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NON-ABELIAN GRADINGS OF LIE ALGEBRAS

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Abstract. We introduce non-abelian gradings of Lie algebras as their isotypic decompositions with respect to reductive groups of automorphisms. The main results relate to a special kind of SL_3-gradings, in terms of which the commutation operation admits a simple description. We show that any simple Lie algebra but C_n admits such a grading, and it is unique up to conjugation.

Introduction. Throughout the paper, the base field is supposed to be the field of complex numbers (or, what is essentially the same in this context, any algebraically closed field of characteristic zero). For brevity, we write SL_n, SO_n etc. instead of $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ etc.

Let S be a reductive algebraic group.

Definition 0.1. An *S-structure* in a Lie algebra $\mathfrak{g}$ is a homomorphism $\Phi : S \to \operatorname{Aut}\mathfrak{g}$.

If the group S is abelian, then the weight subspaces of $\mathfrak{g}$ with respect to S constitute a grading of the algebra $\mathfrak{g}$. In particular, the root decompositions of semisimple Lie algebras, various cyclic gradings etc. are obtained in this way.

If S is not abelian, the isotypic decomposition of $\mathfrak{g}$ with respect to S can be viewed as a “non-abelian grading”. In general, the commutation operation in $\mathfrak{g}$ does not admit a reasonable description in terms of such a grading. However, in some cases it does admit a simple description, which provides, in particular, some interesting models of the exceptional simple Lie algebras.

We will consider two types of S-structures: very short SL_2-structures and short SL_3-structures.

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Definition 0.2. A non-trivial SL_2-structure in a Lie algebra $\mathfrak{g}$ is called *very short*, if the representation Φ decomposes into 1- and 3-dimensional irreducible representations.

A very short SL_2-structure in a semisimple Lie algebra $\mathfrak{g}$ gives rise to the isotypic decomposition of the form

$$\mathfrak{g} = \mathfrak{Der}\, J + \mathfrak{sl}_2 \otimes J, \tag{1}$$

where J is a semisimple Jordan algebra and $\mathfrak{Der}\, J$ is the Lie algebra of its derivations acting naturally on the second summand. The commutator of two elements of the second summand is determined by the formula

$$[a \otimes x, b \otimes y] = (a,b)[L_x, L_y] + [a,b] \otimes xy, \tag{2}$$

where $(a,b) = \operatorname{tr} ab$ is an invariant scalar product in $\mathfrak{sl}_2$, and L_z stands for the multiplication by z in the algebra J.

Conversely, any semisimple Jordan algebra J defines a semisimple Lie algebra $\mathfrak{g}(J)$ by the above formulas. The algebra $\mathfrak{g}(J)$ is simple if and only if the algebra J is simple. This connection between Lie and Jordan algebras goes back to J. Tits [T], I. Kantor [K], and M. Koecher [Ko].

Among the exceptional simple Lie algebras, only E_7 admits a very short SL_2-structure. It corresponds to the Jordan algebra of Hermitian (3×3)-matrices over (complex) octonions (the Albert algebra).

Note that the 3-dimensional irreducible representation of the group SL_2 is in fact the tautological representation of the group $\mathrm{SO}_3 = \mathrm{SL}_2/\{\pm E\}$, so very short SL_2-structures can be considered as (very short) SO_3-structures. Under this name they were studied in [V2], where also a wider class of *short* SO_3-structures was considered.

One can also note that very short SL_2-structures are the same as root gradings of type A_1 in the sense of [BM].

The main results of the present paper relate to short SL_3-structures.

Definition 0.3. A non-trivial SL_3-structure in a simple Lie algebra $\mathfrak{g}$ is called *short*, if the representation Φ decomposes into the adjoint representation of SL_3 and 1- and 3-dimensional irreducible representations.

Let Φ be a short SL_3-structure. Then $d\Phi(\mathfrak{sl}_3) = \operatorname{ad}(\mathfrak{s})$, where $\mathfrak{s} \subset \mathfrak{g}$ is a (uniquely defined) subalgebra (isomorphic to $\mathfrak{sl}_3$). The adjoint representation of SL_3 is realized in $\mathfrak{s}$, so the representation of SL_3 in $\mathfrak{g}/\mathfrak{s}$ decomposes into 1- and 3-dimensional irreducible representations. Note that there are two 3-dimensional irreducible representations of SL_3: the tautological representation and its dual representation. The space of the tautological representation will be denoted by V, while the dual space will be denoted by V^*.

We shall show that in each simple Lie algebra but C_n there exists a short SL_3-structure, and it is unique up to conjugation. For $\mathfrak{g} = A_n$, the short SL_3-structure is "degenerate": the corresponding isotypic decomposition is essentially a $\mathbb{Z}$-grading of depth 1.

With any non-degenerate short SL_3-structure in a simple Lie algebra $\mathfrak{g}$, we canonically associate a very short SL_2-structure in a semisimple subalgebra $\mathfrak{h}$ (called the *stock* of $\mathfrak{g}$) and thereby a semisimple Jordan algebra J. The Jordan algebras appearing in this way are characterized by the property that their (weighted) norm N is of degree 3. We call them *cubic* algebras (see Definition 2.5).

The isotypic decomposition for a non-degenerate short SL_3-structure has the form

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{str}_0 J + V \otimes J + V^* \otimes J', \tag{3}$$

where J' is a copy of the corresponding cubic algebra J, and $\mathfrak{str}_0 J$ is the (*weighted*) *reduced structure Lie algebra* of J naturally acting on J (see Definitions 2.1 and 2.3) and acting on J' as on the dual space of J with respect to the canonical scalar product in J (see Definition 2.6).

The subalgebra $\mathfrak{str}_0 J$ coincides with the centralizer $\mathfrak{z}(\mathfrak{s})$ of $\mathfrak{s}$ in $\mathfrak{g}$. It decomposes as

$$\mathfrak{str}_0 J = \mathfrak{Der}\, J + J_0, \tag{4}$$

where $J_0 \subset J$ is the subspace of elements with (weighted) trace 0 acting on J by multiplication operators.

The commutators of elements of the last two summands are given by the formulas

$$[a \otimes x, b \otimes y] = 2(a \times b) \otimes \nu(x,y)' \in V^* \otimes J', \tag{5}$$

$$[a' \otimes x', b' \otimes y'] = -2(a' \times b') \otimes \nu(x,y) \in V \otimes J, \tag{6}$$

$$[a \otimes x, b' \otimes y'] = (a \otimes b')_0(x,y) + 2\langle a, b'\rangle\big([L_x, L_y] + L_{(xy)_0}\big) \in \mathfrak{s} + \mathfrak{str}_0 J, \tag{7}$$

where

$a, b \in V$; $a', b' \in V^*$;

$\langle a, b'\rangle$ is the canonical pairing between V and V^*;

$a \times b \in V^*$, $a' \times b' \in V$ are the vector products;

$(a \otimes b')_0$ is the projection of the linear operator $a \otimes b'$ on $\mathfrak{sl}_3$;

$x, y \in J$; $x', y' \in J'$ are the copies of x, y (and $\nu(x,y)' \in J'$ is the copy of $\nu(x,y)$);

(x,y) is the canonical scalar product in J;

$(xy)_0$ is the projection of xy on J_0;

$\nu : J \times J \to J$ is the commutative bilinear map defined by

$$\nu(x,x) = N(x)x^{-1}. \tag{8}$$

(The right hand side of this equality is consistently defined for all $x \in J$; see formula (14).)

Thus, the Lie algebra $\mathfrak{g}$ is uniquely reconstructed from the cubic Jordan algebra J. Conversely, by the above formulas any cubic Jordan algebra defines a simple Lie algebra.

In particular, in this way the (non-trivial) one-dimensional Jordan algebra corresponds to the Lie algebra G_2, and the Jordan algebras of Hermitian (3×3)-matrices over the four composition algebras correspond to the Lie algebras F_4, E_6, E_7, and E_8.

The constructions of the present paper are closely related to those of the paper [AF]. From the view-point of the present paper, the authors of [AF] consider the SL_2-structures in simple Lie algebras, for which only 1-, 2-, and 3-dimensional irreducible components are allowed. One can show that those of them having only one 3-dimensional irreducible component are obtained by the restriction of short SL_3-structures to the group SL_2 naturally embedded into SL_3.

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1. Preliminaries

1.1. Some definitions. For an S-structure $\Phi : S \to \operatorname{Aut} \mathfrak{g}$, we denote by $\mathfrak{g}^S$ the isotypic component of $\mathfrak{g}$ corresponding to the trivial one-dimensional representation of S, i.e. the subalgebra of S-invariant elements of $\mathfrak{g}$, and by $\mathfrak{g}_S$ the sum of all the other isotypic components. Obviously, $[\mathfrak{g}^S, \mathfrak{g}_S] \subset \mathfrak{g}_S$.

Definition 1.1. A non-trivial S-structure in a Lie algebra $\mathfrak{g}$ is called *reduced* if

(R1) the subalgebra $\mathfrak{g}^S$ does not contain non-zero ideals of $\mathfrak{g}$;
(R2) the subspace $\mathfrak{g}_S$ is not contained in a proper ideal of $\mathfrak{g}$.

Clearly, any non-trivial S-structure in a simple Lie algebra is reduced. In general, one can canonically associate a reduced S-structure to any S-structure, replacing $\mathfrak{g}$ with the factor algebra of the least ideal of $\mathfrak{g}$ containing $\mathfrak{g}_S$ modulo its intersection with the greatest ideal of $\mathfrak{g}$ contained in $\mathfrak{g}^S$.

For a reductive Lie algebra $\mathfrak{g}$, the conditions (R1) and (R2) are equivalent, since $\mathfrak{g}_S$ is the orthogonal complement of $\mathfrak{g}^S$ with respect to an invariant scalar product in $\mathfrak{g}$ (see Subsection 1.3) and the orthogonal complement of an ideal is an ideal. Moreover, if the algebra $\mathfrak{g}$ is reductive and the group S is connected, then the condition (R1) implies that the algebra $\mathfrak{g}$ is semisimple, because its center lies in $\mathfrak{g}^S$.

In what follows we shall assume by default all the considered S-structures to be reduced.

Definition 1.2. An S-structure is called *faithful* if $\operatorname{Ker} \Phi = \{e\}$.

In an obvious way, any S-structure can be considered as a faithful $(S/\operatorname{Ker} \Phi)$-structure.

Definition 1.3. An S-structure is called *inner* if $\Phi(S)$ lies in the group $\operatorname{Int} \mathfrak{g}$ of inner automorphisms of $\mathfrak{g}$.[1]

Note that an S-structure is automatically inner if the algebra $\mathfrak{g}$ is reductive and the group S is connected.

1.2. Equivariant bilinear maps. The following two lemmas play a key role in our study of S-structures throughout the paper.

We denote by V_ρ the space of a (finite-dimensional) representation ρ of the group S and by $\rho\sigma$ the (tensor) product of representations ρ and σ.

Lemma 1.4. *Let ρ, σ, and τ be irreducible representations of the group S. Assume that $\rho\sigma$ contains τ with multiplicity one, and let*

$$p : V_\rho \times V_\sigma \to V_\tau$$

be a fixed non-zero S-equivariant bilinear map (defined up to a scalar multiplication). Let U_ρ, U_σ, and U_τ be some vector spaces, on which the group S acts trivially. Then every S-equivariant bilinear map

$$P : (V_\rho \otimes U_\rho) \times (V_\sigma \otimes U_\sigma) \to (V_\tau \otimes U_\tau)$$

[1] Here $\operatorname{Int} \mathfrak{g} = \operatorname{Ad}(G)$, where G is any connected Lie group with $\operatorname{Lie}(G) = \mathfrak{g}$.

is given by the formula

$$P(a \otimes x, b \otimes y) = p(a,b)\nu(x,y),$$

where

$$\nu : U_\rho \times U_\sigma \to U_\tau$$

is some bilinear map.

Proof. We have

$$(V_\rho \otimes U_\rho) \otimes (V_\sigma \otimes U_\sigma) = (V_\rho \otimes V_\sigma) \otimes (U_\rho \otimes U_\sigma)$$
$$= (V_\tau + V') \otimes (U_\rho \otimes U_\sigma) = V_\tau \otimes (U_\rho \otimes U_\sigma) + V' \otimes (U_\rho \otimes U_\sigma),$$

where $V' \subset V_\rho \otimes V_\sigma$ is the sum of isotypic components distinct from V_τ.

Considering P as an S-equivariant linear map

$$P : (V_\rho \otimes U_\rho) \otimes (V_\sigma \otimes U_\sigma) \to V_\tau \otimes U_\tau,$$

we see that the summand $V' \otimes (U_\rho \otimes U_\sigma)$ must go to zero. This means that

$$P((a \otimes x) \otimes (b \otimes y)) = p(a,b) \otimes \varphi(x \otimes y),$$

where

$$\varphi : U_\rho \otimes U_\sigma \to U_\tau$$

is some linear map. This is exactly what the lemma says. ■

LEMMA 1.5. *Let ρ and τ be irreducible representations of the group S. Assume that both $S^2\rho$ and $\wedge^2\rho$ contain τ with multiplicity at most one, and let*

$$p : V_\rho \times V_\rho \to V_\tau \quad (\textit{resp. } q : V_\rho \times V_\rho \to V_\tau)$$

be a fixed non-zero S-equivariant symmetric (resp. skew-symmetric) bilinear map (defined up to a scalar multiplication), if $S^2\rho$ (resp. $\wedge^2\rho$) contains τ; otherwise set $p = 0$ (resp. $q = 0$). Let U_ρ and U_τ be some vector spaces, on which the group S acts trivially. Then every S-equivariant skew-symmetric bilinear map

$$P : (V_\rho \otimes U_\rho) \times (V_\rho \otimes U_\rho) \to (V_\tau \otimes U_\tau)$$

is given by the formula

$$P(a \otimes x, b \otimes y) = p(a,b)\varphi(x,y) + q(a,b)\psi(x,y),$$

where

$$\varphi : U_\rho \times U_\rho \to U_\tau \quad (\textit{resp. } \psi : U_\rho \times U_\rho \to U_\tau)$$

is some skew-symmetric (resp. symmetric) bilinear map.

Proof. The proof follows the same lines as the proof of the preceding lemma. The only additional argument is the equality

$$\wedge^2(V \otimes U) = S^2V \otimes \wedge^2U + \wedge^2V \otimes S^2U,$$

which holds for any vector spaces V and U. ■

Recall that the group SL_2 has just one irreducible representation ρ_k in each dimension k and

$$\rho_k\rho_l = \rho_{k+l} + \rho_{k+l-2} + \ldots + \rho_{k-l} \quad \text{for } k \geq l,$$
$$S^2\rho_k = \rho_{2k} + \rho_{2k-4} + \ldots, \quad \wedge^2\rho_k = \rho_{2k-2} + \ldots.$$

Thus, the conditions of Lemmas 1.4 and 1.5 automatically hold for $S = \mathrm{SL}_2$.

1.3. Invariant scalar products in reductive Lie algebras. Let G be a reductive algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$. It is well known that there exists an $\mathrm{Ad}(G)$-invariant scalar product in $\mathfrak{g}$, and, for $\mathfrak{g}$ simple, it is unique up to a scalar multiplication. In particular, if $R : G \to \mathrm{GL}(V)$ is a locally faithful linear representation of G, then

$$(X, Y) = \operatorname{tr} dR(X)\, dR(Y)$$

is such a scalar product. The last product has the following property: for each torus $T \subset G$, it is rational and positive definite on $\mathfrak{t}(\mathbb{Q})$, the space of rational points of the subalgebra $\mathfrak{t} = \mathrm{Lie}(T) \subset \mathfrak{g}$. In what follows, when we speak of an invariant scalar product in $\mathfrak{g}$, we always assume that this property holds. Obviously, it retains under the restriction to any reductive subalgebra.[2]

The above property guarantees, in particular, that the orthogonal complement of an algebraic ideal of $\mathfrak{g}$ (the tangent Lie algebra of a normal algebraic subgroup of G) is also an algebraic ideal.

1.4. $\mathfrak{sl}_2$-triples. Here we reproduce some facts about $\mathfrak{sl}_2$-triples in reductive Lie algebras.

Let G be a connected reductive algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$. An $\mathfrak{sl}_2$-*triple* in $\mathfrak{g}$ is a triple $\{e, h, f\}$ of non-zero elements satisfying the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{9}$$

Such elements constitute a basis of a subalgebra isomorphic to $\mathfrak{sl}_2$. It follows from the $\mathfrak{sl}_2$-theory that the operator $\mathrm{ad}(h)$ is semisimple and its eigenvalues are integer. The triple is called *even*, if the eigenvalues of $\mathrm{ad}(h)$ are even.

Let now $h \in \mathfrak{g}$ be an arbitrary semisimple element with even eigenvalues. Then its eigenspaces

$$\mathfrak{g}_k = \{x \in \mathfrak{g} : [h, x] = 2kx\}$$

constitute a $\mathbb{Z}$-grading of the algebra $\mathfrak{g}$. The subspaces $\mathfrak{g}_k$ and $\mathfrak{g}_l$ are orthogonal with respect to any invariant scalar product in $\mathfrak{g}$, unless $k + l = 0$. It follows that the subspaces $\mathfrak{g}_k$ and $\mathfrak{g}_{-k}$ are dual and the subspace $\mathfrak{g}_0$ is non-degenerate, so $\mathfrak{g}_0$ is a reductive subalgebra. Let G_0 be the corresponding connected algebraic subgroup of G. It acts on each subspace $\mathfrak{g}_k$ via the adjoint representation of G. It is known [V1, Subsection 2.6] that G_0 has an open orbit in each g_k, $k \neq 0$.

If h is included in an $\mathfrak{sl}_2$-triple $\{e, h, f\}$, then $e \in \mathfrak{g}_1$, $f \in \mathfrak{g}_{-1}$, and it follows from the $\mathfrak{sl}_2$-theory that $[\mathfrak{g}_0, e] = \mathfrak{g}_1$, which means that e lies in the open orbit of G_0 in $\mathfrak{g}_1$. Let

[2]Under a reductive subalgebra of $\mathfrak{g}$ we always mean the tangent Lie algebra of a reductive subgroup of G.

further $\tilde{\mathfrak{g}}_0$ be the orthogonal complement of $\mathbb{C}h$ in $\mathfrak{g}_0$. It is a reductive ideal of $\mathfrak{g}_0$, and

$$\mathfrak{g}_0 = \mathbb{C}h \oplus \tilde{\mathfrak{g}}_0 \tag{10}$$

(a direct sum of Lie algebras). Denote by $\tilde{G}_0$ the connected (normal) subgroup of G_0 with $\mathrm{Lie}(\tilde{G}_0) = \tilde{\mathfrak{g}}_0$. Note that

$$([\tilde{\mathfrak{g}}_0, e], f) = (\tilde{\mathfrak{g}}_0, [e, f]) = (\tilde{\mathfrak{g}}_0, h) = 0.$$

Hence, $[\tilde{\mathfrak{g}}_0, e] \neq \mathfrak{g}_1$, which means that the $\tilde{G}_0$-orbit of e is not open in $\mathfrak{g}_1$.

Conversely, if the $\tilde{G}_0$-orbit of some element $e \in \mathfrak{g}_1$ of the open G_0-orbit is not open, then $[\tilde{\mathfrak{g}}_0, e] \not\ni e$ and, hence, there exists an element $f \in \mathfrak{g}_{-1}$ such that $[e, f] = ch$ with $c \neq 0$. Replacing f with $c^{-1}f$, we obtain an $\mathfrak{sl}_2$-triple.

Resuming, we have proved the following criterion.

PROPOSITION 1.6. *In the above notation, a semisimple element $h \in \mathfrak{g}$ with even eigenvalues can be included in an $\mathfrak{sl}_2$-triple if and only if the group $\tilde{G}_0$ has no open orbit in $\mathfrak{g}_1$.*

2. Jordan algebras. Here we recall some definitions and facts concerning Jordan algebras. They can be found, e.g., in [A], [BK], and [FK]. We also introduce some new notions.

2.1. The structure group. For any Jordan algebra J, the transformations of the form $[L_x, L_y]$ $(x, y \in J)$ are derivations of J. Their linear combinations are called the *inner derivations*. They constitute an ideal, denoted by $\mathfrak{int}\, J$, in the Lie algebra $\mathfrak{der}\, J$ of all derivations. If the algebra J is semisimple, then $\mathfrak{int}\, J = \mathfrak{der}\, J$. Obviously, $\mathfrak{der}\, J = \mathrm{Lie}(\mathrm{Aut}\, J)$.

Assume now that the algebra J has a unit 1. Then the linear span of $\mathfrak{der}\, J$ and the multiplication operators $L_z\,(z \in J)$ is a Lie subalgebra in $\mathfrak{gl}(J)$, which is called the *structure algebra* of J and is denoted by $\mathfrak{str}\, J$. As a vector space, it is a direct sum of $\mathfrak{der}\, J$ and the space of the multiplication operators. Identifying $z \in J$ with L_z, we will write

$$\mathfrak{str}\, J = \mathfrak{der}\, J + J.$$

The reflection in $\mathfrak{der}\, J$ along J is an involution of the algebra $\mathfrak{str}\, J$.

There is an algebraic subgroup $\mathrm{Str}\, J \subset \mathrm{GL}(J)$ called the *structure group* of J, such that $\mathrm{Lie}(\mathrm{Str}\, J) = \mathfrak{str}\, J$ and the stabilizer of the unit $1 \in J$ in $\mathrm{Str}\, J$ coincides with $\mathrm{Aut}\, J$.

2.2. The norm and the trace. The *rank* (or the *degree*) $r = \mathrm{rk}\, J$ of a semisimple Jordan algebra J is defined as the dimension of a maximal semisimple associative subalgebra of J (isomorphic to the direct sum of r copies of $\mathbb{C}$) or, equivalently, as the degree of the minimal polynomial of a generic element of J. (It is also equal to the rank of the symmetric space $\mathrm{Str}\, J/\mathrm{Aut}\, J$.) Obviously, the rank of a direct sum of simple Jordan algebras is equal to the sum of their ranks.

The minimal polynomial of a generic element x of a semisimple Jordan algebra J has the form

$$m(x, t) = t^r - f_1(x)t^{r-1} + \ldots + (-1)^r f_r(x),$$

where f_k is a homogeneous polynomial of degree k in the coordinates of x. The polynomials f_r and f_1 are called the *norm* and the *trace* of the algebra J. We will denote them by N and T. Note that $N(1) = 1$ and $T(1) = r$. The trace is the differential of the norm at the unit 1 of the algebra J.

The complement of the divisor $N = 0$ is an open orbit of the group $\operatorname{Str} J$. It follows that the norm is a semi-invariant of this group, namely,

$$N(gx) = \chi(g)N(x) \quad \text{for any} \quad g \in \operatorname{Str} J,$$

where χ is some surjective character of $\operatorname{Str} J$ (i.e., $\chi(\operatorname{Str} J) = \mathbb{C}^*$).

DEFINITION 2.1. The kernel of the character χ is called the *reduced structure group* of J and is denoted by $\operatorname{Str}_0 J$. The Lie algebra $\mathfrak{str}_0 J = \operatorname{Lie}(\operatorname{Str}_0 J)$ is called the *reduced structure algebra* of J.

It is easy to see that

$$\mathfrak{str}_0 J = \mathfrak{der} J + J_0, \tag{11}$$

where

$$J_0 = \{x \in J : T(x) = 0\}. \tag{12}$$

Obviously, the norm of a direct sum of simple Jordan algebras is equal to the product of their norms (with disjoint sets of variables). The norm of a simple Jordan algebra is an irreducible polynomial.

2.3. Weighted norms and traces. One can slightly generalize the notions of the norm and the trace of a semisimple Jordan algebra.

Namely, let $J = J_1 \oplus \ldots \oplus J_s$ be the decomposition of a semisimple Jordan algebra J into a direct sum of simple Jordan algebras. For any $x = x_1 + \ldots + x_s \in J$ $(x_k \in J_k)$, denote by $N_k(x)$ and $T_k(x)$ the norm and the trace of the element x_k in J_k. Then

$$N = N_1 \ldots N_s, \quad \text{and} \quad T = T_1 + \ldots + T_s.$$

The divisor $N = 0$ is the union of the irreducible divisors $N_k = 0$, $k = 1, \ldots, s$, each of them being invariant under the connected component $(\operatorname{Str} J)^0$ of the group $\operatorname{Str} J$. (The whole group can permute some of these divisors.) It follows that $N_1, \ldots, N_s$ are semi-invariants of the group $(\operatorname{Str} J)^0$ and, moreover, any semi-invariant N of this group normalized by the condition $N(1) = 1$, is uniquely represented in the form

$$N = N_1^{p_1} \ldots N_s^{p_s}$$

with some non-negative integers $p_1, \ldots, p_s$ called the *weights* of N.

DEFINITION 2.2. Let $p_1, \ldots, p_s > 0$. The functions

$$N = N_1^{p_1} \ldots N_s^{p_s} \quad \text{and} \quad T = p_1 T_1 + \ldots + p_s T_s$$

are called the *weighted norm* and the *weighted trace* of J with weights $p_1, \ldots, p_s$.

It is easy to see that the weighted trace is the differential at 1 of the weighted norm (with the same weights).

DEFINITION 2.3. The subgroup of $\operatorname{Str} J$ leaving a given weighted norm N invariant will be called the *weighted reduced structure group* and denoted by $\operatorname{Str}_0 J$, if it is clear, which weights are meant. Its tangent Lie algebra will be called the *weighted reduced structure algebra* and denoted by $\mathfrak{str}_0 J$.

The weighted reduced structure algebra is given by equations (11) and (12), where the trace should be replaced with the weighted trace.

Moreover, define the *weighted minimal polynomial* $m(x,t)$ (with weights $p_1, \dots, p_s$) of a generic element $x = x_1 + \ldots + x_s \in J$ $(x_k \in J_k)$ as

$$m(x,t) = m_1(x_1,t)^{p_1} \ldots m_s(x_s,t)^{p_s} = t^r - f_1(x)t^{r-1} + \ldots + (-1)^r f_r(x),$$

where $m_k(x_k,t)$ is the minimal polynomial of the element $x_k \in J_k$. The number $r = \sum_k p_k r_k$, where $r_k = \operatorname{rk} J_k$, is called the *weighted rank* of J. The polynomials $f_r(x)$ and $f_1(x)$ are just the weighted norm and the weighted trace of J.

2.4. Associative scalar products

DEFINITION 2.4. A (non-degenerate) scalar product $(\cdot,\cdot)$ in a Jordan algebra J is called *associative* if

$$(xy,z) = (x,yz) \quad \text{for any } x,y,z \in J.$$

An associative scalar product exists if and only if the algebra J is semisimple. In particular, in a simple Jordan algebra J any associative scalar product, up to a scalar factor, has the form

$$(x,y) = T(xy), \tag{13}$$

where T is the trace of the algebra J. The scalar product (13) will be called *canonical.*

In a general semisimple Jordan algebra J, any associative scalar product is a linear combination (with non-zero coefficients) of the canonical scalar products in the simple summands of J.

2.5. Cubic Jordan algebras. The following definition plays a crucial role in the present paper.

DEFINITION 2.5. A *cubic Jordan algebra*, or just a *cubic algebra*, is a semisimple Jordan algebra J with a fixed cubic weighted norm N.

The weights $p_1, \dots, p_s$ of N satisfy the equality

$$\sum_k p_k r_k = 3,$$

where $r_k = \operatorname{rk} J_k$. It is easy to see that there are only the following possibilities:

(C1) $s = 1$; $r_1 = 1$; $p_1 = 3$;
(C2) $s = 2$; $r_1 = r_2 = 1$; $p_1 = 2, p_2 = 1$;
(C3) $s = 3$; $r_1 = r_2 = r_3 = 1$; $p_1 = p_2 = p_3 = 1$;
(C4) $s = 2$; $r_1 = 2, r_2 = 1$; $p_1 = p_2 = 1$;
(C5) $s = 1$; $r_1 = 3$; $p_1 = 1$.

The cubic weighted norm N of a cubic algebra differs from its norm only in the cases (C1) and (C2). In these two cases $N = N_1^3$ and $N = N_1^2 N_2$, respectively.

In the future, when we talk about the weighted norm, the weighted trace, the weighted reduced structure group, and the weighted reduced structure algebra of a cubic algebra J, we shall usually omit the adjective "weighted" and denote these objects just by N, T, $\operatorname{Str} J$, and $\operatorname{Str}_0 J$.

Definition 2.6. The associative scalar product $(x, y) = T(xy)$, where T is the weighted trace of a cubic algebra J, is called the *canonical scalar product* in J.

The minimal polynomial of a generic element of a cubic algebra has the form

$$m(x,t) = t^3 - f_1(x)t^2 + f_2(x)t - f_3(x),$$

where

$$f_1(x) = T(x), \quad f_2(x) = \frac{1}{2}[T(x)^2 - T(x^2)], \quad f_3(x) = N(x).$$

It follows that

$$N(x)x^{-1} = x^2 - f_1(x)x + f_2(x) = x^2 - T(x)x + \frac{1}{2}[T(x)^2 - T(x^2)]. \tag{14}$$

Thus, the map $J \to J$, $x \mapsto \check{x} := N(x)x^{-1}$ is quadratic and thereby is defined for all x. (The element $\check{x}$ is sometimes called the *adjoint* of x.) The corresponding commutative bilinear map

$$\nu : J \times J \to J$$

obtained by polarization is determined by the formula

$$\nu(x,y) = xy - \frac{1}{2}[T(y)x + T(x)y + T(xy) - T(x)T(y)]. \tag{15}$$

It is easy to see that the trilinear form $(\nu(x, y), z)$ is symmetric and

$$(\nu(x,x),x) = 3N(x). \tag{16}$$

3. Very short SL_2-structures

3.1. The isotypic decomposition. Let $\Phi : \mathrm{SL}_2 \to \operatorname{Aut}\mathfrak{g}$ be a (reduced) very short SL_2-structure. Since the 3-dimensional irreducible representation of the group $S = \mathrm{SL}_2$ is its adjoint representation, the isotypic decomposition of $\mathfrak{g}$ has the form

$$\mathfrak{g} = \mathfrak{g}^S + \mathfrak{sl}_2 \otimes J,$$

where J is some vector space, on which the group SL_2 acts trivially. According to Lemma 1.5, commuting elements $a \otimes x$ and $b \otimes y$ of the second summand gives rise to a commutative bilinear operation on J and a skew-symmetric bilinear map $\Delta : J \times J \to \mathfrak{g}^S$ so that

$$[a \otimes x, b \otimes y] = (a,b)\Delta(x,y) + [a,b] \otimes xy, \tag{17}$$

where (a, b) is the invariant scalar product in $\mathfrak{sl}_2$ defined by

$$(a,b) = \operatorname{tr} ab. \tag{18}$$

The action of the subalgebra $\mathfrak{g}^S$ on the second summand of the isotypic decomposition reduces to an action of $\mathfrak{g}^S$ on J. The kernel of the last action is an ideal of $\mathfrak{g}$, which is contained in $\mathfrak{g}^S$ and, hence, equals zero. Thus, we can identify $\mathfrak{g}^S$ with a subalgebra of $\mathfrak{gl}(J)$. Further, the Jacobi identity for one element of $\mathfrak{g}^S$ and two elements of the second summand means that $g^S \subset \mathfrak{der}\, J$ and the map Δ is $\mathfrak{g}^S$-equivariant.

The following useful formula (Lagrange's formula) can be easily checked:

$$[[a, b], c] = 2((b, c)a - (a, c)b).$$

With the help of this formula, one can show that the Jacobi identity for three elements of the second summand is equivalent to the following identities for $x, y, z \in J$:

$$\Delta(xy, z) + \Delta(yz, x) + \Delta(zx, y) = 0, \tag{19}$$

$$\Delta(x, y) = 2[L_x, L_y]. \tag{20}$$

The latter identity shows that the map Δ is expressed in terms of the algebra J (and thereby is automatically $\mathfrak{g}^S$-equivariant). Substituting this expression in (19), we obtain the identity

$$[L_{xy}, L_z] + [L_{yz}, L_x] + [L_{zx}, L_y] = 0.$$

Since the left hand side of the last identity is symmetric in x, y, z, it is equivalent to the identity $[L_{x^2}, L_x] = 0$, which is exactly the identity of Jordan algebras (apart from the commutativity).

It follows from (20) that $\mathfrak{g}^S \supset \mathfrak{int}\, J$. Obviously, the subspace $\mathfrak{int}\, J + \mathfrak{sl}_2 \otimes J$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{g}_S = \mathfrak{sl}_2 \otimes J$. Hence, it coincides with $\mathfrak{g}$, i.e. $\mathfrak{g}^S = \mathfrak{int}\, J$. Thus,

$$\mathfrak{g} = \mathfrak{int}\, J + \mathfrak{sl}_2 \otimes J, \tag{21}$$

so the algebra $\mathfrak{g}$ is uniquely determined by the Jordan algebra J.

The SL_2-structure is inner if and only if the Jordan algebra J has a unit.

Indeed, if the algebra J has a unit 1, then one obtains from (17) and (20):

$$\Delta(1, y) = 0,$$

$$[a \otimes 1, b \otimes y] = [a, b] \otimes y.$$

It is also clear that

$$[a \otimes 1, D] = 0$$

for any $D \in \mathfrak{int}\, J$. Thus, the SL_2-structure is defined by the adjoint action of the subalgebra $\mathfrak{sl}_2 \otimes 1$ isomorphic to $\mathfrak{sl}_2$. Conversely, if the SL_2-structure is inner, then it is defined by the adjoint action of a subalgebra isomorphic to $\mathfrak{sl}_2$. It is easy to see that the last subalgebra must have the form $\mathfrak{sl}_2 \otimes 1$, where 1 is a unit of the algebra J.

Any homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras with very short SL_2-structures (commuting with the action of SL_2) gives rise to a homomorphism $J_1 \to J_2$ of the corresponding Jordan algebras, which is surjective (injective) if and only if φ is surjective (injective).

Conversely, for any Jordan algebra J the formulas (21), (17), and (20) define a Lie algebra $\mathfrak{g} = \mathfrak{g}(J)$. This is the so-called *Tits–Kantor–Koecher construction*. Any surjective homomorphism $J_1 \to J_2$ of Jordan algebras gives rise to a surjective homomorphism $\mathfrak{g}(J_1) \to \mathfrak{g}(J_2)$ of Lie algebras. It follows that the algebra $\mathfrak{g}(J)$ is simple if and only if the algebra J is simple.

Clearly, $\mathfrak{g}(J_1 \oplus J_2) = \mathfrak{g}(J_1) \oplus \mathfrak{g}(J_2)$. It follows that the algebra $\mathfrak{g}(J)$ is semisimple if and only if the algebra J is semisimple.

Making use of this correspondence between Jordan and Lie algebras, one can prove that all derivations of a semisimple Jordan algebra J are inner. Indeed, any derivation of J generates a derivation of the Lie algebra $\mathfrak{g}(J)$ commuting with the action of SL_2. Since the algebra $\mathfrak{g}(J)$ is semisimple, the latter derivation is inner and, moreover, it is defined by some element of $\mathfrak{g}^S = \mathfrak{int}\, J$.

3.2. The invariant scalar products. Assuming that the algebra $\mathfrak{g}$ is semisimple, choose an invariant scalar product in $\mathfrak{g}$. Its restriction to the subspace $\mathfrak{g}_S = \mathfrak{sl}_2 \otimes J$ is non-degenerate and $\mathfrak{sl}_2$-invariant and, hence, has the form

$$(a \otimes x, b \otimes y) = (a, b)(x, y), \tag{22}$$

where (x, y) is a scalar product in J. The identity

$$([a \otimes x, b \otimes y], c \otimes z) = (a \otimes x, [b \otimes y, c \otimes z])$$

implies that the so defined scalar product in J is associative. This establishes a bijection between the invariant scalar products in $\mathfrak{g}$ and associative scalar products in J.

The invariant scalar product in $\mathfrak{g} = \mathfrak{g}(J)$ will be called *canonical*, if the corresponding scalar product in J is canonical (see Subsection 2.4).

3.3. The $\mathbb{Z}$-grading. Very short SL_2-structures can be alternatively viewed as a special kind of $\mathbb{Z}$-gradings. Namely, let $e, h, f \in \mathfrak{sl}_2$ constitute an $\mathfrak{sl}_2$-triple. Then the decomposition (21) can be re-written as follows:

$$\mathfrak{g} = f \otimes J + (\mathfrak{int}\, J + h \otimes J) + e \otimes J.$$

It is easy to see that this is a $\mathbb{Z}$-grading of depth 1 with

$$\mathfrak{g}_0 = \mathfrak{int}\, J + h \otimes J, \quad \mathfrak{g}_1 = e \otimes J, \quad \mathfrak{g}_{-1} = f \otimes J.$$

Assume now that the algebra $\mathfrak{g}$ is semisimple. Then the algebra J is also semisimple and $\mathfrak{int}\, J = \mathfrak{der}\, J$. The subalgebra $\mathfrak{g}_0$ can be identified with the structure algebra $\mathfrak{str}\, J$ of J by means of the isomorphism

$$\mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{str}\, J, \quad D + h \otimes x \mapsto D + 2L_x \quad (D \in \mathfrak{int}\, J,\ x \in J). \tag{23}$$

We will also consider $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ as two copies of the algebra J denoted by J and J', respectively, and write x instead of $e \otimes x$ and x' instead of $f \otimes x$ for any $x \in J$. Under these conventions, the above decomposition is re-written as

$$\mathfrak{g} = J' + \mathfrak{str}\, J + J. \tag{24}$$

The canonical scalar product defines a duality between J and J'. The Lie algebra $\mathfrak{str}\, J$ acts on J by its definition and acts on J' as on the dual space of J.

Finally, as follows from (17), (20), and (23),

$$[x, y'] = 2([L_x, L_y] + L_{xy}) \quad \text{for} \quad x, y \in J. \tag{25}$$

3.4. Classification. In this subsection, we outline the classification of very short SL_2-structures in simple Lie algebras. According to the above, it is equivalent to the classification of simple Jordan algebras.

For an SL_2-structure $\Phi : \mathrm{SL}_2 \to \operatorname{Aut} \mathfrak{g}$ in a simple Lie algebra $\mathfrak{g}$, consider an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset d\Phi(\mathfrak{sl}_2) \simeq \mathfrak{sl}_2$. (Here we identify $\mathfrak{Der}\,\mathfrak{g}$ with $\mathfrak{g}$.) The condition that Φ is very short, means that the eigenvalues of $\mathrm{ad}(h)$ are 0 and ± 2.

Choose a Cartan subalgebra of $\mathfrak{g}$ containing h. Let $\alpha_1, \dots, \alpha_n$ be the simple roots with respect to it. One may assume that h lies in the Weyl chamber. Then the maximal eigenvalue of $\mathrm{ad}(h)$ is equal to $\delta(h)$, where δ is the highest root. It is equal to 2 if and only if

- (S1) all the numbers $\alpha_i(h)$ but one are equal to 0;
- (S2) the remaining one, say, $\alpha_{i_0}(h)$, is equal to 2;
- (S3) the coefficient of α_0 in the decomposition of δ as a linear combination of simple roots, is equal to 1.

This leaves only few possibilities for h.

Among these candidates, we should choose those which can be included into an $\mathfrak{sl}_2$-triple. This can be done with the help of Proposition 1.6. Note that the condition (S1) implies that in our situation the group $\tilde{G}_0$ is nothing else than the commutator subgroup G_0' of G_0.

The result of this classification is presented in Table 1, where the black vertices correspond to the simple roots that do not vanish on h.

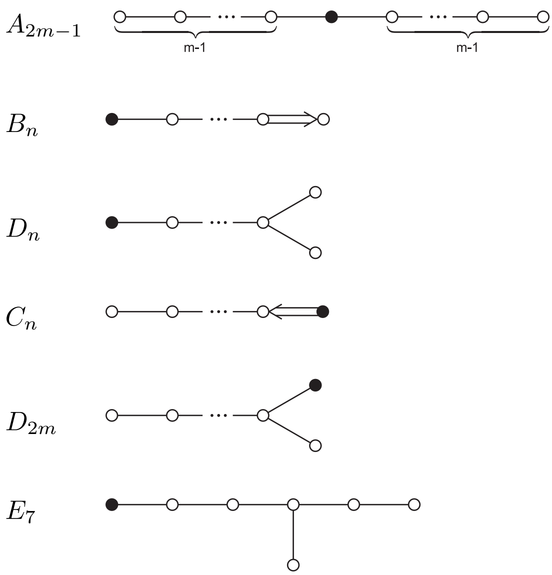


Table 1. Very short SL_2-structures

4. Short SL_3-structures. In this section $\mathfrak{g}$ is a simple Lie algebra and $G = \operatorname{Int} \mathfrak{g}$. The Lie algebra $\mathrm{Lie}(G) = \mathfrak{Der}\,\mathfrak{g}$ is identified with $\mathfrak{g}$.

4.1. The isotypic decomposition. Let $\Phi : S = \mathrm{SL}_3 \to G$ be a short SL_3-structure in $\mathfrak{g}$, and $\mathfrak{s} = d\Phi(\mathfrak{sl}_3)$. Let $Z(\mathfrak{s})$ (resp. $\mathfrak{z}(\mathfrak{s})$) be the centralizer of $\mathfrak{s}$ in G (resp. in $\mathfrak{g}$), and $V = \mathbb{C}^3$ be the space of the tautological representation of SL_3. The isotypic decomposition of $\mathfrak{g}$ with respect to Φ has the form

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{z}(\mathfrak{s}) + V \otimes J + V^* \otimes J',$$

where J and J' are some vector spaces, on which the group SL_3 acts trivially. The group $Z(\mathfrak{s})$ naturally acts on J and J' (and acts trivially on V and V^*).

Define an invariant scalar product in $\mathfrak{g}$ by

$$(X, Y) = \operatorname{tr} XY \quad \text{for} \quad X, Y \in \mathfrak{s} = \mathfrak{sl}_3. \tag{26}$$

The action of the center of SL_3 defines a $\mathbb{Z}_3$-grading of $\mathfrak{g}$, where

$$\mathfrak{g}_0 = \mathfrak{s} + \mathfrak{z}(\mathfrak{s}), \quad \mathfrak{g}_1 = V \otimes J, \quad \mathfrak{g}_{-1} = V^* \otimes J'.$$

The subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are dual with respect to the invariant scalar product. This defines a $Z(\mathfrak{s})$-invariant non-degenerate pairing between J and J' so that

$$(a \otimes x, b' \otimes y') = \langle a, b' \rangle \langle x, y' \rangle \tag{27}$$

for $a \otimes x \in V \otimes J$ and $b' \otimes y' \in V^* \otimes J'$.

The kernel of the action of $Z(\mathfrak{s})$ on J is a normal subgroup of G and, hence, is trivial. Thus, one can identify $Z(\mathfrak{s})$ with a subgroup of $\mathrm{GL}(J)$ (and $\mathfrak{z}(\mathfrak{s})$ with a subalgebra of $\mathfrak{gl}(J)$). It naturally acts on J' as on the dual space of J.

According to Lemma 1.4, the commutator of elements of $\mathfrak{g}_1 = V \otimes J$ with elements of $\mathfrak{g}_{-1} = V^* \otimes J'$ is given by the formula

$$[a \otimes x, b' \otimes y'] = (a \otimes b')_0 \gamma(x, y') + \langle a, b' \rangle \delta(x, y'),$$

where $(a \otimes b')_0$ is the projection of the linear operator $a \otimes b'$ on $\mathfrak{sl}_3$, while $\gamma : J \times J' \to \mathbb{C}$ and $\delta : J \times J' \to \mathfrak{z}(\mathfrak{s})$ are some $Z(\mathfrak{s})$-equivariant bilinear maps.

Proposition 4.1. *$\gamma(x, y') = \langle x, y' \rangle$.*

Proof. It follows from the invariance of the scalar product in $\mathfrak{g}$ that

$$(C, [a \otimes x, b' \otimes y']) = ([C, a \otimes x], b' \otimes y')$$

for any $C \in \mathfrak{s} = \mathfrak{sl}_3$. We have

$$(C, [a \otimes x, b' \otimes y']) = (C, (a \otimes b')_0)\gamma(x, y') = \langle Ca, b' \rangle \gamma(x, y'),$$

and, on the other hand,

$$([C, a \otimes x], b' \otimes y') = (Ca \otimes x, b' \otimes y') = \langle Ca, b' \rangle \langle x, y' \rangle,$$

whence the desired equality follows. ∎

Thus,

$$[a \otimes x, b' \otimes y'] = (a \otimes b')_0 \langle x, y' \rangle + \langle a, b' \rangle \delta(x, y'). \tag{28}$$

Define the *vector product* $V \times V \to V^*$ by

$$\langle a \times b, c \rangle = \det(a, b, c),$$

where det is a fixed non-zero skew-symmetric trilinear form on V.

According to Lemma 1.5, the commutator of two elements of $\mathfrak{g}_1 = V \otimes J$ is given by the formula

$$[a \otimes x, b \otimes y] = 2(a \times b) \otimes \nu(x, y), \tag{29}$$

where $\nu : J \times J \to J'$ is some $Z(\mathfrak{s})$-equivariant commutative bilinear map. (The coefficient can be made arbitrary at the expense of changing ν. The advantage of the coefficient 2 will be clear later.)

Note that the trilinear form $([X, Y], Z)$ on $\mathfrak{g}$ is skew-symmetric. Calculating this form for three elements of $V \otimes J$, we obtain

$$([a \otimes x, b \otimes y], c \otimes z) = \det(a, b, c)\langle \nu(x, y), z\rangle.$$

It follows that $\langle \nu(x, y), z\rangle$ is a symmetric trilinear form on J. Obviously, it is $Z(\mathfrak{s})$-invariant.

Similarly, define the *vector product* $V^* \times V^* \to V$ by

$$\langle a' \times b', c'\rangle = \det{}'(a', b', c'),$$

where $\det'$ is the skew-symmetric trilinear form on V^* normalized by the condition

$$\det(a, b, c)\det{}'(a', b', c') = 1$$

for any dual bases $\{a, b, c\}$ and $\{a', b', c'\}$ of V and V^*. The commutator of two elements of $\mathfrak{g}_{-1} = V^* \otimes J'$ is given by the formula

$$[a' \otimes x', b' \otimes y'] = -2(a' \times b') \otimes \nu'(x', y'), \tag{30}$$

where $\nu' : J' \times J' \to J$ is some $Z(\mathfrak{s})$-equivariant commutative bilinear map. As above, one can prove that $\langle \nu'(x', y'), z'\rangle$ is a symmetric trilinear form on J'.

PROPOSITION 4.2. *There is an involution $\sigma \in \operatorname{Aut}\mathfrak{g}$ leaving invariant the subalgebra $\mathfrak{s}$ and acting on the center of $\Phi(\mathrm{SL}_3)$ as the inversion.*

Proof. Include the center of $\Phi(\mathrm{SL}_3)$ into a maximal torus T of G. Clearly, $\mathfrak{t} = \mathrm{Lie}(T) \subset \mathfrak{g}_0$, so the subalgebra $\mathfrak{s}$ is normalized by $\mathfrak{t}$. One can take for σ a "Weyl involution" of $\mathfrak{g}$ acting on $\mathfrak{t}$ as multiplication by -1. ■

Obviously, σ leaves invariant $\mathfrak{g}_0$ and permutes $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$.

COROLLARY 4.3. *If $\nu = 0$, then $\nu' = 0$, and vice versa.*

DEFINITION 4.4. A short SL_3-structure is called *degenerate* if $\nu = 0$ and *non-degenerate* otherwise.

If a short SL_3-structure is degenerate, then $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$, so the above $\mathbb{Z}_3$-grading is in fact a $\mathbb{Z}$-grading of depth 1. Later we shall see that this happens if and only if $\mathfrak{g} = A_n$.

4.2. The stock. Let $\{v_1, v_2, v_3\}$ be the standard basis of $V = \mathbb{C}^3$, and $\{v_1', v_2', v_3'\}$ be the dual basis of V^*. Let $\mathfrak{m} \simeq \mathfrak{sl}_2$ be the common stabilizer of v_1 and v_1' in $\mathfrak{s} = \mathfrak{sl}_3$ (the "lower right corner").

Let $H = Z(\mathfrak{m})$ (resp. $\mathfrak{h} = \mathfrak{z}(\mathfrak{m})$) be the centralizer of $\mathfrak{m}$ in G (resp. in $\mathfrak{g}$). Clearly,

$$\mathfrak{h} = \mathbb{C}h + \mathfrak{z}(\mathfrak{s}) + J + J',$$

where $h = \mathrm{diag}(2, -1, -1) \in \mathfrak{s}$, and J (resp. J') stands for $v_1 \otimes J$ (resp. $v_1' \otimes J'$). Being the centralizer of a reductive subalgebra, the subalgebra $\mathfrak{h}$ is also reductive. It intersects every minimal SL_3-invariant subspace of $\mathfrak{g}$ in a one-dimensional subspace.

The element h defines a $\mathbb{Z}$-grading of depth 1 of the algebra $\mathfrak{h}$, where

$$\mathfrak{h}_0 = \mathfrak{z}(h) = \mathbb{C}h + \mathfrak{z}(\mathfrak{s}),$$
$$\mathfrak{h}_1 = \{x \in \mathfrak{h} : [h, x] = 2x\} = J,$$
$$\mathfrak{h}_{-1} = \{x \in \mathfrak{h} : [h, x] = -2x\} = J'.$$

The subalgebra $\mathfrak{h}_0$ is also reductive and decomposes into an orthogonal direct sum of the central ideal $\mathbb{C}h$ and the ideal $\mathfrak{z}(\mathfrak{s})$. Denote by H_0 the corresponding connected subgroup of H.

DEFINITION 4.5. The $\mathbb{Z}$-graded algebra $\mathfrak{h} = \mathfrak{z}(\mathfrak{m})$ is called the *stock* of the Lie algebra $\mathfrak{g}$ with a given short SL_3-structure.

Assume now that the SL_3-structure in $\mathfrak{g}$ is non-degenerate. Then the group $Z(\mathfrak{s})$ has no open orbit in J, since it leaves the cubic form $\langle \nu(x, x), x\rangle$ invariant, and, by Proposition 1.6, the element h can be included in an $\mathfrak{sl}_2$-triple $\{e, h, f\}$. Moreover, since the representation of $Z(\mathfrak{s})$ in J is faithful and its image does not contain the whole group of scalar multiplications, the representation of the algebra $\mathfrak{h}_0$ in J is also faithful. In particular, $\mathfrak{h}_0$ does not contain non-zero ideals of $\mathfrak{h}$. Therefore, the algebra $\mathfrak{h}$ is semisimple (but not necessarily simple) and the $\mathfrak{sl}_2$-triple $\{e, h, f\}$ defines a (reduced) very short SL_2-structure in it.

According to Subsection 3.3, the subspaces $\mathfrak{h}_1 = J$ and $\mathfrak{h}_{-1} = J'$ can be considered as two copies of a (semisimple) Jordan algebra J so that the elements e and f are the units of these two copies. Moreover, the subalgebra $\mathfrak{h}_0$ can be identified with the structure algebra $\mathfrak{str}\, J$ so that the formula (25) holds. Under this identification, the element h (defining the $\mathbb{Z}$-grading of the algebra $\mathfrak{h}$) corresponds to $h \otimes 1$ and acts on J as $2E$.

4.3. The norm. Consider the cubic form

$$N(x) = \frac{1}{3}\langle \nu(x, x), x\rangle \tag{31}$$

on the algebra J. It is invariant under the action of the group $Z(\mathfrak{s})$ and, hence, semi-invariant under the action of the group

$$H_0 = \exp \mathbb{C}h \cdot Z(\mathfrak{s})^0 = (\mathrm{Str}\, J)^0.$$

Since the H_0-orbit of the unit $1 \in J$ is open, $\langle \nu(1, 1), 1\rangle \neq 0$. At the expense of a suitable choice of the form det on V (defined up to proportionality), one can achieve that $\langle \nu(1, 1), 1\rangle = 3$, i.e. $N(1) = 1$.

PROPOSITION 4.6. *Under the above normalization, the cubic form N is a weighted norm of the algebra J in the sense of Definition* 2.2.

Proof. We are to prove that all the weights of N are positive. Suppose that some weight is equal to 0. Then the structure algebra of the corresponding simple ideal $\mathfrak{a}$ of $\mathfrak{h}$ lies in $\mathfrak{z}(\mathfrak{s})$ and, hence, the very ideal is orthogonal to h. But then

$$([h, \mathfrak{a}], \mathfrak{g}) = (h, [\mathfrak{a}, \mathfrak{g}]) = 0.$$

Therefore, $[h, \mathfrak{a}] = 0$, that is, $\mathfrak{a} \subset \mathfrak{h}_0$, which is impossible. ∎

Thus, the pair (J, N) is a cubic algebra.

Proposition 4.7. *Let the invariant scalar product in* $\mathfrak{g}$ *be normalized by the condition* (13)*. Then its restriction to* $\mathfrak{h}$ *is the invariant scalar product in* $\mathfrak{h}$ *corresponding* (*in the sense of Subsection* 3.2) *to the canonical scalar product in the cubic algebra* J*.*

Proof. First, under the identification of $\mathfrak{h}_0$ with the structure algebra of J, the subalgebra $\mathfrak{z}(\mathfrak{s})$ is identified with the restricted structure algebra. The element $h \in \mathfrak{s}$ is orthogonal to $\mathfrak{z}(\mathfrak{s})$. This means that the restriction to $\mathfrak{h}$ of the invariant scalar product in $\mathfrak{g}$ corresponds to the canonical scalar product in J up to a scalar factor. To prove the proposition, it remains to calculate the scalar square of h in $\mathfrak{g}$ and $\mathfrak{h}$.

In view of (13), $(h,h) = 6$ in $\mathfrak{g}$. On the other hand, by formula (22), with the scalar product (18) in $\mathfrak{sl}_2$ and the canonical scalar product in J, we obtain

$$(h \otimes 1, h \otimes 1) = 2 \cdot 3 = 6. \ ∎$$

4.4. The commutation relations. By formula (28) we obtain the following relation in the algebra $\mathfrak{g}$:

$$[v_1 \otimes x, v_1' \otimes y'] = \frac{1}{3} h \langle x, y' \rangle + \delta(x, y').$$

Comparing the projections of the right hand side of this equality on $\mathbb{C}h$ and on $\mathfrak{z}(\mathfrak{s})$ with those of the right hand side of (25), we obtain

$$\langle x, y' \rangle = (x, y), \quad \delta(x, y') = 2([L_x, L_y] + L_{(xy)_0}),$$

where $(xy)_0$ is the projection of xy on J_0. Thus, the pairing between J and J' defined by (27) coincides with the canonical scalar product in J, and the commutation relation (7) holds.

Besides, we see that

$$(\nu(x,x), x) = 3N(x),$$

so $\nu(x,x) = N(x)x^{-1}$, and the commutation relation (5) holds.

Finally, making use of the identity

$$(a \times b) \times c' = \langle a, c' \rangle b - \langle b, c' \rangle a \quad \text{for} \quad a, b \in V, \ c' \in V^*, \tag{32}$$

one can prove that the Jacobi identity for elements $a \otimes 1$, $b \otimes 1 \in \mathfrak{g}_1 = V \otimes J$ and $c' \otimes 1' \in \mathfrak{g}_{-1} = V^* \otimes J'$ yields $\nu'(1', 1') = 1$. Since both $(\nu(x,x), x)$ and $(\nu'(x', x'), x')$ are $Z(\mathfrak{s})$-invariant cubic forms on J (under the identification of J and J'), it follows that they coincide, and, therefore, the commutation relation (6) holds.

4.5. Classification. In this subsection, we classify all the short SL_3-structures in simple Lie algebras. In particular, we prove the following theorem.

Theorem 4.8. *In any simple Lie algebra but* C_n *there is a short* SL_3*-structure, and such a structure is unique up to conjugation. It is degenerate if and only if* $\mathfrak{g} = A_n$*.*

Recall that a subalgebra of a semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is called *regular* if it is normalized by a maximal torus of G.

Proposition 4.9. *If* $\Phi : \mathrm{SL}_3 \to G = \mathrm{Int}\,\mathfrak{g}$ *is a short* SL_3*-structure in a simple Lie algebra* $\mathfrak{g}$*, then the subalgebra* $\mathfrak{s} = d\Phi(\mathfrak{sl}_3) \subset \mathfrak{g}$ *is regular.*

Proof. The subalgebra $\mathfrak{g}_0 = \mathfrak{s} + \mathfrak{z}(\mathfrak{s})$ is regular, because it is normalized by any maximal torus of G containing the center of $\Phi(\mathrm{SL}_3)$. The subalgebra $\mathfrak{s}$ is regular, because it is an ideal of $\mathfrak{g}_0$ and, hence, is normalized by the same torus. ∎

Consider first the case $\mathfrak{g} = \mathfrak{sl}_n (= A_{n-1})$, $n \geq 3$. It is easy to see that any regular subalgebra of $\mathfrak{sl}_n$ isomorphic to $\mathfrak{sl}_3$ is conjugate to the standard embedding of $\mathfrak{sl}_3$ into $\mathfrak{sl}_n$ as the "left upper corner" of order 3. This defines a short SL_3-structure in $\mathfrak{g}$, whose isotypic components $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are the right upper corner of the size $3 \times (n-3)$ and the left lower corner of the size $(n-3) \times 3$, respectively. Obviously,

$$[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0,$$

so this SL_3-structure is degenerate.

Consider now the case $\mathfrak{g} = \mathfrak{sp}_{2n} (= C_n)$, $n \geq 2$. We will assume that in the standard basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ of $\mathbb{C}^{2n}$ the symplectic form ω is defined by the equations

$$\omega(e_i, e_{n+i}) = -\omega(e_{n+i}, e_i) = 1 \quad \text{for} \quad i = 1, \dots, n,$$
$$\omega(e_i, e_j) = 0 \quad \text{for all other pairs}\,(i,j).$$

Then the matrices of the form $\mathrm{diag}(A, -A^\top)\,(A \in \mathfrak{gl}_n)$ constitute a subalgebra in $\mathfrak{sp}_{2n}$ isomorphic to $\mathfrak{gl}_n$, and it is easy to see that any regular subalgebra of $\mathfrak{sp}_{2n}$ isomorphic to $\mathfrak{sl}_3$ is conjugate to the standard embedding of $\mathfrak{sl}_3$ into $\mathfrak{gl}_n \subset \mathfrak{sp}_{2n}$. (In particular, if $n = 2$, there are no such regular subalgebras.) However, the subalgebra $\mathfrak{gl}_n$ acts on the right upper corner of order n of the algebra $\mathfrak{sp}_{2n}$ as on the symmetric square of the space $\mathbb{C}^n$, which gives a 6-dimensional irreducible component of the adjoint representation of the subalgebra $\mathfrak{sl}_3$ in $\mathfrak{sp}_{2n}$. Thus, the considered SL_3-structure is not short.

In all the other cases we will prove the theorem making use of the $\mathbb{Z}_3$-grading associated with a short SL_3-structure as described in Subsection 4.1. It has the following properties:

(A1) it is defined by an inner automorphism (of order 3);
(A2) the subalgebra $\mathfrak{g}_0$ has an ideal $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_3$;
(A3) all the irreducible components of the representation of $\mathfrak{s}$ in $\mathfrak{g}_1$ are isomorphic 3-dimensional representations.

The automorphisms of finite order of simple Lie algebras are easily determined in terms of Kac diagrams: see, e.g., [GOV]. In particular, the Kac diagram of an inner automorphism of order 3 of a simple Lie algebra $\mathfrak{g}$ of rank n is the extended Dynkin diagram of $\mathfrak{g}$ with non-negative integer labels $l_0, l_1, \dots, l_n$ (where l_0 is the label of the vertex corresponding to the lowest root) such that

(L1) the labels are relatively prime together;
(L2) $\sum_{i=0}^n k_i l_i = 3$, where $k_0, k_1, \dots, k_n$ are the coefficients of the linear dependence between the extended set of simple roots with $k_0 = 1$.

For a given Kac diagram, the Dynkin diagram of the (semisimple part of) the subalgebra $\mathfrak{g}_0$ is the subdiagram of the extended Dynkin diagram of $\mathfrak{g}$ formed by the vertices with zero labels, and the lowest weights of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ are the simple roots of $\mathfrak{g}$ corresponding to the vertices with label 1. (See [GOV, Subsection 3.3.7] or [V1, Section 8] for more details.)

A simple analysis shows that the only Kac diagrams of the automorphisms of order 3 associated with short SL_3-structures in the simple Lie algebras G_2, B_n $(n \geq 3)$, D_n $(n \geq 4)$, F_4, E_6, E_7, E_8 are those presented in Table 2 (where the absence of a label means the zero label). This proves Theorem 4.8.

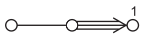
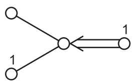
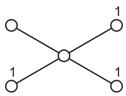
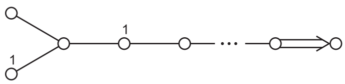
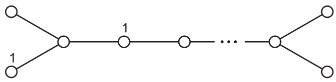
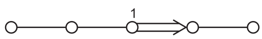
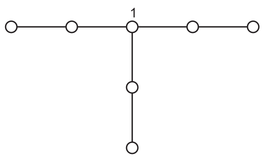
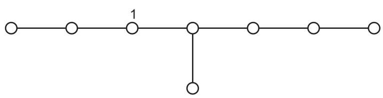
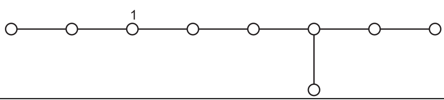
$\mathfrak{g}$	Kac diagram	$\mathfrak{h}$	J	$\dim J$
G_2		A_1	$\mathbb{C}$	1
B_3		$A_1 + A_1$	$\mathbb{C} \oplus \mathbb{C}$	2
D_4		$A_1 + A_1 + A_1$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	3
B_n $(n \geqslant 4)$		$B_{n-2} + A_1$	$Q_{2n-6} \oplus \mathbb{C}$	$2n-4$
D_n $(n \geqslant 5)$		$D_{n-2} + A_1$	$Q_{2n-7} \oplus \mathbb{C}$	$2n-5$
F_4		C_3	$H_3(\mathfrak{A}_0)$	6
E_6		A_5	$H_3(\mathfrak{A}_1)$	9
E_7		D_6	$H_3(\mathfrak{A}_2)$	15
E_8		E_7	$H_3(\mathfrak{A}_3)$	27

Table 2. Non-degenerate short SL_3-structures

In Table 2, we also indicate the stock $\mathfrak{h}$ and the corresponding cubic algebra J. We use the following notation:

Q_n: the *Clifford Jordan algebra* with a basis $\{1, e_1, \ldots, e_n\}$ and the multiplication law $e_i e_j = \delta_{ij}$;

$\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$: the four composition algebras, the complexifications of the real alternative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$;

$H_3(\mathfrak{A}_k)$, $k = 0, 1, 2, 3$: the Jordan algebra of Hermitian matrices of order 3 over $\mathfrak{A}_k$.

Note that the norm of the algebra Q_n is given by the formula

$$N(x_0 + x_1 e_1 + \ldots + x_n e_n) = x_0^2 - x_1^2 - \ldots - x_n^2.$$

The norm of the algebra $H_3(\mathfrak{A}_k)$ is the determinant (which is consistently defined in all these cases).

One can observe that all cubic Jordan algebras occur in Table 2. This means that, for any such algebra J, the algebra $\mathfrak{g} = \mathfrak{g}(J)$ defined by formulas (3)–(8) is a Lie algebra, i.e., satisfies the Jacobi identity. Of course, it is also possible to check the Jacobi identity by direct calculations.

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