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INHOMOGENEOUS KLEINIAN SINGULARITIES AND QUIVERS

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Abstract. The purpose of this article is to generalize a construction by H. Cassens and P. Slodowy of the semiuniversal deformations of the simple singularities of types A_r, D_r, E_6, E_7 and E_8 to the singularities of inhomogeneous types B_r, C_r, F_4 and G_2 defined in 1978 by P. Slodowy. Let Γ be a finite subgroup of SU_2. Then $\mathbb{C}^2/\Gamma$ is a simple singularity of type $\Delta(\Gamma)$. By studying the representation space of a quiver defined from Γ via the McKay correspondence, and a well chosen finite subgroup Γ' of SU_2 containing Γ as normal subgroup, we will use the symmetry group $\Omega = \Gamma'/\Gamma$ of the Dynkin diagram $\Delta(\Gamma)$ and explicitly compute the semiuniversal deformation of the singularity $(\mathbb{C}^2/\Gamma, \Omega)$ of inhomogeneous type. The fibers of this deformation are all equipped with an induced Ω-action. By quotienting we obtain a deformation of a singularity $\mathbb{C}^2/\Gamma'$ with some unexpected fibers.

1. Introduction. In [8] F. Klein showed that if Γ is a finite subgroup of SU_2, then the quotient $\mathbb{C}^2/\Gamma$ is a surface S in $\mathbb{C}^3$ defined by a polynomial equation $R(X, Y, Z) = 0$. The surface has an isolated singularity and is called a Kleinian (or simple) singularity. P. Du Val showed in [5] that the simply-laced Dynkin diagrams can be obtained from the Kleinian singularities. This connection between Lie theory and Kleinian singularities has since been extensively studied, especially by E. Brieskorn and P. Slodowy.

In [10] J. McKay discovered another connection between the finite subgroups of SU_2 and the simply-laced Lie algebras. From this correspondence P. B. Kronheimer constructed in [9] a semiuniversal deformation of $\mathbb{C}^2/\Gamma$ using hyper-Kähler reduction. Then in [4] H. Cassens and P. Slodowy worked on P. B. Kronheimer's results to obtain the semiuniversal deformation and the minimal resolution of $\mathbb{C}^2/\Gamma$ in an algebro-geometric context.

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Dynkin diagrams can be separated in two classes: the simply-laced (or homogeneous) ones, namely A_r, D_r, E_6, E_7 and E_8, and the non-simply-laced (or inhomogeneous) ones B_r, C_r, F_4 and G_2. In his thesis, P. Slodowy extended in 1978 the definition of a simple singularity to the inhomogeneous types by adding a second finite subgroup Γ' of SU_2 containing Γ as normal subgroup. Then $\Gamma'/\Gamma = \Omega$ acts on $\mathbb{C}^2/\Gamma$ and this action can be lifted to the minimal resolution of the singularity and induces an action on the exceptional divisors which corresponds to a group of automorphisms of the Dynkin diagram of $\mathbb{C}^2/\Gamma$. P. Slodowy also generalized the McKay correspondence to the inhomogeneous types (cf. [13]).

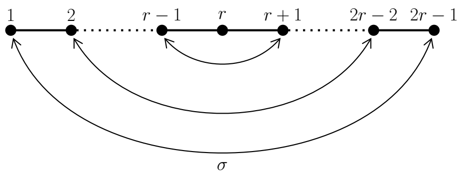
The aim of this article is to generalize the construction by H. Cassens and P. Slodowy [4] to the inhomogeneous cases. In the second section we present the folding of a Dynkin diagram, the definitions of the simple singularities of homogeneous and inhomogeneous types and the McKay correspondence. In the third section we present the construction by H. Cassens and P. Slodowy. The fourth and fifth sections are devoted to the generalization of the construction as well as computations.

Throughout this article the base field is the complex number field $\mathbb{C}$.

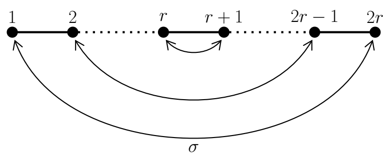
2. Lie theory background

2.1. Folding of a Dynkin diagram. Let $\mathfrak{g}$ be a simple Lie algebra of finite dimension over $\mathbb{C}$ with a root system Φ. Any automorphism σ of the Dynkin diagram of $\mathfrak{g}$ can be extended to a unique outer automorphism $\dot{\sigma}$ of $\mathfrak{g}$. One can verify that the Dynkin diagrams that have a non-trivial outer automorphism group are those of type A_r ($r \geq 2$), D_r ($r \geq 3$) and E_6. It is illustrated below:

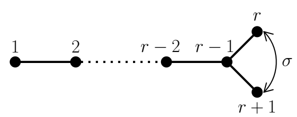
A_{2r-1}:



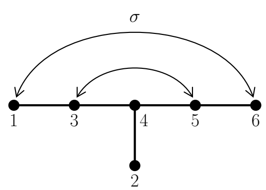
A_{2r}:



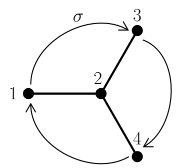
D_{r+1}:



E_6:



D_4:



The *folding of a Dynkin diagram* consists in computing the invariants $\mathfrak{g}_0$ in $\mathfrak{g}$ of the automorphism $\dot{\sigma}$. It is for example studied by V. Kac in [6]. One can also compute the invariants Q^σ of the root lattice Q by the action of σ on its corresponding Dynkin diagram.

We summarize the results we obtained in the following table:

Type of $\mathfrak{g}$	A_{2r-1}	A_{2r}	D_{r+1}	E_6	D_4
Type of $\mathfrak{g}_0$	C_r	B_r	B_r	F_4	G_2
Type of Q^σ	B_r	C_r	C_r	F_4	G_2
Order of σ	2	2	2	2	3

Table 1

One notices that, in all five cases, the types of $\mathfrak{g}_0$ and Q^σ are dual to each other. This is due to the fact that the short roots and the long roots are switched when we go from the Lie algebra to the root lattice.

2.2. Simple singularities and Dynkin diagrams

2.2.1. *Simple singularities of type A_r, D_r, E_6, E_7 and E_8.* Let Γ be a finite subgroup of SU_2. F. Klein showed in [8] that Γ is isomorphic to the cyclic group $\mathcal{C}_n$ of order n, the binary dihedral group $\mathcal{D}_n$ of order $4n$, the binary tetrahedral group $\mathcal{T}$ of order 24, the binary octahedral group $\mathcal{O}$ of order 48 or the binary icosahedral group $\mathcal{I}$ of order 120.

The next theorem is due to F. Klein [8].

THEOREM 2.1. *Let Γ be a finite subgroup of SU_2. Then $\mathbb{C}^2/\Gamma$ injects into $\mathbb{C}^3$ as the zero set of a polynomial $R \in \mathbb{C}[X,Y,Z]$, which presents an isolated singularity. The quotient $\mathbb{C}^2/\Gamma$ is called a* Kleinian (*or* simple) *singularity.*

In the case when $S = \mathbb{C}^2/\Gamma$ is a simple singularity, P. Du Val [5] proved that if s is the singular point and $\pi_0 : \widetilde{S} \to S$ is the minimal resolution of S, then the preimage of s is a union of projective lines whose intersection matrix is the additive inverse of a Cartan matrix of type $\Delta(\Gamma) = A_r$, D_r or E_r. The results by F. Klein and P. Du Val are summarized in the following table:

Γ	R	Type of $\Delta(\Gamma)$
$\mathcal{C}_n$	$X^n + YZ$	A_{n-1}
$\mathcal{D}_n$	$X(Y^2 - X^n) + Z^2$	D_{n+2}
$\mathcal{T}$	$X^4 + Y^3 + Z^2$	E_6
$\mathcal{O}$	$X^3 + XY^3 + Z^2$	E_7
$\mathcal{I}$	$X^5 + Y^3 + Z^2$	E_8

Table 2

2.2.2. *Simple singularities of type B_r, C_r, F_4 and G_2.* The definition of the Kleinian singularities of inhomogeneous types is due to P. Slodowy [13].

DEFINITION 2.2. A *simple singularity* of type B_r $(r \geq 2)$, C_r $(r \geq 3)$, F_4 or G_2 is a pair (X_0, Ω) of a simple singularity X_0 (in the former sense) and a group Ω of automorphisms of X_0 according to the following list:

Type of (X_0, Ω)	Type of X_0	Γ	Γ'	Ω
B_r $(r \geq 2)$	A_{2r-1}	$\mathcal{C}_{2r}$	$\mathcal{D}_r$	$\mathbb{Z}/2\mathbb{Z}$
C_r $(r \geq 3)$	D_{r+1}	$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	$\mathcal{D}_2$	$\mathcal{O}$	$\mathfrak{S}_3$

Table 3

The inhomogeneous type is commonly referred as $\Delta(\Gamma, \Gamma')$.

A simple singularity of inhomogeneous type is then a simple homogeneous singularity with a symmetry of the Dynkin diagram. One notices from Subsection 2.1 that the type of (X_0, Ω) is the same as the type of the folding of a root lattice of the same type as X_0.

Remark 2.3. The type $(A_{2r}, \mathbb{Z}/2\mathbb{Z})$ is the only case that appears in Table 1 but not in Table 3. This is because the action of the symmetry group fails to lift to the exceptional locus of the minimal resolution of X_0.

The notion of symmetry has been added to simple singularities, therefore it is necessary to include this symmetry in the definition of deformations of singularities of type B_r, C_r, F_4 and G_2 ([13]).

Let Δ be a Dynkin diagram of type A_{2r-1}, D_r, or E_6, $\mathfrak{g}$ a Lie algebra of type Δ with adjoint simple group G, $e \in \mathfrak{g}$ a subregular nilpotent element of $\mathfrak{g}$, (e, f, h) an $\mathfrak{sl}_2(\mathbb{C})$-triple of $\mathfrak{g}$ and $S_e = e + \mathfrak{z}_{\mathfrak{g}}(f)$ a *Slodowy slice* at e. Then the restriction of the adjoint quotient $\delta := \chi_{|_{S_e}} : S_e \to \mathfrak{h}/W$ is $\mathrm{Aut}(\Delta)$-equivariant. As a result, there is an action of $\mathrm{Aut}(\Delta)$ on the special fiber $X = \delta^{-1}(0)$. Now let Δ_0 be the unique inhomogeneous Dynkin diagram such that $\mathrm{folding}(\Delta) = \Delta_0$ and $AS(\Delta_0) = \mathrm{Aut}(\Delta)$ with $AS(\Delta_0)$ being the associated symmetry group of Δ_0 defined by

$$AS(\Delta_0) = \begin{cases} \mathfrak{S}_3 & \text{if } \Delta_0 = G_2, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise.} \end{cases}$$

The following results come from [13]:

Theorem 2.4. *$(X, AS(\Delta_0))$ is a simple singularity of type Δ_0.*

Let G_0 denote the simple adjoint group of type Δ_0 with Lie algebra $\mathfrak{g}_0$. Let (e_0, f_0, h_0) be an $\mathfrak{sl}_2(\mathbb{C})$-triple with e_0 a subregular nilpotent element of $\mathfrak{g}_0$ and $S_0 = e_0 + \mathfrak{z}_{\mathfrak{g}_0}(f_0)$. Let $\delta_0 : S_0 \to \mathfrak{h}_0/W_0$ denote the restriction to S_0 of the adjoint quotient map of $\mathfrak{g}_0$.

Theorem 2.5. *The $AS(\Delta_0)$-equivariant deformation $\delta : S_e \to \mathfrak{h}/W$ of X is $AS(\Delta_0)$-semiuniversal, and the restriction $\delta^{AS(\Delta_0)}$ of δ over the fixed point space $(\mathfrak{h}/W)^{AS(\Delta_0)}$ is isomorphic to δ_0.*

Remark 2.6. Theorem 2.5 allows an identification of $\mathfrak{h}_0/W_0$ with $(\mathfrak{h}/W)^{AS(\Delta_0)}$. However P. Slodowy also showed that if $\mathfrak{h}_1 := \mathfrak{h}^{AS(\Delta_0)}$ and $W_1 := \{w \in W \,|\, w\gamma = \gamma w, \forall \gamma \in AS(\Delta_0)\}$, then $\mathfrak{h}_1/W_1 \to (\mathfrak{h}/W)^{AS(\Delta_0)}$ is an isomorphism (cf. Chapters 7, 8 of [13]).

2.3. McKay correspondence

2.3.1. *Homogeneous correspondence.* In 1980, J. McKay noticed in [10] a link between the irreducible representations of the finite subgroups of SU_2 and the extended Dynkin diagrams of types A_r, D_r and E_r.

Let Γ be a finite subgroup of SU_2. As such Γ acts naturally on $V_{\mathrm{nat}} := \mathbb{C}^2$. For every irreducible representation V_i, $0 \leq i \leq r$, of Γ, one has

$$V_{\mathrm{nat}} \otimes V_i = \bigoplus_{j=0}^{r} V_j^{m_{ij}}, \qquad 0 \leq i \leq r,$$

with $m_{ij} \in \mathbb{Z}$, for all $0 \leq i, j \leq r$. J. McKay observed the following:

McKay correspondence. The matrix $2I - M$ with $M = (m_{ij})_{0 \leq i,j \leq r}$ and I the $(r+1) \times (r+1)$ identity matrix is the Cartan matrix of the extended Dynkin diagram $\widetilde{\Delta}(\Gamma)$ associated to Γ.

This correspondence was obtained by J. McKay through explicit computation. R. Steinberg has since proved the result in a more abstract way in [14].

2.3.2. *Inhomogeneous correspondence.* The following theorem is due to P. Slodowy [13].

Theorem 2.7. *Let $\Gamma \lhd \Gamma'$ be a pair of finite subgroups of SU_2 as in the table in Definition 2.2. By restriction, the irreducible representations of Γ' may be regarded as representations of Γ. Let $S_1, \ldots, S_l$ denote the equivalence classes (with respect to Γ) of these representations and let N be the natural representation of Γ as a subgroup of SU_2, which can be seen as the restriction of the natural representation of Γ'. It follows that the tensor product $N \otimes S_i$ decomposes as*

$$N \otimes S_i = \bigoplus_{j=1}^{l} S_j^{b_{ji}}, \qquad 1 \leq i \leq l,$$

which defines an $l \times l$ matrix $B = (b_{ij})_{1 \leq i,j \leq l}$. One can check explicitly that the matrix

$$C = 2I - B$$

is the Cartan matrix of the extended Dynkin diagram $\widetilde{\Delta^{\vee}}(\Gamma, \Gamma')$ of the dual of $\Delta(\Gamma, \Gamma')$.

Remarks 2.8.

1. In the case of D_4, the group $\Gamma' = \mathcal{O}$ can be replaced with the smaller group $\mathcal{T}$ and the theorem remains valid. The difference will be $\Omega = \mathbb{Z}/3\mathbb{Z}$.
2. The preceding theorem is called “by restriction”. A similar construction can be made by inducing representations of Γ' from the irreducible representations of Γ. The Cartan matrix thus obtained is then the transpose of the one obtained by the restriction process.

3. Deformations of homogeneous simple singularities. In [4] H. Cassens and P. Slodowy gave a construction of the semiuniversal deformations of the simple singularities based on quiver theory, P. B. Kronheimer’s work [9] and H. Cassens’ Ph.D. thesis [3]. Their construction is presented in this section.

Let Γ be a finite subgroup of SU_2, R its regular representation and $\Delta(\Gamma)$ the associated Dynkin diagram (cf. Subsection 2.2.1). Using McKay correspondence P. B. Kronheimer proved that $M(\Gamma) := (\operatorname{End}(R) \otimes N)^\Gamma$ is the representation space for a quiver Q whose vertices are the vertices of the extended Dynkin diagram $\widetilde{\Delta}(\Gamma)$, with two arrows (one in each direction) for any edge in $\widetilde{\Delta}(\Gamma)$. It is called a *McKay quiver*. For every arrow $a : i \to j$ of Q, the opposite arrow $j \to i$ is denoted by $\bar{a}$.

The group $G(\Gamma) = (\prod_{i=0}^r \mathrm{GL}_{d_i}(\mathbb{C}))/\mathbb{C}^*$, with $(d_0, \dots, d_r)$ the dimension vector of $M(\Gamma)$, acts on $M(\Gamma)$ by simultaneous conjugation.

By fixing an orientation of Q, i.e. a function $\epsilon : Q_1 \to \mathbb{C}^*$ (Q_1 is the set of arrows of the quiver Q) such that $\epsilon(\overline{a}) = -\epsilon(a) = -1$ for every arrow a belonging to a fixed orientation of the edges of Q and its opposite arrow $\bar{a}$, one is able to define a non-degenerate $G(\Gamma)$-invariant symplectic form $\langle ., . \rangle$ on $M(\Gamma)$ that induces a moment map

$$\mu_{CS} : M(\Gamma) \to (\operatorname{Lie} G(\Gamma))^* \subset \bigoplus_{i=0}^r M_{d_i}(\mathbb{C}).$$

Here $\operatorname{Lie} G(\Gamma)$ is identified with its dual $(\operatorname{Lie} G(\Gamma))^*$.

Let Z be the dual of the center of $\operatorname{Lie} G(\Gamma)$. As the moment map is $G(\Gamma)$-equivariant, for all $z \in Z$, $G(\Gamma)$ acts on the fiber $\mu_{CS}^{-1}(z)$. According to results by G. Kempf and L. Ness [7] and P. B. Kronheimer [9], one obtains:

$$\mu_{CS}^{-1}(Z)//G(\Gamma) \longrightarrow Z$$

is the pullback of the semiuniversal deformation of the Kleinian singularity $\mathbb{C}^2/\Gamma$, where $\mu_{CS}^{-1}(Z)//G(\Gamma)$ signifies the GIT quotient (cf. [11]).

4. Deformations of inhomogeneous simple singularities. This section aims to extend the construction of Section 3 to the inhomogeneous simple singularities of type B_r, C_r, F_4 and G_2.

Let us start with a Dynkin diagram $\Delta(\Gamma)$ of type A_{2r-1}, D_r or E_6 with Γ being the associated finite subgroup of SU_2. The notation and results of Section 3 give the diagram

$$\begin{array}{ccccc} M(\Gamma) \supset \mu_{CS}^{-1}(Z) \twoheadrightarrow \mu_{CS}^{-1}(Z)//G(\Gamma) = & X_\Gamma \times_{\mathfrak{h}/W} \mathfrak{h} & \xrightarrow{\psi} & X_\Gamma \\ & \downarrow{\scriptstyle \widetilde{\alpha}} & \circlearrowleft & \downarrow{\scriptstyle \alpha} \\ & Z \cong \mathfrak{h} & \xrightarrow[\pi]{} & \mathfrak{h}/W \end{array}$$

with α the semiuniversal deformation of the singularity $\mathbb{C}^2/\Gamma$ of type $\Delta(\Gamma)$, $\mathfrak{h}$ a Cartan subalgebra of type $\Delta(\Gamma)$ and W the associated Weyl group. Let Γ' be the finite subgroup of SU_2 such that there exists a simple singularity of inhomogeneous type $\Delta(\Gamma, \Gamma')$ (cf. Definition 2.2). Then $\Omega = \Gamma'/\Gamma$ acts on the singularity $X_{\Gamma,0} = \alpha^{-1}(0)$. Our aim is to define natural actions of Ω on X_Γ and $\mathfrak{h}/W$ such that α becomes Ω-equivariant. Indeed, if α is Ω-equivariant, we obtain the next theorem which is a direct consequence of Theorem 2.5:

Theorem 4.1. *Set $X_{\Gamma,\Omega} := \alpha^{-1}((\mathfrak{h}/W)^\Omega)$ and $\alpha^\Omega := \alpha_{|X_{\Gamma,\Omega}}$. Assume $\alpha : X_\Gamma \to \mathfrak{h}/W$ is Ω-equivariant. Then $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ is the semiuniversal deformation of an inhomogeneous singularity of type $\Delta(\Gamma, \Gamma')$.*

A natural way to accomplish this is to make $\widetilde{\alpha}$ an Ω-equivariant map. One can show that it is the case when the action of Ω on $M(\Gamma)$ is symplectic. The following theorems are proved in [2].

Theorem 4.2.

1. *For the case* $(A_{2r-1}, \mathbb{Z}/2\mathbb{Z})$*, the action of* $\Omega = \Gamma'/\Gamma$ *on* $M(\Gamma)$ *is symplectic when* Ω *reverses the orientation of the McKay quiver.*
2. *For the other cases, the action of* Ω *on* $M(\Gamma)$ *is symplectic when* Ω *preserves the orientation of the McKay quiver.*

Theorem 4.3. *For any McKay quiver built on a Dynkin diagram of type* A_{2r-1}*,* D_{r+1} *or* E_6*, there exists an action of* $\Omega = \Gamma'/\Gamma$ *on* $M(\Gamma)$ *that is both symplectic and induces the natural action* (*of Theorem* 2.1) *on the singularity* $\mathbb{C}^2/\Gamma$*.*

Using K. Saito's flat coordinates [12] on $\mathfrak{h}/W$, which make the action of Ω linear on $\mathfrak{h}/W$, we are able to compute explicitly the semiuniversal deformations of the simple singularities of inhomogeneous types B_r $(r \geq 2)$, C_3, F_4 and G_2. The explicit expressions can be found in [2].

5. Quotients of the deformations of inhomogeneous types. It was shown in the previous section that the morphism $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ is Ω-invariant. Hence Ω acts on each fiber of α^Ω and the fibers can be quotiented. It is known that $(\alpha^\Omega)^{-1}(\overline{0}) = X_{\Gamma,0} = \mathbb{C}^2/\Gamma$. Hence the fiber above the origin of the quotient map is also a Kleinian singularity (see Theorem 4.3). Indeed, $(\alpha^\Omega)^{-1}(\overline{0})/\Omega = X_{\Gamma,0}/\Omega \cong (\mathbb{C}^2/\Gamma)/(\Gamma'/\Gamma) \cong \mathbb{C}^2/\Gamma'$. As Γ' is a finite subgroup of SU_2, $\mathbb{C}^2/\Gamma'$ is a Kleinian singularity. Therefore the family given by the quotient map $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ is a deformation of the simple singularity $\mathbb{C}^2/\Gamma'$.

In [2] we computed the explicit expression of $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ for the types B_r $(r \geq 2)$, C_3, F_4 and G_2. The results are as follows:

Type of α^Ω	Type of $\overline{\alpha^\Omega}$	Rank of $\overline{\alpha^\Omega}$
B_r	D_{r+2}	r
C_3	D_6	3
F_4	E_7	4
G_2	E_7	2

Table 4

Because of a theorem by E. Brieskorn [1], it is known that the semiuniversal deformation of a simple singularity of type X_r $(X = A$, D or $E)$ is of rank r. But one can see that it is not the case for $\overline{\alpha^\Omega}$. It follows that $\overline{\alpha^\Omega}$ is not a semiuniversal deformation in any of the cases.

Studying the discriminant of $\overline{\alpha^\Omega}$ gives unexpected results for the types C_3 and G_2 ([2]):

Proposition 5.1.

1. *When* α^Ω *is the semiuniversal deformation of a singularity of type* C_3*, every fiber of the family* $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ *is singular.*
2. *When* α^Ω *is the semiuniversal deformation of a singularity of type* G_2*, every fiber of the family* $\overline{\alpha^\Omega} : X_{\Gamma,\Omega}/\Omega \to (\mathfrak{h}/W)^\Omega$ *is singular.*

Consider the diagram with S_e a Slodowy slice to a subregular nilpotent element e of the simply-laced simple Lie algebra $\mathfrak{g}$ with root system Φ, χ the adjoint quotient of $\mathfrak{g}$, the reflection hyperplanes H_α with respect to the roots $\alpha \in \Phi$ and the discriminant $\mathbb{D}$ of χ. P. Slodowy proved in [13] that the type of singularities that appear in S_e above a point $\pi(h) \in \mathbb{D}$ is given by the sub-root-system $\{\alpha \in \Phi \,|\, h \in H_\alpha\}$. It might be interesting to see whether the singularities in the fibers of $\alpha^\Omega : X_{\Gamma,\Omega} \to (\mathfrak{h}/W)^\Omega$ can be described in a similar manner using the morphism $\pi_1 : \mathfrak{h}_1 \to \mathfrak{h}_1/W_1 \xrightarrow{\cong} (\mathfrak{h}/W)^\Omega$ from Remark 2.6.

$$\begin{array}{ccc} & & S_e \\ & & \big\downarrow \chi|_{S_e} \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W \\ \bigcup_{\alpha\in\Phi^+} H_\alpha & \mapsto & \mathbb{D} \end{array}$$

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