one of those parts have an eigenvalue slightly larger than λ, the operators in the other part have an eigenvalue slightly smaller than λ. We shall call these parts *positive* and *negative*. Resonance hyper-surfaces divide a small enough neighbourhood of a resonance operator H_0 into several parts, which will be called *cells*. If V is a regular direction at a resonance point H_0, then it belongs to one and only one of those cells, by which we mean $\exists \varepsilon > 0\ \forall s \in (0, \varepsilon)$ the operator $H_0 + sV$ belongs to the cell.

THEOREM 6.5.2. *If a plane section of the resonance set consists of only simple curves, then the number of curves in a neighbourhood of* H_0 *is not greater than the geometric multiplicity* m.

Proof. Each curve divides the plane into two parts: positive and negative. If there are $M > m$ curves, then some points of the plane section will be positive for all M curves and negative for none, and some points will be negative for all M curves and positive for none. The total resonance index of a continuous piecewise linear path from one of the latter points to one of the former points will be greater than m. Those two points can also be connected by a continuous piecewise linear path which has only one resonance point H_0. By the homotopy stability of the total resonance index (Theorem 6.3.5), this path will have total resonance index greater than m too. This contradicts the U-turn inequality (5.8.2). ∎

We say that a plane section of the resonance set is *simple* if the section does not have non-simple resonance curves.

COROLLARY 6.5.3. *In any simple plane section of the resonance set there are no more than $2m$ resonance cells in a neighbourhood of a resonance point H_0, where m is the geometric multiplicity of H_0.*

THEOREM 6.5.4. *The resonance set does not have cusps.*

Proof. If a cusp exists, then at its vertex there exists a simple direction V which can be continuously deformed to the direction $-V$ in the set of simple directions. This implies that the resonance indices of V and $-V$ are equal, which is false. ∎

7. Resonance index and Fredholm index

In this section we consider the relationship of the total resonance index with a traditional definition of spectral flow, the total Fredholm index.

7.1. Resonance matrix as direction reduction. In this subsection for convenience we denote a resonance point by H_0, instead of the usual H_{r_λ}.

Given a λ-resonant operator H_0 and a regular direction V, to a triple $(\lambda; H_0, V)$ we can assign a finite-rank self-adjoint operator VP_λ (see (1.3.13)). Results of this subsection demonstrate that it is the operator VP_λ which is responsible for spectral flow generated by the direction V.

THEOREM 7.1.1. *If V is a regular direction at a resonance point H_0, then the direction VP_λ is also regular.*

Proof. Assume the contrary. Then for any $s \in \mathbb{R}$ there exists a non-zero vector $f(s)$ such that

$$(H_0 + sVP_\lambda)f(s) = \lambda f(s). \tag{7.1.1}$$

This equality can be rewritten as $(H_0 - \lambda)f(s) = -sVP_\lambda f(s)$, which implies that $VP_\lambda f(s)$ is orthogonal to the eigenspace $\mathcal{V}_\lambda$. It also implies that

$$f(s) = -sS_\lambda P_\lambda f(s) + \text{order 1 vector}.$$

Since $VP_\lambda f(s) \perp \mathcal{V}_\lambda$ and since $P_\lambda f(s)$ is a resonance vector, it follows from Theorem 2.3.1 and a remark after Corollary 2.3.3 that $S_\lambda P_\lambda f(s)$ is a resonance vector. Hence, by the last equality, so is $f(s)$. That is, $P_\lambda f(s) = f(s)$. Combined with (7.1.1), this gives $(H_0 + sV)f(s) = \lambda f(s)$, contradicting the regularity of V. ∎

THEOREM 7.1.2. *For any regular direction V at a resonance point H_0 and for any non-resonance $s \in \mathbb{C}$,*

$$R_\lambda(H_0 + sVP_\lambda)VP_\lambda = R_\lambda(H_0 + sV)VP_\lambda, \tag{7.1.2}$$

where $P_\lambda = P_\lambda(H_0, V)$.

Proof. Applying the second resolvent identity (1.3.2) to the pair of self-adjoint operators $H_0 + sV$ and $H_0 + sVP_\lambda = H_0 + sV - sV(1 - P_\lambda)$ gives

$$(E) := R_\lambda(H_0 + sVP_\lambda)VP_\lambda = [1 - sR_\lambda(H_0 + sV)V(1 - P_\lambda)]^{-1} R_\lambda(H_0 + sV)VP_\lambda.$$

Using the notation $A_\lambda(s) = R_\lambda(H_s)V$, we can rewrite this equality as

$$(E) = [1 - sA_\lambda(s)(1 - P_\lambda)]^{-1}A_\lambda(s)P_\lambda.$$

It follows from (1.3.8) that $A_\lambda(s)(1 - P_\lambda) = \tilde{A}_\lambda(s)$, where $\tilde{A}_\lambda(s)$ is the holomorphic (at $s = 0$) part of the meromorphic function $A_\lambda(s)$. Hence, for all small enough s,

$$(E) = [1 - s\tilde{A}_\lambda(s)]^{-1}A_\lambda(s)P_\lambda = [1 + s\tilde{A}_\lambda(s) + s^2\tilde{A}^2_\lambda(s) + \cdots]A_\lambda(s)P_\lambda.$$

Since $A_\lambda(s)$ and P_λ commute and $\tilde{A}_\lambda(s)P_\lambda = 0$, it follows that $(E) = A_\lambda(s)P_\lambda$. Since both sides of this equality are holomorphic, it holds for all, not necessarily small, s, as required. ∎

THEOREM 7.1.3. *For any regular direction V at a resonance point H_0,*

$$P_\lambda(H_0, VP_\lambda(H_0, V)) = P_\lambda(H_0, V) \quad \textit{and} \quad \mathbf{A}_\lambda(H_0, VP_\lambda(H_0, V)) = \mathbf{A}_\lambda(H_0, V).$$

Proof. Using the definition (1.3.5) of the idempotent P_λ and Theorem 7.1.2, we have

$$\begin{aligned} P_\lambda(H_0, VP_\lambda(H_0, V)) &= \frac{1}{2\pi i}\oint_C R_\lambda(H_0 + sVP_\lambda)VP_\lambda\, ds \\ &= \frac{1}{2\pi i}\oint_C R_\lambda(H_0 + sV)VP_\lambda\, ds \;=\; P_\lambda^2 = P_\lambda, \end{aligned}$$

where C is a contour enclosing the resonance point $s = 0$. The proof of the second equality is the same, but uses (1.3.6) instead of (1.3.5). ∎

THEOREM 7.1.4. *Let V be a regular direction at a resonance point H_0. The resonance matrices of the directions V and VP_λ are equal.*

Proof. The resonance matrix of V is VP_λ and the resonance matrix of VP_λ is $VP_\lambda \cdot P_\lambda(H_0, VP_\lambda)$. By Theorem 7.1.3, these matrices are equal. ∎

Since the resonance index of a direction is equal to the signature of its resonance matrix (Theorem 5.10.3), we have the following corollary of Theorem 7.1.4.

THEOREM 7.1.5. *Let V be a regular direction. The resonance indices of the directions V and VP_λ are equal.*

This theorem is important in that it often allows one to replace a direction V by a finite-rank direction VP_λ.

DEFINITION 7.1.6. We say that two directions V_1 and V_2 at a resonance point H_0 are *plane homotopic* if there exists $\varepsilon > 0$ such that for all $(s_1, s_2) \in [0, \varepsilon]^2 \setminus \{(0, 0)\}$ the operators $H_0 + s_1V_1 + s_2V_2$ and $H_0 - s_1V_1 - s_2V_2$ are regular at λ.

Geometrically, two directions V_1 and V_2 are plane homotopic if one of them can be deformed to the other within the affine plane they generate so that the half-interval being deformed stays outside the resonance set.

THEOREM 7.1.7. *If V is a regular direction at a resonance point H_0, then the directions V and VP_λ are plane homotopic.*

Proof. Firstly, by Theorem 7.1.1, the direction VP_λ is regular. Let

$$H_{s,t} = H_0 + sV + tVP_\lambda.$$

Since V is regular, for all small enough s the operator $H_0 + sV - \lambda$ is invertible. We need to show that for all small enough s and t the operator $H_0 + sV + tVP_\lambda - \lambda$ is also invertible. By the second resolvent identity, we have

$$R_\lambda(H_{s,t}) = (1 + tR_\lambda(H_s)VP_\lambda)^{-1}R_\lambda(H_s) = (1 + tA_\lambda(s)P_\lambda)^{-1}R_\lambda(H_s), \tag{7.1.3}$$

whenever the right hand side makes sense. Hence to prove the claim it is enough to prove that if $1+tA_\lambda(s)$ is invertible (which is equivalent to the existence of $R_\lambda(H_0+(s+t)V)$), then so is $1 + tA_\lambda(s)P_\lambda$. Assume the contrary. Then, since $A_\lambda(s)P_\lambda$ is compact, there exists a non-zero vector χ such that

$$(1 + tA_\lambda(s)P_\lambda)\chi = 0.$$

Since $A_\lambda(s)$ and P_λ commute, this implies $\chi = -tP_\lambda A_\lambda(s)\chi$. It follows that $P_\lambda\chi = \chi$, and therefore $(1 + tA_\lambda(s))\chi = 0$. Hence, $1 + tA_\lambda(s)$ is not invertible. ∎

Theorem 7.1.7 provides another proof of the equality of the resonance index and of the signature of the resonance matrix for the case where λ does not belong to the essential spectrum.

Proposition 7.1.8. *Any two regular non-negative* (*or non-positive*) *directions are plane homotopic.*

Proof. Let V_1 and V_2 be two regular and non-negative directions at a resonance point H_0. Since V_1 is regular, for all small enough $s > 0$ the operator $H_0 + sV_1$ is non-resonance, and near λ there are only eigenvalues of $H_0 + sV_1$ which are larger than λ. Adding tV_2 can only increase these eigenvalues. Choosing s and t small enough we can also ensure that there are no other eigenvalues $H_0 + sV_1 + tV_2$ near λ. ∎

Theorem 7.1.9. *Plane homotopic directions have equal resonance indices.*

This is a special case of homotopy stability of resonance index, Theorem 6.3.5.

7.2. Resonance index and Fredholm index. In this section we consider the relationship between the resonance index and the Fredholm index. For the reader's convenience we recall here some well-known definitions and theorems.

A bounded operator T acting from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is *Fredholm* if T has closed range and if the kernels of both T and T^* are finite-dimensional. In this case the *index* of T is the integer

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\ker(T^*).$$

"Bounded" in the definition of a Fredholm operator can be replaced by "closed", and "Hilbert" by "Banach", but we do not need this. Since the index is sensitive to the choice of domain and range, one may also write $\operatorname{ind}_{\mathcal{H},\mathcal{K}}(T)$.

We denote by $\mathcal{F}(\mathcal{H},\mathcal{K})$ the set of all (bounded) Fredholm operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, then one writes $\mathcal{F}(\mathcal{H})$ for $\mathcal{F}(\mathcal{H},\mathcal{H})$. The set $\mathcal{F}(\mathcal{H},\mathcal{K})$ has the following properties, proofs of which can be found in e.g. [Hör3, Chapter XIX].

(1) For any compact operator K on a Hilbert space $\mathcal{H}$ the operator $1+K$ is Fredholm and $\operatorname{ind}(1+K)=0$.
(2) The set of Fredholm operators is open in the norm topology. The index $\operatorname{ind}(T)$ is stable in the norm topology, that is, ind is a locally constant function on $\mathcal{F}(\mathcal{H},\mathcal{K})$.
(3) If $T\in\mathcal{F}(\mathcal{H},\mathcal{K})$ and $K\colon\mathcal{H}\to\mathcal{K}$ is compact, then $T+K\in\mathcal{F}(\mathcal{H},\mathcal{K})$ and $\operatorname{ind}(T)=\operatorname{ind}(T+K)$. Item (1) is a special case of this property.
(4) If $T\in\mathcal{F}(\mathcal{H}_1,\mathcal{H}_2)$ and $S\in\mathcal{F}(\mathcal{H}_2,\mathcal{H}_3)$, then $ST\in\mathcal{F}(\mathcal{H}_1,\mathcal{H}_3)$ and $\operatorname{ind}(ST)=\operatorname{ind}(S)+\operatorname{ind}(T)$.
(5) $T\in\mathcal{F}(\mathcal{H},\mathcal{K})$ if and only if there exists a bounded operator $S\colon\mathcal{K}\to\mathcal{H}$ such that $ST-1_{\mathcal{H}}$ and $TS-1_{\mathcal{K}}$ are compact. Such an operator S is called a *parametrix* of T and it is also Fredholm. Moreover, $\operatorname{ind}(S)=-\operatorname{ind}(T)$.

A pair of orthogonal projections (P,Q) is called a *Fredholm pair* if the operator $PQ\colon Q\mathcal{H}\to P\mathcal{H}$ is Fredholm. The *essential co-dimension* of (P,Q) is the index of PQ; it is denoted by $\operatorname{ec}(P,Q)$.

A pair (P,Q) is Fredholm iff $\|\pi(P-Q)\|<1$, where $\pi\colon\mathcal{B}(\mathcal{H})\to\mathcal{Q}(\mathcal{H})$ is the canonical epimorphism of the C^*-algebra of bounded operators $\mathcal{B}(\mathcal{H})$ onto the Calkin C^*-algebra $\mathcal{Q}(\mathcal{H})=\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is the norm-closed ideal of compact operators; for a proof see e.g. [BCPRSW, Lemma 4.1]. If (P_1,P_2) and (P_2,P_3) are Fredholm pairs, then so are the pairs (P_1,P_3) and (P_2,P_1) and

$$\operatorname{ec}(P_1,P_3)=\operatorname{ec}(P_1,P_2)+\operatorname{ec}(P_2,P_3),\qquad \operatorname{ec}(P_2,P_1)=-\operatorname{ec}(P_1,P_2). \tag{7.2.1}$$

Now we proceed to a discussion of J. Phillips' definition [Ph, Ph2] of spectral flow as total Fredholm index. The theory of spectral flow was developed for self-adjoint operators H_0 with compact resolvent and with some summability condition such as p-summability and θ-summability (though p-summability implies θ-summability which in its turn implies compactness of resolvent, we choose to mention both). This assumes that the spectrum of H_0 is discrete, and so H_0 has no essential spectrum. The spectral flow theory originates in the analysis of elliptic differential operators $\mathcal{D}$ acting on sections of vector bundles over compact manifolds, such as Dirac operators on spin manifolds, and these operators have compact resolvent and satisfy some summability assumptions. Our aim here is to demonstrate directly that the spectral flow as total Fredholm index and the total resonance index are identical notions. But since the theory of the former was developed for operators with compact resolvent, we assume here this condition. An inspection of proofs of basic theorems of spectral flow theory shows that the summability conditions are used essentially. It is quite possible that double operator integral techniques may allow one to adjust the theory so that it becomes applicable to operators with essential spectrum as long as zero (or more generally a point λ) does not belong to it, but carrying out this plan may or may not be straightforward. In any case, as was demonstrated in [ACS], it is sufficient to assume only compactness of resolvent without summability conditions.

For the rest of this subsection, we assume that H_0 has compact resolvent and that $\mathcal{A}_0$ is a subspace of the algebra of bounded operators with the operator norm. If H_s is a continuous path in $\mathcal{A}$ such that H_0 and H_1 are not λ-resonant, then the *spectral flow* of

$\{H_s\}_{s\in[0,1]}$ through λ, by the definition of J. Phillips, is the number

$$\mathrm{sf}(\{H_s\}) = \sum_{j=1}^{n} \mathrm{ec}(P_{s_{j-1}}, P_{s_j}),$$

where P_s is the spectral projection $E^{H_s}_{[\lambda,\infty)}$, and $\{s_j\}_{j=0}^n$ is a partition of the interval $[0,1]$. The spectral flow is well-defined for all partitions with small enough diameter $\max_j |s_j - s_{j-1}|$, and it does not depend on the choice of such a partition, which easily follows from the additivity property (7.2.1) of the essential co-dimension. Moreover, spectral flow is homotopy invariant. Since this is one of several definitions of spectral flow, we shall call it here *total Fredholm index*.

This preliminary material can be found in e.g. [Ph, Ph2, BCPRSW], [Az, §§1.5, 1.6] and [Hör3, Chapter XIX].

We give a direct proof of the equality of the total resonance index and total Fredholm index, which does not allude to the Robbin–Salamon uniqueness Theorem 6.4.2.

By the *Fredholm index of a regular direction* V at H_0 we mean the number

$$\mathrm{ec}(E^{H_0-\varepsilon V}_{[\lambda,\infty)}, E^{H_0+\varepsilon V}_{[\lambda,\infty)}),$$

where $\varepsilon > 0$ is a small enough number. General theory [Ph, Ph2] shows that this essential co-dimension is independent of the choice of sufficiently small $\varepsilon > 0$.

THEOREM 7.2.1. *Total resonance index coincides with total Fredholm index.*

Proof. Let H_r be a continuous piecewise linear path which connects two λ-regular operators H_0 and H_1. Since both the total resonance index (Theorem 6.3.5) and the total Fredholm index are homotopy stable, using a small perturbation of this path, we can modify it in such a way that all crossings of this path with the resonance set will occur at simple points and at simple directions. The Concatenation Axiom (which trivially holds for both the total resonance index and the total Fredholm index) reduces the matter to the case of a path H_r which intersects the resonance set only once at a simple point and at a simple direction. Let V be the direction. By Theorem 7.1.7, V is plane homotopic to VP_λ. By the Homotopy Axiom (or by Theorem 7.1.9), the directions V and VP_λ have the same resonance index and Fredholm index. Since V is a simple direction at a simple point, the resonance matrix VP_λ is a rank 1 self-adjoint operator. Depending on whether the resonance index of V is $+1$ or -1, the operator VP_λ is plane deformable to $\langle\chi,\cdot\rangle\chi$ or $-\langle\chi,\cdot\rangle\chi$, where χ is an eigenvector of the simple point where H_r crosses the resonance set. Finally, it remains to note that the Fredholm index of the regular direction $\langle\chi,\cdot\rangle\chi$ (respectively, $-\langle\chi,\cdot\rangle\chi$) is obviously 1 (respectively, -1). ∎

8. Resonance index and spectral shift function

The aim of this section is to demonstrate the equality

$$\text{spectral shift function} = \text{total resonance index}$$

outside the essential spectrum. Since this is a special case of an essentially stronger result which asserts that the total resonance index is equal to the singular spectral shift function

for a.e. λ [Az6, §6], [Az5], we do not formulate any theorems here. It is also well known that the spectral shift function (SSF) outside essential spectrum satisfies the Robbin–Salamon axioms, and therefore the total resonance index and spectral shift function coincide by the uniqueness Theorem 6.4.2. Nevertheless, here we demonstrate the argument of the proof of the above-mentioned more general result in this special case, where it simplifies quite significantly while retaining one of the key points of the proof.

Given two self-adjoint operators H_0 and H_1 with trace class difference $V = H_1 - H_0$, the *spectral shift function* of the pair (H_0, H_1) is the unique real-valued integrable function $\xi \in L_1(\mathbb{R})$ such that for all compactly supported functions φ of class C^2 the *Lifshitz–Krein trace formula* [Kr, L] holds:

$$\operatorname{Tr}(\varphi(H_1) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(\lambda)\xi(\lambda)\, d\lambda.$$

The Birman–Solomyak formula [BS] gives another remarkable representation for the spectral shift function, but this formula treats SSF as a distribution:

$$\xi(\varphi) = \int_0^1 \operatorname{Tr}(V\varphi(H_r))\, dr, \tag{8.0.1}$$

where $H_r = H_0 + rV$. This formula indicates that SSF is an integral of a one-form $V \mapsto \operatorname{Tr}(V\varphi(H))$ on the affine space of trace class perturbations of H_0. This form is *exact*, so that the straight line $H_0 + rV$ connecting the operators H_0 and H_1 can be replaced by a piecewise smooth path [AzS]. In spectral flow theory there exist analytic integral formulas for spectral flow (due to Getzler [Ge] and Carey–Phillips [CP, CP2]) which can be considered as analogues of the Birman–Solomyak formula with specifically chosen distribution φ, though the settings of the operator-theoretic spectral shift function and the differential geometric spectral flow were different (the former with relative trace class conditions imposed on the perturbation V with more or less arbitrary self-adjoint H_0, while in the latter the perturbation V is an arbitrary bounded self-adjoint operator with summability conditions imposed on H_0). Alan Carey indicated many times that the idea of expressing the spectral flow as an integral of a one-form belongs to I. M. Singer (1974).

Starting with the Birman–Solomyak formula as the definition of SSF, one can show that it satisfies Krein's trace formula in the form

$$\operatorname{Tr}(\varphi(H_1) - \varphi(H_0)) = \int_{-\infty}^{\infty} \varphi'(\lambda)\, dm_\xi(\lambda),$$

where m_ξ is the spectral shift measure. Though the proof of the Birman–Solomyak formula is somewhat simpler than that of the Lifshitz–Krein formula, it does not allow one to prove that SSF is absolutely continuous, so the original proof of the Lifshitz–Krein formula is indispensable. Nevertheless, there are sufficient indications, some of which were mentioned above, that the Birman–Solomyak formula is more fundamental.

Outside the essential spectrum the spectral shift function coincides with the spectral flow. This connection was demonstrated in [ACS], though it seems unlikely not to have been known in some form before. Since the spectral flow is an inherently integer-valued function, SSF is also integer-valued outside the essential spectrum, but inside the essential spectrum this is not the case. The celebrated Birman–Krein formula, which connects SSF

and the scattering matrix:

$$\det S(\lambda; H_1, H_0) = e^{-2\pi i \xi(\lambda)},$$

indicates that the reason for non-integrality of SSF is the existence of a non-trivial scattering matrix. It turns out however [Az2, Az4] that SSF admits a natural decomposition as a sum of two components, the *absolutely continuous* $\xi^{(a)}$ and *singular* $\xi^{(s)}$, *spectral shift functions*, such that the second component $\xi^{(s)}$ is integer-valued almost everywhere inside the essential spectrum too. The definitions of $\xi^{(a)}$ and $\xi^{(s)}$ are obtained by modified Birman–Solomyak formulas:

$$\xi^{(a)}(\varphi) = \int_0^1 \operatorname{Tr}(V\varphi(H_r^{(a)}))\, dr \quad \text{and} \quad \xi^{(s)}(\varphi) = \int_0^1 \operatorname{Tr}(V\varphi(H_r^{(s)}))\, dr,$$

where $H^{(a)}$ and $H^{(s)}$ stand for the absolutely continuous and singular parts of a self-adjoint operator H respectively. It is shown in [Az4] that for trace class perturbations the measure $\xi^{(a)}$ is absolutely continuous and its density $\xi^{(a)}(\lambda)$ satisfies a modified Birman–Krein formula

$$\det S(\lambda; H_1, H_0) = e^{-2\pi i \xi^{(a)}(\lambda)}.$$

This formula combined with the original Birman–Krein formula implies the integrality of $\xi^{(s)}$.

The definitions of $\xi^{(a)}$ and $\xi^{(s)}$ indicate that the generalised spectral flow one-form $\operatorname{Tr}(V\delta(H))$ naturally splits into two components: absolutely continuous $\operatorname{Tr}(V\delta(H^{(a)}))$ and singular $\operatorname{Tr}(V\delta(H^{(s)}))$. But unlike the spectral flow one-form, these two components are not exact: see a counter-example in [Az4, §8.3]. Outside the essential spectrum the absolutely continuous spectral flow one-form drops out, and thus in this case the remaining singular part becomes exact. In terms of the resonance set $\mathcal{R}(\lambda)$ the exactness of the singular spectral flow one-form is equivalent to the equality $\operatorname{codim} \mathcal{R}(\lambda) = 1$, and while outside the essential spectrum the resonance set $\mathcal{R}(\lambda)$ always has co-dimension one, inside the essential spectrum this is not the case.

The definition of $\xi^{(s)}$ is hardly suitable for calculation of this function. In [Az5] (see also [Az6, Section 6]) it was shown that the singular spectral shift function is equal to the total resonance index, which is something far easier to work with. The proof of this equality for values of λ inside the essential spectrum cannot be explained in a short space, but if λ is outside the essential spectrum then the proof simplifies sufficiently to be presented here.

It is probably worth stressing that the notion of resonance index was discovered in the course of work on the singular spectral shift function and scattering theory. Since the singular spectral shift function coincides with the spectral shift function outside the essential spectrum, it was immediately clear that the total resonance index coincides with the spectral flow, and this is how this paper originated.

We assume that a self-adjoint perturbation V of a self-adjoint operator H_0 is trace class. This assumption is not necessary for the equality in question:

$$\text{if} \quad \lambda \notin \sigma_{\text{ess}} \quad \text{then} \quad \xi(\lambda) = \sum_{r \in [0,1]} \operatorname{ind}_{\text{res}}(\lambda; H_r, V),$$

but it will simplify the proof. For relatively trace class perturbations the proof, which includes the main essential spectrum case too, will appear in [AzD].

The Birman–Solomyak formula (8.0.1) implies the equality

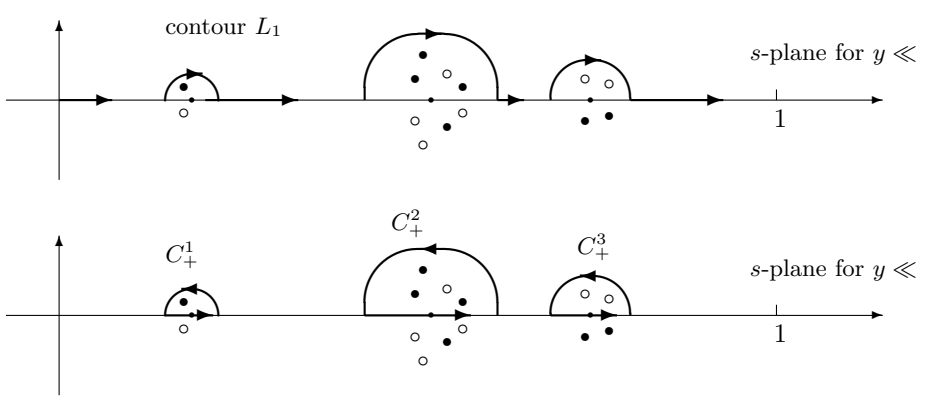
$$\frac{1}{\pi}\int_{\mathbb{R}} \operatorname{Im} R_{\lambda+iy}(x)\xi(x)\,dx = \frac{1}{\pi}\int_0^1 \operatorname{Tr}(V \operatorname{Im} R_{\lambda+iy}(H_r))\,dr.$$

The left hand side is the Poisson integral of the function ξ. Hence, a well-known property of the Poisson integral implies that as $y \to 0^+$, the left hand side converges to $\xi(\lambda)$ for a.e. λ. Hence, we will be done if we show that the right hand side converges to the total resonance index. Since λ lies outside the common essential spectrum of the operators H_r, the operator $\operatorname{Im} R_{\lambda+iy}(H_r)$ has zero limit for all values of r from the interval $[0,1]$, except some special values of r for which the resolvent $R_{\lambda+i0}(H_r) = R_\lambda(H_r)$ does not exist. Since λ is outside the essential spectrum, these values of r are those for which λ is an eigenvalue of H_r, that is, they are resonance points. In other words, a hindrance for taking the limit $y \to 0^+$ on the right hand side is the presence of resonance points. If there were no resonance points, the limit of the right hand side would be zero, which agrees with the fact that no eigenvalue of H_r reached the point λ and therefore the spectral flow through λ is zero.

So, the presence of resonance points in the domain $[0,1]$ of integration is a hindrance, but it makes the RHS non-zero and interesting. To overcome this hindrance we note that the integrand of the RHS,

$$\operatorname{Tr}(V \operatorname{Im} R_{\lambda+iy}(H_r)),$$

is a meromorphic function of r. A small neighbourhood of the interval $[0,1]$ in the coupling constant complex plane for sufficiently small $y > 0$ contains poles of $VR_{\lambda+iy}(H_s)$ and $VR_{\lambda-iy}(H_s)$, and these poles converge to the poles r_λ^j of $VR_\lambda(H_s)$ from $[0,1]$ as $y \to 0$. We represent the path of integration $[0,1]$ as the sum of a path L_1, shown below, which circumvents the poles from above, and closed contours C_+^j encircling the poles of the group of r_λ^j lying in $\mathbb{C}_+$. Clearly, as $y \to 0^+$ the integral over L_1 vanishes: there are no obstructions in the form of poles on L_1 in the limit and the integrand is zero when $y = 0$. (The following figures are taken from [Az7].)



We show that

$$\frac{1}{\pi}\oint_{C_+^j}\operatorname{Tr}(V\operatorname{Im}R_{\lambda+iy}(H_s))\,ds=\operatorname{ind}_{\mathrm{res}}(\lambda;H_{r_\lambda^j},V),$$

and this will complete the proof. It is enough to do this for one contour, so we omit the superscript index. We have

$$\begin{aligned}LHS&=\frac{1}{2\pi i}\oint_{C_+}\operatorname{Tr}\big(VR_{\lambda+iy}(H_s)-VR_{\lambda-iy}(H_s)\big)\,ds\\&=\operatorname{Tr}\left(\frac{1}{2\pi i}\oint_{C_+}R_{\lambda+iy}(H_s)V\,ds\right)-\operatorname{Tr}\left(\frac{1}{2\pi i}\oint_{C_+}R_{\lambda-iy}(H_s)V\,ds\right).\end{aligned}$$

We note that by (1.3.5) these two integrals are equal to $P^{\uparrow}_{\lambda+iy}(r_\lambda)$ and $P^{\uparrow}_{\lambda-iy}(r_\lambda)$, where $P^{\uparrow}_{\lambda\pm iy}(r_\lambda)$ is the sum of the idempotents $P_{\lambda\pm iy}(r^{(k)}_{\lambda\pm iy})$ corresponding to the resonance points $r^{(k)}_{\lambda\pm iy}$ of the group of r_λ taken from $\mathbb{C}_+$, which is indicated by the arrow $\uparrow$.

It remains to note that the ranks, and therefore the traces, of the idempotents $P^{\uparrow}_{\lambda+iy}(r_\lambda)$ and $P^{\uparrow}_{\lambda-iy}(r_\lambda)$ are N_+ and N_- respectively.

References

[APS] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. 79 (1976), 71–99.

[Az] N. A. Azamov, *Spectral shift function in von Neumann algebras*, thesis, Flinders Univ. 2008, 199 pp.

[Az2] N. A. Azamov, *Infinitesimal spectral flow and scattering matrix*, arXiv:0705.3282v4.

[Az3] N. A. Azamov, *Pushnitski's μ-invariant and Schrödinger operators with embedded eigenvalues*, arXiv:0711.1190v1 (2007).

[Az4] N. A. Azamov, *Absolutely continuous and singular spectral shift functions*, Dissertationes Math. 480 (2011), 102 pp.

[Az5] N. A. Azamov, *Resonance index and singular spectral shift function*, arXiv:1104.1903.

[Az6] N. A. Azamov, *Spectral flow inside essential spectrum*, Dissertationes Math. 518 (2016), 156 pp.

[Az7] N. A. Azamov, *Resonance index and singular -invariant*, arXiv:1501.07673 (2015).

[ACDS] N. A. Azamov, A. L. Carey, P. G. Dodds and F. A. Sukochev, *Operator integrals, spectral shift and spectral flow*, Canad. J. Math. 61 (2009), 241–263.

[ACS] N. A. Azamov, A. L. Carey and F. A. Sukochev, *The spectral shift function and spectral flow*, Comm. Math. Phys. 276 (2007), 51–91.

[AzD] N. A. Azamov and T. Daniels, *Singular spectral shift function for Schrödinger operators*, arXiv:1608.04184 v2 (2017).

[AzS] N. A. Azamov and F. A. Sukochev, *Spectral averaging for trace compatible operators*, Proc. Amer. Math. Soc. 136 (2008), 1769–1778.

[BCPRSW] M.-T. Benameur, A. L. Carey, J. Phillips, A. Rennie, F. A. Sukochev and K. P. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras*, in: Analysis, Geometry and Topology of Elliptic Operators, World Sci., Hackensack, NJ, 2006, 297–352.

[BS] M. Sh. Birman and M. Z. Solomyak, *Remarks on the spectral shift function*, J. Soviet Math. 3 (1975), 408–419.

[CP] A. L. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canad. J. Math. 50 (1998), 673–718.

[CP2] A. L. Carey and J. Phillips, *Spectral flow in Fredholm modules, eta invariants and the JLO cocycle*, K-Theory 31 (2004), 135–194.

[Di] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1960.

[Ge] E. Getzler, *The odd Chern character in cyclic homology and spectral flow*, Topology 32 (1993), 489–507.

[Hör3] L. Hörmander, *The Analysis of Linear Partial Differential Operators, III*, Springer, Berlin, 1985.

[Ka] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1980.

[Kr] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sb. 33 (75) (1953), 597–626 (in Russian).

[LL] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon Press, London, 1958.

[L] I. M. Lifshitz, *On a problem in perturbation theory connected with quantum statistics*, Uspekhi Mat. Nauk 7 (1952), no. 1, 171–180 (in Russian).

[Ph] J. Phillips, *Self-adjoint Fredholm operators and spectral flow*, Canad. Math. Bull. 39 (1996), 460–467.

[Ph2] J. Phillips, *Spectral flow in type I and II factors—a new approach*, Cyclic Cohomology and Noncommutative Geometry, 1997, Fields Inst. Commun. 17, Amer. Math. Soc., Providence, RI, 1997, 137–153.

[Pu] A. Pushnitski, *The spectral flow, the Fredholm index, and the spectral shift function* in: Spectral Theory of Differential Operators, Amer. Math. Soc. Transl. (2) 225, Adv. Math. Sci. 62, Amer. Math. Soc., Providence, RI, 2008, 141–155.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York, 1978.

[RoSa] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. 27 (1995), 1–33.

[Ya] D. R. Yafaev, *Mathematical Scattering Theory: General Theory*, Amer. Math. Soc., Providence, RI, 1992.

Index